## Lorentz Group

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## Lorentz group

The Lorentz group is a collection of linear transformations of space-time coordinates

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

which leaves the proper time

$$\tau^2 = (x^o)^2 - (\vec{x})^2 = x^{\mu} x^{\nu} g_{\mu\nu} = x^2$$

invariant. This requires  $\Lambda^{\mu}_{\nu}$  to satisfy the pseudo-orthogonality relation,

$$\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}g_{\mu\nu}=g_{\alpha\beta}$$

## Generators

For infinitesmal transformation, write

$$\Lambda^{\mu}_{\ lpha} = g^{\mu}_{lpha} + \epsilon^{\mu}_{\ lpha} \qquad \qquad {
m with} \quad |\epsilon^{\mu}_{lpha}| \ll 1$$

Pseudo-orthogonality relation implies

$$\left(g^{\mu}_{\alpha}+\epsilon^{\mu}_{\ \alpha}\right)\left(g^{\nu}_{\beta}+\epsilon^{\nu}_{\ \beta}\right)g_{\mu\nu}=g_{\alpha\beta}$$

Or

$$\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0$$

We see that  $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ . There are 6 independent group parameters.

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**Example**: Boost along x - axis

$$\Lambda^{\mu}_{\ \alpha} = \left(\begin{array}{ccc} \cosh \omega & \sinh \omega & 0 & 0 \\ \sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & \omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This implies that

$$\varepsilon_1^0 = \omega, \quad \varepsilon_0^1 = \omega$$
 (1)

Or

$$\varepsilon_{01} = \omega$$
,  $\varepsilon_{10} = -\omega$ 

Similarly,  $\varepsilon_{02}$ , and  $\varepsilon_{03}$  correspond to boosts in y and z directions respectively. **Example**: rotation about z - axis

$$\Lambda^{\mu}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\varepsilon_2^1 = -\theta, \qquad \varepsilon_1^2 = \theta$$

Or

$$\varepsilon_{12} = heta$$
,  $\varepsilon_{21} = - heta$ 

Similarly,  $\epsilon_{23}$ , and  $\epsilon_{31}$  correspond to rotations about x- and y axis respectively.

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Consider  $f(x^{\mu})$ , function of  $x^{\mu}$ . Under the infinitesimal Lorentz transformation, the change in f is

$$\begin{aligned} f(x^{\mu}) & \to \quad f(x'^{\mu}) = f(x^{\mu} + \varepsilon^{\mu}_{\alpha} x^{\alpha}) \approx f(x^{\mu}) + \varepsilon_{\alpha\beta} x^{\beta} \partial_{\alpha} f + \cdots \\ & = \quad f(x^{\mu}) + \frac{1}{2} \varepsilon_{\alpha\beta} [x^{\beta} \partial^{\alpha} - x^{\alpha} \partial^{\beta}] f(x) + \cdots \end{aligned}$$

Introduce operators  $M_{\mu\nu}$  to represent this change,

$$f(x') = f(x) - \frac{i}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta} f(x) + \cdots$$

then

$$M^{\alpha\beta} = -i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha})$$
<sup>(2)</sup>

generators  $M_{\mu\nu}$  are called the generators of Lorentz group. Note that for  $\alpha, \beta = 1, 2, 3$  these are just the angular momentum operator  $L_{ij} = i (x_i \partial_j - x_j \partial_i)$ .

Using the generators given in Eq(2) it is straightforward to work out commutators of these generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\delta}M_{\beta\gamma})$$

Define

$$M_{ij} = \epsilon_{ijk} J_k, \quad M_{oi} = K_i$$

where  $J'_k s$  correspond to rotations and  $K_i$  the Lorentz boost operators. We can solve for  $J_i$  to get

$$J_i = \frac{1}{2} \epsilon_{ijk} M_{jk}$$

The commutator of  $J'_i s$  are,

$$[J_i, J_j] = \left(\frac{1}{2}\right)^2 \varepsilon_{ikl} \varepsilon_{jmn} [M_{kl}, M_{mn}] = (-i) \left(\frac{1}{2}\right)^2 \varepsilon_{ikl} \varepsilon_{jmn} (g_{lm} M_{kn} - g_{km} M_{ln} - g_{ln} M_{km} + g_{kn} M_{lm})$$
$$= \left(\frac{1}{2}\right)^2 (-i) \left[-\varepsilon_{ikl} \varepsilon_{jln} M_{kn} + \varepsilon_{ikl} \varepsilon_{jkn} M_{ln} + \varepsilon_{ikl} \varepsilon_{jml} M_{km} - \varepsilon_{ikl} \varepsilon_{jmk} M_{lm}\right]$$

Using identity

$$\epsilon_{abc}\epsilon_{alm} = \left(\delta_{bl}\delta_{cm} - \delta_{bm}\delta_{cl}\right)$$

we get

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{3}$$

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Thus we can identify  $J_i$  as the angular momentum operator. Similarly, we can derive

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k \tag{4}$$

Eqs(3,4) are called the Lorentz algebra.

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To simplify the Lorentz algebra, we define the combinations

$$A_i = rac{1}{2}(J_i + iK_i)$$
 ,  $B_i = rac{1}{2}(J_i - iK_i)$ 

Then it is straightforward to derive the following commutation relations,

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0$$

Thus algebra of Lorentz generators factorizes into 2 independent SU(2) algebra. The representations are tensor products of the representation of SU(2) algebra. We label irrep by  $(j_1, j_2)$  which transforms as  $(2j_1 + 1)$ -dim rep under  $A_i$  algebra and  $(2j_2 + 1)$ -dim rep under  $B_i$  algebra.

## Simple representations

(1/2, 0) representation  $\chi_a$ This 2-component object has the following properties,

$$A_{i}\chi_{a} = \left(\frac{\sigma_{i}}{2}\right)_{ab}\chi_{b} \qquad \Longrightarrow \qquad \frac{1}{2}\left(J_{i} + iK_{i}\right)\chi_{a} = \left(\frac{\sigma_{i}}{2}\right)_{ab}\chi_{b}$$
$$B_{i}\chi_{a} = 0 \qquad \Longrightarrow \qquad \frac{1}{2}\left(J_{i} - iK_{i}\right)\chi_{a} = 0$$

Combining these realtions we get

$$\vec{J}\chi = (rac{ec{\sigma}}{2})\chi, \qquad \qquad \vec{K}\chi = -i(rac{ec{\sigma}}{2})\chi$$

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2 (0,  $\frac{1}{2})$  representation  $\eta_{\rm a}$ 

Similarly, we can get

$$\begin{aligned} A_i \eta_a &= 0 \qquad \Rightarrow \qquad \frac{1}{2} (J_i + iK_i) \eta_a &= 0 \\ B_i \eta_a &= \left(\frac{\sigma_i}{2}\right)_{ab} \qquad \Longrightarrow \qquad \frac{1}{2} (J_i - iK_i) \eta_a &= \left(\frac{\sigma_i}{2}\right)_{ab} \eta_b \\ \vec{J} \eta &= \left(\frac{\vec{\sigma}}{2}\right) \eta, \qquad \quad \vec{K} \eta &= i \left(\frac{\vec{\sigma}}{2}\right) \eta \end{aligned}$$

If we define a 4-component  $\psi$  by putting togather these 2 representations,

$$\psi = \left(\begin{array}{c} \chi \\ \eta \end{array}\right)$$

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Then the action of the Lorentz generators are

$$\vec{J}\psi = \begin{pmatrix} \vec{e} & 0\\ 0 & \vec{e} \end{pmatrix} \psi, \qquad \vec{K}\psi = \begin{pmatrix} -i\vec{e} & 0\\ 0 & i\vec{e} \end{pmatrix} \psi$$
(5)

 $\psi$  are related to the 4-component Dirac field we studied before, but with different representation for the  $\gamma$  matrices. This can be seen as follows.

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Consider Dirac matrices in the following form

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix} \text{ where } \sigma^{\mu} = (1, \vec{\sigma}) \text{ , } \overline{\sigma}^{\mu} = (1, -\vec{\sigma})$$

More explicitly,

$$\gamma^{o} = \left( egin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} 
ight) \qquad \vec{\gamma} = \left( egin{array}{cc} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{array} 
ight)$$

It is straightforward to check that in this case.

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{cc} 1 & 0\\ 0 & -1 \end{array}\right)$$

This means that in 4-component field  $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$ ,  $\chi$  is **right-handed** and  $\eta$  is **left-handed**. In this representation, usually called the **Weyl representation**, it is easy to check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i\begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0\\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

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In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = \exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\} = \exp\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\}$$

Write  $arepsilon^{0i}=eta^i$ ,  $arepsilon^{ij}=arepsilon^{ijk} heta^k$ 

$$\sigma_{ij}\epsilon^{ij} = \epsilon^{ijk}\theta^k\epsilon_{ijl} \left(\begin{array}{cc}\sigma_l & 0\\ 0 & \sigma_l\end{array}\right) = 2 \left(\begin{array}{cc}\vec{\sigma}\cdot\vec{\theta} & 0\\ 0 & \vec{\sigma}\cdot\vec{\theta}\end{array}\right)$$

$$\sigma_0 i \varepsilon^0 i = \left(\begin{array}{cc} -i \vec{\sigma} \cdot \vec{\beta} & 0\\ 0 & i \vec{\sigma} \cdot \vec{\beta} \end{array}\right)$$

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$$-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i}+\sigma_{ij}\varepsilon^{ij})=\frac{-i}{2}\left(\begin{array}{cc}\vec{\sigma}\cdot\vec{\theta}-i\vec{\sigma}\cdot\vec{\beta}&0\\0&\vec{\sigma}\cdot\vec{\theta}+i\vec{\sigma}\cdot\vec{\beta}\end{array}\right)$$

More precisely,

$$\psi'(\mathbf{x}') = S\psi = \exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\}\psi = \exp\left[\frac{-i}{2}\begin{pmatrix}\vec{\sigma}\cdot\vec{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0\\ 0 & \vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta}\end{pmatrix}\right]\psi \quad (6)$$

If we write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$

then in terms of the generators  $\overrightarrow{J}$ ,  $\overrightarrow{K}$ 

$$L = \exp\left[ (-i) \left( \vec{J} \cdot \vec{\theta} + \vec{K} \cdot \vec{\beta} \right) \right]$$

We then see from Eq(6) that for this  $\psi$ ,  $\overrightarrow{J}$ ,  $\overrightarrow{K}$  are of the form,

$$\vec{J} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0\\ 0 & \vec{\sigma} \end{pmatrix}, \qquad \vec{K} = \frac{1}{2} \begin{pmatrix} -i\vec{\sigma} & 0\\ 0 & i\vec{\sigma} \end{pmatrix}$$

These are the same as those in Eq(5). This demonstrate that the wavefunction which satisfies Dirac equation is just the representation  $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$  under the Lorentz group. Futhermore, the right-handed components transform as  $\left(\frac{1}{2},0\right)$  representation while left-handed components transform as  $\left(0,\frac{1}{2}\right)$  representation.

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Alternative choice is to use  $\psi_R$  and the complex conjugate  $\psi_R^*$  ( sometime dotted indice are used for this basis) instead of  $\psi_R$  and  $\psi_L$ . Since

$$\vec{J}\psi_R = (rac{\vec{\sigma}}{2})\psi_R$$
,  $\vec{K}\psi_R = -i(rac{\vec{\sigma}}{2})\psi_R$ 

we get for the complex conjuate

$$ec{J}\psi_{\mathcal{R}}^{*}=(rac{ec{\sigma}^{*}}{2})\psi_{\mathcal{R}}^{*}, \qquad \qquad ec{\kappa}\psi_{\mathcal{R}}^{*}=i(rac{ec{\sigma}^{*}}{2})\psi_{\mathcal{R}}^{*}$$

It is probably more clearer to use some other notation for  $\psi_R^*$ ,

$$\vec{J}\chi = (rac{\vec{\sigma}^*}{2})\chi, \qquad \qquad \vec{K}\chi = i(rac{\vec{\sigma}^*}{2})\chi$$

Then

$$\vec{A}\chi = \frac{1}{2}(\vec{J} + i\vec{K})\chi = 0, \qquad \vec{B}\chi = \frac{1}{2}(\vec{J} - i\vec{K})\chi = (\frac{\vec{\sigma}^*}{2})\chi$$

and indeed  $\chi$  belongs to the irrep  $(0, \frac{1}{2})$ .