Group Theory in Physics
Note 2 Theory of Representation

October 26, 2012

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1 Group Representation

In physics application, the group representation are very useful in deducing the consequence of the symmetries of the system. Roughly speaking, representation of a group is just some way to realize the same group operation other than the original definition of the group. Of particular interest to most phyical application is the realization of group operation by the matrices whose multiplication operation can be naturally associated with group multiplication. As usual the matrices can be associated with linear operators acting on vector space.

1.1 Definition of Representation

Given a group G, if for each $A_i \in G$, there is $n \times n$ matrice $D(A_i), i = 1, 2, \ldots$ such that

$$D(A_i) D(A_j) = D(A_i A_j) \tag{1}$$

then D's forms a n-dimensional representation of the group G. In other words, the correspondence $A_i \to D(A_i)$ is a homomorphism. The condition in Eq(1) means that the matrices $D(A_i)$ satisfy the same multiplication law as the group elements. If this homomorphism turns out to be an isomorphism $(1-1)$ then the representation iscalled faithful. Note that a matrix M_{ij} can be viewed as linear operator M acting on some vector space V with repect to some choice of basis $\{e_i\}$,

$$Me_i = \sum_j e_j M_{ji}$$

Note that the choice of the basis is not unique. If we make a change to a new basis,

$$e_i = \sum_j f_n S_{ni}, \qquad S \quad \text{non-singular}$$

then

$$\sum_n M f_n S_{ni} = \sum_{k,j} f_k S_{kj} M_{ji}$$

Multiply by S^{-1}_{im},

$$M f_m = \sum_{k,j,i} f_k S_{kj} M_{ji} S^{-1}_{im} = \sum_k f_k \left(SMS^{-1}\right)_{km} = \sum_k f_k \left(M'\right)_{km}$$

where

$$M' = SMS^{-1}$$

Thus under the change of basis, the matrix M is changed by a similarity transformation.

One way to generate such matrices for the symmetry of certain geometric objects is to use the group induced transformations, discussed before. Recall that each group element A_a will induce a transformation of the coordinate vector $\vec{r}$,

$$\vec{r} \to A_a \vec{r}$$

Then for any function of $\vec{r}$, say $\varphi(\vec{r})$ and for any group element A_a define a new transformation P_{A_a} by

$$P_{A_a}\varphi(\vec{r}) = \varphi\left(A_a^{-1}\vec{r}\right)$$

Among the functions obtained, $P_{A_1}\varphi(\vec{r})$, $P_{A_2}\varphi(\vec{r})$, $\cdots P_{An}\varphi(\vec{r})$, we select the linearly independent set $\varphi_1(\vec{r})$, $\varphi_2(\vec{r})\cdots\varphi_\ell(\vec{r})$.Then it is clear that $P_A\varphi_a$ can be expressed as linear combination of φ_i. This is because

$$P_{A_b}\varphi_a = P_{A_b}P_{A_a}\varphi(\vec{r}) = P_{A_b}\varphi\left(A_a^{-1}\vec{r}\right) = \varphi\left(A_a^{-1}\left(A_b^{-1}\vec{r}\right)\right) = \varphi\left(\left(A_bA_a\right)^{-1}\vec{r}\right)$$

Thus we can write

$$P_{A_i}\varphi_a = \sum_{b=1}^{\ell}\varphi_b D_{ba}(A_i)$$

and $D_{ba}(A_i)$ forms a representation of G. This can be seen as follows.

$$P_{A_iA_j}\varphi_a = P_{A_i}P_{A_j}\varphi_a = P_{A_i}\sum_b \phi_b D_{ba}(A_j) = \sum_{b.c.}\phi_c D_{cb}(A_i)D_{ba}(A_j)$$

On the other hand,

$$P_{A_iA_j}\varphi_a = \sum_c \varphi_c D\left(A_iA_j\right)_{ca}$$

This gives

$$D\left(A_iA_j\right)_{ca} = D_{cb}(A_i)D_{ba}(A_j)$$

which means that $D(A_i)'s$ form representation of the group.

Example: Group D_3, symmetry of the triangle.

As we have discussed before, choosing a coordinate system on the plane, we can represent the group elements by the following matrices,

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Choose $f(\vec{r}) = f(x,y) = x^2 - y^2$, we get

$$P_A f(\vec{r}) = f\left(A^{-1}\vec{r}\right) = \frac{1}{4}\left(x + \sqrt{3}y\right)^2 - \frac{1}{4}\left(\sqrt{3x} - y\right)^2 = -\frac{1}{2}\left(x^2 - y^2\right) + \sqrt{3}xy.$$

We now have a new function $g(x,y) = 2xy$. We can operate on $g(r)$ to get,

$$P_A g(\vec{r}) = g\left(A^{-1}\vec{r}\right) = 2\left(-\frac{1}{2}\right)\left(x + \sqrt{3}y\right)\frac{1}{2}\left(\sqrt{3x} - y\right) = -\frac{1}{2}\left[\left(\sqrt{3}\right)\left(x^2 - y^2\right) - 2xy\right] = -\frac{\sqrt{3}}{2}\left(x^2 - y^2\right) - \frac{1}{2}(2xy)$$

The linearly independent set of induced functions consists of $f = x^2 - y^2$, $g = \frac{1}{2}xy$.Using f, g as basis, we get

$$P_A(f,g) = (f,g)\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

The matrix generated this way is the same as A if we change g to -g.

Similarly

$$\begin{aligned} P_B f(\vec{r}) &= f\left(B^{-1}\vec{r}\right) = \frac{1}{4}\left(x - \sqrt{3}y\right)^2 - \frac{1}{4}\left(\sqrt{3}x + y\right)^2 = -\frac{1}{2}\left(x^2 - y^2\right) - \sqrt{3}xy \\ P_B g(\vec{r}) &= g\left(B^{-1}\vec{r}\right) = 2\left(\frac{1}{2}\right)\left(x - \sqrt{3}y\right)\left(-\frac{1}{2}\right)\left(\sqrt{3}x + y\right) = \sqrt[\dagger]{\frac{1}{2}}\left[\sqrt[\dagger]{3}\left(x^2 - y^2\right) - 2xy\right] = \frac{\sqrt{3}}{2}\left(x^2 - y^2\right) - \frac{1}{2}(2xy) \end{aligned}$$

$$P_B(f,g) = (f,g)\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

same as B given before.

Remarks

1. If $D^{(1)}(A)$ and $D^{(2)}(A)$ are both representation of the group, then

$$D^{(3)}(A) = \begin{pmatrix} D^{(1)}(A) & 0 \\ 0 & D^{(2)}(A) \end{pmatrix} \qquad \text{(block diagonal form)}$$

   also forms a representation. We will denote it as a direct sum $\oplus$,

$$D^{(3)}(A) = D^{(1)}(A) \oplus D^{(2)}(A) \qquad \text{direct sum}$$

2. If $D^{(1)}(A)$ and $D^{(2)}(A)$ are 2 representations of G with same dimension and there exists a square matrix U such that

$$D^{(1)}(A_i) = UD^{(2)}(A_i)U^{-1} \qquad \text{for all } A_i \in G.$$

   then $D^{(1)}$ and $D^{(2)}$ are said to be equivalent representations . Recall that if we change the basis used to represent the linear operator, the corresponding matrix undergo similar transformation. Since they represent the same operator, we consider them the "same" representation but with respect to different choice of basis.

2 Reducible and Irreducible Representations

A representation D of a group G is called **irreducible representation** (irrep) if it is defined on a vector space $V(D)$ which has no non-trivial invariant subspace. Otherwise, it is reducible. In essence this definition simply means that for a reducible representation, the linear opertors correponding to the group elements will leave some smaller vector space invariant. In other words, all the group actions can be realized in this subspace.

We need to convert these statment into more practical criterion. Suppose the representaion D is reducible on the vector space V. Then there exists a subspace S which is invariant under D. For any vector $v \in V$, we can decompos it as,

$$v = s + s_\perp$$

where $s \in S$ and $s_\perp$ belongs to the complement $S_\perp$ of S. If we write the vector v in the block form,

$$v = \begin{pmatrix} s \\ s_\perp \end{pmatrix}$$

then the representation matrix can be written as

$$Av = D(A)v = \begin{pmatrix} D_1(A) & D_2(A) \\ D_3(A) & D_4(A) \end{pmatrix}\begin{pmatrix} s \\ s_\perp \end{pmatrix}$$

For the space S to be invariant under group operators means that

$$D_3(A_i) = 0, \qquad \forall\ A_i \in G$$

i.e. the matrices $D(A_i)$ are all of the upper triangular form,

$$D(A_i) = \begin{pmatrix} D_1(A_i) & D_2(A_i) \\ 0 & D_4(A_i) \end{pmatrix}, \qquad \forall\ A_i \in G \tag{2}$$

Note that multiplication of upper triangular matrices gives again upper triangular matrix,

$$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} \begin{pmatrix} D_1' & D_2' \\ 0 & D_4' \end{pmatrix} = \begin{pmatrix} D_1 D_1' & D_1 D_2' + D_2 D_4' \\ 0 & D_4 D_4' \end{pmatrix}$$

This just reflects the fact the invariant subspace is still invariant if you operate on it many times.

A representation is **completely reducible** if all the matrices in the representations $D(A_i)$ can be simultaneously brought into block diagonal form by the same similarity transformation U,

$$UD(A_i)U^{-1} = \begin{pmatrix} D_1(A_i) & 0 \\ 0 & D_2(A_i) \end{pmatrix}, \qquad \text{for all } A_i \in G$$

i.e. $D_2(A_i) = 0$ in the upper triangular matrices given in Eq (2). In other words, the space complement to S is also invariant under the group operation. This will be the case if the represnetation matrices are unitary as stated in the following theorem;

Theorem: Any **unitary** reducible representation is completely reducible.

Proof: For simplicity we assume that the vector space V is equipped with a scalar product (u, v). In this case we can choose the complement space $S\perp$ to be perpendicular to S, i.e.

$$(u, v) = 0, \qquad \text{if } \ u \in S, v \in S_\perp$$

Recall that the scalar product is invariant under the unitary transformation,

$$0 = (u, v) = (D(A_i)u, D(A_i)v)$$

Thus if $D(A_i)u \in S$, then $D(A_i)v \in S_\perp$ which implies that $S_\perp$ is also invariant under the group operation.■

In physical applications, we deal mostly with unitary representations and they are completely reducible.

2.1 Unitary Representation

Since unitary operators preserve the scalar product of a vector space, representation by unitary matrices will simplify the analysis of group theory. In the realm of finite groups, we can always transform the representation into unitay one. This is the content of the following theorem.

<u>**Fundamental Theorem**</u>

Every irrep of a finite group is equivalent to a unitary irrep (rep by unitary matrices)

<u>Proof</u>:

Let $D(A_r)$ be a representation of the group $G = \{E, A_2 \cdots A_n\}$

Consider the sum

$$H = \sum_{r=1}^{n} D(A_r) D^\dagger(A_r) \qquad \text{then } H^\dagger = H$$

Since H is positive semidefinite, we can define squre root h by

$$h^2 = H, \quad h^\dagger = h$$

This can be achieved by diagonizing this hermitian matrix H by unitary transformation and then transforming it back after taking the square root of the eigvalues. Define new set of matrices by

$$\bar{D}(A_r) = h^{-1} D(A_r) h \qquad r = 1, 2, \cdots, n$$

Since this is a similarity transformation, $\bar{D}(A_r)$ also forms a rep equivalent to $D(A_r)$. We will now show that $\bar{D}(A_r)$ is unitary,

$$\begin{aligned} \bar{D}(A_r)\bar{D}^\dagger(A_r) &= \left[h^{-1}D(A_r)h\right]\left[hD^\dagger(A_r)h^{-1}\right] = h^{-1}D(A_r) \sum_{s=1}^{n}\left[D(A_S)D^\dagger(A_S)\right]D^\dagger(A_r)h^{-1} \\ &= h^{-1}\left[\sum_{s=1}^{n} D(A_r A_s) D^\dagger(A_r A_s)\right]h^{-1} = h^{-1}\sum_{s'=1}^{n} D(A_{S'})D^\dagger(A_{s'})h^{-1} = h^{-1}h^2h^{-1} = 1 \end{aligned}$$

where we have used the rearrangement theorem.■

2.2 Schur's Lemma

One of the most important theorems in the study of the irreducible reprentation is the following lemma.

Schur's Lemma

(i) Any matrix which commutes with all matrices of irrep is a multiple of identity matrix.

Proof: Assume $\exists\, M$ such that

$$MD(A_r) = D(A_r)M \qquad \forall\ A_r \in G$$

and the hermitian conjugate is

$$D^\dagger(A_r)M^\dagger = M^\dagger D^\dagger(A_r)$$

As shown above, $D(A_r)$ can be taken to be unitary, then

$$M^\dagger = D(A_r)M^\dagger D^\dagger(A_r) \quad \text{or} \quad M^\dagger D(A_r) = D(A_r)M^\dagger$$

Thus $M^\dagger$ also commutes with all D's and so are the combination $M+M^\dagger$ and $i\left(M - M^\dagger\right)$, which are hermitian. We only have to consider the case where M is hermitian. Start by diagonalizing M by unitary matrix U,

$$M = UdU^\dagger \qquad\qquad d: \text{diagonal}$$

Define $\bar{D}(A_r) = U^\dagger D(A_r)U$, then

$$d\bar{D}(A_r) = \bar{D}(A_r)d$$

or in terms of matrix elements,

$$\sum_\beta d_{\alpha\beta}\bar{D}_{\beta r}(A_s) = \sum_\beta \bar{D}_{\alpha\beta}(A_s)d_{\beta\gamma}$$

Since d is diagonal, we get

$$(d_{\alpha\alpha} - d_{\gamma\gamma})\bar{D}_{\alpha\gamma}(A_s) = 0 \Longrightarrow \qquad \text{if } d_{\alpha\alpha} \neq d_{\gamma\gamma}, \text{ then } \bar{D}_{\alpha\gamma}(A_s) = 0$$

This means if diagonal elements d_{ii} are all different, then the off-diagonal elements of $\bar{D}$ are all zero. The only possible non-zero off-diagonal elements of $\bar{D}$ can arise when some of $d'_{\alpha\alpha}s$ are equal. For example, if $d_{11} = d_{22}$, then $\bar{D}_{12}$ can be non-zero. Thus $\bar{D}$ will be in the block diagonal form, i.e.

$$\text{if } d = \begin{bmatrix} d_1 & & & & & & & \\ & d_1 & & & & & & \\ & & \ddots & & & & & \\ & & & d_1 & & & & \\ & & & & d_2 & & & \\ & & & & & \ddots & & \\ & & & & & & d_2 & \\ & & & & & & & \ddots \end{bmatrix} \quad \text{then} \quad \bar{D} = \begin{bmatrix} \boxed{D_1} & 0 & \\ 0 & \boxed{D_2} & \\ & & \ddots \end{bmatrix}$$

This is true for every matrix in the representation. Thus all the matrices in the representation are in the block diagonal form. But D is irreducible and not all matrices can be brought into block diagonal form. Thus all d_i's have to be equal

$$d = cI. \qquad \text{or} \qquad M = UdU^\dagger = dUU^\dagger = d = cI \qquad \blacksquare$$

(ii) If the only matrix that commutes with all the matrices of a representation is a multiple of identity, then the representation is irrep.

Proof: Suppose D is reducible, then we can transform them into

$$D(A_i) = \begin{bmatrix} D^{(1)}(A_i) & \\ & D^{(2)}(A_i) \end{bmatrix} \qquad \text{for all } A_i \in G$$

construct $M = \begin{bmatrix} I & 0 \\ 0 & 2I \end{bmatrix}$ then clearly

$$D(A_i)M = MD(A_i) \qquad \text{for all } i$$

But M is not a multiple of identity (contradiction). Therefore D must be irreducible.

Remarks

1. Any irrep of Abelian group is 1−dimensional. This is because for any element $A, D(A)$ commutes with all $D(A_i)$. Then Schur's lemma $\Longrightarrow D(A) = cI \quad \forall A \in G$. But D is irrep, so D has to be 1×1 matrix.

2. In any irrep, the identity element E is always represented by identity matrix. This follows Schur's lemma.

3. From $D(A) D(A^{-1}) = D(E) = I.$, we see that $D(A^{-1}) = [D(A)]^{-1}$and for unitary representation $D(A^{-1}) = D^{\dagger}(A)$.

(iii) If $D^{(1)}$ and $D^{(2)}$ are irreps of dimensions l_1, l_2 and

$$MD^{(1)}(A_i) = D^{(2)}(A_i) M. \tag{3}$$

then (a) if $l_1 \neq l_2 \quad M = 0$

(b) if $l_1 = l_2$, then either $M = 0$ or $\det M \neq 0$ and reps are equivalent.

Proof: : Without loss of generality we can take $l_1 \leq l_2$. Hermitian conjugate of Eq(3) gives

$$D^{(1)\dagger} M^{\dagger} = M^{\dagger} D^{(2)\dagger}$$

then

$$MM^{\dagger} D^{(2)}(A_i)^{\dagger} = MD^{(1)}(A_i)^{\dagger} M^{\dagger} = D^{(2)}(A_i)^{\dagger} MM^{\dagger}$$

Or

$$\left(MM^{\dagger}\right) D^{(2)}(A_i) = D^{(2)}(A_i)\left(MM^{\dagger}\right) \qquad \forall (A_i) \in G$$

Then from Schur's lemma (i) we get $MM^{\dagger} = cI$,where I is a l_2−dimensional identity matrix.

First consider the case $l_1 = l_2$, where we get $|\det M|^2 = c^{\ell_1}$. Then either $\det M \neq 0$,which implies M is non-singular and from Eq(3)

$$D^{(1)}(A_i) = M^{-1} D^{(2)}(A_i) M \quad \forall (A_i) \in G$$

This means $D^{(1)}(A_i)$ and $D^{(2)}(A_i)$ are equivalent. Otherwise, if the determinant is zero,

$$\det M = 0 \quad \Longrightarrow c = 0 \quad \text{or } MM^{\dagger} = 0 \quad \Longrightarrow \quad \sum_{\gamma} M_{\alpha\gamma} M^{*}_{\beta\gamma} = 0 \quad \forall \alpha.\beta.$$

In particular, for $\alpha = \beta \quad \sum_{\gamma} |M_{\alpha\gamma}|^2 = 0 \quad M_{\alpha\gamma} = 0$ for all $\alpha.\gamma \Longrightarrow M = 0$.

Next, if $l_1 < l_2$, then M is a rectangular $l_2 \times l_1$, matrix

$$M = \underbrace{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}}_{l_1} l_2$$

We can define a square matrix by adding colums of zeros

$$N = \overbrace{[M, 0]}^{l_2}\} l_2 \cdot \qquad\qquad l_2 \times l_2 \text{ square matrix}$$

then

$$N^{\dagger} = \begin{pmatrix} M^{\dagger} \\ 0 \end{pmatrix} \quad \text{and} \quad NN^{\dagger} = (M, 0)\begin{pmatrix} M^{\dagger} \\ 0 \end{pmatrix} = MM^{\dagger} = cI$$

where I is the $l_2 \times l_2$ identity matrix. But from construction we see that $\det N = 0$, Hence $c = 0, \Longrightarrow NN^{\dagger} = 0$ or $M = 0$ identically. ■

3 Great Orthogonality Theorem

The most important theorem for the representation of the finite group is the following one.

Theorem (Great orthogonality theorem): Suppose G is a group with n elements, $\{A_i, i = 1, 2, \cdots n\}$, and $D^{(\alpha)}(A_i)$, $\alpha = 1, 2 \cdots$ are all the inequivalent irreps of G with dimension l_α.

Then

$$\sum_{\alpha=1}^{n} D^{(\alpha)}_{ij}(A_\alpha) D^{(\beta)*}_{k\ell}(A_\alpha) = \frac{n}{l_\alpha} \delta_{\alpha\beta} \delta_{ik} \delta_{j\ell}$$

Proof: Define

$$M=\sum_a D^{(\alpha)}\left(A_a\right)XD^{(\beta)}\left(A_a^{-1}\right)$$

where X is an arbitrary $l_\alpha\times l_\beta$ matrix. Then multiplying M by representation matrices, we get

$$\begin{aligned}D^{(\alpha)}\left(A_b\right)M &= D^{(\alpha)}\left(A_b\right)\sum_a D^{(\alpha)}\left(A_a\right)XD^{(\beta)}\left(A_a^{-1}\right)\left[D^{(\beta)}\left(A_b^{-1}\right)D^{(\beta)}\left(A_b\right)\right]\\ &= \sum_a D^{(\alpha)}\left(A_bA_a\right)XD^{(\beta)}\left(\left(A_bA_a\right)^{-1}\right)D^{(\beta)}\left(A_b\right)=MD^{(\beta)}\left(A_b\right)\end{aligned}$$

(i) If $\alpha\neq\beta$, then $M=0$ from Schur's lemma, we get

$$M=\sum_a D_{ir}^{(\alpha)}\left(A_a\right)X_{rs}D_{sk}^{(\beta)}\left(A_a^{-1}\right)=\sum_a D_{ir}^{(\alpha)}\left(A_a\right)X_{rs}D_{ks}^{(\beta)*}\left(A_a\right)=0$$

Choose $X_{rs}=\delta_{rj}\delta_{sl}$ (i.e. X is zero except the jl element). Then we have

$$\sum_a D_{ij}^{(\alpha)}\left(A_\alpha\right)D_{k\ell}^{(\beta)*}\left(A_\alpha\right)=0$$

This shows that for different irreducible representations, the matrix elements, after summing over group elements, are orthogonal to each other.

(ii) $\alpha=\beta$ then we can write $M=\sum_a D^{(\alpha)}\left(A_a\right)XD^{(\alpha)}\left(A_a^{-1}\right)$.This implies

$$D^{(\alpha)}\left(A_a\right)M=MD^{(\alpha)}\left(A_b\right)\qquad\Longrightarrow\qquad M=cI$$

Then

$$\sum_a T_r\left[D^{(\alpha)}\left(A_a\right)XD^{(\alpha)}\left(A_a^{-1}\right)\right]=cl_2\quad\text{or } nT_rX=cl_2,\quad\text{or } c=\frac{\left(T_rX\right)n}{l_\alpha}$$

Take $X_{rs}=\delta_{rj}\delta_{s\ell}$ then $T_rX=\delta_{j\ell}$ and

$$\sum_a D^{(\alpha)}\left(A_a\right)_{ij}D^{(\alpha)}\left(A_a\right)_{k\ell}^*=\frac{n}{l_\alpha}\delta_{ik}\delta_{j\ell}$$

This gives the orthogonality for different matrix elements within a given irreducible representation. ■

Geometric Interpretation

Imagine a complex $n-$dimensional vector space in which axes (or componenets) are labeled by group elements $E,A_2.A_3...A_n$(Group element space). Consider the vector in this space with componets made out of the matrix element of irreducible representation matrix $D^{(\alpha)}\left(A_a\right)_{ij}$.Each vector in this $n-$dimensionl space is labeled by 3 indices, $i,\mu.\nu$

$$\vec{D}_{\mu\nu}^{(i)}=\left(D_{\mu\nu}^{(i)}\left(E\right),D_{\mu\nu}^{(i)}\left(A_2\right),\cdots D_{\mu\nu}^{(i)}\left(A_n\right)\right)$$

Great orthogonality theorem says thtat all these vectors are $\perp$ to each other. As a result

$$\sum_i l_i^2\leq n$$

because there can be no more than n mutually $\perp$ vectors in n-dimensional vector space.

As an example, we take the 2-dimensional representation we have work out before,

$$E=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad A=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix},\qquad B=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}\tag{4}$$

$$K=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\qquad L=\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix},\qquad M=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}\tag{5}$$

Label the axises by the groupl elements in the order (E,A,B,K,L,M). Then we can construct four 6-dimensional ectors from these 2×2 matrices,

$$\begin{array}{lllllll}
D_{11}^{(2)} & =(1 & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-1 & ,\frac{1}{2} & ,\frac{1}{2}) \\
D_{12}^{(2)} & =(0 & ,\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & ,0 & ,-\frac{\sqrt{3}}{2} & ,\frac{\sqrt{3}}{2}) \\
D_{21}^{(2)} & =(0 & ,-\frac{\sqrt{3}}{2} & ,\frac{\sqrt{3}}{2} & ,0 & -\frac{\sqrt{3}}{2} & ,\frac{\sqrt{3}}{2}) \\
D_{21}^{(2)} & =(1 & ,-\frac{1}{2} & ,-\frac{1}{2} & ,1 & ,-\frac{1}{2} & ,-\frac{1}{2})
\end{array}$$

It is straightforward to check that these 4 vectors are perpendicular to each other.

Note that the other two vectors which are orthogonal to these vectors are of the form,

$$\begin{array}{lllllll}
D_E & =(1 & ,1 & ,1 & ,1 & ,1 & ,1) \\
D_A & =(1 & ,1 & ,1 & ,-1 & ,-1 & ,-1)
\end{array} \tag{6}$$

coming from the identity representation and other 1-dimensional representation.

4 Character of Representation

The matrices in irrep are not unique, becuase we can generate another equivalent irrep by similairty transformation. However, the trace of a matrix is invariant under similarity transformation,

$$Tr\left(SAS^{-1}\right)=TrA$$

We can use the trace, or character, to characterize the irrep.

$$\chi^{(\alpha)}\left(A_i\right)\equiv T_r\left[D^{(\alpha)}\left(A_i\right)\right]=\sum_a D_{aa}^{(\alpha)}\left(A_i\right)$$

Useful Properties

1. If $D^{(\alpha)}$ and $D^{(\beta)}$ are equivalent, then

$$\chi^{(\alpha)}\left(A_i\right)=\chi^{(\beta)}\left(A_i\right) \qquad \forall A_i\in G$$

2. If A and B are in the same class, then

$$\chi^{(\alpha)}(A)=\chi^{(\alpha)}(B)$$

Proof: If A and B are in same class $\Longrightarrow \exists x\in G$ such that $xAx^{-1}=B\Longrightarrow D^{(\alpha)}(x)\,D^{(\alpha)}(A)\,D^{(\alpha)}\left(x^{-1}\right)=D^{(\alpha)}(B)$

Using

$$D^{(\alpha)}\left(x^{-1}\right)=D^{(\alpha)}(x)^{-1}$$

we get

$$T_r\left[D^{(\alpha)}(x)\,D^{(\alpha)}(A)\,D^{(\alpha)}(x)^{-1}\right]=T_r\left[D^{(\alpha)}(B)\right] \quad \text{or} \quad \chi^{(\alpha)}(A)=\chi^{(\alpha)}(B)$$

Hence $\chi^{(\alpha)}$ is a function of class, not of each element

3. Denote $\chi_i=\chi\left(\mathcal{C}_i\right)$, the character of ith class. Let n_c be the number of classes in G, and n_i the number of group elements in class $\mathcal{C}_i$.

From great orthogonality theorem

$$\sum_r D_{ij}^{(\alpha)}\left(A_r\right)D_{k\ell}^{(\beta)*}\left(A_r\right)=\frac{n}{l_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}$$

we get

$$\sum_r \chi^{(\alpha)}\left(A_r\right)\chi^{(\beta)*}\left(A_r\right)=\frac{n}{l_\alpha}\cdot\delta_{\alpha\beta}l_\alpha=n\delta_{\alpha\beta}$$

or

$$\sum_i n_i\chi^{(\alpha)}\left(\mathcal{C}_i\right)\chi^{(\beta)*}\left(\mathcal{C}_i\right)=n\delta_{\alpha\beta} \tag{7}$$

This is the great orthogonality theorem for the characters.

Define $\quad U_{\alpha i}=\sqrt{\frac{n_i}{n}}\chi^{(\alpha)}\left(\mathcal{C}_i\right)$, then great orthogonality theorem implies,

$$\sum_{i=1}^{n_c}U_{\alpha i}U_{\beta i}^{*}=\delta_{\alpha\beta}$$

Thus, if we consider $U_{\alpha i}$ as components in n_c dimensional vector space, $\vec{U}_\alpha=\left(U_{\alpha 1}U_{\alpha 2}\cdots U_{\alpha n_c}\right)$, then $\vec{U}_\alpha$ $\alpha=1,2,3\cdots n_r$ (n_r: # of indep irreps) form an othornormal set of vectors, i.e.

$$U_\beta U_\alpha=\sum_{i=1}^{n_c}U_{\alpha i}U_{\beta i}^{*}=\delta_{\alpha\beta}$$

This implies that

$$n_r\leq n_c$$

As an illustration, the characters for the representation give in Eqs(4,5) are

$$\chi^{(3)}\left(E\right)=2,\qquad\chi^{(3)}\left(A\right)=\chi^{(3)}\left(B\right)=-1,\qquad\chi^{(3)}\left(K\right)=\chi^{(3)}\left(L\right)=\chi^{(3)}\left(M\right)=0$$

From these we can form a 3-dimensional vector,

$$\chi^{(3)}=\left(2,-1,0\right)\tag{8}$$

Similarly for the representations in Eq(6) we get

$$\chi^{(1)}\left(E\right)=1,\qquad\chi^{(1)}\left(A\right)=\chi^{(1)}\left(B\right)=1,\qquad\chi^{(1)}\left(K\right)=\chi^{(1)}\left(L\right)=\chi^{(1)}\left(M\right)=1$$

$$\chi^{(2)}\left(E\right)=1,\qquad\chi^{(2)}\left(A\right)=\chi^{(2)}\left(B\right)=1,\qquad\chi^{(2)}\left(K\right)=\chi^{(2)}\left(L\right)=\chi^{(2)}\left(M\right)=-1$$

From these we can form another two 3-dimensional vectors,

$$\chi^{(1)}=\left(1,1,1\right)$$

$$\chi^{(2)}=\left(1,1,-1\right)$$

The orthogonality relations in Eq (7) these 3-dimensional vectors are orthogonal to each other when weighted by the number of elements in the class. For example,

$$\left(\chi^{(2)},\chi^{(3)}\right)=2\times 1+\left(-1\right)\times 1\times 2+0\times\left(-1\right)\times 3=0$$

$$\left(\chi^{(1)},\chi^{(3)}\right)=2\times 1+\left(-1\right)\times 1\times 2+0\times 1\times 3=0$$

4.1 Decomposition of Reducible Representation

For a reducible representation, we can write

$$D=D^{(1)}\oplus D^{(2)}\qquad\text{i.e. }D\left(A_i\right)=\begin{pmatrix}D^{(1)}\left(A_i\right)&\\&D^{(2)}\left(A_i\right)\end{pmatrix}\qquad\forall A_i\in G$$

Then we have for the trace,

$$\chi\left(A_i\right)=\chi^{(1)}\left(A_i\right)+\chi^{(2)}\left(A_i\right)$$

Denote by $D^{(\alpha)},\alpha=1,2\cdots n_r$, all the inequivalent unitary irrep. Then any rep D can be decomposed as

$$D=\sum_\alpha c_\alpha D^{(\alpha)}\qquad c_\alpha\text{: some integer },\left(\#\text{ of time }D^{(\alpha)}\text{ appears}\right)$$

In terms of traces, we get

$$\chi\left(\mathcal{C}_i\right)=\sum_\alpha c_\alpha\chi^{(\alpha)}\left(\mathcal{C}_i\right)$$

where we indicate that the trace is a function of class $\mathcal{C}_i$.The coefficient can be calculated as follows (by using orthogonity theorem). Multiply by $n_i\chi_i^{(\beta)*}$ and sum over i

$$\sum_i\chi_i\chi_i^{(\beta)*}n_i=\sum_i\sum_\alpha c_\alpha\chi_i^{(\alpha)}\chi_i^{(\beta)*}n_i=\sum_\alpha c_\alpha\cdot n\delta_{\alpha\beta}=nc_\beta$$

or

$$\boxed{c_\beta=\frac{1}{n}\sum_i \chi_i \chi_i^{(\beta)*} n_i}$$

From this we also get,

$$\sum_i n_i \chi_i \chi_i^* = \sum_i n_i \sum_{\alpha,\beta} c_\alpha \chi_i^{(\alpha)} c_\beta \chi_i^{(\beta)*} = n \sum_\alpha |c_\alpha|^2$$

This leads to the following theorem:

Theorem: If the rep D with character χ_i satifies the relation,

$$\sum_i n_i \chi_i \chi_i^* = n$$

then the representation D is irreducible. For example for the 2-dimensional representation, the character given in Eq (8), we have

$$\sum_i n_i \chi_i \chi_i^* = 2^2 + 1 \times 2 + 3 \times 0 = 6$$

This shows that the 2-dim representation is irreduceble.

5 Regular Representation

Given a group $G = \{A_1 = E, A_2 ... A_n\}$ .We can construct the regular rep as follows:
Take any $A \in G$. If

$$AA_2 = A_3 = 0A_1 + 0A_2 + 1 \cdot A_3 + 0A_4 + \cdots$$

i.e. we write the product "formally" as linear combination of group elements,

$$AA_s = \sum_{r=1}^{n} C_{rs} A_r = \sum_{r=1}^{n} A_r D_{rs}(A), \qquad \text{i.e. } C_{rs} = D_{rs}(A) \text{ is either 0 or1.} \tag{9}$$

$$\begin{aligned} \text{i.e.} \quad D_{rs}(A) &= 1 \quad \text{if } AA_s = A_r \quad \text{or} \quad A = A_r A_s^{-1} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Strictly speaking, the sum over group elments is undefined. But here only one group element shows up in the right-hand side in Eq(9), we do not need to define the sum of group elements. Then $D(A)$'s form a rep of G: regular representation with dimensional n. This can be seen as follows:

$$\sum_r A_r D_{rs}(AB) = ABA_s = A \sum_t A_t D_{ts}(B) = \sum_{t \cdot r} D_{ts}(B) A_r D_{rt}(A)$$

or

$$D_{rs}(AB) = D_{rt}(A) D_{ts}(B)$$

We now want to compute the trace of the regular representation. From the definition of the regular representation

$$D_{rs}(A) = 1 \qquad \text{iff} \qquad AA_s = A_r$$

we see that the diagonal elements are of the form,

$$D_{rr}(A) = 1 \qquad iff \qquad AA_r = A_r \quad or \quad A = E$$

Therefore every character is zero except for identity class,

$$\begin{aligned} \chi^{(reg)}(\mathcal{C}_i) &= 0 \qquad i \neq 1 \\ \chi^{(reg)}(\mathcal{C}_i) &= n \qquad i = 1 \end{aligned} \tag{10}$$

From this we can work out how $D^{(reg)}$ reduces to irreps. Write

$$D^{(reg)} = \sum_\alpha c_\alpha D^{(\alpha)}$$

then

$$c_\alpha = \frac{1}{n}\sum_i \chi_i^{(reg)} \chi_i^{(\alpha)*} n_i = \frac{1}{n}\chi_1^{(reg)}\chi_1^{(\alpha)*} = \frac{1}{n}\cdot n l_\alpha = l_\alpha$$

This means that D_{reg} contains the irreps as many times as its dimension,

$$\chi_i^{(reg)} = \sum_\alpha^{n_r} l_\alpha \chi_i^{(\alpha)} \qquad \text{or} \quad \chi_i^{(reg)} = \sum_{\alpha=1}^{n_r} \chi_1^{(\alpha)*}\chi_i^{(\alpha)} = n\delta_{i1}$$

For the identity class $\chi_1^{reg} = n, \quad \chi_1^{(\alpha)} = l_\alpha$, then we get

$$\boxed{\sum_\alpha l_\alpha^2 = n}$$

This severely constraints the possible dimensionalities of irreps becuase both n and l_α have to be integers. For D_3, with $n = 6$, the only possible solution for $\sum_\alpha l_\alpha^2 = 6$ is $l_1 = 1.\ l_2 = 1.\ l_3 = 2$, and their permutations.

We now want to show that

$$\boxed{n_c = n_r}$$

i.e. # of classes = # of irreps. We will derive another orthogonal relations of the characters $\chi_i^{(\alpha)}$ with summation over irreps rather than classes,

$$\sum_\alpha \chi_i^{(\alpha)}\chi_j^{(\alpha)}$$

The derivation is quite involved. We separate them into several steps.

1. Define $D_i^{(\alpha)}$ by adding up all matrices corresponding to elements in the same class $\mathcal{C}_i$,

$$D_i^{(\alpha)} = \sum_{A\in\mathcal{C}_i} D^{(\alpha)}(A)$$

Then we can show that $D_i^{(\alpha)}$ is a multiple of identity. First we see that

$$\begin{aligned} D^{(\alpha)}(A_j)\, D_i^{(\alpha)} D^{(\alpha)}\left(A_j^{-1}\right) &= \sum_{A\in\mathcal{C}_i} D^{(\alpha)}(A_j)\, D^{(\alpha)}(A)\, D^{(\alpha)}\left(A_j^{-1}\right) \\ &= \sum_{A\in\mathcal{C}_i} D^{(\alpha)}\left(A_j A A_j^{-1}\right) = D_i^{(\alpha)} \end{aligned}$$

Using

$$D^{(\alpha)}\left(A_j^{-1}\right) = D^{(\alpha)}(A_j)^{-1}$$

we get

$$D^{(\alpha)}(A_j)\, D_i^{(\alpha)} = D_i^{(\alpha)} D^{(\alpha)}(A_j)$$

i.e. $D_i^{(\alpha)}$ commutes with all matrice in the irrep. From Schur's lemma, we get $D_i^{(\alpha)} = \lambda_i^{(\alpha)} 1$ where $\lambda_i^{(\alpha)}$ is some number. Taking the trace, we get

$$n_i\chi_i^{(\alpha)} = \lambda_i^{(\alpha)} l_2 \qquad or \qquad \lambda_i^{(\alpha)} = \frac{n_i\chi_i^{(\alpha)}}{l_2} = \frac{n_i\chi_i^{(\alpha)}}{\chi_1^{(\alpha)}}$$

where $\chi_1^{(\alpha)}$ is the character of identity class.

2. From the property of the class multiplication, we have

$$\mathcal{C}_i\mathcal{C}_j = \sum_k \mathcal{C}_{ijk}\mathcal{C}_k$$

LHS $\mathcal{C}_i\mathcal{C}_j$ is a collections of product of group element of the type A_iA_j where $A_i \in \mathcal{C}_i$ and $A_j \in \mathcal{C}_j$. We can map these products into irrep matrices and sum over to get

$$D_i^{(\alpha)} D_j^{(\alpha)} = \sum_k \mathcal{C}_{ijk} D_k^{(\alpha)} \qquad \text{or} \quad \lambda_i^{(\alpha)}\lambda_j^{(\alpha)} = \sum_k \mathcal{C}_{ijk}\lambda_k^{(\alpha)} \tag{11}$$

For example, in the group D_3, we have classes, $\mathcal{C}_1 = \{E\}, \mathcal{C}_2 = \{A, B\}, \mathcal{C}_3 = \{K, L, M\}$. Then

$$\begin{aligned} \mathcal{C}_2\mathcal{C}_3 &= \{A, B\}\{K, L, M\} = \{AK, AL, AM, BK, BL, BM\} \\ &= 2\{K, L, M\} = 2\mathcal{C}_3 \end{aligned}$$

Map these elements to their matrix representation and sum over,

$$\begin{aligned} LHS &= D(AK) + D(AL) + D(AM) + D(BK) + D(BL) + D(BM) \\ &= D(A)D(K) + D(A)D(L) + D(A)D(M) + D(B)D(K) + D(B)D(L) + D(B)D(M) \\ &= [D(A) + D(B)][D(L) + D(M) + D(K)] \end{aligned}$$

and

$$RHS = 2[D(L) + D(M) + D(K)]$$

This illustrate the relation in Eq(11).
Using the values of $\lambda_i^{(\alpha)}$ in Eq(**??**) we get,

$$\frac{n_i\chi_i^{(\alpha)}}{\chi_1^{(\alpha)}}\frac{n_j\chi_j^{(\alpha)}}{\chi_1^{(\alpha)}} = \sum_k \mathcal{C}_{ijk}\frac{n_k\chi_k^{(\alpha)}}{\chi_1^{(\alpha)}} \qquad \text{or} \quad n_i n_j \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \chi_1^{(\alpha)}\sum_k \mathcal{C}_{ijk} n_k \chi_k^{(\alpha)}$$

We can now sum over the irrep α,

$$\sum_\alpha n_i n_j \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \sum_\alpha\sum_k \mathcal{C}_{ijk} n_k \chi_1^{(\alpha)}\chi_k^{(\alpha)}$$

3. We now compute the coefficients $\mathcal{C}_{ijk}$. If any element A belongs to ith class, let the class which contains A^{-1} be denoted by $i^{'} - th$ class. We then get

$$\mathcal{C}_{ij1} = \begin{cases} 0 & \text{for} \quad j \neq i' \\ n_i & \text{for} \quad j = i' \end{cases}$$

We make use of the property of regular rep, $\sum_\alpha \chi_1^\alpha \chi_k^\alpha = n\delta_{k1}$, in Eq(**??**) to get,

$$\sum_\alpha n_i n_j \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \sum_\alpha\sum_k \mathcal{C}_{ijk} n_k \chi_1^{(\alpha)}\chi_k^{\alpha} = n\mathcal{C}_{ij1} = \begin{cases} 0 & \text{for} \quad j \neq i' \\ nn_i & \text{for} \quad j = i' \end{cases}$$

Then

$$\sum_{\alpha=1}^{n_r} \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \frac{n}{n_j}\delta j i'$$

Since rep is unitary

$$D^{(\alpha)}(A_i)^\dagger = D^{(\alpha)}(A_i)^{-1} = D^{(\alpha)}\left(A_i^{-1}\right) = D^{(\alpha)}(A_{i'})$$

we get

$$\chi_{i'}^{(\alpha)} = \chi_i^{(\alpha)*}$$

and

$$\sum_{\alpha=1}^{n_r} \chi_i^{(\alpha)}\chi_j^{(\alpha)*} = \frac{n}{n_j}\delta j i$$

This is the orthogonal relation we were looking for.

If we now consider $\chi_i^{(\alpha)}$ as a vector in n_r dim space $\vec{\chi}_i = \left(\chi_i^{(1)}, \chi_i^{(2)}, \ldots \chi_i^{(n_r)}\right)$ we get

$$n_c \leq n_r$$

Combine this with the result $n_r \leq n_c$, we have derived before, we get

$$n_r = n_c$$

5.1 Character Table

For a finite group, the essential information about the irreducible representations can be summarized in a table which lists the characters of each irreducible representation in terms of the classes. This table has many useful applications. To construct such table we can use the following useful information:

1. # of columns = # of rows = # of classes
2. $\sum\limits_{\alpha} l_\alpha^2 = n$
3. $\sum\limits_{i} n_i \chi_i^{(\alpha)} \chi_i^{(\beta)*} = n\delta_{\alpha\beta}$ and $\sum\limits_{\alpha} \chi_i^{(\alpha)} \chi_j^{(\alpha)*} = \frac{n}{n_i}\delta_{ij}$
4. If $l_\alpha = 1$, χ_i is itself a rep.
5. $\chi^{(\alpha)}\left(A^{-1}\right) = T_r\left(D^{(\alpha)}\left(A^{-1}\right)\right) = T_r\left(D^{(\alpha)^\dagger}\left(A^{-1}\right)\right) = \chi^{(\alpha)*}(A)$
   If A and A^{-1} are in the same class then $\chi(A)$ is real.
6. $D^{(\alpha)}$ is a rep $\Longrightarrow$ $D^{(\alpha)*}$ is also a rep
   so if $\chi^{(\alpha)}$'s are complex numbers, another row will be their complex conjugate
7. If $l_\alpha > 1, \chi_i^{(\alpha)} = 0$ for at least one class. This follows from the relation
$$\sum_i n_i \left|\chi_i\right|^2 = n \quad \text{and} \quad \sum_i n_i = n$$
8. For physical symmetry group, $x.y$ and z form a basis of a rep.

Example : D_3 character table

			E	$2C_3$	$3C_2'$
$x^2\dagger y^2, z^2$		A_1	1	1	1
	$R_z, z.$	A_2	1	1	-1
(xz, yz)	(x, y)	E	2	-1	0
$x^2 - y^2, xy$	$(R_x.R_y)$				

In this table, the typical basis functions up to quadratic in coordinate system are listed.

Remark: the basis functions listed in the usual character table are not necessarily normalized. In particular, the quadratic functions have to be handled carefully. The danger is that if we use the basis functions given in the character table, we might not generate unitary matrices.

Using the transformation properties of the coordinate, we can also infer the transformation properties of any vectors.

For example, the usual coordinates have the transformation property,

$$\vec{r} = (x, y, z) \sim A_2 \oplus E \quad in \quad D_3$$

This means that electric field of $\vec{E}$ or magnetic field $\vec{B}$ will have same transformation property,

$$\vec{B} \sim \vec{E} \sim A_2 \oplus E$$

because they all transform the same way under the rotation.

6 Product Representation (Kronecker product)

Let x_i be the basis for $D^{(\alpha)}$, i.e. $\quad x_i' = \sum\limits_{j=1}^{\ell_\alpha} x_j D_{ji}^{(\alpha)}(A)$

y_ℓ be the basis for $D^{(\beta)}$, i.e. $\quad y_k' = \sum\limits_{\ell=1}^{\ell_\beta} y_\ell D_{\ell k}^{(\beta)}(A)$

then the products $x_j y_l$ transform as

$$x_i' y_k' = \sum_{j\cdot\ell} D_{ji}^{(\alpha)}(A) D_{k\ell}^{(\beta)}(A) x_j y_\ell \equiv \sum_{j\cdot\ell} D_{j\ell;ik}^{(\alpha\times\beta)}(A) x_j y_\ell$$

where

$$\boxed{D^{(\alpha\times\beta)}_{j\ell;ik}(A) = D^{(\alpha)}_{ij}(A)\, D^{(\beta)}_{\ell k}(A)}$$

Note that in these matrices, row and column are labelled by 2 indices, instead of one. It is easy to show that $D^{(\alpha\times\beta)}$ forms a rep of the group.

$$\left[D^{(\alpha\times\beta)}(A)\, D^{(\alpha\times\beta)}(B)\right]_{ij;k\ell} = \sum_{s.t} D^{(\alpha\times\beta)}(A)_{ij,st}\, D^{(\alpha\times\beta)}(B)_{st;k\ell}$$

$$= \sum_{s.t} \quad D^{(\alpha)}_{is}(A)\;\; D^{(\beta)}_{jt}(A)\;\; D^{(\alpha)}_{sk}(B)\;\; D^{(\beta)}_{t\ell}(B) = D^{(\alpha)}_{ik}(AB)\, D^{(\beta)}_{j\ell}(AB) = D^{(\alpha\times\beta)}(AB)_{ik;k\ell}$$

or

$$\boxed{D^{(\alpha\times\beta)}(A)\, D^{(\alpha\times\beta)}(B) = D^{(\alpha\times\beta)}(AB)}$$

The basis functions for $D^{(\alpha\times\beta)}$ are $x_i y_j$

The character of this new rep can be calculated by making the row and colum indices the same and sum over,

$$\chi^{(\alpha\times\beta)}(A) = \sum_{j\cdot\ell} D^{(\alpha\times\beta)}_{j\ell;j\ell}(A) = \sum_{j\cdot\ell} D^{(\alpha)}_{jj}(A)\, D^{(\beta)}_{\ell\ell}(A) = \chi^{(\alpha)}(A)\,\chi^{(\beta)}(A)$$

$$\chi^{(\alpha\times\beta)}(A) = \chi^{(\alpha)}(A)\,\chi^{(\beta)}(A)$$

If $\alpha = \beta$, we can further decompose the product rep by symmetrization or antisymmetrization;

$$D^{\{\alpha\times\alpha\}}_{ik,j\ell}(A) = \frac{1}{2}\left[D^{(\alpha)}_{ij}(A)\, D^{(\alpha)}_{k\ell}(A) + D^{(\alpha)}_{i\ell}(A)\, D^{(\alpha)}_{kj}(A)\right] \qquad \text{basis } \frac{1}{\sqrt{2}}\left(x_i y_k + x_k y_i\right)$$

$$D^{[\alpha\times\alpha]}_{ik,j\ell}(A) = \frac{1}{2}\left[D^{(\alpha)}_{ij}(A)\, D^{(\alpha)}_{k\ell}(A) - D^{(\alpha)}_{i\ell}(A)\, D^{(\alpha)}_{kj}(A)\right] \qquad \text{basis } \frac{1}{\sqrt{2}}\left(x_i y_k - x_k y_i\right)$$

These matrices also form rep of G and the characters are given by

$$\chi^{\{\alpha\times\alpha\}}(A) = \frac{1}{2}\left[\left(\chi^{(\alpha)}(A)\right)^2 + \chi^{(\alpha)}\left(A^2\right)\right], \qquad \chi^{[\alpha\times\alpha]}(A) = \frac{1}{2}\left[\left(\chi^{(\alpha)}(A)\right)^2 - \chi^{(\alpha)}\left(A^2\right)\right]$$

Example $\quad D_3$

			$E.$	$2\mathcal{C}_3$	$3C_2^{'}$		
		Γ_1	1	1	1		
	$R_z.z$	Γ_2	1	1	-1		
(xz, yz)	(x, y)	Γ_3	2	-1	0		
$\left(x^2-y^2, xy\right)$		$\Gamma_3\times\Gamma_3$	4	1	0	$=$	$\Gamma_1\oplus\Gamma_2\oplus\Gamma_3$
		$\left(\Gamma_3\times\Gamma_3\right)_s$	3	0	1	$=$	$\Gamma_1\oplus\Gamma_3$
		$\left(\Gamma_3\times\Gamma_3\right)_a$	1	1	-1	$=$	Γ_2

For illustration, consider the reduction of $\Gamma_3\times\Gamma_3$. The characters are

$$\chi^{(\Gamma_3\times\Gamma_3)}(E) = 4, \qquad \chi^{(\Gamma_3\times\Gamma_3)}(C_3) = 1, \qquad \chi^{(\Gamma_3\times\Gamma_3)}(C_2) = 0$$

If we write the reduction as

$$\Gamma_3\times\Gamma_3 = c_1\Gamma_1 \oplus c_2\Gamma_2 \oplus c_3\Gamma_3$$

then

$$c_\alpha = \frac{1}{n}\sum_{\alpha} \chi^{(\Gamma_3\times\Gamma_3)}(C_i)\,\chi^{(\alpha)}_i n_i$$

From the character table, we see that

$$c_1 = \frac{1}{6}\left[4\times 1 + 1\times 1\times 2 + 0\times 1\times 3\right] = 1$$

$$c_2 = \frac{1}{6}\left[4\times 1 + 1\times 1\times 2 + 0\times(-1)\times 3\right] = 1$$

$$c_3 = \frac{1}{6}\left[4\times 2 + (-1)\times 1\times 2 + 0\times 1\times 3\right] = 1$$

7 Direct Product Group

Given 2 groups $G_1 = \{E, A_2 \cdots A_n\}$, $G_2 = \{E, B_2 \cdots B_m\}$, we can define the product group as $G_1 \otimes G_2 = \{A_i B_j; i = 1 \cdots n, j = 1 \cdots m\}$ with multiplication law

$$(A_k B_\ell) \times (A_{k'} B_{\ell'}) = (A_k B_{k'}) (B_\ell B_{\ell'})$$

It turns out that irrep of $G_1 \otimes G_2$ are just direct product of irreps of G, and G_2. Let $D^{(\alpha)}(A_i)$ be an irrep of $G_1$and $D^{(\beta)}(B_j)$ an irrep of G_2 then the matrices defined by

$$D^{(\alpha\times\beta)}(A_i B_j)_{ab;cd} \equiv D^{(\alpha)}(A_i)_{ac} D^{(\beta)}(B_j)_{bd}$$

will have the property

$$\begin{aligned}
\left[D^{(\alpha\times\beta)}(A_i B_j) D^{(\alpha\times\beta)}(A_k B_\ell)\right]_{ab;cd} &= \sum_{e\cdot f} \left[D^{(\alpha\times\beta)}(A_i B_j)\right]_{ab;ef} \left[D^{(\alpha\times\beta)}(A_k B_\ell)\right]_{ef;cd} \\
&= \sum_{e\cdot f} \left[D^{(\alpha)}(A_i)_{ac} D^{(\alpha)}(A_k)_{ec}\right] \left[D^{(\beta)}(B_j)_{bf} D^{(\beta)}(B_e)_{fd}\right] \\
&= D^{(\alpha)}(A_i A_k)_{ac} D^{(\beta)}(B_j B_\ell)_{bd} = D^{(\alpha\times\beta)}(A_i A_k . B_j B_\ell)_{ab;cd}
\end{aligned}$$

This means that the matrice $D^{(\alpha\times\beta)}(A_i B_j)$ form a representation of the product group $G_1 \otimes G_2$. The characters can be calculated,

$$\chi^{(\alpha\times\beta)}(A_i B_j) = \sum_{ab} D^{(\alpha\times\beta)}(A_i B_j)_{ab;ab} = \sum_{a\cdot b} D^{(\alpha)}(A_i)_{aa} D^{(\beta)}(B_j)_{bb} = \chi^{(\alpha)}(A_i)\chi^{(\beta)}(B_j)$$

Then

$$\sum_{i\cdot j} \left|\chi^{(\alpha\times\beta)}(A_i B_j)\right|^2 = \left(\sum_i \left|\chi^{(\alpha)}(A_i)\right|^2\right) \left(\sum_j \left|\chi^{(\beta)}(B_j)\right|^2\right) = nm \quad \Longrightarrow \quad D^{(\alpha\times\beta)} \text{ is irrep.}$$

Example, $G_1 = D_3 = \{E, 2C_3, 3C_2'\}, G_2 = \{E, \sigma_h\} = \varphi$ where σ_h: reflection on the plane of triangle.
Direct product group is then$D_{3h} \equiv D_3 \otimes \varphi = E, A, B = \{E, 2C_3, 3C_2', \sigma_h, 2C_3\sigma_h, 3C_2'\sigma_h\}$

Character Table

φ	E	σ_h
$\Gamma^\dagger$	1	1
Γ^-	1	-1

D_3	E	$2C_3$ AB	$2C_2'$ KLM
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	2	-1	0

Character Table

	E	$2C_3$	$2C_2'$	σ_h	$2C_3\sigma_h$	$2C_2'\sigma_h$
$\Gamma_1^\dagger$	1	1	1	1	1	1
$\Gamma_2^\dagger$	1	1	-1	1	1	-1
$\Gamma_3^\dagger$	2	-1	0	2	-1	0
$\Gamma_1^\dagger$	1	1	1	-1	-1	-1
$\Gamma_2^\dagger$	1	1	-1	-1	-1	1
Γ_3^{-1}	2	-1	0	-2	1	0