

Tensor Method in $SU(n)$

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Tensor Analysis

The analysis we have discussed for $SU(2)$ and $SU(3)$ shows that, as the group gets larger, the elementary techniques used to dissect the representation structure becomes very complicate. The tensor method we will discuss here provides a handle which is very useful for low rank representations.

1 Tensor analysis in $O(3)$

1.1 Rotation in R_3

Rotation: coordinate axes are fixed and the physical system is undergoing a rotation. Let x'_a, x_b be the components of new and old vectors. Then we have

$$x'_a = \sum_b R_{ab} x_b$$

where R_{ab} are elements of matrix which represents rotation. For example, the matrix for the rotation about z -axis is of the form,

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the relation between x'_a and x_b is linear and homogeneous.

Important properties of transformation:

1. R is an orthogonal matrix,

$$RR^T = R^T R = 1, \quad \text{or} \quad R_{ab} R_{ac} = \delta_{bc}, \quad R_{ab} R_{cb} = \delta_{ac}$$

More conveniently, we will write the orthogonality relations as,

$$R_{ab} R_{cd} \delta_{ac} = \delta_{bd}, \quad R_{ab} R_{cd} \delta_{bd} = \delta_{ac}$$

i.e. in the product of 2 rotation matrix elements, making row(or colm) indices the same and summed over will give Kronecker δ .

2. The combination $\vec{x}^2 = x_a x_a$ is invariant under rotations,

$$x'_a x'_a = R_{ac} R_{ab} x_c x_b = x_b x_b$$

This is the same as saying the length of a vector is invariant under rotation. We can generalize to the case of 2 arbitrary vectors, \vec{A}, \vec{B} with transformation property,

$$A'_a = R_{ab} A_b, \quad B'_c = R_{cd} B_d$$

Then $\vec{A} \cdot \vec{B} = A_a B_a$ is invariant under rotation. It is more convenient to write this as

$$\vec{A} \cdot \vec{B} = A_a B_a = A_a B_b \delta_{ab}$$

which is sometimes called the **contraction** of indices.

3. Transformation of the gradient operators is

$$\frac{\partial}{\partial x'_a} = \frac{\partial}{\partial x_c} \frac{\partial x_c}{\partial x'_a}$$

From $x_b = (R^{-1})_{ba} x'_a$, we get then

$$\frac{\partial}{\partial x'_a} = (R^{-1})_{ca} \frac{\partial}{\partial x_c}$$

Thus gradient operator transforms by $(R^{-1})^T$. However, for rotations, R is orthogonal, $(R^{-1})^T = R$,

$$\frac{\partial}{\partial x'_a} = R_{ac} \frac{\partial}{\partial x_c}$$

i.e. $\partial_\alpha = \frac{\partial}{\partial x_a}$ transforms the same way as x_a .

Note that for the case the coordinate transformation is not orthogonal the coordinates and the partial derivative transform differently.

1.2 Tensors

Tensors are essentially those quantities which transform in the same way as the products of vectors under the coordinate transformation, the rotation in our case. Suppose we have two vectors, i.e. they have the transformation properties,

$$A_a \rightarrow A'_a = R_{ab} A_b, \quad B_c \rightarrow B'_c = R_{cd} B_d$$

then the product will have the transformation,

$$A'_a B'_c = R_{ab} R_{cd} A_b B_d$$

The **second rank tensors** are those objects which have the same transformation properties as the product of 2 vectors, i. e.,

$$T_{ac} \rightarrow T'_{ac} = (R_{ab} R_{cd}) T_{bd}$$

Definition of n -th rank tensors (Cartesian tensors)

$$T_{i_1 i_2 \dots} \rightarrow T'_{i_1 i_2 \dots i_n} = (R_{i_1 j_1}) (R_{i_2 j_2}) \dots (R_{i_n j_n}) T_{j_1 j_2 \dots j_n}$$

Note again that these transformations are linear and homogeneous which implies that

$$if \quad T_{j_1 j_2 \dots j_n} = 0, \quad for \text{ all } j_m$$

then they all zero in other coordinate system.

1.3 Tensor operations

1. Multiplication by constants

$$(cT)_{i_1 i_2 \dots i_n} = cT_{i_1 i_2 \dots i_n}$$

2. Add tensors of same rank

$$(T_1 + T_2)_{i_1 i_2 \dots i_n} = (T_1)_{i_1 i_2 \dots i_n} + (T_2)_{i_1 i_2 \dots i_n}$$

3. Multiplication of 2 tensors

$$(ST)_{i_1 i_2 \dots i_n j_1 j_2 \dots j_m} = S_{i_1 i_2 \dots i_n} T_{j_1 j_2 \dots j_m}$$

This will give a tensor of rank which is the sum of the ranks of 2 constituent tensors.

4. Contaction

$$S_{abc} T_{ae} = S_{abc} T_{de} \delta_{ad} \rightarrow \text{3rd rank tensor}$$

This will reduce the rank of tensor by 2.

5. Symmetrization

$$\text{if } T_{ab} \text{ 2nd rank tensor} \Rightarrow T_{ab} \pm T_{ba} \text{ are also 2nd rank tensors}$$

6. Special numerical tensors

$$RR^T = 1, \quad \Rightarrow R_{ij} R_{kj} = \delta_{ik}, \quad \text{or} \quad R_{ij} R_{kl} \delta_{jl} = \delta_{ik}$$

This means that δ_{ij} can be treated as 2nd rank tensor. Similarly,

$$(\det R) \varepsilon_{abc} = \varepsilon_{ijk} R_{ai} R_{bj} R_{ck}$$

ε_{abc} a 3rd rank tensor. Useful identities for ε_{abc}

$$\varepsilon_{ijk} \varepsilon_{ijl} = \delta_{kl}, \quad \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

One can use these numerical tensors to construct other tensors. For example,

$$\varepsilon_{ijk} A_i B_k \quad \text{vector}$$

$$r_i r_j - \delta_{ij} r^2 \quad \text{second rank tensor}$$

More general notation for tensor transformation (Jackson),

$$x'_a = R_{ab} x_b, \quad \Rightarrow R_{ab} = \frac{\partial x'_a}{\partial x_b}$$

Then we can write

$$x'_a = \frac{\partial x'_a}{\partial x_b} x_b$$

This can be used for more general coordinate transformations other than the simple rotation and is used extensively in general relativity.

2 Transformation Law of Tensors in SU(N)

The $SU(n)$ group consists of $n \times n$ unitary matrices with unit determinant. We can regard these matrices as linear transformations in an n -dimensional complex vector space C_n . Thus any vector

$$\psi_i = (\psi_1, \psi_2, \dots, \psi_n)$$

in C_n is mapped by an $SU(n)$ transformation U_{ij} , as

$$\psi_i \rightarrow \psi'_i = U_{ij} \psi_j \quad (1)$$

Thus ψ'_i also belong to C_n , with $UU^\dagger = U^\dagger U = 1$ and with $\det U = 1$. Clearly for any two vectors we can define a scalar product

$$(\psi, \phi) \equiv \psi_i^* \phi_i$$

which is invariant under $SU(n)$ transformation. The transformation law for the conjugate vector is given by,

$$\psi_i^* \rightarrow \psi_i'^* = U_{ij}^* \psi_j^* = \psi_j^* U_j^\dagger \quad (2)$$

Here we see that ψ_i and ψ_i^* transform differently under the unitary transformation. It is convenient to introduce upper and lower indices to write

$$\psi^i \equiv \psi_i^*, \quad U_i^j \equiv U_{ij}, \quad U_j^i \equiv U_{ij}^*$$

Thus complex conjugation just change the lower indices to upper ones, and vice versa. In these notation, the transformation law in Eqs(1,2) become

$$\psi_i \rightarrow \psi_i' = U_i^j \psi_j$$

and

$$\psi^i \rightarrow \psi'^i = U_j^i \psi^j$$

The $SU(n)$ invariant scalar product can be written as

$$(\psi, \phi) = \psi^i \phi_i$$

and the unitary condition becomes

$$U_k^i U_j^k = \delta_j^i \quad (3)$$

where the Kronecker delta is defined as

$$\delta_j^i = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Note that in this notation the summation is always over a pair of upper and lower indices. We call this a **contraction of indices**. The ψ_i are the basis for the $SU(n)$ **defining representation** (also called the **fundamental** or **vector representation** and denoted as \mathbf{n}), while the ψ^i are the basis for the conjugate representation, \mathbf{n}^* .

Higher-rank tensors are defined as those quantities which have the same transformations properties as the direct products of vectors. Thus tensors generally have both upper and lower indices with the transformation law,

$$\psi_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p} = \left(U_{k_1}^{i_1} U_{k_2}^{i_2} \dots U_{k_p}^{i_p} \right) \left(U_{j_1}^{l_1} U_{j_2}^{l_2} \dots U_{j_q}^{l_q} \right) \psi_{l_1 l_2 \dots l_q}^{k_1 k_2 \dots k_p}$$

They correspond to the basis for higher-dimensional representations.

2.1 Invariant tensors

The Kronecker delta and Levi-Civita symbol are invariant tensors under $SU(n)$ transformations. They play important role in the study of irreducible tensors.

1. From the unitarity condition of Eq(3) we immediately have

$$\delta_j^i = U_k^i U_j^k \delta_l^k \quad (4)$$

Hence δ_j^i , even though do not change under the $SU(n)$ transformations, behaves as if they are second rank tensors. They can be used to contract indices of other tensor to produce a tensor of lower rank. For example, if ψ_{ij}^k is a 3rd rank tensor,

$$\psi_{ij}^k \rightarrow \psi'^k_{ij} = U_a^k U_i^b U_j^c \psi_{bc}^a$$

the contracting with δ_i^k gives

$$\psi'^k_{ij} \delta_i^k = \delta_i^k U_a^k U_i^b U_j^c \psi_{bc}^a = U_j^c \delta_a^b \psi_{bc}^a$$

where we have used Eq(4). This gives a tensor of rank 1(vector).

2. The Levi-Civita symbol is defined as the totally antisymmetric quantity,

$$\varepsilon^{i_1 i_2 \dots i_n} = \varepsilon_{i_1 i_2 \dots i_n} = \begin{cases} 1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an even permutation of } (1, 2, \dots, n) \\ -1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an odd permutation of } (1, 2, \dots, n) \\ 0 & \text{otherwise} \end{cases}$$

This is also an invariant tensor, because from the property of the determinant we have

$$(\det U) \varepsilon_{i_1 i_2 \dots i_n} = U_{i_1}^{j_1} U_{i_2}^{j_2} \dots U_{i_n}^{j_n} \varepsilon_{j_1 j_2 \dots j_n}$$

Since $\det U$ in $SU(n)$, $\varepsilon_{i_1 i_2 \dots i_n}$ can be treated as $n - th$ rank tensor. We can use this to change the rank of a tensor. For example,

$$\psi^{i_2 \dots i_n} \varepsilon_{i_1 i_2 \dots i_n} \sim \psi_{i_1}$$

is a vector.

3 Irreducible Representations and Young Tableaux

3.1 Permutation symmetry and tensors

Generally the tensors we have just defined are basis for reducible representation of $SU(n)$. To decompose them into irreducible representations we use the following property of these tensors. The permutation of upper(or lower) indices commutes with the $SU(n)$ transformations, as the latter consists of product of identical U_{ij} 's(or U_{ij}^* 's). We will illustrate this with a simple example. Consider the second rank tensor ψ_{ij} whose transformation is given by

$$\psi'_{ij} = U_i^a U_j^b \psi_{ab}$$

Since U 's are the same, we can relabel the indices to get

$$\psi'_{ji} = U_j^b U_i^a \psi_{ba}$$

Thus the permutation of indices in the tensor does not change the transformation law. If P_{12} is the permutation operator which interchanges the two indices,

$$P_{12}\psi_{ij} = \psi_{ji},$$

then P_{12} commutes with the group transformation,

$$P_{12}\psi'_{ij} = U_i^a U_j^b (P_{12}\psi_{ab})$$

This property can be used to decompose ψ_{ij} as follows. First we form eigenstates of the permutation operator P_{12} by symmetrization or antisymmetrization,

$$S_{ij} = \frac{1}{2}(\psi_{ij} + \psi_{ji}), \quad A_{ij} = \frac{1}{2}(\psi_{ij} - \psi_{ji})$$

so that

$$P_{12}S_{ij} = S_{ij}, \quad P_{12}A_{ij} = -A_{ij}$$

In group theory language, S_{ij} form basis of an one-dimensional representation of the permutation group S_2 and A_{ij} the basis for another representation. It is clear that S_{ij} , and A_{ij} will not mix under the $SU(n)$ transformations,

$$S'_{ij} = U_i^a U_j^b S_{ab}, \quad A'_{ij} = U_i^a U_j^b A_{ab}$$

This shows that the second rank tensor ψ_{ij} decomposes into S_{ij} , and A_{ij} in such a way that group transformations never mix parts with different symmetries. It turns out that S_{ij} , and A_{ij} can not be decomposed any further and they thus form the basis of irreducible representations of $SU(n)$. This can be generalized to tensors of higher rank (hence the possibility of mixed symmetries) with the result that the basis for irreducible representations of $SU(n)$ correspond to tensors with definite permutation symmetry among (the positions of) its indices.

3.2 Young tableaux

The task of finding irreducible tensors of an arbitrary rank f (i.e. number of indices) involves forming a complete set permutation operations on these indices. The problem of finding irreducible representations of the permutation group has a complete solution in terms of the **Young tableaux**. They are pictorial representations of the permutation operations of f objects as a set of f boxes each with an index number in it. For example, for the second rank tensors, the symmetrization of indices i and j in S_{ij} is represented by $\boxed{i \ j}$; the antisymmetrization operation in A_{ij} is

represented by $\boxed{\begin{smallmatrix} i \\ j \end{smallmatrix}}$. For the third rank tensors ψ_{ijk} , we have

$$\boxed{i \ j \ k} \quad \text{for totally symmetric combination } S_{ijk} = \frac{1}{6}(\psi_{ijk} + \psi_{jki} + \psi_{kij} + \psi_{jik} + \psi_{kji} + \psi_{ikj}) \quad (5)$$

$$\boxed{\begin{smallmatrix} i \\ j \\ k \end{smallmatrix}}, \quad \text{for the totally antisymmetric combination } A_{ijk} = \frac{1}{6}(\psi_{ijk} + \psi_{jki} + \psi_{kij} - \psi_{jik} - \psi_{kji} - \psi_{ikj}) \quad (6)$$

$$\boxed{\begin{smallmatrix} i & j \\ k \end{smallmatrix}}, \quad \text{for the tensor with mixed symmetry } \chi_{ijk} = \frac{1}{4}(\psi_{ijk} + \psi_{jik} - \psi_{kij} - \psi_{kji}) \quad (7)$$

A general Young tableau is shown in the following figure. It is an arrangement of f boxes in rows and columns such that the length of rows should not increase from top to bottom: $f_1 \geq f_2 \geq f_3 \cdots$ and $f_1 + f_2 + f_3 \cdots = f$. Each box has an index $i_k = 1, 2, \cdots n$.

i_1	i_2	\cdots	\cdots	i_{f_1}
i_{f_1+1}	\cdots	\cdots	i_{f_2}	
\vdots				
i_{\dots}				

To this tableau we associate the tensor,

$$\psi_{i_1 \cdots i_{f_1}, i_{f_1+1} \cdots}$$

with the following operations.

1. First apply symmetrization operators which leave invariant the indices appearing in each row.
2. Then apply antisymmetrization operators which leave indices in each column invariant.

A tableau where the index numbers do not decrease when going from left to right in a row and always increase from top to bottom is a **standard tableau**. For example, the $n = 3$ mixed symmetry tensors are given here with the respective Young tableaux.

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where we have made use of the tensor $\chi_{ij;k}$ given in Eq (7). Thus we have 8 independent tensors which correspond to the 8-dimensional irrep we discussed before. Note that non-standard tableaux give tensors that, by symmetrization or antisymmetrization, either vanish or are not independent of the standard tableaux. Thus for a given pattern of the Young tableaux the number of independent tensors is equal to the number of standard tableaux which can be formed. It is not hard to see that this number for the simplest case of a tensor with k antisymmetric indices is

$$\left. \begin{array}{c} \square \\ \square \\ \square \\ \vdots \\ \square \end{array} \right\} k, \quad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \times 2 \times 3 \cdots k}$$

and that for a tensor with k symmetric indices the number is

$$\underbrace{\square \square \square \cdots \square}_k \quad \binom{n+k-1}{k} = \frac{n(n+1)\cdots(n+k-1)}{1 \times 2 \times 3 \cdots k}$$

One should note that because of antisymmetrization there are not more than n rows in any Young tableau. Also if there are n rows, we can use $\varepsilon_{i_1 \cdots i_n}$ to contract the indices in the columns with n entries. Pictorially, we can simply cross out any columns with n rows without changing the transformation property of the tensor.

Fundamental theorem (See for example, Hammermesh 1963.) A tensor corresponding to the Young tableaux of a given pattern forms the basis of an irrep of $SU(n)$. Moreover if we enumerate all possible Young tableaux under the restriction that there should be no more than $n-1$ rows, the corresponding tensors form a complete set, in the sense that all finite-dimensional irreps of the group are counted only once.

We next give two formulae of the **dimensionality of irreps**. If the Young tableau is characterized by the length of its rows $(f_1, f_2, \cdots, f_{n-1})$, define the length differences of adjacent rows as

$$\lambda_1 = f_1 - f_2, \quad \lambda_2 = f_2 - f_3, \quad \cdots \lambda_{n-2} = f_{n-2} - f_{n-1}, \quad \lambda_{n-1} = f_{n-1}$$

The dimension of an $SU(n)$ irreps will then be the number of standard tableaux for a given pattern and is given by

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) &= (1 + \lambda_1)(1 + \lambda_2) \cdots (1 + \lambda_{n-1}) \\
 &\times \left(1 + \frac{\lambda_1 + \lambda_2}{2}\right) \left(1 + \frac{\lambda_2 + \lambda_3}{2}\right) \cdots \left(1 + \frac{\lambda_{n-2} + \lambda_{n-1}}{2}\right) \\
 &\times \left(1 + \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\right) \left(1 + \frac{\lambda_2 + \lambda_3 + \lambda_4}{3}\right) \cdots \left(1 + \frac{\lambda_{n-3} + \lambda_{n-2} + \lambda_{n-1}}{3}\right) \\
 &\cdots \\
 &\times \left(1 + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}}{n-1}\right)
 \end{aligned} \tag{8}$$

On can easily check that the special results of totally symmetric and anti-symmetric tensors are recovered from this formula. Here we give some simple examples.

Example 1. $SU(2)$ group. The Young tableaux can have only one row

$$d(\lambda_1) = (1 + \lambda_1)$$

Thus $\lambda_1 = 2j$. It follows that

$$\square \quad \text{doublet}, \quad \square \square \quad \text{triplet}, \dots$$

Example 2. $SU(3)$ group. The Young tableaux can now have two rows and

$$d(\lambda_1, \lambda_2) = (1 + \lambda_1)(1 + \lambda_2) \left(1 + \frac{\lambda_1 + \lambda_2}{2}\right)$$

This equivalent to the other formula for the dimensionality of $SU(n)$ representations with the identification, $\lambda_1 = p$ and $\lambda_2 = q$. Simple cases are given below

$$\begin{array}{ccc}
 \square(1, 0) \mathbf{3}, & \square \square(2, 0) \mathbf{6}, & \square \square \square(3, 0) \mathbf{10}, \\
 \begin{array}{|c|} \hline \square \\ \hline \end{array}(0, 1) \mathbf{3}^*, & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}(0, 2) \mathbf{6}^*, & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}(1, 1) \mathbf{8},
 \end{array}$$

The formula given in Eq(8) is rather cumbersome to use for large value of n ; in such case the second formula is perhaps more useful. For this we need to introduce two definitions—**hook length** and **distance to the first box**. For any box in the tableaux, draw two perpendicular lines, in the shape of a hook, one going to the right and another going downward. The total number of boxes that this hook passes, including the original box itself, is the hook length (h_i) associated with i th box. For example,

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} h_i = 3, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} h_i = 1$$

The distance to the first box (D_i) is defined to be the number of steps going from the box in the upper left-handed corner of the tableaux to the i th box with each step toward the right counted as +1 unit and each downward step as -1 unit. For example, we have

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 & \\ \hline -2 & & \\ \hline \end{array}$$

The dimension of the $SU(n)$ irrep associated with the Young tableaux is given by

$$d = \prod_i \left(\frac{n + D_i}{h_i} \right)$$

where the products are taken over all boxes in the tableaux. We illustrate this with a simple example

$$\text{Young tableaux } \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \quad \text{hook lengths } \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & \\ \hline \end{array}, \quad \text{distance to the first box } \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & \\ \hline \end{array}$$

and the dimension is

$$d = \binom{n}{3} \binom{n+1}{1} \binom{n-1}{1} = \frac{n(n^2-1)}{3}$$

This gives $d = 8$ for $n = 3$ as expected.

Similarly, if

$$u_i \sim \mathbf{3}, \quad w_j \sim \mathbf{3}$$

then

$$u_i w_j = \frac{1}{2} (u_i w_j + u_j w_i) + \frac{1}{2} (u_i w_j - u_j w_i)$$

which corresponds to

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \mathbf{3}^*$$