Homework set 3, Due Fri Nov 23

1. The coplanar molecule boron trifluoride BF_3 has 3 flourine atomes at the vertices of equilateral triangle and the boron atom lies at the center, thus having D_{3h} symmetry. Classify the normal modes of vibration and point out any degeneracies.

2. Consider a crystal with D_4 symmetry. Discuss the splitting of atomic levels with $l = 1, 2, 3, 4$.

3. For any 4-vector $x^\mu = (t, x, y, z)$ define a 2×2 hermitian matrix by

$$h = t + \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli matrices.

   (a) Compute the determinant of h

   (b) Let U be any 2×2 matrix with $\det U = 1$ and define a new matrix by

$$h' = U h U^\dagger$$

   Show that h' is hermitian and $\det h' = \det h$.

   (c) If we write $h' = t' + \vec{\sigma} \cdot \vec{r}\,'$, show that the relation between $x^\mu = (t, x, y, z)$ and $x'^\mu = (t', x', y', z')$ is a Lorentz transformation.

   (d) Show that if

$$U = \begin{pmatrix} e^\omega & 0 \\ 0 & e^{-\omega} \end{pmatrix}$$

   this will induce a Lorentz boost in $z-$direction.

4. The Hamiltonian for a 2-dimensional isotropic harmonic oscillator is of the form,

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2} m\omega^2 \left(x_1^2 + x_2^2\right)$$

   (a) Find the energy eigenvalues and degeneracies.

   (b) Introduce creation and destruction operators a and $a^\dagger$ to write the Hamiltonian as

$$H = \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

   Show that H is invariant under the following transformation,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = U \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

   where U is any 2×2 unitary matrix.

   (c) The commutation relations between creation and destruction operators are of the usual form,

$$\left[a_i^\dagger, a_j\right] = -\delta_{ij}, \qquad [a_i, a_j] = 0, \qquad \left[a_i^\dagger, a_j^\dagger\right] = 0$$

   Define

$$J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad J_+ = a_1^\dagger a_2, \qquad J_- = (J_+)^\dagger$$

   Show that they satisfy the angular momentum commutation relations,

$$[J_\pm, J_3] = \mp J_\pm, \qquad [J_+, J_-] = 2J_3$$