Introduction to Group Theory

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INTRODUCTION

Group theory : framework for studying symmetry. The representation theory of the group simplifies the physical solutions. For example, suppose *one*-dim Hamiltonian has the symmetry under parity $x \to -x$, i.e.

$$H(x) = H(-x)$$

Then from Schrödinger's equation,

$$H(x)\psi(x) = E\psi(x)$$

we get

$$H(-x)\psi(-x) = H(x)\psi(-x) = E\psi(-x)$$

$\implies \psi(-x)$ is eigenstate with same E. Form the linear combinations,

$$\psi_\pm = \frac{1}{\sqrt{2}}\left(\psi(x) \pm \psi(-x)\right)$$

which are parity eigenstates. Note that this means only that the eigenstates can be chosen to be either symmetric or antisymmetric and does not imply that the system has degenerate eigenstates.

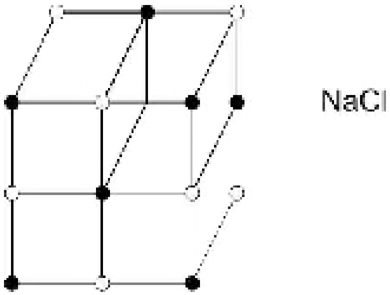
REMARK: Symmetry of H does not necessarily imply the symmetry of the eigenfunctions. It only says that given an eigenfunction, the symmetry operation will generate other solutions which may or may not be independent of the original eigenfunction.

Examples OF Symmetry Groups in Physics

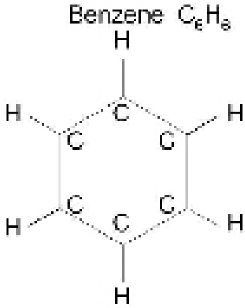
1. **Finite groups**
    1. Crystallographic groups (symmetry group of crystals)

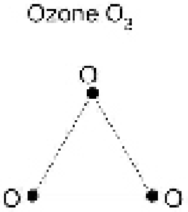
       $$\text{Symmetry operations} \left\{ \begin{array}{l} \text{translations - periodic} \\ \text{rotations - space group} \end{array} \right\}$$

    2. Example:
        1. *NaCl*



Molecules





3. Permutation group
   In QM, wavefunction of identical particles is either symmetric of antisymmetric under the permutation of the coordinates. These permutations form a group, permutation group $S_{n.}$ Permutation group also important in study of representations of unitary groups.

2. **Continuous groups**
   1. $O(3)$ or $SU(2)-$Rotation group in $3-$dimension
      This group describes theory of angular momentum in quantum mechanics. Structure of this group also provides the foundation for more complicated groups.

2. $SU(3)-$Special unitary 3×3 matrices
   This has been used to describe the spectrum of hadrons in terms of quark model. Symmetry here is only approximate. $SU(3)$ has also been used in QCD. Here symmetry is exact but has the peculiarity of confinement.
3. $SU(2)_L \times U(1)_Y$
   This symmetry group is used in standard model of electromagnetic and weak interactions. However the symmetry here is also broken (spontaneously).
4. $SU(5)$, $SO(10)$, $E(6)$
   These symmetries unify electromagnetic, weak and strong interactions, grand unified theories (GUT). Of course these groups are badly broken.
5. $SL(2, C)$ Lorentz group
   This group describes tspace-time structure in special relativity. It plays a crucial role in the relativistic field theory.

ELEMENT OF GROUP THEORY

Definition of Group

A group G consists of elements $(a, b, c\ldots)$ with an operstion * with properties:

(i) Closure : If $a, b, \in G \implies c = a * b$ is also in G.

(ii) Associative: $a * (b * c) = (a * b) * c$

(iii) Identity : $\exists$ an element e such that $a * e = e * a = a \quad \forall\, a \in G$

(iv) Inverse : for every $a \in G$, $\exists$ an element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$

We will denote the group operation $a * b$ by ab.

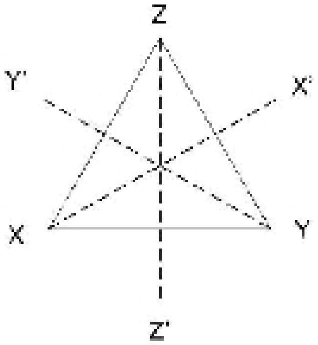
Examples of Group

1. All real numbers under "$+$"
2. All real numbers without "0" under "$\times$"
3. All integers under "$+$"
4. All rotations in 3-dimensional space: $O(3)$
5. All $n \times n$ matrices under "$+$"
6. All non-singular $n \times n$ matrices under "$\times$": $GL(n)$ (General Linear Group in n-dimension)
7. All $n \times n$ matrices with determinant 1: $SL(n)$ (Special Linear Group in n-dimension)
8. All $n \times n$ unitary matrices under "$\times$": $U(n)$ (Unitary Group in n-dimension)
9. All $n \times n$ unitary matrices with determinant 1: $SU(n)$ (Special Unitary Group in n-dimension)

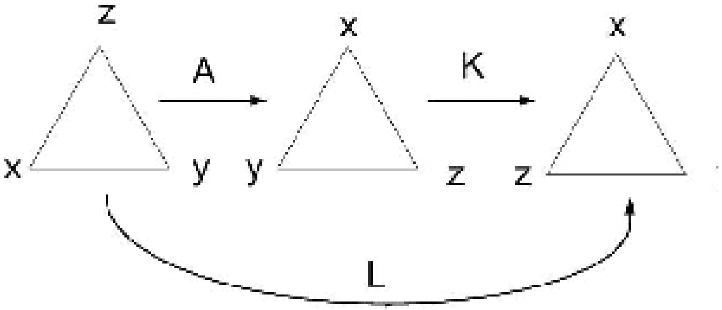
10. All $n \times n$ orthogonal matrices: $O(n)$
11. All $n \times n$ orthogonal matrices with determinant 1: $SO(n)$
12. Permutations of n objects: S_n

Abelian Group - If group multiplication is commutative, i.e. $ab = ba \quad \forall\, a, b \in G$.
Finite Group - If number of elements in G is finite.
Order of the Group - $\#$ of elements in the group.
Subgroup - a subset of the group which is also a group. *e.g.* $SL(n)$ is a subgroup of $GL(n)$.

Simple Example : Symmetry of a regular triangle (called D_3 group)



Operations
A: rotation $\curvearrowright$ by 120°in the plane of triangle
B: rotation $\curvearrowright$ by 240°in the plane of triangle
K: rotation $\curvearrowright$ by 180° about zz'
L: rotation $\curvearrowright$ by 180° about yy'
M: rotation $\curvearrowright$ by 180° about xx'
E: no rotation
Group Multiplication : consider the product KA



Thus we have $KA = L$

This way we can work out the multiplication of any 2 group elements and summarize the result in **multiplication table**.

	E	A	B	K	L	M
E	E	A	B	K	L	M
A	A	B	E	M	K	L
B	B	E	A	L	M	K
K	K	L	M	E	A	B
L	L	M	K	B	E	A
M	M	K	L	A	B	E

Clearly $\{E, A, B\}$, $\{E, L\}$, $\{E, K\}$, $\{E, M\}$ are subgroups

Isomorphism: Two groups $G = \{x_1, x_2, \ldots\}$ and $G' = \{x'_1, x'_2, \ldots\}$ are isomorphic if $\exists$ a one-to-one mapping $x_i \to x'_i$ such that

$$x_i x_j = x_k \implies x'_i x'_j = x'_k$$

In other words, the groups G and $G^{'}$, which might operate on different physical system, have the same structure as far as group theory is concerned.
Symmetry group S_3: permutation symmetry of 3 objects. S_3 has 6 group elements.

$$\begin{pmatrix} 123 \\ 123 \end{pmatrix}, \quad \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \quad \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \quad \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \quad \begin{pmatrix} 123 \\ 321 \end{pmatrix}, \quad \begin{pmatrix} 123 \\ 213 \end{pmatrix}$$

S_3 is isomorphic to D_3 by associate the vertices of the triangle with $1, 2$ and 3.

Rearrangement Theorem

Theorem : Each element of G appears exactly once in each row or column of multiplication table.

Proof: Take group elements to be $E, A_2, A_3, \ldots, A_h$.
Multiply by arbitrary A_k, we get $A_kE, A_kA_2, A_kA_3, \ldots, A_kA_h$.
If 2 elements e.g. $A_kA_i = A_kA_j$ where $A_i \neq A_j$.
Multiply this by A_k^{-1} to get $A_i = A_j$ which contradicts the initial assumption.
Hence all elements in each row after multiplication are different. $\Longrightarrow$ each group element occurs only once in each row. Thus multiplication of the group by a fixed element of the group, simply rearrange them. This is called the **rearrangement theorem**. ■

Applications of Rearrangement Theorem

1. Suppose we are summing over the group elements of some functions of group elements, $\sum_{A_i} f(A_i)$. Then rearrangement theorem implies that

$$\sum_{A_i} f(A_i) = \sum_{A_i} f(A_iA_k)$$

   for any $A_k \in G$. This result is central to many important result of the representation theory of finite groups. The validity of this theorem for continuous group is then an important requirement in generalized the results from the finite groups to continuous groups.

2. Using this theorem, we can show that there is only one group of order 3. From multiplication table

	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

In fact, this group is of the form $A, B = A^2, E = A^3$. This is the cyclic group of order 3.

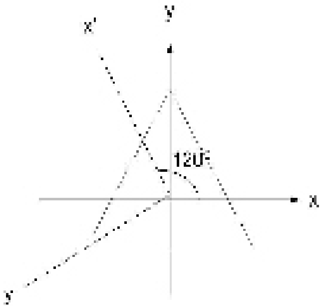
Cyclic Group of order n, is of the form, $Z_n = \left\{A, A^2 A^3, \ldots A^n = E\right\}$
Clearly, all cyclic groups are Abelian.
Examples of cyclic groups:

1. 4th roots of unity; $1, -1, i, -i = \left\{i, i^2 = -1, i^3 = -i, i^4 = 1, \ \right\}$
2. Benzene molecule

$$Z_6 = \left\{A, A^2 \ldots A^6 = E\right\} \qquad A = \text{ rotation by } \frac{\pi}{3}$$

Group Induced Transformations

In physics, many group transfomations are geometrical. Then group transfomations can be represented as operations in the coordinate space. As an example, take a coordinate system for the triangle as shown,



Keep the triangle fixed and rotate the coordinate system, we get the relations between the old and new coordinates as

$$x' = \cos\frac{2\pi}{3}x + \sin\frac{2\pi}{3}y = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

$$y' = -\sin\frac{2\pi}{3}x + \cos\frac{2\pi}{3}y = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y$$

or

$$\begin{pmatrix} x\,' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \vec{x}' = \mathbf{A}\vec{x} \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus group element A is represented by matrix $\mathbf{A}$ acting on the coordinate system (x, y). We can do this for other group elements to get,

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{L} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

The product of the group elements can also be expressed in terms of matrices, e.g

$$\vec{x}' = \mathbf{A}\vec{x} \qquad \vec{x}'' = \mathbf{K}\vec{x}' \Longrightarrow \vec{x}'' = \mathbf{KA}\vec{x}$$

$$\mathbf{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \mathbf{KA} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \mathbf{L}$$

Thus, the matrix multiplication gives same result as the multiplication table. This is an isomorphism between the symmetry group and the set of 6 matrices. This is an example of representation - group elements are represented by a set of matrices (does not have to be 1-1 correspondence).

Transformation of Functions

We can generalize the transformations of coordinates to functions of (x, y), $f(x, y)$.

(i) Take $A \in G$, which generate matrix A on (x, y).

(ii) Replace $\vec{x}$ by $A^{-1}x'$ in f. This defines a new function $g(x', y')$. Denote $g(x, y)$ by $g(x, y) = P_A f(x, y)$ or more simply $P_A f(x) = f\left(A^{-1}x\right)$.

Example: $f(x, y) = x^2 - y^2$, take

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

then

$$\begin{aligned} x' &= -\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y \\ y' &= -\tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= -\tfrac{1}{2}x' - \tfrac{\sqrt{3}}{2}y' \\ y &= \tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y' \end{aligned}$$

and

$$f(x, y) = \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 - \left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 = -\frac{1}{2}\left(x'^2 - y'^2\right) + \sqrt{3}x'y' = g\left(x', y'\right)$$

Or

$$g(x, y) = -\frac{1}{2}\left(x^2 - y^2\right) + \sqrt{3}xy = P_A f(x, y)$$

Symbolically, we have

$$f(\vec{x}) \to f\left(A^{-1}\vec{x}\right) = g(\vec{x}) \qquad \text{or } P_A f(\vec{x}) = g(\vec{x}) = f\left(A^{-1}\vec{x}\right)$$

<u>Theorem</u>: If A's form a group G, the P_A's defined on certain function $f(\vec{x})$ also form a group G_P.
Proof:

$$P_A f(\vec{x}) = f\left(A^{-1}x\right) = g(\vec{x})$$

$$P_B P_A f(\vec{x},) = P_B g(\vec{x}) = g\left(B^{-1}\vec{x}\right) = f\left(A^{-1}\left(B^{-1}\vec{x}\right)\right) = f\left(A^{-1}B^{-1}\vec{x}\right) = f\left((BA)^{-1}\vec{x}\right) = P_{BA} f(\vec{x}$$

Thus we have $P_B P_A = P_{BA} \implies G$ is homomorphic to G_P. But the correspondence $A \to P_A$ is not necessarily one-to-one.
Note that we define P_A in terms of A^{-1} in order to get the homomorphism $P_B P_A = P_{BA}$.
For example if we have the function $f(x, y) = x^2 + y^2 \implies P_A = P_B = P_E = \ldots = 1$.

Coset

Coset is a useful tool to decompose the group into disconnect sets. Let $H = \{E, S_2, S_3 \dots S_g\}$ be a subgroup of G.
For $x \in G$ but $\notin H$, if we multiply the whoe subgroup by x on the right we get

$$\{Ex, S_2x, S_3x, \dots S_gx\} \qquad \underline{\text{right coset of } x}\text{, denoted by } Hx$$

and the left multiplication gives,

$$\{xE, xS_2, xS_3, \dots xS_g\} \qquad \underline{\text{left coset of } x}\text{, denoted by } xH$$

Note that a coset can not form a group, because identity is not in the set.

Properties of Cosets

(i) Hx and H have no elements in common.
Suppose there is one element in common,

$$S_k = S_jx, \qquad \text{where } x \notin H$$

then

$$x = S_j^{-1}S_k \in H$$

a contradiction because $x \notin H$ by construction.

(ii) Two right (or left) cosets either are identical or have no element in common. Consider Hx and Hy, with $x \neq y$.Suppose there is one element in common between these 2 cosets

$$S_k x = S_j y$$

then

$$xy^{-1} = S_k^{-1} S_j \in H$$

But $Hxy^{-1} = H$ by rearrangement theorem which implies that $Hx = Hy$

Theorem: Order of a subgroup H is a factor of order of G.
Proof: Consider all distinct right cosets

$$H, \; Hx_{2,} \; Hx_3, \ldots, Hx_l$$

Each element of G must appear in exactly one of cosets. Since no elements in common among cosets, we must have $g = l \times h$ where h the order of H, l some integer and g order of G.■
Remark: For example, a group of order 6, like D_3, the only non-trivial subgroups are those with orde 2 or 3.

Conjugate: B and A are conjugate to each other if $\exists\, x \in G$ such that

$$xAx^{-1} = B \qquad \text{(similarity transformation)}$$

Remark: Replacing each element by its conjugate under some fixed element x is an isomorphism under x. This can be seen as follows. From

$$A' = xAx^{-1} \qquad B' = xBx^{-1}$$

$$A'B' = \left(xAx^{-1}\right)\left(xBx^{-1}\right) = xABx^{-1} = (AB)'$$

we see that $g_i \rightarrow g_i' = xg_ix^{-1}$ is an isomorphism because the correspondence is one-to-one. In coordinate transformations, similarity transformations correspond to change of basis and represents same operation.

Coset Space

$G/H = \{\text{cosets } Hx,\ x \in G \text{ but not in } H\}$

Coset space is obtained by grouping together elements which are related by left (or right) multiplication of elements in the subgroup H. This decomposition is useful in reducing the structure of the group to a smaller structure.

Class

All group elements conjugate to a given element is called a *class*. These are group elements which are the same operation with reapect to differenent basis

Denote the group elements by $G = \{E, x_2, \ldots, x_n\}$

Take $A \in G$, then

$$\left.\begin{array}{c} EAE^{-1} \\ x_2 A x_2^{-1} \\ \vdots \\ x_h A x_h^{-1} \end{array}\right\} \text{class (all group elements conjugate to } A\text{). Note that these elements are not}$$

necessarily all different.

<u>Example:</u> symmetry group of triangle D_3 From the multiplication table,

$$\begin{aligned} AAA^{-1} &= A & AKA^{-1} &= L \\ BAB^{-1} &= A & BKB^{-1} &= M \\ KAK^{-1} &= B & KKK^{-1} &= K \\ LAL^{-1} &= B & LKL^{-1} &= M \\ MAM^{-1} &= B & MKM^{-1} &= L \end{aligned}$$

the classes are $\{E\}, \{A, B\}, \{K, L, M\}$.

Note: E is always in a class by itself because $A_i^{-1} E A_i = E A_i^{-1} A_i = E, \quad \forall A_i \in G$. Here $\{A, B\}-$ rotations by $\frac{2\pi}{3}$ and $\{K, L, M\}-$ rotations by π. This is a very general feature–all elements in the same class have same angle of rotation.

Invariant Subgroup: If a subgroup H of G consists entirely of complete classes. For example $H = \{E, A, B\}$ is an **invariant subgroup** while $\{E, K\}$ is not. Invariant subgroup is also called **normal subgroup** or normal divisor. Symbolically for invariant subgroups, we have $xHx^{-1} = H$ for any $x \in G$. which implies $xH = Hx$, i.e. left cosets are the same as right cosets.
For any G, there are at two trivial invariant subgroups, $\{E\}$ and the group G itself. If a group only has these two invariant subgroups, then it is called a *simple* group. Examples of simple groups are cyclic groups of prime order.

Factor Group (or Quotient Group)

Consider the invariant subgroup $H = \{E, h \ldots h_\ell\}$ of G and the collection of all distinct left (or right) cosets $[a] \equiv aH$, $[b] \equiv bH, \ldots$ (where in this notation $[E] = H$).
Define the multiplication of cosets as follows: Suppose $r_1 \in [a], r_2 \in [b]$ and $r_1 r_2 = R'$. Then we define $[a][b] = [R']$. Let $a, b \in G$ but not in H, then the product of elements from these two cosets can be written as

$$(ah_i)(bh_j) = ab\left(b^{-1}h_i b\right)h_j = ab\left(h_k h_j\right) \in \text{coset containing } ab$$

where $h_k = b^{-1}h_i b$. Notation: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two classes, then $\mathcal{C}_1 = \mathcal{C}_2$ means that $\mathcal{C}_1$ and $\mathcal{C}_2$ have same collection of group elements. Thus the coset multiplication is well-defined and is analogous to multiplication of the group elements. It is not hard to see that the collection of these cosets of H forms a group, called the **Quotient Group** and is denoted by G/H.
Theorem: If $\mathcal{C}$ is a class, for any $x \in G$, we have $x\mathcal{C}x^{-1} = \mathcal{C}$.
Proof: Write $\mathcal{C} = \{A_1, A_2, \ldots, A_j\}$ then $x\mathcal{C}x^{-1} = \left\{xA_1x^{-1}, xA_2x^{-1}, \ldots, xA_jx^{-1}\right\}$.
Take any element in $x\mathcal{C}x^{-1}$ say xA_ix^{-1}. This is related to A_i by conjugation. xA_ix^{-1} is in the class containing A_i and hence $xA_ix^{-1} \in \mathcal{C}$. Thus, each element in $x\mathcal{C}x^{-1}$ must appear in $\mathcal{C}$ because $\mathcal{C}$ is a class. But all elements in $x\mathcal{C}x^{-1}$ are different. Therefore $x\mathcal{C}x^{-1} = \mathcal{C}$.
Theorem: Any collection $\mathcal{C}$ which satisfies $x\mathcal{C}x^{-1} = \mathcal{C}$ for all $x \in G$ consists wholly of complete classes.
Proof: First we can subtract out all complete classes from both sides of the equation. Let remainder by R. So we have $xRx^{-1} = R$. Suppose R is not a complete class. This means that there exists some element A_i which is related to some element $R_j \in R$ by conjugate and $A_i \notin R$, i.e.

$$A_i = yR_jy^{-1} \quad \text{and} \quad A_i \notin R$$

But this violates the assumption $xRx^{-1} = R$ for $\underline{\text{all } x \in G}$.

Class Multiplication

For 2 classes $\mathcal{C}_{i,} \mathcal{C}_j$ in G, we have from the previous theorems

$$\mathcal{C}_i \, \mathcal{C}_j = \left(x^{-1} \, \mathcal{C}_i x\right) \left(x^{-1} \, \mathcal{C}_j x\right) = x^{-1} \left(\mathcal{C}_i \, \mathcal{C}_j\right) x \qquad \forall \, x \in G \, .$$

Thus $\mathcal{C}_i \, \mathcal{C}_j$ consists of complete classes, and we can write

$$\mathcal{C}_i \, \mathcal{C}_j = \sum_k c_{ijk} \, \mathcal{C}_k$$

where c_{ijk} are some integers.

Direct Product of Two Groups

Given two groups $G = \{x_i, i = 1, \ldots, n\}$, $G' = \{y_j, j = 1, \ldots, m\}$, the direct product group is defined as

$$G \otimes G' = \{(x_i, y_j)\,;\, i = 1, \ldots n, j = 1, \ldots, m\}$$

with group multiplication defined by

$$(x_i, y_j) \times (x_{i'}, y_{j'}) = (x_i x_{i'}, y_j y_{j'})$$

It is clear that $G \otimes G'$ forms a group. Note that $(E, y_j) \times (x_i, E') = (x_i, y_j) = (x_i, E') \times E$, $\Longrightarrow$ G and G' are subgroups of $G \otimes G'$ with the property that group elements from G commutes with group elements from G'.

We can generalize this to define direct product of 2 subgroups. Let S and T be subgroups of G such that S and T commute with each other,

$$s_i t_j = t_j s_i \qquad \forall\; s_i \in S, t_j \in T.$$

Then we can define the direct product $S \otimes T$ as

$$S \otimes T = \{s_i t_j \,|\, s_i \in S, t_j \in T\}$$

Example:

$$Z_2 = \{1, -1\} \qquad\qquad Z_3 = \left\{1, e^{2\pi i/3}, e^{4\pi i/3}\right\}$$

$$
\begin{aligned}
Z_2 \otimes Z_3 &= \left\{1, e^{2\pi i/3}, e^{4\pi i/3}, -1, -e^{2\pi i/3}, -e^{4\pi i/3}\right\} \\
&= \left\{1, e^{2\pi i/3}, e^{4\pi i/3}, e^{i\pi}, e^{5\pi i/3}, e^{\pi i/3}\right\}
\end{aligned}
$$

Clearly, this is isomorphic to Z_6.