Algebraic Approach

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Algerbric Approach

For groups larger than $SU(2)$, it is more efficient to use the algebric approach, in terms of commutators of the generators, similar to the study of rotaion group in terms of the angular momentum algebra. In fact, the angular momentum algebra forms the basis for the more general groups.

Lie Algebra

Paremetrize group elements in the exponential form,

$$g = e^{i\alpha_a X_a}$$

where $\alpha_1, \alpha_{2,} \cdots$ group parameters and $X_1, X_2, \cdots$ are generators . Write the group multiplication as

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

Clearly $\delta_a' s$ are functions of α and β. To explore these relations we write

$$i\delta_a X_a = \ln\left(e^{i\alpha_a X_a} e^{i\beta_b X_b} + 1 - 1\right) = \ln(1 + K)$$

where

$$K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$$

Suppose α and β are small and expand K to second order terms,

$$\begin{aligned} K &= \left[1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + \cdots\right] \times \left[1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + \cdots\right] - 1 \\ &= i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 + \cdots \end{aligned}$$

This gives

$$\begin{aligned} i\delta_a X_a &= K - \frac{1}{2}K^2 + \cdots \\ &= i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 + \frac{1}{2}(\alpha_a X_a + \beta_b X_b)^2 + \cdots \\ &= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + \cdots \end{aligned}$$

and

$$[\alpha_a X_a, \beta_b X_b] = -2i(\delta_c - \alpha_c - \beta_c) X_c + \cdots \equiv i\gamma_c X_c$$

Since this must be true for all α and β, we must have

$$\gamma_c = \alpha_a \beta_b f_{abc}, \qquad f_{abc} \quad \text{some constant}$$

and

$$[X_a, X_b] = i f_{abc} X_c$$

Thus generators form an algebra under commutation.
One simply way to see that the structure of the group is controlled by the commutation is to note that

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i\alpha_a X_a + i\beta_b X_b}$$

The equality holds only for Abelian group. So for non-Abelian group the commutators of generators are non zero and responsible for non-commuting property of group multiplication. Furthermore, from the Baker-Campell-Hausdroff formula

$$\exp(A)\exp(B) = \exp\left[A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]]\right] + \cdots$$

if the commuatator gives back a linear combination of generators then the result of multiplication of 2 group elements in the exponentiatl form produce another element in the exponential form.

SU(n) Algebra

The $SU(n)$ is the group formed by $n \times n$ unitary matrices with $det = 1$. The number of independent real group parameters needed to for group elements can be calculated as follows. Condition $UU^\dagger = 1$, implies

$$\sum_k U_{ik} U_{jk}^* = \delta_{ij}$$

Define n-dim vectors by

$$\vec{U_i} = (U_{i1}, U_{i2}, \ldots U_{in}), \qquad i = 1, 2, 3, \ldots, n$$

Then unitary conditions $\Longrightarrow$

$$\left(\vec{U}_i, \vec{U}_j\right) = \sum_k U_{ik} U_{jk}^* = \delta_{ij}$$

$\vec{U}_i{}'s$ form an orthonomal set in n-dim complex vector space. From the magnitudes

$$\left(\vec{U}_i, \vec{U}_i\right) = 1$$

gives n conditions on the real components of matrix elements and

$$\left(\vec{U}_i, \vec{U}_j\right) = 0 \qquad \text{for } i \neq j \text{ give } \frac{n(n-1)}{2} \times 2 = n^2 - n \qquad \text{conditions}$$

$$\det U = 1 \qquad \text{gives one condition}$$

Hence, independent real parameters in $SU(n)$ is

$$2n^2 - \left(n^2 - n\right) - n - 1 = n^2 - 1$$

Another way is to explore the relation between unitaty matrix U hermitian matrix H by writing

$$U = e^{iH}$$

Using the identity

$$\det\left(e^A\right) = e^{TrA}$$

we get

$$\det U = 1, \qquad T_r H = 0$$

Since there are $n^2 - 1$ traceless hermitian matrices, $SU(n)$ has $n^2 - 1$ parameters.

SU(2) Algebra

Physical examples of $SU(2)$ group are the rotation group in 3-dimension and isospin symmetry in strong interaction. Since there are 3 group parameters. Parametrize 2×2 unitary matrices by

$$U(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \exp(i\varepsilon_i \sigma_i)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are real parameters, and $\sigma_1, \sigma_2, \sigma_3$ are the ususal Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The commutation relations for Pauli matrices are of the form,

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}, \qquad i, j, k = 1, 2, 3$$

where ε_{ijk} is the totally anti-symmetric Levi-Civita symbol with $\varepsilon_{123} = 1$. The operators J_1, J_2, J_3 which satisfy the same commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \tag{1}$$

are called the generators of $SU(2)$ and the commutation relations are called the **Lie algebra of** $SU(2)$. These commutation relations are the same as angular momentum operators $\vec{L} = \vec{r} \times \vec{p}$,

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \tag{2}$$

Thus these two algebras (1)(2) are in one to one correspondance (isomorphism). Recall that in coordinate space, angular momentum operators are of the form,

$$L_1 = -i\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right), \qquad L_2 = -i\left(x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\right), \qquad L_3 = -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

It is clear that these operator will leave the quadratic form,

$$x_1^2 + x_2^2 + x_3^2$$

invariant. For example,

$$\begin{aligned} L_1\left(x_1^2 + x_2^2 + x_3^2\right) &= -i\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\left(x_1^2 + x_2^2 + x_3^2\right) \\ &= -i\left(x_2 2x_3 - x_3 2x_2\right) = 0 \end{aligned}$$

Representation of $SU(2)$ algebra

Any set of matrices D_1, D_2, D_3 satisfying the same algebra,

$$[D_i, D_j] = i\varepsilon_{ijk} D_k$$

are called the **representation** of generators J_1, J_2, J_3. For example $D_i = \frac{\sigma_i}{2}$ is a representation (defing representation or fundamental representation)
To finother representations, define

$$J_\pm = J_1 \pm iJ_2$$

then

$$[J_\pm, J_3] = \mp J_\pm, \qquad [J_+, J_-] = 2J_3$$

Suppose

$$J_3 |m\rangle = m |m\rangle$$

then from these commutation relations,

$$J_3 (J_+ |m\rangle) = (m+1)(J_+ |m\rangle), \qquad J_3 (J_- |m\rangle) = (m-1)(J_- |m\rangle)$$

$\Longrightarrow$ J_+ (J_-) is the raising(lowering) operator which increases (decrease) the eigenvalues of J_3 by one unit. Define

$$J^2 \equiv J_1^2 + J_2^2 + J_3^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \geq 0 \qquad (3)$$

Then

$$\left[J^2, J_i\right] = 0, \qquad \text{for} \quad i = 1, 2, 3$$

and J^2 is called **Casmir operator**, the operator which commutes with all generators in the group. From Schur's lemma, the Casmir operators will be represented by a multiple of identity matrix in any irreps. Choose the states to be eigenstates of J^2, J_3, with eigenvalues, λ, m

$$J^2 \left|\lambda, m\right\rangle = \lambda \left|\lambda, m\right\rangle, \qquad J_3 \left|\lambda, m\right\rangle = m \left|\lambda, m\right\rangle$$

with normalization

$$\left\langle \lambda', m' \mid \lambda, m \right\rangle = \delta_{\lambda\lambda'}\delta_{mm'} \tag{4}$$

From Eq(3) m, the eigenvalue of J_3, is bounded by

$$m^2 \leq \lambda$$

Thus representation matrices are finite dimensional. Since $J_\pm \left|\lambda, m\right\rangle$ are eigenstates of J_3 with eigenvalues $m \pm 1$, we can write

$$J_\pm \left|\lambda, m\right\rangle = C_\pm(\lambda, m) \ \left|\lambda, m \pm 1\right\rangle,$$

$C_\pm(\lambda, m)$ are to be determined by the normalization conditions in Eq(4). Since $J^2 - J_3^2 \geq 0$, we get $\lambda - m^2 \geq 0$, i.e. eigenvalue m^2 is bounded. Thus largest value of m, say $m = j$, will have the property,

$$J_+ \left|\lambda, j\right\rangle = 0,$$

Write J^2 operator as

$$J^2 = \frac{1}{2} J_+ J_- + J_- J_+) + J_3^2 = J_- J_+ + J_3^2 + J_3$$

Applying this to $|\lambda, j\rangle$,

$$\left(\lambda - j^2 - j\right)|\lambda, j\rangle = 0, \qquad \Rightarrow \qquad \lambda = j\,(j+1)$$

Similarly, for smallest value of m, say $m = j'$,

$$J_-\left|\lambda, j'\right\rangle = 0, \quad \text{and} \quad \lambda = j'\left(j'-1\right)$$

Combing these two, we get

$$j\,(j+1) = j^{'}\left(j'-1\right), \quad \Rightarrow \quad j = -j', \quad \text{or} \quad j' = j+1$$

The solution $j' = j+1$ violates the assumption that j is the largest value of m. Thus we will take

$$j = -j'.$$

Since J_- decreases value of m by 1 each time it acts on the eigenstate of J_3, the maximum and minimum values of m should differ by an integer,

$$j - j' = 2j = \text{integer}, \qquad \Rightarrow \qquad j \ \text{ integer or half integer}$$

We will use the parameter j to label the state instead of λ, with

$$J^2\,|j, m\rangle = j\,(j+1)\,|j, m\rangle$$

The coefficients $C_{\pm}(\lambda, m)$ can be calculated as follows,

$$J_+ |j,m\rangle = C_+(j,m) |j,m+1\rangle \,, \qquad \langle j,m| J_- = \langle j,m+1| C_+^*(j,m)$$

$$\langle j,m| J_- J_+ |j,m\rangle = |C_+(j,m)|^2$$

On the other hand,

$$\langle j,m| J_- J_+ |j,m\rangle = \langle j,m| \left(J^2 - J_3^2 - J_3\right) |j,m\rangle = \left[j(j+1) - m^2 - m\right] = (j-m)(j+m+1)$$

We can then take

$$C_+(j,m) = \sqrt{(j-m)(j+m+1)}$$

Similarly,

$$C_-(j,m) = \sqrt{(j+m)(j-m+1)}$$

To summarize, the states $|j,m\rangle$, $m = -j, -j+1, \ldots j-1, j$ form the basis of the irreducible representation characterized by j. These states have the following properties,

$$J^2 |j,m\rangle = j(j+1) |j,m\rangle \,, \quad J_3 |j,m\rangle = m |j,m\rangle \tag{5}$$

$$J_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)} |j,m+1\rangle \,, \qquad J_- |j,m\rangle = \sqrt{(j+m)(j-m+1)} |j,m-1\rangle \tag{6}$$

We can get the matrix elements of J_1, J_2 by using,

$$J_1 = \frac{1}{2}\left(J_+ + J_-\right), \qquad J_2 = \frac{1}{2i}\left(J_+ - J_-\right)$$

Thus for a given j, the matrices contructed for J_1, J_2 and J_3 will satisfy the angular momentum algebra given in Eq(1) and they are the irreps of $SU(2)$ group.

Remark:

For Lorentz group in 2-dim, the generators are,

$$L_3 = -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right), \qquad K_1 = -i\left(x_0\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_0}\right), \qquad K_2 = -i\left(x_0\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_0}\right)$$

These operators leave the quadratic form,

$$x_1^2 + x_2^2 - x_0^2$$

invariant. Commutators for these generators are,

$$[L_3, K_1] = iK_2, \qquad [L_3, K_2] = -iK_1, \qquad [K_1, K_2] = -iL_3$$

We can also define the raising and lowering operators by

$$K_+ = K_1 + iK_2, \qquad K_- = K_1 - iK_2$$

to get the

$$[L_3, K_\pm] = \mp K_\pm, \qquad [K_+, K_-] = 2L_3$$

We also get that

$$\left[K_1^2 + K_2^2 - L_3^2,\ L_3\right] = 0, \qquad \left[K_1^2 + K_2^2 - L_3^2,\ K_1\right] = 0, \qquad \left[K_1^2 + K_2^2 - L_3^2,\ K_2\right] = 0$$

The Casmir operator is now of the form $B = K_1^2 + K_2^2 - L_3^2$. Now the eigenvalue of L_3 is no longer bounded by that of B. The main difference between Lorentz group in 2 dimension and the 3-dimensional rotation group $O(3)$ is that $O(3)$ is a compact group while the Lorentz group is non-compact.

Product Representation

Back to $SU(2)$ group to study product representations. Consider, 2 spin 1/2 particles, we want to know the total spin J of the product of the two wavefunctions. In this simple case, the answer is $J = 0$, or 1. We now want to study this problem in terms of group theory. Denote the spin-up and spin-down states of the first particle by r_1 and r_2, and for the second particles, s_1 and s_2. Under $SU(2)$ matrices, they transform according to

$$r'_i = U\left(\vec{\epsilon}\right)_{ij} r_j, \qquad s'_k = U\left(\vec{\epsilon}\right)_{kl} s_l$$

where

$$U\left(\vec{\epsilon}\right) = \exp\left(i\vec{\epsilon}\cdot\vec{J}\right), \qquad \text{and} \qquad \vec{J} = \frac{\vec{\sigma}}{2}$$

Then for the product

$$\left(r'_i s'_k\right) = U\left(\vec{\epsilon}\right)_{ij} U\left(\vec{\epsilon}\right)_{kl} \left(r_j s_l\right) = D\left(\vec{\epsilon}\right)_{ik,jl} \left(r_j s_l\right)$$

where

$$D\left(\vec{\epsilon}\right)_{ik,jl} = U\left(\vec{\epsilon}\right)_{ij} U\left(\vec{\epsilon}\right)_{kl}$$

In general, $D\left(\vec{\epsilon}\right)$ is reducible. To see the decomposition, work with the generators directly by taking $\varepsilon_i \ll 1$,

$$r'_i \simeq \left(1 + i\vec{\epsilon}\cdot\vec{J}\right)_{ij} r_j = \left(1 + i\vec{\epsilon}\cdot\vec{J}^{(1)}\right)_{ij} r_j$$

$$s'_k \simeq \left(1 + i\vec{\epsilon}\cdot\vec{J}\right)_{kl} s_l = \left(1 + i\vec{\epsilon}\cdot\vec{J}^{(2)}\right)_{kl} s_l$$

where $\vec{J}^{(1)}$ operates only on r_i and does not affect s_k; while $\vec{J}^{(2)}$ operates on s_i. Define the total angular mometum operator as

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

Let α_i (β_i) denote the spin-up (spin-down) state of the ith particle. There are four combinations of two-particle states :

$$\alpha_1\alpha_2, \quad \alpha_1\beta_2, \quad \beta_1\alpha_2, \quad \beta_1\beta_2$$

Start with state with largest value of J_3,

$$J_3\left|\alpha_1\alpha_2\right\rangle = J_3^{(1)}\left|\alpha_1\alpha_2\right\rangle + J_3^{(2)}\left|\alpha_1\alpha_2\right\rangle = \left|\alpha_1\alpha_2\right\rangle$$

To find the total J, we write

$$\left(\vec{J}\right)^2 = \left(\vec{J}^{(1)}\right)^2 + \left(\vec{J}^{(2)}\right)^2 + \left[\left(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)}\right) + 2J_3^{(1)}J_3^{(2)}\right]$$

and find that

$$\left(\vec{J}\right)^2\left|\alpha_1\alpha_2\right\rangle = 2\left|\alpha_1\alpha_2\right\rangle$$

$\Longrightarrow$ the state $|\alpha_1\alpha_2\rangle$ has $J = 1$ and $J_3 = 1$,

$$|1,1\rangle = |\alpha_1\alpha_2\rangle \tag{7}$$

Use the lowering operator $J_- = J_-^{(1)} + J_-^{(2)}$, to reach all other states in the $J = 1$ irrep,

$$J_- |1,1\rangle = J_- |\alpha_1\alpha_2\rangle = \left(J_-^{(1)} + J_-^{(2)}\right) |\alpha_1\alpha_2\rangle = |\alpha_1\beta_2\rangle + |\beta_1\alpha_2\rangle$$

On the other hand, from Eq(6) we get,

$$J_- |1,1\rangle = \sqrt{2}\,|1,0\rangle$$

Then

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_1\beta_2\rangle + |\beta_1\alpha_2\rangle\right) \tag{8}$$

Obviously,

$$|1,-1\rangle = |\beta_1\beta_2\rangle$$

The remaining state with $J = 0$ can be obtained by orthogonality with respect to the state $|1,0\rangle$ given in Eq(8),

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_1\beta_2\rangle - |\beta_1\alpha_2\rangle\right)$$

More generally, the product representations, $|j_1, m_1\rangle \times |j_2, m_2\rangle$ can be combined into eigenstates $|J, M\rangle$ of total angular momentum, $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$,

$$|J, M\rangle = \sum_{m_1, m_2} |j_1, m_1\rangle\, |j_2, m_2\rangle\, \langle j_1, m_1, j_2, m_2 | JM\rangle$$

The coefficients $\langle j_1, m_1, j_2, m_2 | J, M\rangle$ are called **Clebsch-Gordon coefficients**. Thus for the above case (Eqs(7,8))

$$\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} | 1, 1 \right\rangle = 1, \qquad \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} | 1, 0 \right\rangle = \frac{1}{\sqrt{2}}, \quad etc$$

Note that the J_3 quantum number is additive,

$$M = m_1 + m_2$$

The procedure of working out the irrep of the product representations can be summarized as follows.

1. Start with the combination of states with largest J_3. Clearly this is also an eigenstate with largest total J.
2. Use the lowering operator $J_- = J_-^{(1)} + J_-^{(2)}$ to get all the other states in the same irrep.
3. Find the orthogonal combination to $|J_m, J_m - 1\rangle$ where J_m is the maximum value of J in the product. This orthogonal state should be $|J_m - 1, J_m - 1\rangle$. Then use the lowering operator to reach othe other $J = J_m - 1$ states.
4. Repeat these steps until $J = |j_1 - j_2|$.

$SU(3)$ **Algebra**

Physical example of this group is the "eightfold way" of mesons and baryons. This also forms the basis for the quark model. Denote the group parameters by α_i, $i = 1, 2, \ldots 8$ and write the group elements as

$$U(\alpha_i) = \exp(i\alpha_i\lambda_i), \qquad \lambda_i : \text{ hermitian traceless } 3 \times 3 \text{ matrices}$$

The standard form of λ_i are given by Gell-Mann,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

These matrices are normalized as

$$Tr(\lambda_i\lambda_j) = 2\delta_{ij}$$

The commutators for these matrices, the Lie algebrs, are

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = if_{ijk}\left(\frac{\lambda_k}{2}\right),$$

where f_{ijk} are the totally anti-symmetric structure constants and those which are non-zero are,

$$f_{123} = 1, \quad f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

Generators F_i of $SU(3)$ satisfy the same commutators,

$$[F_i, F_j] = if_{ijk}F_k$$

Since λ_3 and λ_8 are diagonal, we have

$$[\lambda_3, \lambda_8] = 0$$

which implies that

$$[F_{3,} F_8] = 0$$

Hence, F_3 and F_8 can be diagonalized simultaneously and their eigenvalues, will be used to label states in the representation. The largest number of mutually commuting generators in the algebra is called the **rank of the algebra**. Thus $SU(3)$ is a rank 2 group while $SU(2)$ is a rank 1 group. .

Representation of $SU(3)$

Define the raising and lowering operators by

$$T_{\pm} = F_1 \pm iF_2, \quad U_{\pm} = F_6 \pm iF_7, \quad V_{\pm} = F_4 \pm iF_5$$

If we write $T_3 = F_3$, $Y = \frac{2}{\sqrt{3}} F_8$, where T_3 is the usual isospin and Y is the hypercharge, the commutation relations become

$$\begin{array}{lll}
[T_3, T_{\pm}] = \pm T_{\pm} & [Y, T_{\pm}] = 0 & [T_+, T_-] = 2T_3 \\
[T_3, U_{\pm}] = \mp \frac{1}{2} U \pm & [Y, U_{\pm}] = \pm U_{\pm} & [U_+, U_-] = \frac{3}{2} Y - T_3 \equiv 2U_3 \\
[T_3, V_{\pm}] = \pm \frac{1}{2} V \pm & [Y, V_{\pm}] = \pm V_{\pm} & [V_+, V_-] = \frac{3}{2} Y + T_3 \equiv 2V_3 \\
[T_+, V_+] = 0 & [T_+, U_-] = 0 & [U_+, V_+] = 0 \\
[T_+, V_-] = -U_- & [T_+, U_+] = V_+ & [U_+, V_-] = T_-
\end{array}$$

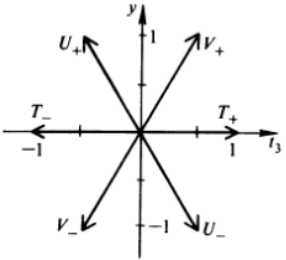
Clearly, these raising and lowering operators move the states on the plane labeled by (T_3, Y),

T_+ raises T_3 by 1 unit and leaves y unchanged;

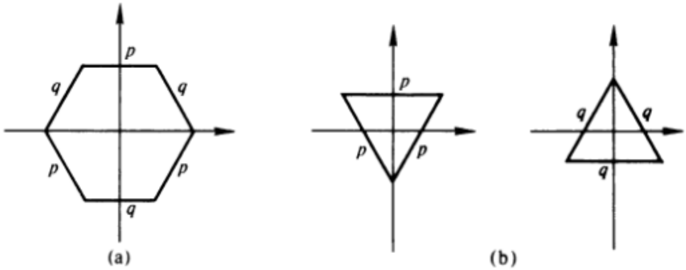
U_+ raises T_3 by $\frac{1}{2}$ unit and raises y by 1 unit;

V_+ raises T_3 by $\frac{1}{2}$ unit and raises y by 1 unit; etc $\cdots$

If the units of T_3 and Y are appropriately scaled in the graph, these raising and lowering operators connect points along lines that are multiple of 60^0 with repsect to each other.



Each irrep of $SU(3)$ is characterized by a set of two integers (p, q) . Graphically it shows up as a figure with a hexagonal boundary on the $T_3 - Y$ plane: three sides having p units of length and other sides having q units: the hexagon collapses into an equilateral triangle when either p or q vanishes. The boundary is symmetric under reflection about $Y-$axis. We recall that an $SU(2)$ irreps is characterized by one integer j; graphically it is a straight line of $2j$ length. There are $2j+1$ sites, each of them singly occupied by one state. For the $SU(3)$ representation (p, q) the multiplicity of states on each site in the $T_3 - Y$ plane form the following pattern : the sites in the boundary are singly occupied, on the next layer they are doubly occupied, on the third layer they are triply occupied, etc., until a triangular layer is reached beyond which the multiplicity ceases to increase and remians to be $q+1$ for $p > q$ $(p+1$ for $q > p)$.



To get all the states within a given irrep, start with state with largest value of T_3, ψ_{max}, which can be characterized as

$$T_+\psi_{\text{max}} = V_+\psi_{\text{max}} = U\_\psi_{\text{max}} = 0$$

Repeatly apply V_- on ψ_{max}, it has to vanish at some point, say after $p+1$ steps,

$$(V_-)^{p+1}\,\psi_{\text{max}} = 0$$

At the step $(V_-)^p\,\psi_{\text{max}}$, we can apply T_- until it reach another corner,

$$(T_-)^{q+1}\,(V_-)^p\,\psi_{\text{max}} = 0$$

Then we use U_+, V_+, T_+, U_- until it comes back ψ_{max}. This completes the outer layer of the representation. For the next layer we can start with state ψ'_{max} which has same properties as ψ_{max}, *i.e.*

$$T_+\psi'_{\text{max}} = V_+\psi'_{\text{max}} = U\_\psi'_{\text{max}} = 0$$

but with T_3 quantum number one unit less than ψ_{max}. Repeat the procedures as described above.

Remarks :

1. The boundary is always convex.
   This can be seen as follows. If there is a state at site C as shown in the figure, we can write

$$U_+\psi_C = \lambda\psi_A, \qquad \lambda \text{ some constant and } \psi_A = V_-\psi_B$$

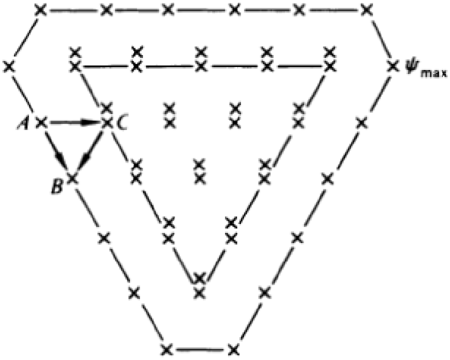
   Then

$$\lambda(\psi_A, \psi_A) = (V_-\psi_B, U_+\psi_C) = (\psi_B, V_+U_+\psi_C) = (\psi_B, U_+V_+\psi_C) = 0$$

   This implies $\lambda = 0$ and there is no state at site C.

2. The boundary will in general be a six-sided figure and is symmetric under rotation of 120^0. The boundary is also symmetric with repect to reflection in $Y - axis$.

3. There is one state at each site on the boundary, 2 states at each site on the next layer, 3 states at each site on the next layer, etc, until a triangluar layer is reached, beyond which the multiplicity ceases to increase. Denote the representation by the integers (p, q)



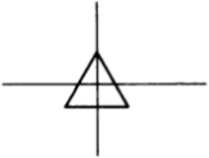
Then the number of states in the inner most triangle is

$$\sum_{l=1}^{p-q+1} l = \frac{1}{2}\left(p-q+1\right)\left(p-q+2\right)$$

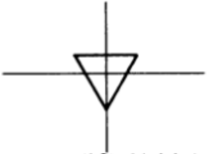
and the multiplicity for each state in the triangle is $(q+1)$. The number of states in the next layer is $3(p-q+2)$ with multiplicity q. Contiuing the counting of states in this fashsion, we can get the dimensionality of the irrep as

$$
\begin{aligned}
d(p,q) &= \frac{1}{2}(q+1)(p-q+1)(p+q+2) + 3\sum_{\nu=0}^{q}(q-\nu)(p-q+2\nu+2) \\
&= \frac{1}{2}(p+1)(q+1)(p+q+2)
\end{aligned}
$$

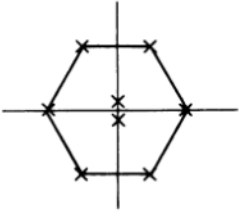
4. In general, there will be several states for a given value of (T_3, Y). The states on the $Y=$ constant line form isospin multiplet. For example, there are $p+1$ states on the top line and the isospin of these states is $T=\frac{p}{2}$. Simliarly the next line will have isospin $T=\frac{1}{2}(p-1)$, etc. It is not hard to see that $(T_3)_{\max}=\frac{1}{2}(p+q)$



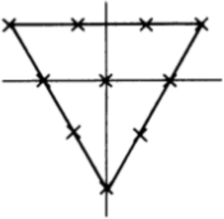
$(p,q)=(0,1)$, $\mathbf{3}^*$ (triplet)



$(p,q)=(1,0)$, $\mathbf{3}$ (triplet)



$(p,q)=(1,1)$, $\mathbf{8}$ (octet)



$(p,q)=(3,0)$, $\mathbf{10}$ (decuplet)