Rotation group $R(3)$

Ling-Fong Li

Infinite group : group with infinite number of elements.
Label group elements by real parameters (group parameters)

$$A(\alpha_1, \alpha_2, \cdots \alpha_n), \qquad \alpha_1, \alpha_2, \cdots \alpha_n \quad \text{group parameters}$$

Continuous group : group parameters continuous
Compact group: group parameters vary over compact domain.
For example, 3-dim rotational group is compact because angles of rotation, varies over compact interval $[0, 2\pi]$
Lorentz group is not compact because , $\beta = \frac{v}{c}$ varies over non-compact interval, $[0, 1)$.
Example: $SO(2)$, 2×2 real orthogonal matrices with $det = 1$ is an one-parameter continuous group. Choose parameter to be the angle of rotation ϕ, $0 \leq \phi \leq 2\pi$.and is a compact group.
Denote group element by, $R(\phi)$ The group multiplication is

$$R(\phi_1) R(\phi_2) = R(\phi_1 + \phi_2)$$

Visualize group elements as points on the unit circle and are labeled by the angle. This is called group parameter space.
Group Integration
Group invariant measure
In finite group, rearrangement theorem,

$$\sum_{j=1}^{n} f(A_j) = \sum_{j=1}^{n} f(A_j B) = \sum_{j=1}^{n} f(BA_j), \qquad B, A_j \in G$$

plays essential role in representation theory.
For continuous group, sum over group elements $\Longrightarrow$ integration in group parameter space,

$$\int dA \equiv \int W\left(\alpha_1, \cdots \alpha_n\right)\ d\alpha_1 \cdots d\alpha_n$$

where $W\left(\alpha_1, \cdots \alpha_n\right)$ is a measure (or weight function).
Define group integration (or choose a measure W) such that rearrangement theorem holds.
This is called the **group invariant integration** (measure).
The measure $W\left(\alpha_1, \cdots \alpha_n\right)$ should be chosen such that

$$\int u(A)\, dA = \int u(AB)\, dA = \int u(BA)\, dA$$

where

$$\int u(A)\, dA \equiv \int u\left(\alpha_1, \cdots \alpha_n\right) W\left(\alpha_1, \cdots \alpha_n\right) d\alpha_1 \cdots d\alpha_n$$

and $u(A)$ is some arbitray function of group elements. Use notation,

$$A = A\left(\alpha_1, \cdots \alpha_n\right) = A\left(\vec{\alpha}\right), \qquad B = A\left(\beta_1, \cdots \beta_n\right) = A\left(\vec{\beta}\right)$$

Then

$$BA = A\left(\vec{\beta}\right) A\left(\vec{\alpha}\right) = A\left(\vec{\gamma}\right)$$

where

$$\vec{\gamma} = \vec{\gamma}\left(\vec{\beta}, \vec{\alpha}\right)$$

are some functions determined by the group multiplication. Write the integrations as,

$$dA = W\left(\vec{\alpha}\right) d^n\alpha = W\left(\alpha_1, \cdots \alpha_n\right) d\alpha_1 \cdots d\alpha_n$$

$$d\left(BA\right) = W\left(\vec{\gamma}\right) d^n\gamma = W\left(\gamma_1, \cdots \gamma_n\right) d\gamma_1 \cdots d\gamma_n$$

Thus group invariant measure should have the property

$$dA = d\left(BA\right), \qquad \text{or} \quad W\left(\vec{\alpha}\right) d^n\alpha = W\left(\vec{\gamma}\right) d^n\gamma$$

Note that left (or right) multiplication by a fixed group element, say B, is a 1-1 mapping of G onto itself.
Thus giving a set of group elements in some region V of the parameter space, under the left (or right) multiplication by B, these elements will move to other region V'. Since the total number of element in V is the same as those in V', we get,

$$\rho V = \rho' V',$$

where $\rho\,(\rho')$ is the density of elements at $V\,(V')$. Thus if we take the measure $W\,(\alpha_1, \cdots \alpha_n)$ to be the density of the group elements at $\overrightarrow{\alpha}$, then

$$W\left(\overrightarrow{\alpha}\right) d^n\alpha = W\left(\overrightarrow{\gamma}\right) d^n\gamma, \qquad \text{where } \overrightarrow{\gamma} = \overrightarrow{\gamma}\left(\overrightarrow{\beta}, \overrightarrow{\alpha}\right)$$

i.e. $W\left(\overrightarrow{\alpha}\right)$ is a group invariant measure.

To get the density of elements $W\left(\overrightarrow{\alpha}\right)$ consider an infinitesmal volume element $V_0 = d\alpha_1 \cdots d\alpha_n$ in the neighborhood of the origin(identity) i.e. $I = A\,(0)$. Under left multiplication by B, they move to V_1,

$$W\left(\overrightarrow{\alpha}\right) V_0 = W\left(\overrightarrow{\gamma}\right) V_1$$

Or

$$\frac{W\left(\overrightarrow{\alpha}\right)}{W\left(\overrightarrow{\gamma}\right)} = \frac{V_1}{V_0}$$

Thus ratio of weight functions are ratio of the volume elements. We will normalize the density such that $W\left(\overrightarrow{0}\right) = 1$, i.e. density is 1 at origin. Setting $\overrightarrow{\alpha} = \overrightarrow{0}$, we get

$$W\left(\overrightarrow{\alpha}\right) = W\left(\overrightarrow{0}\right) = 1, \qquad \text{and} \qquad W\left(\overrightarrow{\gamma}\right) = W\left(\overrightarrow{\beta}\right)$$

Then

$$W\left(\vec{\beta}\right)=\frac{V_0}{V_1}$$

Recall that the change in volume elements under the transformation induced by $B=A\left(\vec{\beta}\right)$ is given by the Jacobian of the change of the variables $\vec{\alpha}\rightarrow\vec{\gamma}\left(\vec{\beta},\vec{\alpha}\right)$. Thus we get

$$V_0=d\alpha_1\cdots d\alpha_n=\left.\frac{\partial\left(\alpha_1,\cdots\alpha_n\right)}{\partial\left(\gamma_1,\cdots\gamma_n\right)}\right|_{\vec{\alpha}=\vec{0}}d\gamma_1\cdots d\gamma_n=\left.\frac{\partial\left(\alpha_1,\cdots\alpha_n\right)}{\partial\left(\gamma_1,\cdots\gamma_n\right)}\right|_{\vec{\alpha}=\vec{0}}V_1$$

Thus the invariant group measure is given by

$$W\left(\vec{\beta}\right)=\left[\left.\frac{\partial\left(\gamma_1,\cdots\gamma_n\right)}{\partial\left(\alpha_1,\cdots\alpha_n\right)}\right|_{\vec{\alpha}=\vec{0}}\right]^{-1}$$

Thus to find the group invariant measure, need to know the change of group parameters under the multiplication of group element.

$SO(2)$ **group.**

The group multiplication is given by

$$R(\alpha)\,R(\beta) = R(\gamma)\,, \qquad \text{with } \gamma = \alpha + \beta$$

where α, β, γ are angles of rotations. The group invariant measure is

$$W(\beta) = \left[\frac{\partial \gamma}{\partial \alpha}\right]^{-1} = 1$$

i.e.group elements of $SO(2)$ are uniformly populated along the unit circle $\Longrightarrow$ the points per unit length is same everywhere. This explains the feature $W(\beta) = 1$ in this case. The group integration is

$$\int_0^{2\pi} d\alpha$$

Since this is an Abelian group, all irreps are 1-dimensional and of the form

$$e^{\pm im\alpha}, \qquad m = \text{integer}$$

The great orthogonality theorem is of the form,

$$\int_0^{2\pi} d\alpha\; e^{im\alpha}\left(e^{im'\alpha}\right)^* = \delta_{mm'}2\pi$$

If we had chosen any other measure $f(\alpha) \neq$ constant, then we would not have the great orthogonality theorem.

$SU(2)$ group- 2×2 unitary matrice with determinant=1,

$$UU^{\dagger}=U^{\dagger}U=1,\qquad \det U=1$$

Write unitary matrix U as

$$U=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$U^{-1}=\frac{1}{\det U}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},\qquad U^{\dagger}=\begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix}$$

The conditions

$$U^{-1}=U^{\dagger},\ \text{and}\ \det U=1$$

imply that

$$a^{*}=d,\qquad c^{*}=-b,\qquad |a|^{2}+|b|^{2}=1$$

Most general 2×2 unitary matrix is

$$U=\begin{pmatrix} a & -b \\ b^{*} & a^{*} \end{pmatrix},\qquad \text{with}\quad |a|^{2}+|b|^{2}=1$$

Parametrize a and b in terms of real variables,

$$a=u_1+iu_2,\qquad b=u_3+iu_4,\qquad u_i\quad \text{real}$$

then we have

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$$

Thus group elements of $SU(2)$ are now on the surface of a sphere in 4-dimensional space. This suggests the invariant measure as

$$\int dR = \int du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right)$$

It is intuitively clear that multiplication by an $SU(2)$ element corresponds to a rotation on the surface of sphere and this measure is invariant. To show this, we need to prove that

$$\int dR' f(R) = \int dR f(R),$$

where $R' = SR$ and f is some arbitrary function. Using the parametrizations,

$$\begin{aligned} R' &= \begin{pmatrix} u_1' + iu_2' & -u_3' - iu_4' \\ u_3' - iu_4' & u_1' - iu_2' \end{pmatrix}, \qquad S = \begin{pmatrix} s_1 + is_2 & -s_3 - is_4 \\ s_3 - is_4 & s_1 - is_2 \end{pmatrix}, \\ R &= \begin{pmatrix} u_1 + iu_2 & -u_3 - iu_4 \\ u_3 - iu_4 & u_1 - iu_2 \end{pmatrix}, \end{aligned}$$

we get from the relation $R' = SR$,

$$\begin{cases} u_1' = s_1 u_1 - s_2 u_2 - s_3 u_3 - s_4 u_4 \\ u_2' = s_2 u_1 + s_1 u_2 - s_4 u_3 + s_3 u_4 \\ u_3' = s_3 u_1 + s_4 u_2 + s_1 u_3 - s_2 u_4 \\ u_4' = s_4 u_1 - s_3 u_2 + s_2 u_3 + s_1 u_4 \end{cases}$$

The Jacobian of the transformation is then

$$J = \frac{\partial\left(u_1', u_2', u_3', u_4'\right)}{\partial\left(u_1, u_2, u_3, u_4\right)} = \begin{vmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & -s_4 & s_3 \\ s_3 & s_4 & s_1 & -s_2 \\ s_4 & -s_3 & s_2 & s_1 \end{vmatrix}$$

It is easy to see that $J = 1$ because it is the determinant of an orthogonal matrix using the fact that $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$. Then we have

$$\begin{aligned} du_1' \cdots du_4' \delta\left(u_1'^2 + u_2'^2 + u_3'^2 + u_4'^2 - 1\right) &= \frac{\partial\left(u_1', u_2', u_3', u_4'\right)}{\partial\left(u_1, u_2, u_3, u_4\right)} du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right) \\ &= du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right) \end{aligned}$$

Rotation Group $O(3)$

Homomorphism to $SU(2)$ Group

Elements of rotation group $O(3)$ will be denoted by $P_{\vec{n}}(\theta)$, the operator which rotates the system by angle θ about the axis $\vec{n}$.
Write

$$P_{\vec{n}}(\theta) = \exp\left(-i\frac{\theta\vec{J}\cdot\vec{n}}{\hbar}\right)$$

then $\vec{J}$ will be called the generator of rotation. we can also parametrize the rotation operator in terms of the familiar Euler rotations,

$$R(\alpha,\beta,\gamma) = P_z(\alpha)P_y(\beta)P_z(\gamma) = \exp\left(-\frac{iJ_z}{\hbar}\alpha\right)\exp\left(-\frac{iJ_y}{\hbar}\beta\right)\exp\left(-\frac{iJ_z}{\hbar}\gamma\right),$$

where $0 \le \alpha,\gamma \le 2\pi$, $0 \le \beta \le \pi$. We will now show that $O(3)$ is homomorphic to $SU(2)$ group. For any vector $\vec{r} = (x,y,z)$ define a 2×2 hermitian matrix by

$$h = \vec{\sigma}\cdot\vec{r} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli matrices. It is easy to see that h has the properties

$$Tr(h) = 0, \qquad \det h = -\left(x^2+y^2+z^2\right) = -r^2$$

Let U any 2×2 unitary matrix U and define a new matrix h' by

$$h' = UhU^{\dagger} \tag{1}$$

Then h' is also hermitan, traceless and has the same determinant as h,

$$h^{'} = \left(h'\right)^{\dagger}, \qquad Tr\left(h'\right) = 0, \qquad \det h' = \det h$$

If we expand h' in terms of Pauli matrices,

$$h' = \vec{\sigma} \cdot \vec{r}'$$

then the relation between $\vec{r}$ and $\vec{r}'$ is just a 3-dimensional rotation becuse of the determinants,

$$\det h = -\left(x^2 + y^2 + z^2\right) = \det h' = -\left(x'^2 + y'^2 + z'^2\right)$$

i.e. relation between $\vec{r}$ and $\vec{r}'$ is a linear transformation and can be written as

$$r_i' = R_{ij} r_j$$

where R is an orthogonal matrix. This establishes the correspondence between $SU(2)$ matrix and 3-dimensional rotation.
Note that from Eq(1) we see that U and $-U$, give the same h' and hence the same rotation. So the correspondence

$$\pm U \rightarrow R$$

is a homomorphism rather than isomorphism.

Rotation about z-axis

Suppose U is diagonal. Then the general form is given by,

$$U = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

and

$$h^{'} = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} = UhU^{\dagger} = \begin{pmatrix} z & (x - iy)\, e^{i\alpha} \\ (x + iy)\, e^{-i\alpha} & -z \end{pmatrix}$$

This gives the relation

$$\begin{cases} x' = \cos\alpha x + \sin\alpha y \\ y' = -\sin\alpha x + \cos\alpha y \\ z' = z \end{cases}$$

which is clearly a rotation around $z - axis$. Thus a diagonal U corresponds to rotation about z-axis.

Rotation about y-axis
If U is real

$$U = \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Then

$$\begin{aligned} h' &= \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \\ &= \begin{pmatrix} z\cos\beta + x\sin\beta & x\cos\beta - iy - z\sin\beta \\ iy + x\cos\beta - z\sin\beta & -z\cos\beta - x\sin\beta \end{pmatrix} \end{aligned}$$

and

$$\begin{cases} x' = \cos\beta x - \sin\beta z \\ y' = y \\ z' = \sin\beta x + \cos\beta z \end{cases}$$

i.e. a rotation about $y-axis$.

Rotation in terms of Euler angles

The 2×2 unitary matrix corresponding to rotation by Euler angles, is then

$$\begin{aligned} U(\alpha,\beta,\gamma) &= P_z(\alpha)\,P_y(\beta)\,P_z(\gamma) && (2)\\ &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} -e^{i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} && (3)\\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha+\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}, \end{aligned}$$

where

$$a = e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2}$$
$$b = e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2}$$

write

$$a = u_1 + iu_2 = e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2}, \qquad b = u_3 + iu_4 = e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2}$$

the group invariant integration can be converted to the integration over Euler angles by computing the Jacobian.

$$\int du_1 du_2 du_3 = \int d\alpha d\beta d\gamma J$$

First write

$$\begin{aligned}\int dR &= \int du_1 \cdots du_4 \delta\left(u_1^2+u_2^2+u_3^2+u_4^2-1\right) \\ &= \int du_1 du_2 du_3 \frac{1}{2\left|u_4\right|}\left[\delta\left(u_4-a\right)+\delta\left(u_4+a\right)\right]\end{aligned}$$

where

$$a=\sqrt{1-\left(u_1^2+u_2^2+u_3^2\right)}$$

The Jacobian is then of the form

$$\begin{aligned}J &= \frac{\partial\left(u_1, u_2, u_3\right)}{\partial(\alpha, \beta, \delta)} \\ &= \frac{1}{8}\left[\begin{array}{ccc}-\sin(\alpha+\gamma)/2\cos\beta/2 & -\cos(\gamma+\gamma)/2\sin\beta/2 & -\sin(\alpha+\gamma)/2\cos\beta/2 \\ -\cos(\alpha+\gamma)/2\cos\beta/2 & \sin(\alpha+\gamma)/2\sin\beta/2 & -\cos(\alpha+\gamma)/2\cos\beta/2 \\ -\sin(\alpha-\gamma)/2\sin\beta/2 & \cos(\alpha-\gamma)/2\cos\beta/2 & \sin(\alpha-\gamma)/2\sin\beta/2\end{array}\right] \\ &= \frac{1}{8} 2\sin(\alpha-\gamma)/2\left[\cos\beta/2\sin^2\beta/2\right]\end{aligned}$$

The integration measure in terms of Euler angles is of the form,

$$\int dR=\int du_1 \cdots du_4 \delta\left(u_1^2+u_2^2+u_3^2+u_4^2-1\right)=\frac{1}{16} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\gamma$$

The factor $\frac{1}{16}$ is an overall normalization factor and is usually neglected for conveience.

Irreducible Representation of $SU(2)$

The 2×2 matrice of $SU(2)$ can be viewed as a rotations in the complex 2−dimensional space C_2. We can use the induced transformation on functions of these 2–dimensional coordinates to generate other representations.

ξ ,η basis vector for 2×2 matrice of $SU(2)$

$$P_U(\xi,\eta) = (\xi,\eta)\,U = (\xi,\eta)\begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} = (a\xi + b^*\eta, -b\xi + a^*\eta)$$

i.e.

$$\begin{aligned} \xi' &= P_U\xi = a\xi + b^*\eta \\ \eta' &= P_U\eta = -b\xi + a^*\eta \end{aligned}$$

Consider the action of P_U on the mononomial $\xi^\lambda\eta^\mu$, with $\lambda + \mu = 2j$, where j is an integer or half integer. As an example, take $j = 1$. There are 3 different monomials,

$$\xi^2, \qquad \xi\eta, \qquad \eta^2$$

Then under the induced transformation

$$\xi'^2 = (a\xi + b^*\eta)^2 = a^2\xi^2 + 2ab^*\xi\eta + b^{2*}\eta^2$$

$$\xi'\eta' = (a\xi + b^*\eta)(-b\xi + a^*\eta) = -ab\xi^2 + (aa^* - bb^*)\,\xi\eta + a^*b^*\eta^2$$

$$\eta'^2 = (-b\xi + a^*\eta)^2 = b^2\xi^2 - 2a^*b\xi\eta + a^{2*}\eta^2$$

Write this in matrix notation

$$\begin{pmatrix} \frac{\xi'^2}{\sqrt{2}} \\ \xi'\eta' \\ \frac{\eta'^2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} a^2 & \sqrt{2}ab^* & b^{2*} \\ -\sqrt{2}ab & (aa^* - bb^*) & \sqrt{2}a^*b^* \\ b^2 & -\sqrt{2}a^*b & a^{2*} \end{pmatrix} \begin{pmatrix} \frac{\xi^2}{\sqrt{2}} \\ \xi\eta \\ \frac{\eta^2}{\sqrt{2}} \end{pmatrix}$$

We have inserted the factor of $\frac{1}{\sqrt{2}}$ in order to get a unitary matrix. In terms of Euler angles α, β, γ

$$D^{(1)} = \begin{pmatrix} \frac{1}{2}(1+\cos\beta)\, e^{-i(\alpha+\gamma)} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \frac{1}{2}(1-\cos\beta)\, e^{-i(\alpha-\gamma)} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ \frac{1}{2}(1-\cos\beta)\, e^{i(\alpha-\gamma)} & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1}{2}(1+\cos\beta)\, e^{i(\alpha+\gamma)} \end{pmatrix}$$

As discussed before these matrice for different values of a, b will form the representation matrices of the $SU(2)$ group. For case $\beta = \gamma = 0$, we have rotation about $z - axis$,

$$D^{(1)}(\alpha) = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix}$$

However, in Cartesian coordinates the rotation around $z - axis$ is,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotation matrix $D^{(1)}(\alpha)$ corresponding to basis $(x + iy, z, x - iy)$, spherical basis, related to the Cartesian coordinates by a unitary transformation,

$$\begin{pmatrix} x + iy \\ z \\ x - iy \end{pmatrix} = \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Generalize this to arbitrary mononomials

$$\phi^j_m = \frac{1}{n_{jm}} (\xi)^{j+m} (\eta)^{j-m}, \qquad m = -j, -j+1, \cdots, j$$

where n_{jm} is a normalization factor to be determined later. Since the transformation from (ξ, η) to (ξ', η') is linear and homogeneous the transform of ϕ^j_m will have same j but different m. More explicitly,

$$\begin{aligned} P_U \phi^j_m &= \frac{1}{n_{jm}} \left(a\xi + b^* \eta\right)^{j+m} \left(-b\xi + a^* \eta\right)^{j-m} \\ &= \frac{1}{n_{jm}} \sum_{s=0}^{j-m} \sum_{r=0}^{j+m} \left[\frac{(j+m)!}{r!\,(j+m-r)!}\right] \left[\frac{(j-m)!}{s!\,(j-m-s)!}\right] \\ & \quad (-b)^s \left(a^*\right)^{j-m-s} a^r \left(b^*\right)^{j+m-r} \xi^r \xi^s \left(\eta\right)^{j+m-r} \left(\eta\right)^{j-m-s} \end{aligned}$$

Define m' by

$$s = j + m' - r, \qquad j \leq m' \leq j$$

Then

$$\begin{aligned} P_U \phi^j_m &= \sum_{r,m'} \left(\frac{n_{jm'}}{n_{jm}}\right) \left[\frac{(j+m)!\,(j-m)!\,(-1)^{j+m'-r}}{r!\,(j+m-r)!\,(j+m'-r)!\,(r-m-m')!}\right] \\ & \quad a^r \left(b^*\right)^{j+m-r} (b)^{j+m'-r} \left(a^*\right)^{r-m-m'} \phi^j_{m'} \end{aligned}$$

In terms of Euler angles,

$$a = e^{-i(\alpha+\gamma)/2} \cos\frac{\beta}{2}, \qquad b = e^{-i(\alpha-\gamma)/2} \sin\frac{\beta}{2}$$

and we can write

$$P_U \phi^j_m = \sum_{m'} \phi^j_{m'} D^j_{m'm} (\alpha, \beta, \gamma)$$

where

$$D^j_{m'm} (\alpha, \beta, \gamma) = e^{-im'\alpha} d^j_{m'm} (\beta) e^{-im\gamma} \tag{4}$$

with

$$d^j_{m'm} (\beta) = \frac{(j+m)! (j-m)! n_{jm'}}{n_{jm}} \sum_k \frac{(-1)^{j+m'-k}}{k! (j+m-k)! (j+m'-k)! (k-m-m')!} \left(\cos \frac{\beta}{2}\right)^{2k-m-m'} \left(\sin \frac{\beta}{2}\right)^{2j-2k+m+m'} \tag{5}$$

Note that sum over k covers all thos values for which the argument of the factorial functions are positive.

Properties of $D^j_{m'm}(\alpha,\beta,\gamma)$

1. $D^j_{m'm}(\alpha,\beta,\gamma)$'s form $(2j+1)$ dim rep of $SU(2)$, because group induced transformation always generates a representation of the group as discussed in Note 2.
2. $D^j_{m'm}(0,0,0)=\delta_{mm'}$, the identiy matrix. To see this we set $\beta=0$ in Eq(5) and the non-zero term is where $2k=2j+m+m'$. This implies that

$$j+m-k=\frac{1}{2}(m-m'), \qquad j+m'-k=\frac{1}{2}(m'-m)$$

   and the positivity of the arguments of the factorial functions gives $m=m'$ and $d^j_{m'm}(0)=\delta_{mm'}$.
3. For matrix $D^j_{m'm}(\alpha,\beta,\gamma)$ to be unitary, we require

$$D^j_{m'm}(\alpha,\beta,\gamma)^{\dagger}=D^j_{m'm}(-\gamma,-\beta,-\alpha) \qquad \Rightarrow \qquad d^j_{m'm}(-\beta)=d^j_{mm'}(\beta)$$

   It is straightforward to show that this fixes the constant n_{jm} to be

$$n_{jm}=\sqrt{(j+m)!\,(j-m)!}$$

and the $d-$finction is then

$$d^j_{m'm}(\beta) = \sum_k \frac{(-1)^{j+m'-k}\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{k!\,(j+m-k)!\,(j+m'-k)!\,(k-m-m')!} \quad (6)$$

$$\left(\cos\frac{\beta}{2}\right)^{2k-m-m'}\left(\sin\frac{\beta}{2}\right)^{2j-2k+m+m'} \quad (7)$$

4. For each j, the rep $D^j_{m'm}(\alpha,\beta,\gamma)$ is irreducible. This can seen as follows. Suppose $\exists$ a matrix M such that

$$MD^j(\alpha,\beta,\gamma) = D^j(\alpha,\beta,\gamma)\,M \quad (8)$$

Consider following cases:

1. $\alpha \neq 0$, $\beta = \gamma = 0$
From Eq(6), only non-zero term is

$$2j - 2k + m + m' = 0$$

which gives

$$d^j_{m'm}(0) = \delta_{mm'}$$

Then

$$D^j_{m'm}(\alpha,0,0) = e^{-im'\alpha}\delta_{mm'}$$

and Eq(8) implies that

$$\left(e^{-im\alpha} - e^{-im'\alpha}\right) M_{mm'} = 0$$

Thus $M_{mm'} = 0$ if $m \neq m'$, i.e. M is diagonal.

2. $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$
   Since M is diagonal, Eq(8) gives

$$M_{mm} D^j_{m'm} = D^j_{m'm} M_{m'm'} \qquad \text{(no sum)}$$

   But for arbitrary (α, β, γ), $D^j_{m'm}$ is not zero. Thus $M_{mm} = M_{m'm'}$, or M is a multiple of identity. Schur's lemma implies that $D^j_{m'm}(\alpha, \beta, \gamma)$ is irreducible.

5. If we replace β by $\beta + 2\pi$, we see that from Eq(2) that the 2×2 matrix U has the property,

$$U \rightarrow -U$$

For the representation matrice this implies that

$$d^j(\beta + 2\pi) = (-1)^{2j} d^j(\beta), \qquad \text{or} \qquad D^j(-U) = (-1)^{2j} D^j(U)$$

Thus for $j =$integer $\quad D^j(-U) = D^j(U) \quad$ single valued representation
$j =$ half integer $\quad D^j(-U) = D^j(U) \quad$ double valued representation

6. Representation of generators J_i, $i = 1, 2, 3$
Recall that

$$R(\alpha, \beta, \gamma) = \exp\left(-\frac{iJ_z}{\hbar}\alpha\right)\exp\left(-\frac{iJ_y}{\hbar}\beta\right)\exp\left(-\frac{iJ_z}{\hbar}\gamma\right)$$

Take $\beta = \gamma = 0$, $\alpha \ll 1$, we get

$$R(\alpha, \beta, \gamma) \simeq \left(1 - \frac{iJ_z}{\hbar}\alpha\right)$$

From

$$D^j_{m'm}(\alpha, 0, 0) \simeq (1 - im'\alpha)\,\delta_{mm'}$$

we get

$$D^j_{m'm}(J_z) = \hbar m \delta_{mm'}$$

Similarly, take $\alpha = -\gamma = -\frac{\pi}{2}$, and β small, we can get $D(J_x)$
$\alpha = \gamma = 0$ and β small, we can get $D(J_y)$
In particular, for $j = 1/2$, we have

$$D^{1/2}(j_x) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D^{1/2}(j_y) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D^{1/2}(j_z) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are the Pauli matrices up to $\frac{\hbar}{2}$.

7. Great Orthogonality Theorem reads

$$\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{\pi} d\gamma D^{j}_{m'm}(\alpha,\beta,\gamma)\, D^{k*}_{n'n}(\alpha,\beta,\gamma) = \delta_{jk}\delta_{mn}\delta_{m'n'} \frac{\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{\pi} d\gamma}{(2j+1)}$$

Using $D^{j}_{m'm}(\alpha,\beta,\gamma)$ given in Eq(4) we can integrate over angles α and β to reduce the integral to orthogonality relation on $d^{j}_{m'm}(\beta)$,

$$\int_0^{\pi} \sin\beta \, d\beta d^{j}_{m'm}(\beta)\, d^{k}_{m'm}(\beta) = \frac{2\delta_{jk}}{(2j+1)}$$

8. Characters of irreps
Since all rotations of same angle are in the same class, choose rotation about z-axis to compute the trace

$$\chi^{(j)}(\theta) = Tr\left[D^{j}\left(\vec{n},\theta\right)\right] = Tr\left[D^{j}(\theta,0,0)\right] = \sum_{m=-j}^{j} e^{-im\theta} = \frac{\sin\left(j+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Note that

$$\chi^{(j)}(\theta) - \chi^{(j-1)}(\theta) = 2\cos j\theta$$

Theorem: There are no irreps of $SU(2)$ group other than $D^j(\alpha, \beta, \gamma)$.
Proof: Suppose D is another irrep with character$\chi(\theta)$, not contained in D^j. Then

$$\chi(\theta) = \chi(-\theta)$$

since they are in the same class. Then from orhogonality theorem

$$\chi(\theta) \perp \chi^{(j)}(\theta)$$

for all j. $\Longrightarrow$

$$\chi(\theta) \perp \left[\chi^{(j)}(\theta) - \chi^{(j-1)}(\theta)\right] = 2\cos j\theta$$

From the property of Fourier series, $\cos j\theta$, $2j = 1, 2, 3, \cdots$ form a complete set of even functions in the range $0 < \theta < \pi$. Thus $\chi(\theta) = 0$. for all θ. ■

Basis Functions

Suppose we define for $j = l =$integer

$$D^{(l)}_{0m}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2l+1}} Y^m_l(\beta, \gamma)$$

then we have

$$D^{(l)}_{m0}(\alpha, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y^{m*}_l(\beta, \alpha)$$

Suppose we have the relation for multiplication of group elements

$$R(\alpha_2, \beta_2, \gamma_2)\, R(\alpha_1, \beta_1, \gamma_1) = R(\alpha, \beta, \gamma)$$

The corresponding representation matrices will satisfy

$$D^l_{m' m}(\alpha, \beta, \gamma) = \sum_{m''} D^l_{m' m''}(\alpha_2, \beta_2, \gamma_2)\, D^l_{m'' m}(\alpha_1, \beta_1, \gamma_1)$$

Setting $m' = 0$, the relation is

$$Y_l^m(\beta, \gamma) = \sum_{m''} Y_l^{m''}(\beta_2, \gamma_2)\, D^l_{m'' m}(\alpha_1, \beta_1, \gamma_1)$$

Thus functions $\{Y_l^m(\beta, \gamma)\}$ form the basis for irrep $D^l(\alpha, \beta, \gamma)$. We can rewrite this in a more familiar spherical angles notation as,

$$P_R Y_l^m(\theta, \phi) = Y_l^m(\theta', \phi') = \sum_{m''} Y_l^{m''}(\theta, \phi)\, D^l_{m'' m}(\alpha, \beta, \gamma) \qquad (9)$$

where

$$\cos\theta' = \cos\theta\cos\beta + \sin\theta\sin\phi\cos(\phi - \gamma)$$

Example :

1. $l = 1$

$$
\begin{aligned}
Y_1^1(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\frac{(x+iy)}{r}, \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}\cos\theta = \sqrt{\frac{3}{8\pi}}\frac{z}{r},
\end{aligned}
$$

$$
Y_1^{-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} = -\sqrt{\frac{3}{8\pi}}\frac{(x-iy)}{r}
$$

2. $l = 2$

$$
Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi} \sim \frac{\left(x^2-y^2+2ixy\right)}{r^2} \sim \frac{(x+iy)^2}{r^2}
$$

$$
Y_2^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi} \sim \frac{z(x+iy)}{r^2}
$$

$$
Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{2\pi}}\left(3\cos^2\theta-1\right) \sim \frac{\left(2z^2-x^2-y^2\right)}{r^2} \sim \frac{2z^2-(x+iy)(x-iy)}{r^2}
$$

$$
Y_2^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi} \sim \frac{z(x-iy)}{r^2}
$$

$$
Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi} \sim \frac{\left(x^2-y^2-2ixy\right)}{r^2} \sim \frac{(x-iy)^2}{r^2}
$$

3. For the special case of $m = 0$ we have

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

and from Eq(9) we get the addition theorem,

$$P_l(\cos\theta') = \left(\frac{4\pi}{2l+1}\right) \sum_m Y_l^m(\theta,\phi) Y_l^{m*}(\beta,\gamma)$$

where

$$\cos\theta' = \cos\theta\cos\beta + \sin\theta\sin\beta\cos(\phi-\gamma)$$

Product Representations

Addition of Angular Momentum

Suppose $D^{(j_1)}$ and $D^{(j_2)}$ are 2 irreps of $SU(2)$. How to reduce the product representation $D^{(j_1)} \otimes D^{(j_2)} \equiv D^{(j_1 \times j_2)}$, i.e. write $D^{(j_1 \times j_2)}$ as sum of irreps?. Using the characters of irreps (take $j_1 > j_2$),

$$\begin{aligned}
\chi^{(j_1 \times j_2)}(\theta) &= \chi^{(j_1)}(\theta)\,\chi^{(j_2)}(\theta) \\
&= \frac{\sin\left(j_1 + \frac{1}{2}\right)\theta \sin\left(j_2 + \frac{1}{2}\right)\theta}{\sin^2\frac{\theta}{2}} = \frac{\cos(j_1 - j_2)\theta - \cos(j_1 + j_2 + 1)\theta}{\sin^2\frac{\theta}{2}} \\
&= \frac{1}{\sin^2\frac{\theta}{2}} \left\{ \begin{array}{c} [\cos(j_1 - j_2)\theta - \cos(j_1 - j_2 + 1)\theta] \\ + [\cos(j_1 - j_2 + 1)\theta - \cos(j_1 - j_2 + 2)\theta] \\ + \cdots [\cos(j_1 + j_2)\theta - \cos(j_1 + j_2 + 1)\theta] \end{array} \right\} \\
&= \frac{1}{\sin\frac{\theta}{2}} \left\{ \sin\left(j_1 - j_2 + \frac{1}{2}\right)\theta + \sin\left(j_1 - j_2 + \frac{3}{2}\right)\theta + \cdots \sin(j_1 + j_2)\theta \right\}
\end{aligned}$$

the character of $D^{(j_1)} \otimes D^{(j_2)}$ has the decomposition,

$$\chi^{(j_1 \times j_2)}(\theta) = \sum_{J=|j_1 - j_2|}^{j_1 + j_2} \chi^{(J)}(\theta)$$

which implies the reduction,

$$D^{(j_1 \times j_2)} = D^{(j_1)} \otimes D^{(j_2)} = \sum_{J=|j_1-j_2|}^{j_1+j_2} D^{(J)} \tag{10}$$

We can relate this to the addition of angular momenta. Let $f_{j_1}^{m_1}(\hat{r})$ be the basis for $D^{(j_1)}$ irrep and $f_{j_2}^{m_2}(\hat{r})$ be the basis for $D^{(j_2)}$. Then the product $f_{j_1}^{m_1}(\hat{r})\, f_{j_2}^{m_2}(\hat{r})$ are the basis for $D^{(j_1)} \otimes D^{(j_2)}$, i.e.

$$P_R\left[f_{j_1}^{m_1} f_{j_2}^{m_2}\right] = \left[P_R f_{j_1}^{m_1}\right]\left[P_R f_{j_2}^{m_2}\right] = \left(\sum_{m_1'} f_{j_1}^{m_1'} D_{m_1 m_1'}^{(j_1)}\right)\left(\sum_{m_2'} f_{j_2}^{m_2'} D_{m_2 m_2'}^{(j_2)}\right)$$

For infinitesmal rotation around, say z-axis, (setting $\hbar = 1$)

$$P_R = e^{-i\alpha J_z} \simeq (1 - i\alpha J_z), \qquad \alpha \ll 1$$

Let $\vec{J}_1$ acting on $f_{j_1}^{m_1}$ and leaving $f_{j_2}^{m_2}$ alone and $\vec{J}_2$ acting on $f_{j_2}^{m_2}$ and leaving $f_{j_1}^{m_1}$ alone. Then we can write

$$\begin{aligned} P_R\left[f_{j_1}^{m_1} f_{j_2}^{m_2}\right] &= \left[(1 - i\alpha J_z)\, f_{j_1}^{m_1}\right]\left[(1 - i\alpha J_z)\, f_{j_2}^{m_2}\right] \simeq \left[1 - i\alpha\left(J_{1z} + J_{2z}\right)\right] f_{j_1}^{m_1} f_{j_2}^{m_2} \\ &= (1 - i\alpha J_z)\, f_{j_1}^{m_1} f_{j_2}^{m_2} \end{aligned}$$

where

$$J_z = J_{1z} + J_{2z}$$

We can extend this to other components to write

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

which is just the total angular momentum. $\Longrightarrow$ infinitesmal rotation of product of the basis can be written in terms of total angular momenta. From Eq(10) we see that decomposition of the product of irreps is related to the additon of angular momenta.

Clebsch-Gordon Coefficients

In the decomposition of product representations

$$D^{(j_1 \times j_2)} = D^{(j_1)} \otimes D^{(j_2)} = D^{(|j_1 - j_2|)} + D^{(|j_1 - j_2|+1)} + \cdots + D^{(j_1 + j_2)}$$

$\Longrightarrow$representation matrices $D^{(j_1 \times j_2)}$ can be changed into a direct sum of irreps D^j. In terms of matrix elements we have

$$D^{(j_1 \times j_2)}_{m_1' m_2'; m_1 m_2} = D^{(j_1)}_{m_1 m_1'} D^{(j_2)}_{m_2 m_2'}$$

On the other hand,

$$\sum_J D^J \equiv \Delta = \begin{pmatrix} D^{(|j_1-j_2|)} & & & \\ & D^{(|j_1-j_2|+1)} & & \\ & & \ddots & \\ & & & D^{(j_1+j_2)} \end{pmatrix} \equiv \Delta_{J'M';JM} = \delta_{JJ'} D^J_{MM'}$$

where J, J' label the boxes and M, M' labels the row and colums within each box.
Let A be the unitary matrix which transform $D^{(j_1)} \otimes D^{(j_2)}$ into Δ,

$$D^{(j_1)} \otimes D^{(j_2)} = A^\dagger \Delta A$$

Writing out in terms of matrix elements,

$$D^{(j_1 \times j_2)}_{m_1' m_2'; m_1 m_2} = \left[A^\dagger\right]_{m_1' m_2'; J' M'} \Delta_{J' M; JM} \left[A\right]_{JM; m_1 m_2} \tag{11}$$

New notation for the matrix elements of the similarity transformation,

$$\left[A\right]_{JM; m_1 m_2} \equiv \langle JM | j_1 m_1 j_2 m_2 \rangle$$

These are called **Clebsch-Gordon coefficients**. Since A is unitary, $AA^{\dagger:} = 1$, and $A^\dagger A = 1$ we have

$$\sum_{J' M'} \langle j_1 m_1' j_2 m_2' | J' M' \rangle \langle J' M' | j_1 m_1 j_2 m_2 \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}$$

$$\sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | J' M' \rangle = \delta_{JJ'} \delta_{MM'}$$

Note that unitary matrix A is not uniquely defined. Let B be a unitary matrix which commutes with Δ,

$$B\Delta = \Delta B, \qquad \text{or} \qquad \Delta = B^\dagger \Delta B$$

Then

$$D = A^\dagger \Delta A = (BA)^\dagger \Delta BA$$

Thus if A block diagonalizes D, so does $A' = BA$. Since Δ is of the form

$$\Delta = \begin{pmatrix} D^{(J_1)} & & & \\ & D^{(J_2)} & & \\ & & \ddots & \\ & & & D \end{pmatrix}$$

From Schur's lemma, B must be of the form,

$$B = \begin{pmatrix} c_1 I_1 & & & \\ & c_2 I_2 & & \\ & & \ddots & \\ & & & I \end{pmatrix}, \qquad \text{where} \quad c_i = e^{i\theta_r}$$

Or

$$B_{JM;J'M'} = \delta_{JJ'}\delta_{MM'}e^{i\theta_j}$$

and

$$[A']_{JM;m_1 m_2} = e^{i\theta_J}\,[A]_{JM;m_1 m_2}$$

Thus similarity transformation is defined up to J dependent phases.
Condon-Shortly convention : Choose

$$\langle JJ|j_1 j_1 j_2 J - j_1\rangle$$

to be real and positive, then it turns out that all Clebsch-Gordon coefficients are real.
In terms of Clebsch-Gordon coefficients Eq 11) becomes

$$D^{(j_1)}_{m_1 m_1'} D^{(j_2)}_{m_2 m_2'} = \sum_{J,M,M'} D^J_{M'M}\,\langle JM'|j_1 m_1' j_2 m_2'\rangle\,\langle JM|j_1 m_1 j_2 m_2\rangle$$

Theorem: Let $\psi_{j_1}^{m_1}(\hat{r})$ be the basis for $D^{(j_1)}$ irrep and $\psi_{j_2}^{m_2}(\hat{r})$ be the basis for $D^{(j_2)}$. Then

$$\phi_M^J = \sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle \psi_{j_1}^{m_1} \psi_{j_2}^{m_2}$$

are basisi for D^J.
Proof:

$$\begin{aligned} P_R \phi_M^J &= \sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle \left(P_R \psi_{j_1}^{m_1} \right) \left(P_R \psi_{j_2}^{m_2} \right) = \sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle D_{m_1 m_1'}^{(j_1)} D_{m_2 m_2'}^{(j_2)} \psi_{j_1}^{m_1'} \psi_{j_2}^{m_2'} \\ &= \sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle \sum_{J, M, M'} D_{M' M"}^{J'} \langle J' M' | j_1 m_1' j_2 m_2' \rangle \langle J' M' | j_1 m_1 j_2 m_2 \rangle \psi_{j_1}^{m_1'} \psi_{j_2}^{m_2'} \\ &= \sum_{M" m_1 m_2} D_{M" M}^{J} \langle JM" | j_1 m_1' j_2 m_2' \rangle \psi_{j_1}^{m_1'} \psi_{j_2}^{m_2'} \end{aligned}$$

Thus ϕ_M^J does transform according to the representation D^J.■
As a consequence if $\vec{j}_3 = \vec{j}_1 + \vec{j}_2$ then

$$\sum_{m_1 m_2 m_3} \psi_{j_1}^{m_1} \psi_{j_2}^{m_2} \psi_{j_3}^{m_3} \langle j_1 m_1 j_2 m_2 | j_3, -m_3 \rangle \langle j_3, -m_3 j_3 m_3 | 00 \rangle$$

is invariant under $SU(2)$ transformations.

Rotation group and Quantum Mechanics

In quantum mechanics, implement symmetry transformations by unitary operator U on the states $|\psi\rangle$

$$|\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle, \qquad \text{for all states}$$

so that

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$$

At the same time the operators change as

$$A \longrightarrow A' = UAU^{\dagger}$$

so that

$$\langle\phi'|A'|\psi'\rangle = \langle\phi|A|\psi\rangle$$

If $A' = A$, or $[U, A] = 0$, we say that A is invariant under the transformation U. In particular, if Hamiltonian H is invariant under the symmetry transformation U

$$[U, H] = 0 \tag{12}$$

Suppose ,

$$[U_1, H] = 0, \qquad [U_2, H] = 0$$

then

$$[U_1 U_2, H] = [U_1, H]\, U_2 + U_1\, [U_2, H] = 0$$

Collection of all such operators form a group. Suppose $|\psi\rangle$ is an eigenstate of H with energy E,

$$H|\psi\rangle = E|\psi\rangle$$

Then from Eq(**??**) we see that

$$HU|\psi\rangle = UH|\psi\rangle = E(U|\psi\rangle)$$

which means that $U|\psi\rangle$ is also an eigenstate of H with same energy E. If we run the operator U through the whole group G, we get

$$U_1|\psi\rangle, \qquad U_2|\psi\rangle, \qquad U_3|\psi\rangle, \qquad \cdots$$

If we select a linear independent set out of these, then we have degeneracy of energy levels.On the other hands, if we consider the eigenfunction $\psi(x)$ of time independent Schrodinger equation,

$$H\psi(x) = E\psi(x)$$

we can use induced tranformation on this eignefunctions to get

$$P_{R_1}\psi(x) = \psi\left(R_1^{-1}x\right), \qquad P_{R_2}\psi(x) = \psi\left(R_2^{-1}x\right), \qquad \cdots$$

As discussed before, if we select a linearly independent set out of these transformed functions, they will form a basis of irrep. Since they are also eigenfunctions of Hamiltonian with same E, we get the result,

$$\text{degenercy} = \text{dim of irrep}$$

For example, if choose $\psi\left(\overrightarrow{x}\right) = z$, then the basis for the transformed functions will be, x, y, z. and they form the basis of $l = 1$ irrep of rotational group.
In the case of hydrogen atom, the Hamiltonian is of the form

$$H = -\frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon r}$$

and is invariant under rotation

$$[R(\alpha, \beta, \gamma), H] = 0$$

where R is an arbitrary 3-dimensional rotation. As a consequence, the degeneracy of the energy levels is $2l + 1$.
Another important result is that for matrix elements between states belong to different irrep are zero,

$$\left\langle \phi_{j'}^{m'} | \phi_j^m \right\rangle \sim \delta_{jj'}\delta_{mm'}$$

This follows from the great orthogonal theorem as follows. We can perform a rotation on this matrix element to get

$$\left\langle \phi_{j'}^{m'} | \phi_j^m \right\rangle = \left\langle P_R\phi_{j'}^{m'} | P_R\phi_j^m \right\rangle = \sum_{l,l',} D_{l'm'}^{j*}(R)\, D_{lm}^{j}(R) \left\langle \phi_{j'}^{l'} | \phi_j^l \right\rangle$$

In order to use great orthogonality theorem we sum over all group elements,

$$\left\langle \phi_{j'}^{m'} | \phi_j^m \right\rangle \int dR = \int dR \left\langle P_R\phi_{j'}^{m'} | P_R\phi_j^m \right\rangle = \sum_{l,l',} \int dR\;\; D_{l'm'}^{j*}(R)\, D_{lm}^{j}(R) \left\langle \phi_{j'}^{l'} | \phi_j^l \right\rangle$$

Then from

$$\int dR D^{j*}_{l'm'}(R)\, D^{j}_{lm}(R) \sim \delta_{jj'}\delta_{mm'}\delta_{ll'}$$

we get the result

$$\left\langle \phi^{m'}_{j'} | \phi^{m}_{j} \right\rangle \sim \delta_{jj'}\delta_{mm'}$$

Wigner-Eckart Theorem

In quantum mechanics, need to compute matrix elements of certain operators. We can use the representation theory to simplify the computations. The strategy is to write the operator and wave functions in terms of irrep of group and use the rep theory.

Tensor operators

Suppose $D^{(\alpha)}$ is an irrep of group G and dimension of $D^{(\alpha)}$ is d_α. A set of operators $T_i^{(\alpha)}, i = 1, 2, \cdots d_\alpha$, transforming under the symmetry group G as

$$P_R T_i^{(\alpha)} P_R^{-1} = \sum_j T_j^{(\alpha)} D_{ji}^{(\alpha)}(R)$$

is said to be irreducible tensor operators corresponding to $D^{(\alpha)}$ irreps.

For the case of $SO(3)$ (or $SU(2)$ group) , if the operators $T_j^m, m = -j, -j+1, \cdots j$ satisfy the relation,

$$P_R(\alpha, \beta, \gamma) T_i^m P_R^{-1}(\alpha, \beta, \gamma) = \sum_{m'} T_i^{m'} D^j_{m'm}(\alpha, \beta, \gamma)$$

then we say T_j^m are irreducible tensors of rank j in $SO(3)$. For example,

$$T_1^{(1)} \sim x + iy, \qquad T_0^{(1)} \sim z, \qquad T_{-1}^{(1)} \sim x - iy$$

are tensor operators of rank 1.

Remarks :

1. If $T_{j_1}^{m_1}$ are irreducible tensor of rank j_1 and $T_{j_2}^{m_2}$ are irreducible tensor of rank j_2 then

$$\sum_{m_1,m_2} \langle jm|j_1 m_1 j_2 m_2 \rangle \, T_{j_1}^{m_1} T_{j_2}^{m_2}$$

are irreducible tensor of rank j. In particular, since

$$\langle 00|jmj-m\rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}}$$

we see that the combination

$$\sum_m T_j^m S_j^{-m} (-1)^m$$

is an invariant operator under rotations. Here S_j^m is another tensor operator of rank j.

2. If T_j^m are irreducible tensor operator of rank j, so is

$$S_j^m = (-1)^m \left[T_j^{-m} \right]^{\dagger}$$

This follows from the fact that

$$D^{j*}_{m'm}(\alpha,\beta,\gamma) = (-1)^{m'-m} D^{j*}_{-m',-m}(\alpha,\beta,\gamma)$$

3. Combining remarks (1) and (2) we get that

$$\sum_m T_j^m \left(T_j^m\right)^\dagger$$

is a $SO(3)$ invariant operator.

Theorem $(Wigner - Eckart)$

If ϕ_j^m are basis for D^j , $\phi_{j'}^{m'}$ are basis for $D^{j'}$ and T_k^q are irreducible tensor operator of rank k, then

$$\left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle = \langle j'm' | kqjm \rangle \frac{\left\langle \phi_{j'} || T_k || \phi_j \right\rangle}{\sqrt{2j+1}}$$

where $\left\langle \phi_{j'} || T_k || \phi_j \right\rangle$ are the reduced matrix elements and are independent of m, m', q.

Proof:

Since T_k^q are irreducible tensor operator, we can write

$$\begin{aligned}
\left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle &= \left\langle P_R \phi_{j'}^{m'} | P_R T_k^q P_R^{-1} | P_R \phi_j^m \right\rangle \\
&= \sum_{l,l'q',} D_{l'm'}^{j*}(R)\, D_{lm}^{j}(R) \left\langle \phi_{j'}^{l'} | T_k^{q'} | \phi_j^l \right\rangle D_{q'q}^{k}(R) \\
&= \sum_{J,M,M'l,l'q',} D_{l'm'}^{j*}(R)\, D_{M'M}^{J}(R) \langle JM' | kq'jl' \rangle \langle JM | kqjm \rangle \left\langle \phi_{j'}^{l'} | T_k^{q'} | \phi_j^l \right\rangle
\end{aligned}$$

Summing over all group elements in $SU(2)$, we get

$$\begin{aligned}\int dR \left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle &= \left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle \int dR \\ &= \int dR \sum_{J,M,M'l,l'q',} D_{l'm'}^{j*}(R) D_{M'M}^{J}(R) \langle JM'|kq'jl'\rangle \langle JM|kqjm\rangle \left\langle \phi_{j'}^{l'} | T_k^{q'} | \phi_j^{l} \right\rangle\end{aligned}$$

we have used the fact $\left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle$ is independent of R. Using great orthogonality theorem

$$\int dR D_{l'm'}^{j*}(R) D_{M'M}^{J}(R) = \delta_{Jj'} \delta_{l'M'} \delta_{m'M} \int dR$$

we get

$$\begin{aligned}\left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle &= \langle j'm'|kqjm\rangle \sum_{l,l'q'} \langle j'l'|kq'jl\rangle \left\langle \phi_{j'}^{l'} | T_k^{q'} | \phi_j^{l} \right\rangle \\ &= \langle j'm'|kqjm\rangle \frac{\left\langle \phi_{j'} || T_k || \phi_j \right\rangle}{\sqrt{2j+1}}\end{aligned}$$

where

$$\left\langle \phi_{j'} || T_k || \phi_j \right\rangle = \sqrt{2j+1} \sum_{l,l'q'} \langle j'l'|kq'jl\rangle \left\langle \phi_{j'}^{l'} | T_k^{q'} | \phi_j^{l} \right\rangle$$

is the reduced matrix element and is independent of $m, m\prime, q$.

————————————

Remarks:

1. The $m, m\prime, q$ dependence in the matrix element $\left\langle \phi_{j'}^{m'} | T_k^q | \phi_j^m \right\rangle$ are all contained in $\langle j'm' | kqjm \rangle$ which are universal and independent of details of $\phi_j^m, \phi_{j'}^{m'}$ and T_k^q. Then in the ratios of matrix elements we have

$$\frac{\left\langle \phi_{j'}^{m_2} | T_k^q | \phi_j^{m_1} \right\rangle}{\left\langle \phi_{j'}^{m_4} | T_k^q | \phi_j^{m_3} \right\rangle} = \frac{\langle j'm_2 | kqjm_1 \rangle}{\langle j'm_4 | kqjm_3 \rangle}$$

   Thus , we can calculate only one such matrix element explicitly and use Clebsch-Gordon coefficients to obatin other matrix elements with same j, j' and k.

2. Since Clebsch-Gordon coefficients $\langle j'm' | kqjm \rangle$ has the property that it vanishes unless, $|k-j| \le j' \le j+k$, and $m' = q+m$, the matrix elements will satify the same selection rules independent of the nature of the wavefunctions $\phi_j^m, \phi_{j'}^{m'}$ or the operator T_k^q .

Example: Dipole matrix elements $\left\langle \phi_{j'}^{m'} | \vec{r} | \phi_j^m \right\rangle$, $\vec{r} = (x, y, z)$. Note that the linear combinations,

$$r_+ = -\frac{1}{\sqrt{2}}(x+iy), \qquad r_0 = z, \qquad r_- = \frac{1}{\sqrt{2}}(x-iy)$$

are the basis for the irrep $D^1_{mm'}$. Thus we have the selection rules: matrix elements vanish unless

$$j' = j \pm 1, \qquad \text{but if } j = 0 \quad \text{then } j' = 1$$

and

$$m' = m \text{ or } m \pm 1$$