Group Theory in Physics

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Theory of Group Representation

In phyiscal application, group representation is important in deducing the consequence of the symmetries of the system. Most important one is realization of group operation by matrices.

Definition of Representation

Given a group $G = \{A_{i,} i = 1 \cdots n\}$. If for each $A_i \in G$, there are an $n \times n$ matrices $D(A_i)$ such that

$$D(A_i) D(A_j) = D(A_i A_j) \tag{1}$$

then D's forms a n-dimensional representation of the group G i.e. correspondence $A_i \to D(A_i)$ is a homomorphism. If this is an $1-1$ isomorphism, representation is called faithful. A matrix M_{ij} can be viewed as linear operator M acting on some vector space V with repect to some basis e_i,

$$Me_i = \sum_j e_j M_{ji}$$

Note that the choice of the basis is not unique. If we make a change to a new basis,

$$e_i = \sum_j f_n S_{ni}, \qquad S \quad \text{non-singular}$$

then

$$\sum_n M f_n S_{ni} = \sum_{k,j} f_k S_{kj} M_{ji}$$

Multiply by S^{-1}_{im},

$$Mf_m = \sum_{k,j,i} f_k S_{kj} M_{ji} S^{-1}_{im} = \sum_k f_k \left(SMS^{-1}\right)_{km} = \sum_k f_k \left(M'\right)_{km}$$

where

$$M' = SMS^{-1}$$

To generate such matrices for the symmetry of certain geometric objects is to use the group induced transformations, discussed before. Recall that each element A_a induces a transformation of the coordinate vector $\vec{r}$,

$$\vec{r} \rightarrow A_a \vec{r}$$

Then for any function of $\vec{r}$, say $\varphi(\vec{r})$ and for any group element A_a define a new transformation P_{A_a} by

$$P_{A_a} \varphi(\vec{r}) = \varphi\left(A_a^{-1} \vec{r}\right)$$

Among the transformed functions obtained this way, $P_{A_1}\varphi(\vec{r}), P_{A_2}\varphi(\vec{r}), \cdots P_{An}\varphi(\vec{r})$, we select the linearly independent set $\varphi_1(\vec{r}), \varphi_2(\vec{r}) \cdots \varphi_\ell(\vec{r})$.Then it is clear that $P_A \varphi_a$ can be expressed as linear combination of φ_i.This is because

$$P_{A_b}\varphi_a = P_{A_b} P_{A_a} \varphi(\vec{r}) = P_{A_b} \varphi\left(A_a^{-1}\vec{r}\right) = \varphi\left(A_a^{-1}\left(A_b^{-1}\vec{r}\right)\right) = \varphi\left(\left(A_b A_a\right)^{-1} \vec{r}\right)$$

Thus we can write

$$P_{A_i}\varphi_a = \sum_{b=1}^{\ell} \varphi_b D_{ba}(A_i)$$

and $D_{ba}(A_i)$ forms a representation of G. This can be seen as follows.

$$P_{A_iA_j}\varphi_a = P_{A_i}P_{A_j}\varphi_a = P_{A_i}\sum_b \phi_b D_{ba}(A_j) = \sum_{b.c.} \phi_c D_{cb}(A_i) D_{ba}(A_j)$$

On the other hand,

$$P_{A_iA_j}\varphi_a = \sum_c \varphi_c D(A_iA_j)_{ca}$$

This gives

$$D(A_iA_j)_{ca} = D_{cb}(A_i) D_{ba}(A_j)$$

which means that $D(A_i)'s$ form representation of the group.

Example: Group D_3, symmetry of the triangle.
As seen before, choosing a coordinate system on the plane, we can represent the group elements by the following matrices,

$$A=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},\qquad B=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},\qquad E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$K=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\qquad L=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},\qquad M=\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

Choose $f\left(\vec{r}\right)=f\left(x,y\right)=x^2-y^2$, we get

$$P_A f\left(\vec{r}\right)=f\left(A^{-1}\vec{r}\right)=\frac{1}{4}\left(x+\sqrt{3}y\right)^2-\frac{1}{4}\left(\sqrt{3x}-y\right)^2=-\frac{1}{2}\left(x^2-y^2\right)+\sqrt{3}xy.$$

We now have a new function $g\left(x,y\right)=2xy$ showing up. We can operate on $g\left(r\right)$ to get,

$$\begin{aligned}P_A g\left(\vec{r}\right) &= g\left(A^{-1}\vec{r}\right)=2\left(-\frac{1}{2}\right)\left(x+\sqrt{3}y\right)\frac{1}{2}\left(\sqrt{3x}-y\right)\\ &= -\frac{1}{2}\left[\left(\sqrt{3}\right)\left(x^2-y^2\right)-2xy\right]=-\frac{\sqrt{3}}{2}\left(x^2-y^2\right)-\frac{1}{2}\left(2xy\right)\end{aligned}$$

Thus we have

$$P_A\left(f,g\right)=\left(f,g\right)\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

The matrix generated this way is the same as A if we change g to -g.
Similarly

$$
\begin{aligned}
P_B f\left(\vec{r}\right) &= f\left(B^{-1}\vec{r}\right) = \frac{1}{4}\left(x-\sqrt{3}y\right)^2 - \frac{1}{4}\left(\sqrt{3}x+y\right)^2 = -\frac{1}{2}\left(x^2-y^2\right) - \sqrt{3}xy \\
P_B g\left(\vec{r}\right) &= g\left(B^{-1}\vec{r}\right) = 2\left(\frac{1}{2}\right)\left(x-\sqrt{3}y\right)\left(-\frac{1}{2}\right)\left(\sqrt{3}x+y\right) \\
&= \sqrt[+]{\frac{1}{2}}\left[\sqrt[+]{3}\left(x^2-y^2\right) - 2xy\right] = \frac{\sqrt{3}}{2}\left(x^2-y^2\right) - \frac{1}{2}\left(2xy\right)
\end{aligned}
$$

$$
P_B\left(f,g\right) = \left(f,g\right)\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
$$

same as B
Remarks

1. If $D^{(1)}\left(A\right)$ and $D^{(2)}\left(A\right)$ are both representation of the group, then

$$
D^{(3)}\left(A\right) = \begin{pmatrix} D^{(1)}\left(A\right) & 0 \\ 0 & D^{(2)}\left(A\right) \end{pmatrix} \qquad \text{(block diagonal form)}
$$

also forms a representation. We will denote it as a direct sum $\oplus$,

$$
D^{(3)}\left(A\right) = D^{(1)}\left(A\right) \oplus D^{(2)}\left(A\right) \qquad \text{direct sum}
$$

2. If $D^{(1)}(A)$ and $D^{(2)}(A)$ are 2 representations of G with same dimension and $\exists$ a non-singular matrix U such that

$$D^{(1)}(A_i) = UD^{(2)}(A_i)\,U^{-1} \qquad \text{for all } A_i \in G.$$

then $D^{(1)}$ and $D^{(2)}$ are equivalent representations .

Reducible and Irreducible Representations

A representation D of a group G is called **irreducible representation** (irrep) if it is defined on a vector space $V(D)$ which has no non-trivial invariant subspace. Otherwise, it is reducible. In other words, all group actions can be realized in this subspace.

Suppose representaion D is reducible on the vector space V. Then $\exists$ subspace S invariant under D. For any vector $v \in V$, we can write,

$$v = s + s_{\perp}$$

where $s \in S$ and $s_{\perp}$ belongs to the complement $S_{\perp}$ of S. If we write,

$$v = \begin{pmatrix} s \\ s_{\perp} \end{pmatrix}$$

then the representation matrix is

$$Av = D(A)\, v = \begin{pmatrix} D_1(A) & D_2(A) \\ D_3(A) & D_4(A) \end{pmatrix}$$

Since S is invariant under group operators, we get

$$D_3(A_i) = 0, \qquad \forall\ A_i \in G$$

i.e. $D(A_i)$ are all of the upper triangular form,

$$D(A_i) = \begin{pmatrix} D_1(A_i) & D_2(A_i) \\ 0 & D_4(A_i) \end{pmatrix}, \qquad \forall\ A_i \in G \tag{2}$$

Note that

$$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} \begin{pmatrix} D_1' & D_2' \\ 0 & D_4' \end{pmatrix} = \begin{pmatrix} D_1 D_1' & D_1 D_2' + D_2 D_4' \\ 0 & D_4 D_4' \end{pmatrix}$$

Thus invariant subspace is still invariant if operate on it many times.
A representation is **completely reducible** if all matrices $D(A_i)$ can be simultaneously brought into block diagonal form by the same similarity transformation U,

$$UD(A_i)U^{-1} = \begin{pmatrix} D_1(A_i) & 0 \\ 0 & D_2(A_i) \end{pmatrix}, \qquad \text{for all } A_i \in G$$

Theorem: Any **unitary** reducible representation is completely reducible.
Proof: For simplicity we assume that the vector space V is equipped with a scalar product (u, v). We can choose the complement space $S\perp$ to be perpendicular to S, i.e.

$$(u, v) = 0, \qquad \text{if} \quad u \in S, v \in S_\perp$$

Since scalar product is invariant under the unitary transformation,

$$0 = (u, v) = (D(A_i)u, D(A_i)v)$$

Thus if $D(A_i)\, u \in S$, then $D(A_i)\, v \in S_\perp$ which implies that $S_\perp$ is also invariant under the group operation.■

In physical applications, we deal mostly with unitary representations and they are completely reducible.

Unitary Representation

Since unitary operators preserve the scalar product, representation by unitary matrices will simplify the analysis of group theory. For finite groups, we can show that rep can always be transformed into unitay one.

Fundamental Theorem

Every irrep of a finite group is equivalent to a unitary irrep (rep by unitary matrices)

Proof:

Let $D(A_r)$ be a representation of the group $G = \{E, A_2 \cdots A_n\}$

Consider the sum

$$H = \sum_{r=1}^{n} D(A_r)\, D^\dagger(A_r) \qquad \text{then } H^\dagger = H$$

Since H is positive semidefinite, define squre root h by

$$h^2 = H, \quad h^+ = h$$

This can be achieved by diagonizing this hermitian matrix H by unitary transformation and then transforming it back after taking the square root of the eigvalues. Define new set of matrices by similarity transformation

$$\bar{D}(A_r) = h^{-1} D(A_r)\, h \qquad r = 1, 2, \cdots, n$$

$\bar{D}(A_r)$ is equivalent to $D(A_r)$. We will now show that $\bar{D}(A_r)$ is unitary,

$$
\begin{aligned}
\bar{D}\left(A_r\right)\bar{D}^{\dagger}\left(A_r\right) &= \left[h^{-1}D\left(A_r\right)h\right]\left[hD^{\dagger}\left(A_r\right)h^{-1}\right] \\
&= h^{-1}D\left(A_r\right)\ \sum_{s=1}^{n}\left[D\left(A_S\right)D^{\dagger}\left(A_S\right)\right]D^{\dagger}\left(A_r\right)h^{-1} \\
&= h^{-1}\left[\sum_{s=1}^{n}D\left(A_rA_s\right)D^{\dagger}\left(A_rA_s\right)\right]h^{-1} \\
&= h^{-1}\sum_{s'=1}^{n}D\left(A_{S'}\right)D^{\dagger}\left(A_{s'}\right)h^{-1} = h^{-1}h^2h^{-1} = 1
\end{aligned}
$$

where we have used rearrangement theorem.■

Schur's Lemma–an important theorems in irreducible reprentation

(i) Any matrix which commutes with all matrices of irrep is a multiple of identity matrix.
Proof: Assume $\exists\ M$

$$MD(A_r) = D(A_r)M \qquad \forall\ A_r \in G$$

Hermitian conjugate

$$D^\dagger(A_r)M^\dagger = M^\dagger D^\dagger(A_r)$$

Since $D(A_r)$ is unitary, we get

$$M^\dagger = D(A_r)M^\dagger D^\dagger(A_r) \quad \text{or} \quad M^\dagger D(A_r) = D(A_r)M^\dagger$$

$\Longrightarrow M^\dagger$ also commutes with all D's and so are $M + M^\dagger$ and $i\left(M - M^\dagger\right)$, which are hermitian. Thus, we take M to be hermitian. Start by diagonalizing M by unitary matrix U,

$$M = UdU^\dagger \qquad\qquad d: \text{diagonal}$$

Define $\quad \bar{D}(A_r) = U^\dagger D(A_r)U,\quad$ then we have

$$d\bar{D}(A_r) = \bar{D}(A_r)d$$

or in terms of matrix elements,

$$\sum_{\beta} d_{\alpha\beta} \bar{D}_{\beta r} \left(A_s \right) = \sum_{\beta} \bar{D}_{\alpha\beta} \left(A_s \right) d_{\beta\gamma}$$

Since d is diagonal, we get

$$\left(d_{\alpha\alpha} - d_{\gamma\gamma} \right) \bar{D}_{\alpha\gamma} \left(A_s \right) = 0 \Longrightarrow \qquad \text{if} \;\; d_{\alpha\alpha} \neq d_{\gamma\gamma}, \;\text{then}\; \bar{D}_{\alpha\gamma} \left(A_s \right) = 0$$

$\Longrightarrow$ if diagonal elements d_{ii} are all different, then off-diagonal elements of $\bar{D}$ are all zero. The only possible non-zero off-diagonal elements of $\bar{D}$ can arise when some of $d'_{\alpha\alpha}s$ are equal. For example, if $d_{11} = d_{22}$, then $\bar{D}_{12}$can be non-zero. Thus $\bar{D}$ is in block diagonal form, i.e.

$$\text{if } d = \begin{bmatrix} d_1 & & & & & & & \\ & d_1 & & & & & & \\ & & \ddots & & & & & \\ & & & d_1 & & & & \\ & & & & d_2 & & & \\ & & & & & \ddots & & \\ & & & & & & d_2 & \\ & & & & & & & \ddots \end{bmatrix} \text{ then}$$

$$\bar{D} = \begin{bmatrix} \boxed{D_1} & 0 & \\ 0 & \boxed{D_2} & \\ & & \ddots \end{bmatrix}$$

This is true for every matrix in the rep. Thus all the matrices in rep are block diagonal. But D is irreducible $\implies$ not all matrices can be block diagonal . Thus all d_i's have to be equal

$$d = cI. \qquad \text{or} \qquad M = UdU^{\dagger} = dUU^{\dagger} = d = cI$$

(ii) If the only matrix that commutes with all the matrices of a representation is a multiple of identity, then the representation is irrep.
Proof: Suppose D is reducible, then we can transform them into

$$D(A_i) = \begin{bmatrix} D^{(1)}(A_i) & \\ & D^{(2)}(A_i) \end{bmatrix} \qquad \text{for all } A_i \in G$$

construct $M = \begin{bmatrix} I & 0 \\ 0 & 2I \end{bmatrix}$ then

$$D(A_i) M = M D(A_i) \qquad \text{for all } i$$

But M is not a multiple of identity (contradiction). Therefore D must be irreducible.

Remarks

1. Any irrep of Abelian group is 1−dimensional. Because for any element A, $D(A)$ commutes with all $D(A_i)$. Then Schur's lemma $\Longrightarrow D(A) = cI \quad \forall A \in G$. But D is irrep, so D has to be 1×1 matrix.
2. In any irrep, the identity element E is always represented by identity matrix. This follows Schur's lemma.
3. From $D(A) D\left(A^{-1}\right) = D(E) = I.$, $\Longrightarrow D\left(A^{-1}\right) = [D(A)]^{-1}$and for unitary representation $D\left(A^{-1}\right) = D^{\dagger}(A)$.

(iii) If $D^{(1)}$ and $D^{(2)}$ are irreps of dimension l_1, and dimension l_2 and

$$MD^{(1)}(A_i) = D^{(2)}(A_i)\,M. \tag{3}$$

then (a) if $l_1 \neq l_2 \quad M = 0$
(b) if $l_1 = l_2$, then either $M = 0$ or $\det M \neq 0$ and reps are equivalent.
Proof: : take $l_1 \leq l_2$. Hermitian conjugate of Eq(3) gives

$$D^{(1)\dagger}M^{\dagger} = M^{\dagger}D^{(2)\dagger}, \quad MM^{\dagger}D^{(2)}(A_i)^{\dagger} = MD^{(1)}(A_i)^{\dagger}M^{\dagger} = D^{(2)}(A_i)^{\dagger}MM^{\dagger}$$

or

$$\left(MM^{\dagger}\right)D^{(2)}(A_i) = D^{(2)}(A_i)\left(MM^{\dagger}\right) \qquad \forall (A_i) \in G$$

Then from Schur's lemma (i) we get $MM^{\dagger} = cI$,where I is a l_2-dimensional identity matrix.
First consider the case $l_1 = l_2$, where we get $\left|\det M\right|^2 = c^{\ell_1}$. Then either $\det M \neq 0$, $\Rightarrow$ M is non-singular and from Eq(3)

$$D^{(1)}(A_i) = M^{-1}D^{(2)}(A_i)\,M \qquad \forall (A_i) \in G$$

This means $D^{(1)}(A_i)$ and $D^{(2)}(A_i)$ are equivalent. Otherwise if the determinant is zero,

$$\det M = 0 \quad \Longrightarrow c = 0 \quad \text{or } MM^{\dagger} = 0 \quad \Longrightarrow \quad \sum_{\gamma} M_{\alpha\gamma}M^{*}_{\beta\gamma} = 0 \quad \forall \alpha.\beta.$$

In particular, for $\alpha = \beta \quad \sum_{\gamma} |M_{\alpha\gamma}|^2 = 0 \quad M_{\alpha\gamma} = 0$ for all $\alpha.\gamma \Longrightarrow M = 0$.
Next, if $l_1 < l_2$, then M is a rectangular $l_2 \times l_1$, matrix

$$M = \underbrace{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}}_{l_1} l_2$$

Define a square matrix by adding colums of zeros

$$N = \overbrace{[M, 0]}^{l_2}\}l_2 \qquad l_2 \times l_2 \text{ square matrix}$$

then

$$N^\dagger = \begin{pmatrix} M^\dagger \\ 0 \end{pmatrix} \quad \text{and} \quad NN^\dagger = (M, 0)\begin{pmatrix} M^\dagger \\ 0 \end{pmatrix} = MM^\dagger = cI$$

where I is the $l_2 \times l_2$ identity matrix. But from construction $\det N = 0$. Hence $c = 0, \Longrightarrow NN^\dagger = 0$ or $M = 0$ identically. ■

Great Orthogonality Theorem–most important theorem for the representation of the finite group.

Theorem(Great orthogonality theorem): Suppose G a group with n elements, $\{A_i, i = 1, 2, \cdots n\}$, and $D^{(\alpha)}(A_i)$, $\alpha = 1, 2 \cdots$ are all the inequivalent irreps of G with dimension l_α.

Then

$$\sum_{\alpha=1}^{n} D_{ij}^{(\alpha)}(A_\alpha) D_{k\ell}^{(\beta)*}(A_\alpha) = \frac{n}{l_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{j\ell}$$

Proof: Define

$$M = \sum_a D^{(\alpha)}(A_a) X D^{(\beta)}(A_a^{-1})$$

where X is an arbitrary $l_\alpha \times l_\beta$ matrix. Multiplying M by representation matrix,

$$\begin{aligned} D^{(\alpha)}(A_b) M &= D^{(\alpha)}(A_b) \sum_a D^{(\alpha)}(A_a) X D^{(\beta)}(A_a^{-1}) \left[D^{(\beta)}(A_b^{-1}) D^{(\beta)}(A_b) \right] \\ &= \sum_a D^{(\alpha)}(A_b A_a) X D^{(\beta)}\left((A_b A_a)^{-1} \right) D^{(\beta)}(A_b) = M D^{(\beta)}(A_b) \end{aligned}$$

(i) If $\alpha \neq \beta$, then $M = 0$ from Schur's lemma, we get

$$M = \sum_a D_{ir}^{(\alpha)}(A_a) X_{rs} D_{sk}^{(\beta)}(A_a^{-1}) = \sum_a D_{ir}^{(\alpha)}(A_a) X_{rs} D_{ks}^{(\beta)*}(A_a) = 0$$

Choose $X_{rs} = \delta_{rj}\delta_{sl}$ (i.e. X is zero except the jl element). Then

$$\sum_a D_{ij}^{(\alpha)}(A_\alpha) D_{k\ell}^{(\beta)*}(A_\alpha) = 0$$

$\Longrightarrow$ for different irreps , the matrix elements, after summing over group elements, are orthogonal to each other.

(ii) $\alpha = \beta$ then we can write $M = \sum_a D^{(\alpha)}(A_a) X D^{(\alpha)}(A_a^{-1})$ .This implies

$$D^{(\alpha)}(A_a) M = M D^{(\alpha)}(A_b) \qquad \Longrightarrow \qquad M = cI$$

Then

$$\sum_a T_r \left[D^{(\alpha)}(A_a) X D^{(\alpha)}(A_a^{-1}) \right] = cl_2 \quad \text{or } n T_r X = cl_2, \quad \text{or } c = \frac{(T_r X)\, n}{l_\alpha}$$

Take $\quad X_{rs} = \delta_{rj}\delta_{s\ell}$ then $T_r X = \delta_{j\ell}$ and

$$\sum_a D^{(\alpha)}(A_a)_{ij} D^{(\alpha)}(A_a)^*_{k\ell} = \frac{n}{l_\alpha}\delta_{ik}\delta_{j\ell}$$

This gives orthogonality for different matrix elements within a given irreducible representation. ■

Geometric Interpretation

Imagine a complex $n-$dim vector space, axes (or componenets) are labeled by group elements $E, A_2.A_3...A_n$(Group element space). Components of vector are made out of matrix element of irreducible representation matrix $D^{(\alpha)}(A_a)_{ij}$ .Each vector in this $n-$dim space is labeled by 3 indices, $i, \mu.\nu$

$$\vec{D}^{(i)}_{\mu\nu} = \left(D^{(i)}_{\mu\nu}(E), D^{(i)}_{\mu\nu}(A_2), \cdots D^{(i)}_{\mu\nu}(A_n) \right)$$

Great orthogonality theorem $\Longrightarrow$ these vectors are $\perp$ to each other. As a result

$$\sum_i l_i^2 \leq n$$

because no more than n mutually $\perp$ vectors in n-dim vector space.
As an example, we take the 2-dimensional representation we have work out before,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{4}$$

$$K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{5}$$

Label the axises by the groupl elements in the order (E, A, B, K, L, M). Then we can construct four 6-dimensional ectors from these 2×2 matrices,

$$
\begin{array}{lllllll}
D_{11}^{(2)} & = (1 & , -\frac{1}{2} & , -\frac{1}{2} & , -1 & , \frac{1}{2} & , \frac{1}{2}) \\
D_{12}^{(2)} & = (0 & , \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & , 0 & , -\frac{\sqrt{3}}{2} & , \frac{\sqrt{3}}{2}) \\
D_{21}^{(2)} & = (0 & , -\frac{\sqrt{3}}{2} & , \frac{\sqrt{3}}{2} & , 0 & -\frac{\sqrt{3}}{2} & , \frac{\sqrt{3}}{2}) \\
D_{21}^{(2)} & = (1 & , -\frac{1}{2} & , -\frac{1}{2} & , 1 & , -\frac{1}{2} & , -\frac{1}{2})
\end{array}
$$

It is straightforward to check that these 4 vectors are perpendicular to each other.
Note that the other two vectors which are orthogonal to these vectors are of the form,

$$
\begin{array}{lllllll}
D_E & = (1 & , 1 & , 1 & , 1 & , 1 & , 1) \\
D_A & = (1 & , 1 & , 1 & , -1 & , -1 & , -1)
\end{array} \tag{6}
$$

coming from the identity representation and other 1-dimensional representation.

Character of Representation

Matrices in irrep are not unique because of similairty transformation. But trace of matrix is invariant under similarity transformation,

$$Tr\left(SAS^{-1}\right) = TrA$$

We can use trace, or character, to characterize the irrep.

$$\chi^{(\alpha)}(A_i) \equiv T_r\left[D^{(\alpha)}(A_i)\right] = \sum_a D^{(\alpha)}_{aa}(A_i)$$

Useful Properties

1. If $D^{(\alpha)}$ and $D^{(\beta)}$ are equivalent, then

$$\chi^{(\alpha)}(A_i) = \chi^{(\beta)}(A_i) \qquad \forall\ A_i \in G$$

2. If A and B are in the same class

$$\chi^{(\alpha)}(A) = \chi^{(\alpha)}(B)$$

Proof: If A and B are in same class $\implies \exists x \in G$ such that

$$xAx^{-1} = B \implies D^{(\alpha)}(x)\, D^{(\alpha)}(A)\, D^{(\alpha)}\left(x^{-1}\right) = D^{(\alpha)}(B)$$

Using

$$D^{(\alpha)}\left(x^{-1}\right) = D^{(\alpha)}(x)^{-1}$$

we get

$$T_r\left[D^{(\alpha)}(x)\,D^{(\alpha)}(A)\,D^{(\alpha)}(x)^{-1}\right]=T_r\left[D^{(\alpha)}(B)\right]\quad \text{or } \chi^{(\alpha)}(A)=\chi^{(\alpha)}(B)$$

Hence $\chi^{(\alpha)}$ is a function of class, not of each element

3. Denote $\chi_i=\chi(\mathcal{C}_i)$, character of ith class. Let n_c : number of classes in G, and n_i: number of group elements in $\mathcal{C}_i$.
From great orthogonality theorem

$$\sum_r D_{ij}^{(\alpha)}(A_r)\,D_{k\ell}^{(\beta)*}(A_r)=\frac{n}{l_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}$$

which implies

$$\sum_r \chi^{(\alpha)}(A_r)\,\chi^{(\beta)*}(A_r)=\frac{n}{l_\alpha}\cdot\delta_{\alpha\beta}l_\alpha=n\delta_{\alpha\beta}$$

or

$$\sum_i n_i\chi^{(\alpha)}(\mathcal{C}_i)\,\chi^{(\beta)*}(\mathcal{C}_i)=n\delta_{\alpha\beta}$$

This is the **great orthogonality theorem for the characters.**

Define $U_{\alpha i} = \sqrt{\frac{n_i}{n}} \chi^{(\alpha)}(\mathcal{C}_i)$, then great orthogonality theorem implies,

$$\sum_{i=1}^{n_c} U_{\alpha i} U_{\beta i}^* = \delta_{\alpha\beta}$$

Thus, if we consider $U_{\alpha i}$ as components in n_c dim vector space, $\vec{U}_\alpha = (U_{\alpha 1}, U_{\alpha 2}, \cdots U_{\alpha n_c})$, then $\vec{U}_\alpha \ \alpha = 1, 2, 3 \cdots n_r$ (n_r: # of indep irreps)form an othornormal set of vectors, i.e.

$$U_\beta U_\alpha = \sum_{i=1}^{n_c} U_{\alpha i} U_{\beta i}^* = \delta_{\alpha\beta}$$

This implies that

$$n_r \leq n_c$$

As an illustration, characters for rep given in Eqs(4,5) are

$$\chi^{(3)}(E) = 2, \qquad \chi^{(3)}(A) = \chi^{(3)}(B) = -1, \qquad \chi^{(3)}(K) = \chi^{(3)}(L) = \chi^{(3)}(M) = 0$$

From these form a 3-dim vector,

$$\chi^{(3)} = (2, -1, 0) \tag{7}$$

Similarly for rep in Eq(6) we get

$$\chi^{(1)}(E) = 1, \qquad \chi^{(1)}(A) = \chi^{(1)}(B) = 1, \qquad \chi^{(1)}(K) = \chi^{(1)}(L) = \chi^{(1)}(M) = 1$$

$$\chi^{(2)}(E) = 1, \qquad \chi^{(2)}(A) = \chi^{(2)}(B) = 1, \qquad \chi^{(2)}(K) = \chi^{(2)}(L) = \chi^{(2)}(M) = -1$$

Form another two 3-dimensional vectors,

$$\chi^{(1)} = (1, 1, 1)$$

$$\chi^{(2)} = (1, 1, -1)$$

The orthogonality relations in Eq (**??**) these 3-dimensional vectors are orthogonal to each other when weighted by # of elements in the class. For example,

$$\left(\chi^{(2)}, \chi^{(3)}\right) = 2 \times 1 + (-1) \times 1 \times 2 + 0 \times (-1) \times 3 = 0$$

$$\left(\chi^{(1)}, \chi^{(3)}\right) = 2 \times 1 + (-1) \times 1 \times 2 + 0 \times 1 \times 3 = 0$$

Decomposition of Reducible Representation

For a reducible rep, we can write

$$D = D^{(1)} \oplus D^{(2)} \qquad \text{i.e. } D(A_i) = \begin{pmatrix} D^{(1)}(A_i) & \\ & D^{(2)}(A_i) \end{pmatrix} \qquad \forall A_i \in G$$

The trace is

$$\chi(A_i) = \chi^{(1)}(A_i) + \chi^{(2)}(A_i)$$

Denote by $D^{(\alpha)}, \alpha = 1, 2 \cdots n_r$, all inequivalent unitary irrep. Then any rep D can be decomposed as

$$D = \sum_\alpha c_\alpha D^{(\alpha)} \qquad c_\alpha: \text{ some integer }, \left(\# \text{ of time } D^{(\alpha)} \text{ appears}\right)$$

In terms of traces,

$$\chi(\mathcal{C}_i) = \sum_\alpha c_\alpha \chi^{(\alpha)}(\mathcal{C}_i)$$

The coefficient can be calculated as follows (by using orthogonity theorem). Multiply by $n_i \chi_i^{(\beta)*}$ and sum over i

$$\sum_i \chi_i \chi_i^{(\beta)*} n_i = \sum_i \sum_\alpha c_\alpha \chi_i^{(\alpha)} \chi_i^{(\beta)*} n_i = \sum_\alpha c_\alpha \cdot n\delta_{\alpha\beta} = nc_\beta$$

or

$$\boxed{c_\beta = \frac{1}{n} \sum_i \chi_i \chi_i^{(\beta)*} n_i}$$

From this,

$$\sum_i n_i \chi_i \chi_i^* = \sum_i n_i \sum_{\alpha,\beta} c_\alpha \chi_i^{(\alpha)} c_\beta \chi_i^{(\beta)*} = n \sum_\alpha |c_\alpha|^2$$

This leads to the following theorem:
Theorem: If rep D with character χ_i satifies,

$$\sum_i n_i \chi_i \chi_i^* = n$$

then the representation D is irreducible.

Regular Representation

Given a group $G = \{A_1 = E, A_2 ... A_n\}$ .We can construct the regular rep as follows:
Take any $A \in G$. If

$$AA_2 = A_3 = 0A_1 + 0A_2 + 1 \cdot A_3 + 0A_4 + \cdots$$

i.e. we write the product "formally" as linear combination of group elements,

$$AA_s = \sum_{r=1}^{n} C_{rs} A_r = \sum_{r=1}^{n} A_r D_{rs}(A), \qquad \text{i.e.} \quad C_{rs} = D_{rs}(A) \text{ is either 0 or1.} \tag{8}$$

$$\begin{aligned} \text{i.e.} \quad D_{rs}(A) &= 1 \quad \text{if } AA_s = A_r \quad \text{or} \quad A = A_r A_s^{-1} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Strictly speaking, the sum over group elments is undefined. But here only one group element shows up in the right-hand side in Eq(8). No need to define the sum of group elements. Then $D(A)$'s form a rep of G: regular representation with dimensional n. This can be seen as follows:

$$\sum_r A_r D_{rs}(AB) = ABA_s = A \sum_t A_t D_{ts}(B) = \sum_{t \cdot r} D_{ts}(B) A_r D_{rt}(A)$$

or

$$D_{rs}(AB) = D_{rt}(A) D_{ts}(B)$$

From definition of regular representation

$$D_{rs}(A) = 1 \qquad \text{iff} \qquad AA_s = A_r$$

diagonal elements are,

$$D_{rr}(A) = 1 \qquad iff \qquad AA_r = A_r \quad or \quad A = E$$

Then every character is zero except for identity class,

$$\begin{aligned} \chi^{(reg)}(\mathcal{C}_i) &= 0 \qquad i \neq 1 \\ \chi^{(reg)}(\mathcal{C}_i) &= n \qquad i = 1 \end{aligned} \tag{9}$$

Reduce $D^{(reg)}$ to irreps. Write

$$D^{(reg)} = \sum_{\alpha} c_{\alpha} D^{(\alpha)}$$

then

$$c_{\alpha} = \frac{1}{n} \sum_i \chi_i^{(reg)} \chi_i^{(\alpha)*} n_i = \frac{1}{n} \chi_1^{(reg)} \chi_1^{(\alpha)*} = \frac{1}{n} \cdot n l_{\alpha} = l_{\alpha}$$

$\Longrightarrow D_{reg}$ contains irreps as many times as its dimension,

$$\chi_i^{(reg)} = \sum_{\alpha}^{n_r} l_{\alpha} \chi_i^{(\alpha)} \qquad \text{or} \quad \chi_i^{(reg)} = \sum_{\alpha=1}^{n_r} \chi_1^{(\alpha)*} \chi_i^{(\alpha)} = n\delta_{i1}$$

For identity class $\chi_1^{reg} = n, \quad \chi_1^{(\alpha)} = l_\alpha$, then

$$\boxed{\sum_\alpha l_\alpha^2 = n}$$

This constraints the possible dimensionalities of irreps becuase both n and l_α have to be integers. For D_3, with $n = 6$, the only possible solution for $\sum\limits_\alpha l_\alpha^2 = 6$ is $l_1 = 1.\ l_2 = 1.\ l_3 = 2$, and their permutations.

We now want to show that

$$\boxed{n_c = n_r}$$

i.e. # of classes = # of irreps. We will derive another orthogonal relations $\chi_i^{(\alpha)}$ with summation over irreps rather than classes,

$$\sum_\alpha \chi_i^{(\alpha)} \chi_j^{(\alpha)}$$

The derivation is separated into several steps.

1. Define $D_i^{(\alpha)}$ by

$$D_i^{(\alpha)} = \sum_{A \in \mathcal{C}_i} D^{(\alpha)}(A)$$

Then we can show that $D_i^{(\alpha)}$ is a multiple of identity. First,

$$\begin{aligned} D^{(\alpha)}(A_j) D_i^{(\alpha)} D^{(\alpha)}\left(A_j^{-1}\right) &= \sum_{A \in \mathcal{C}_i} D^{(\alpha)}(A_j) D^{(\alpha)}(A) D^{(\alpha)}\left(A_j^{-1}\right) \\ &= \sum_{A \in \mathcal{C}_i} D^{(\alpha)}\left(A_j A A_j^{-1}\right) = D_i^{(\alpha)} \end{aligned}$$

Using

$$D^{(\alpha)}\left(A_j^{-1}\right) = D^{(\alpha)}(A_j)^{-1}$$

we get

$$D^{(\alpha)}(A_j)\, D_i^{(\alpha)} = D_i^{(\alpha)} D^{(\alpha)}(A_j)$$

i.e. $D_i^{(\alpha)}$ commutes with all matrice in the irrep. From Schur's lemma, we get $D_i^{(\alpha)} = \lambda_i^{(\alpha)} 1$ where $\lambda_i^{(\alpha)}$ is some number. Taking the trace,

$$n_i \chi_i^{(\alpha)} = \lambda_i^{(\alpha)} l_\alpha, \qquad or \qquad \lambda_i^{(\alpha)} = \frac{n_i \chi_i^{(\alpha)}}{l_\alpha} = \frac{n_i \chi_i^{(\alpha)}}{\chi_1^{(\alpha)}}$$

where $\chi_1^{(\alpha)}$ is the character of identity class.

2. From the property of the class multiplication, we have

$$\mathcal{C}_i \mathcal{C}_j = \sum_k C_{ijk} \mathcal{C}_k$$

LHS $\mathcal{C}_i\mathcal{C}_j$ is a collections of group element of the type A_iA_j where $A_i \in \mathcal{C}_i$ and $A_j \in \mathcal{C}_j$. Map these products into irrep matrices and sum over to get

$$D_i^{(\alpha)} D_j^{(\alpha)} = \sum_k C_{ijk} D_k^{(\alpha)} \qquad \text{or} \quad \lambda_i^{(\alpha)} \lambda_j^{(\alpha)} = \sum_k C_{ijk} \lambda_k^{(\alpha)}$$

For example, in the group D_3, we have classes, $\mathcal{C}_1 = \{E\}, \mathcal{C}_2 = \{A, B\}, \mathcal{C}_3 = \{K, L, M\}$. Then

$$\begin{aligned} \mathcal{C}_2\mathcal{C}_3 &= \{A, B\}\{K, L, M\} = \{AK, AL, AM, BK, BL, BM\} \\ &= 2\{K, L, M\} = 2\mathcal{C}_3 \end{aligned}$$

Map these elements to their matrix representation and sum over,

$$\begin{aligned} LHS &= D(AK) + D(AL) + D(AM) + D(BK) + D(BL) + D(BM) \\ &= D(A)D(K) + D(A)D(L) + D(A)D(M) + D(B)D(K) + D(B)D(L) + D(B)D(M \\ &= [D(A) + D(B)][D(L) + D(M) + D(K)] \end{aligned}$$

and

$$RHS = 2[D(L) + D(M) + D(K)]$$

This illustrates the relation .Using the values of $\lambda_i^{(\alpha)}$ in Eq(**??**),

$$\frac{n_i\chi_i^{(\alpha)}}{\chi_1^{(\alpha)}}\frac{n_j\chi_j^{(\alpha)}}{\chi_1^{(\alpha)}} = \sum_k \mathcal{C}_{ijk}\frac{n_k\chi_k^{(\alpha)}}{\chi_1^{(\alpha)}} \qquad \text{or} \quad n_i n_j \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \chi_1^{(\alpha)}\sum_k \mathcal{C}_{ijk} n_k \chi_k^{(\alpha)}$$

Sum over the irrep α,

$$\sum_\alpha n_i n_j \chi_i^{(\alpha)}\chi_j^{(\alpha)} = \sum_\alpha\sum_k \mathcal{C}_{ijk} n_k \chi_1^{(\alpha)}\chi_k^{(\alpha)}$$

3. We now compute the coefficients $\mathcal{C}_{ijk}$. If any element A belongs to ith class, let the class which contains A^{-1} be denoted by $i^{'} - th$ class. We then get

$$\mathcal{C}_{ij1} = \begin{cases} 0 & \text{for} \quad j \neq i' \\ n_i & \text{for} \quad j = i' \end{cases}$$

We make use of the property of regular rep, $\sum_{\alpha} \chi_1^{\alpha} \chi_k^{\alpha} = n\delta_{k1}$, in Eq(**??**) to get,

$$\sum_{\alpha} n_i n_j \chi_i^{(\alpha)} \chi_j^{(\alpha)} = \sum_{\alpha} \sum_{k} \mathcal{C}_{ijk} n_k \chi_1^{(\alpha)} \chi_k^{\alpha} = n\mathcal{C}_{ij1} = \begin{cases} 0 & \text{for} \quad j \neq i' \\ nn_i & \text{for} \quad j = i' \end{cases}$$

Then

$$\sum_{\alpha=1}^{n_r} \chi_i^{(\alpha)} \chi_j^{(\alpha)} = \frac{n}{n_j} \delta ji'$$

Since rep is unitary

$$D^{(\alpha)}(A_i)^{\dagger} = D^{(\alpha)}(A_i)^{-1} = D^{(\alpha)}\left(A_i^{-1}\right) = D^{(\alpha)}(A_{i'})$$

we get

$$\chi_{i'}^{(\alpha)} = \chi_i^{(\alpha)*}$$

and

$$\sum_{\alpha=1}^{n_r} \chi_i^{(\alpha)} \chi_j^{(\alpha)*} = \frac{n}{n_j} \delta ji$$

This is the orthogonal relation for $\chi_i^{(\alpha)}$.

If we now consider $\chi_i^{(\alpha)}$ as a vector in n_r dim space $\overrightarrow{\chi}_i = \left(\chi_i^{(1)}, \chi_i^{(2)}, \dots \chi_i^{(n_r)}\right)$ we get

$$n_c \leq n_r$$

Combine this with the result $n_r \leq n_c$, we have derived before, we get

$$n_r = n_c$$

Character Table

For a finite group, the essential information about the irrep can be summarized in a table witth characters of each irrep in terms of the classes. To construct such table, use the following useful information:

1. # of columns = # of rows = # of classes
2. $\sum_{\alpha} l_{\alpha}^{2} = n$
3. $\sum_{i} n_i \chi_i^{(\alpha)} \chi_i^{(\beta)*} = n\delta_{\alpha\beta}$ and $\sum_{\alpha} \chi_i^{(\alpha)} \chi_j^{(\alpha)*} = \frac{n}{n_i}\delta_{ij}$
4. If $l_\alpha = 1$, χ_i is itself a rep.
5. $\chi^{(\alpha)}\left(A^{-1}\right) = T_r\left(D^{(\alpha)}\left(A^{-1}\right)\right) = T_r\left(D^{(\alpha)^{+}}\left(A^{-1}\right)\right) = \chi^{(\alpha)*}(A)$
   If A and A^{-1} are in the same class then $\chi(A)$ is real.
6. $D^{(\alpha)}$ is a rep $\Longrightarrow$ $D^{(\alpha)*}$ is also a rep
   so if $\chi^{(\alpha)}$'s are complex numbers, another row will be their complex conjugate
7. If $l_\alpha > 1, \chi_i^{(\alpha)} = 0$ for at least one class. This follows from the relation

$$\sum_{i} n_i \left|\chi_i\right|^2 = n \quad \text{and} \quad \sum_{i} n_i = n$$

8. For physical symmetry group, $x.y$ and z form a basis of a rep.
   Example : D_3 character table

			E	$2C_3$	$3C_2'$
x^2+y^2, z^2		A_1	1	1	1
	$R_z, z.$	A_2	1	1	-1
(xz, yz)	(x, y)	E	2	-1	0
x^2-y^2, xy	$(R_x.R_y)$				

Here, typical basis functions up to quadratic in coordinate system are listed.
Remark: the basis functions are not necessarily normalized.
Using the transformation properties of the coordinate, we can also infer the transformation properties of any vectors.
For example, the usual coordinates have the transformation property,

$$\vec{r} = (x, y, z) \sim A_2 \oplus E \quad in \quad D_3$$

This means that electric field of $\vec{E}$ or magnetic field $\vec{B}$ will have same transformation property,

$$\vec{B} \sim \vec{E} \sim A_2 \oplus E$$

because they all transform the same way under the rotation.

Product Representation (Kronecker product)

Let x_i be the basis for $D^{(\alpha)}$, i.e. $\quad x_i' = \sum_{j=1}^{\ell_\alpha} x_j D_{ji}^{(\alpha)}(A)$

y_ℓ be the basis for $D^{(\beta)}$, i.e. $\quad y_k' = \sum_{\ell=1}^{\ell_\beta} y_\ell D_{\ell k}^{(\beta)}(A)$

then the products $x_j y_l$ transform as

$$x_i' y_k' = \sum_{j \cdot \ell} D_{ij}^{(\alpha)}(A)\, D_{k\ell}^{(\beta)}(A)\, x_j y_\ell \equiv \sum_{j \cdot \ell} D_{j\ell;ik}^{(\alpha\times\beta)}(A)\, x_j y_\ell$$

where

$$\boxed{D_{j\ell;ik}^{(\alpha\times\beta)}(A) = D_{ij}^{(\alpha)}(A)\, D_{\ell k}^{(\beta)}(A)}$$

Note that matrices, row and column are labelled by 2 indices, instead of one. We can show that $D^{(\alpha\times\beta)}$ forms a rep of the group.

$$\left[D^{(\alpha\times\beta)}(A)\, D^{(\alpha\times\beta)}(B)\right]_{ij;k\ell} = \sum_{s.t} D^{(\alpha\times\beta)}(A)_{ij,st}\, D^{(\alpha\times\beta)}(B)_{st;k\ell}$$

$$= \sum_{s.t} \; D_{is}^{(\alpha)}(A)\;\; D_{jt}^{(\beta)}(A)\;\; D_{sk}^{(\alpha)}(B)\;\; D_{t\ell}^{(\beta)}(B) = D_{ik}^{(\alpha)}(AB)\, D_{j\ell}^{(\beta)}(AB) = D^{(\alpha\times\beta)}(AB)_{ik;k\ell}$$

or

$$\boxed{D^{(\alpha\times\beta)}(A)\, D^{(\alpha\times\beta)}(B) = D^{(\alpha\times\beta)}(AB)}$$

The basis functions for $D^{(\alpha\times\beta)}$ are $x_i y_j$

The character of this rep can be calculated by making the row and colum indices the same and sum over,

$$\chi^{(\alpha\times\beta)}(A) = \sum_{j\cdot\ell} D^{(\alpha\times\beta)}_{j\ell;j\ell}(A) = \sum_{j\cdot\ell} D^{(\alpha)}_{jj}(A)\, D^{(\beta)}_{\ell\ell}(A) = \chi^{(\alpha)}(A)\,\chi^{(\beta)}(A)$$

$$\chi^{(\alpha\times\beta)}(A) = \chi^{(\alpha)}(A)\,\chi^{(\beta)}(A)$$

If $\alpha=\beta$, we can further decompose the product rep by symmetrization or antisymmetrization;

$$D^{\{\alpha\times\alpha\}}_{ik,j\ell}(A) = \frac{1}{2}\left[D^{(\alpha)}_{ij}(A)\, D^{(\alpha)}_{k\ell}(A) + D^{(\alpha)}_{i\ell}(A)\, D^{(\alpha)}_{kj}(A)\right] \qquad \text{basis } \frac{1}{\sqrt{2}}(x_i y_k + x_k y_i)$$

$$D^{[\alpha\times\alpha]}_{ik,j\ell}(A) = \frac{1}{2}\left[D^{(\alpha)}_{ij}(A)\, D^{(\alpha)}_{k\ell}(A) - D^{(\alpha)}_{i\ell}(A)\, D^{(\alpha)}_{kj}(A)\right] \qquad \text{basis } \frac{1}{\sqrt{2}}(x_i y_k - x_k y_i)$$

These matrices also form rep of G and the characters are given by

$$\chi^{\{\alpha\times\alpha\}}(A) = \frac{1}{2}\left[\left(\chi^{(\alpha)}(A)\right)^2 + \chi^{(\alpha)}(A^2)\right], \qquad \chi^{[\alpha\times\alpha]}(A) = \frac{1}{2}\left[\left(\chi^{(\alpha)}(A)\right)^2 - \chi^{(\alpha)}(A^2)\right]$$

Example D_3

			$E.$	$2\mathcal{C}_3$	$3C_2'$		
		Γ_1	1	1	1		
	$R_z.z$	Γ_2	1	1	-1		
(xz, yz)	(x, y)	Γ_3	2	-1	0		
(x^2-y^2, xy)		$\Gamma_3\times\Gamma_3$	4	1	0	$=$	$\Gamma_1\oplus\Gamma_2\oplus\Gamma_3$
		$(\Gamma_3\times\Gamma_3)_s$	3	0	1	$=$	$\Gamma_1\oplus\Gamma_3$
		$(\Gamma_3\times\Gamma_3)_a$	1	1	-1		Γ_2

Direct Product Group

Given 2 groups $G_1 = \{E, A_2 \cdots A_n\}$, $G_2 = \{E, B_2 \cdots B_m\}$, define product group as $G_1 \otimes G_2 = \{A_i B_j; i = 1 \cdots n, j = 1 \cdots m\}$ with multiplication law

$$(A_k B_\ell) \times (A_{k'} B_{\ell'}) = (A_k B_{k'}) (B_\ell B_{\ell'})$$

It turns out that irrep of $G_1 \otimes G_2$ are just direct product of irreps of G, and G_2. Let $D^{(\alpha)}(A_i)$ be an irrep of $G_1$and $D^{(\beta)}(B_j)$ an irrep of G_2then the matrices defined by

$$D^{(\alpha\times\beta)}(A_i B_j)_{ab;cd} \equiv D^{(\alpha)}(A_i)_{ac} D^{(\beta)}(B_j)_{bd}$$

will have the property

$$\begin{aligned}
\left[D^{(\alpha\times\beta)}(A_i B_j) D^{(\alpha\times\beta)}(A_k B_\ell)\right]_{ab;cd} &= \sum_{e\cdot f} \left[D^{(\alpha\times\beta)}(A_i B_j)\right]_{ab;ef} \left[D^{(\alpha\times\beta)}(A_k B_\ell)\right]_{ef;cd} \\
&= \sum_{e\cdot f} \left[D^{(\alpha)}(A_i)_{ac} D^{(\alpha)}(A_k)_{ec}\right] \left[D^{(\beta)}(B_j)_{bf} D^{(\beta)}(B_e)_{fd}\right] \\
&= D^{(\alpha)}(A_i A_k)_{ac} D^{(\beta)}(B_j B_\ell)_{bd} = D^{(\alpha\times\beta)}(A_i A_k . B_j B_\ell)_{ab;cd}
\end{aligned}$$

Thus matrice $D^{(\alpha\times\beta)}(A_i B_j)$ form a representation of the product group $G_1 \otimes G_2$. The characters are,

$$\chi^{(\alpha\times\beta)}(A_i B_j) = \sum_{ab} D^{(\alpha\times\beta)}(A_i B_j)_{ab;ab} = \sum_{a\cdot b} D^{(\alpha)}(A_i)_{aa} D^{(\beta)}(B_j)_{bb} = \chi^{(\alpha)}(A_i)\chi^{(\beta)}(B_j)$$

Then

$$\sum_{i\cdot j}\left|\chi^{(\alpha\times\beta)}(A_iB_j)\right|^2 = \left(\sum_i\left|\chi^{(\alpha)}(A_i)\right|^2\right)\left(\sum_j\left|\chi^{(\beta)}(B_j)\right|^2\right) = nm \quad \Longrightarrow \quad D^{(\alpha\times\beta)} \text{ is irrep.}$$

Example, $G_1 = D_3 = \{E, 2C_3, 3C_2'\}$, $G_2 = \{E, \sigma_h\} = \varphi$ where σ_h: reflection on the plane of triangle.

Direct product group is then $D_{3h} \equiv D_3 \otimes \varphi = E, A, B = \{E, 2C_3, 3C_2', \sigma_h, 2C_3\sigma_h, 3C_2'\sigma_h\}$

Character Table

φ	E	σ_h
Γ^+	1	1
Γ^-	1	-1

D_3	E	$2C_3$ AB	$2C_2'$ KLM
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	2	-1	0

Character Table

	E	$2C_3$	$2C_2'$	σ_h	$2C_3\sigma_h$	$2C_2'\sigma_h$
Γ_1^+	1	1	1	1	1	1
Γ_2^+	1	1	-1	1	1	-1
Γ_3^+	2	-1	0	2	-1	0
Γ_1^+	1	1	1	-1	-1	-1
Γ_2^+	1	1	-1	-1	-1	1
Γ_3^{-1}	2	-1	0	-2	1	0