Kramers-Kronig Relations 的應用與哲理

國立中山大學 光電工程研究所

張道源

南台灣光電卓越研究中心

- **Crystalline Fibers** (黃升龍所長)
  - 1.2-1.66μm Cr^{4+}:YAG amplifier
  - Periodically-poled $LiNbO_3$ fiber
- **Photonic IC** (賴聰賢教授, 張道源)
  - QW design and MBE growth
  - Device design and process development
- **TW EA Modulator** (邱逸仁教授)

Refractive Indices in IIIV Semiconductors

- n_r below and above the band gap in alloy semiconductors
  - – –– 林猷穎同學, 賴聰賢教授, 張道源
- Electroabsorption and electrorefraction in modulation-doped MQW's
  - – –– 馮瑞陽同學, 莊貴雅同學, 林猷穎同學, 賴聰賢教授, 張道源

Outline

- n_r below band gap: Sellmeier model
- Kramers-Kronig relations
- Absorption above band gap: broadened Sommerfeld factor and broadened discrete exciton
- Double Lorentzian line-shape function
- Electroabsorption and electrorefraction
- Dispersion and chirp in communications
- Mathematics of causality
- Conclusions

Wave Propagation

$$\exp\left(-i\omega t + ikz\right) = \exp\left(-i\omega t + in\frac{\omega}{c}z\right)$$

$$= \exp\left(-i\omega t + in_r\frac{\omega}{c}z - n_i\frac{\omega}{c}z\right) = \exp\left(-i\omega t + in_r\frac{\omega}{c}z - \frac{\alpha_p}{2}z\right)$$

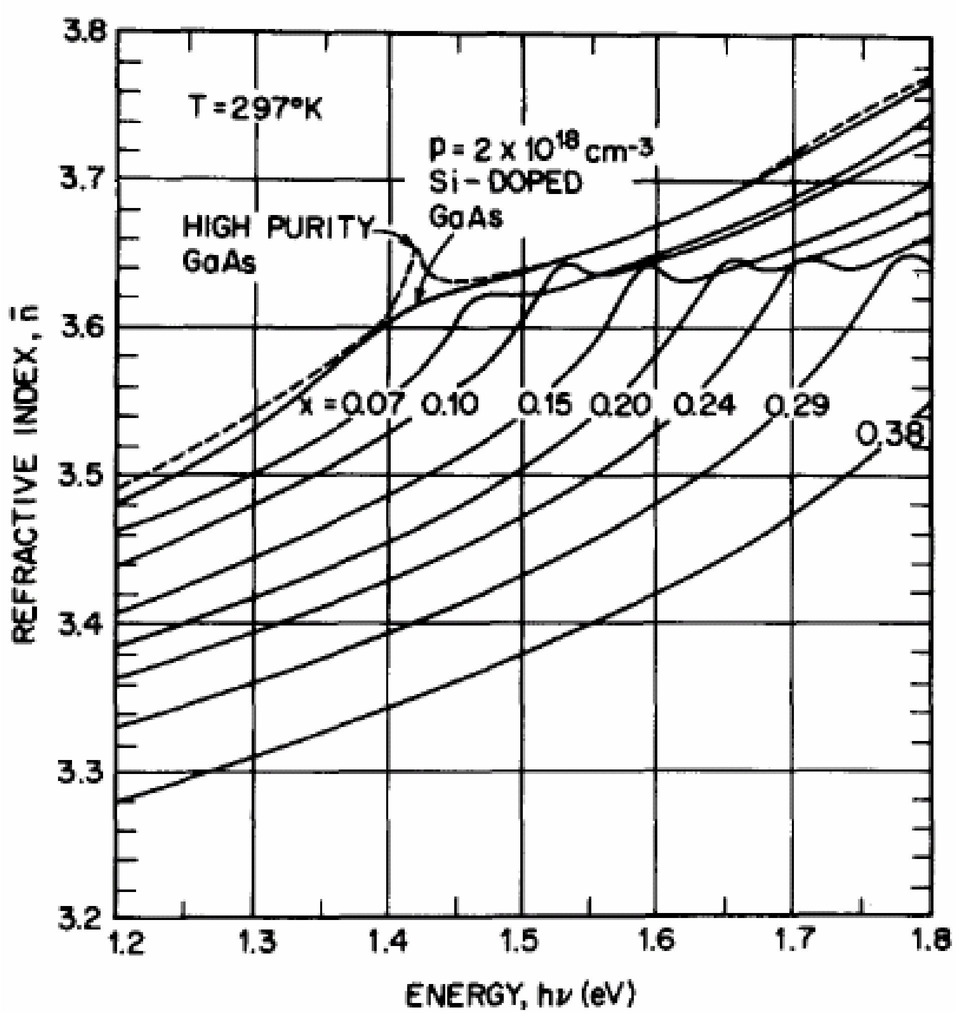
$$n_i = \frac{c}{2\omega}\alpha_p$$

$$\left(\varepsilon_r + i\varepsilon_i\right)/\varepsilon_0 = \left(n_r + in_i\right)^2$$

$$\left(\varepsilon_r/\varepsilon_0\right) = n_r^2 - n_i^2$$

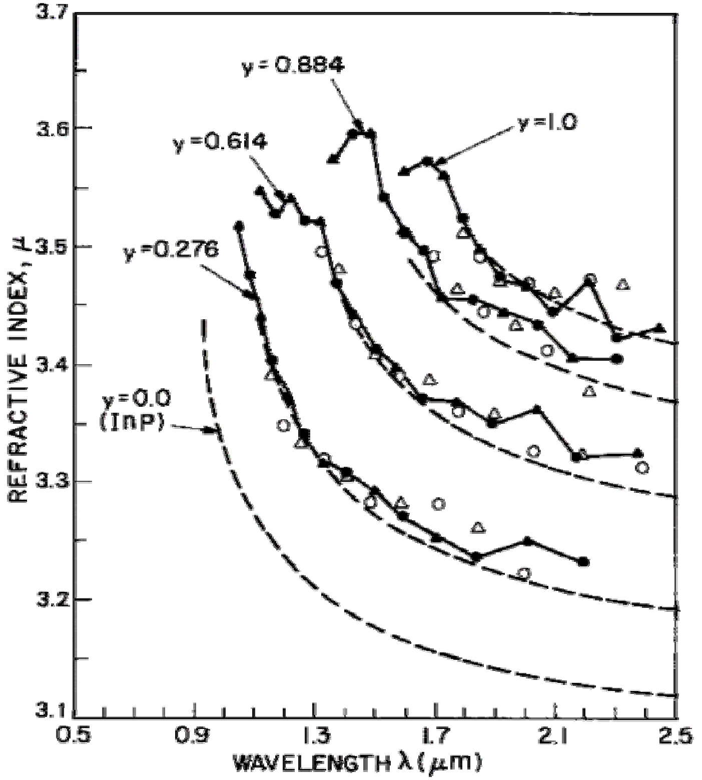
$$\left(\varepsilon_i/\varepsilon_0\right) = 2n_r n_i$$

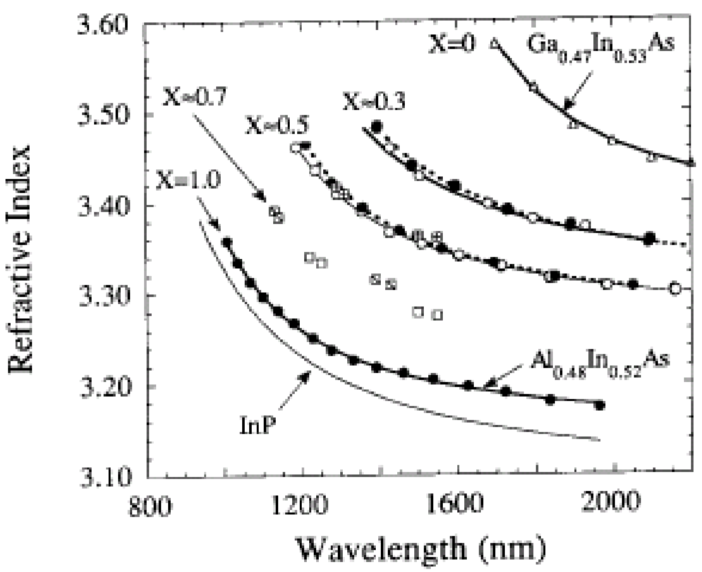
AlGaAs Refractive Indices



Casey, Sell, Panish, Appl. Phys. Lett., 24, 63 (1974)

InGaAlAsP Refractive Indices





$In_{1-x}Ga_xAs_yP_{1-y}$

Chandra, Coldren, Strege, Electron. Lett., 17,6 (1981)

$(Al_{0.48}In_{0.52}As)_x(Ga_{0.47}In_{0.53}As)_{1-x}$

Mondry, Babic, Bowers, Coldren, IEEE Photonics Technol. Lett., 4, 6 (1992)

Below bandgap only !

Goal

- Establish semiempirical close form formulas for n_r of arbitrary, unstrained, ternary and quaternary compounds that is valid from well below band gap to well above band gap
- Extendable to strained and QW layers

Maxwell's Equations

$$\nabla \times \bar{E} = i\omega \bar{B} - \bar{M}$$

$$\nabla \times \bar{H} = -i\omega \bar{D} + \bar{J}$$

$$\nabla \cdot \bar{D} = \rho$$

$$\nabla \cdot \bar{B} = 0$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\bar{B} = \mu \bar{H}$$

$$\varepsilon = \varepsilon_r + i\varepsilon_i = \varepsilon_0 \left(1 + \chi_e\right) = \varepsilon_0 \left(1 + \underline{\chi_r + i\chi_i}\right)$$

$$\chi_i = \frac{\varepsilon_i}{\varepsilon_0} = \underline{\frac{c n_r \alpha_p}{\omega}}$$

Dilute Damped Harmonic Oscillators

$$\chi_r + i\chi_i = \sum_j \left[\frac{e^2 f_j N_j / \varepsilon_0 m_j}{\omega_j^2 - \omega^2 - i2\Gamma_j \omega} \right]$$

$$\chi_r = \sum_j \left[\frac{\left(e^2 f_j N_j / \varepsilon_0 m_j\right)\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + 4\Gamma_j^2 \omega^2} \right] \Leftrightarrow \chi_i = \sum_j \left[\frac{\left(e^2 f_j N_j / \varepsilon_0 m_j\right) 2\Gamma_j \omega}{\left(\omega_j^2 - \omega^2\right)^2 + 4\Gamma_j^2 \omega^2} \right]$$

$$\chi(\omega) = \chi^*(-\omega)$$

As $\omega \Rightarrow \infty$ $\quad \chi_r + i\chi_i \Rightarrow 0$

$\omega \ll \omega_j$

$\Gamma_j \ll \omega_j$

$$\chi_r + i\chi_i \Rightarrow \sum_j \left[\frac{\left(e^2 f_j N_j / \varepsilon_0 m_j\right)}{\left(\omega_j^2 - \omega^2\right)} \right]$$

Multi-oscillator Sellmeier equation

Kramers-Kronig Relations

$$\chi_r(\omega) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\chi_i(\omega')}{\omega' - \omega} d\omega' = \frac{2}{\pi} P\int_{0}^{\infty} \frac{\omega' \chi_i(\omega')}{\omega'^2 - \omega^2} d\omega'$$

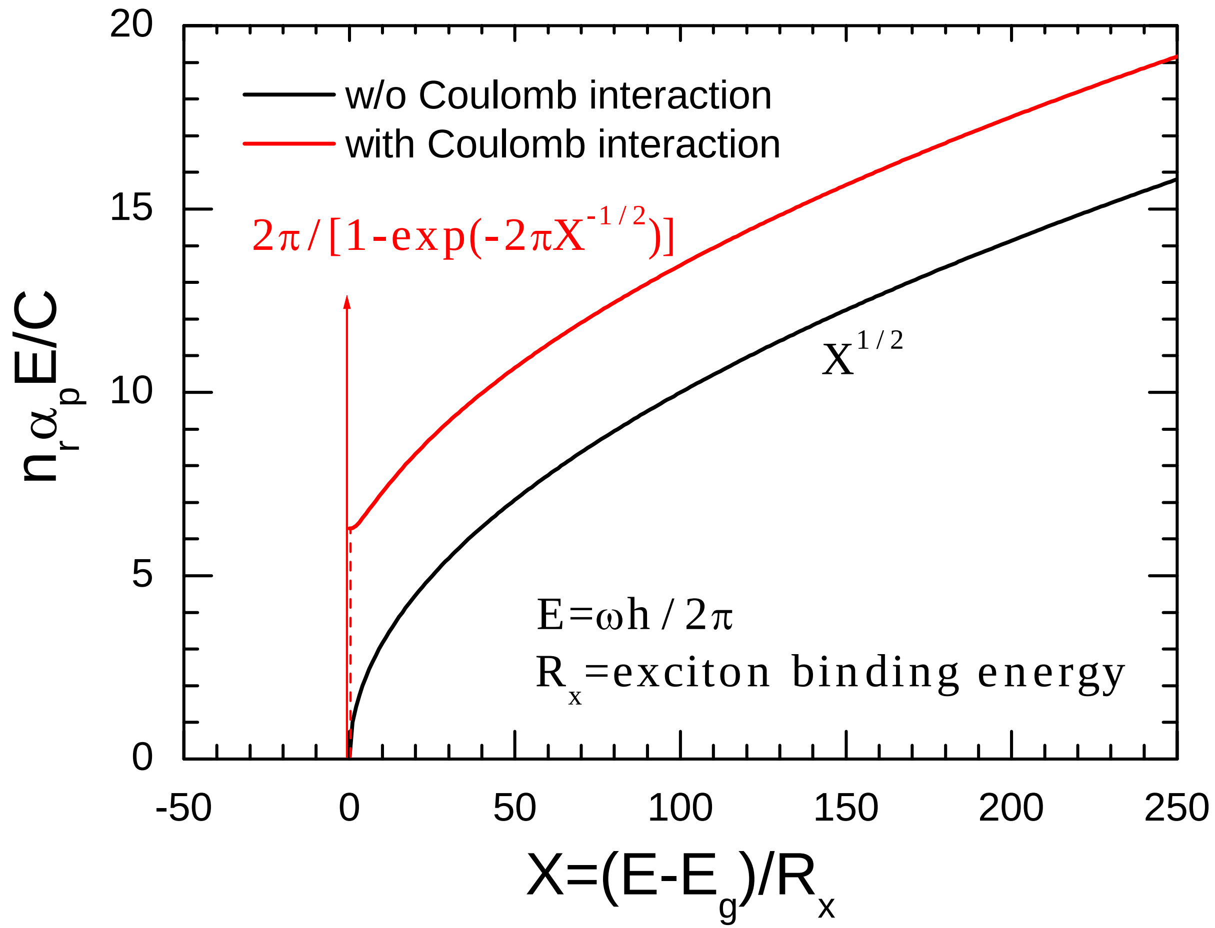
$$\chi_i(\omega) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\chi_r(\omega')}{\omega - \omega'} d\omega' = \frac{2\omega}{\pi} P\int_{0}^{\infty} \frac{\chi_r(\omega')}{\omega^2 - \omega'^2} d\omega'$$

$$\chi_i = \frac{c n_r \alpha_p}{\omega}$$

$$\chi(\omega) = \chi^*(-\omega)$$

Interband Absorption

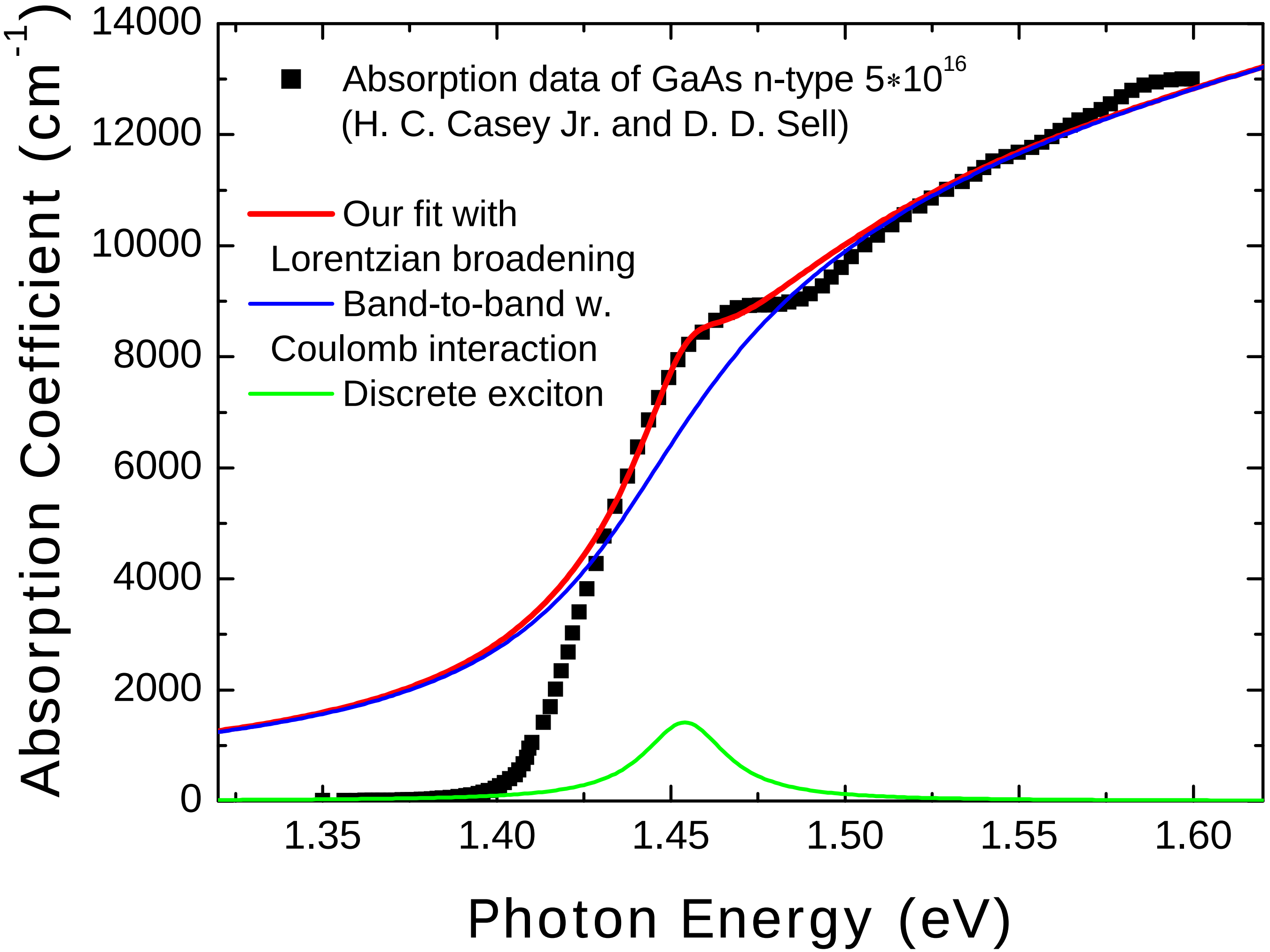
Theoretical models before broadening



Inclusion of Broadening

- The Coulomb enhanced interband absorption must be convoluted with a line-shape function to include broadening
- To obtain close-form results, piecewise linear approximation to the theoretical curve is used

Result for Lorentz broadening



Double-Lorentzian Line Shape

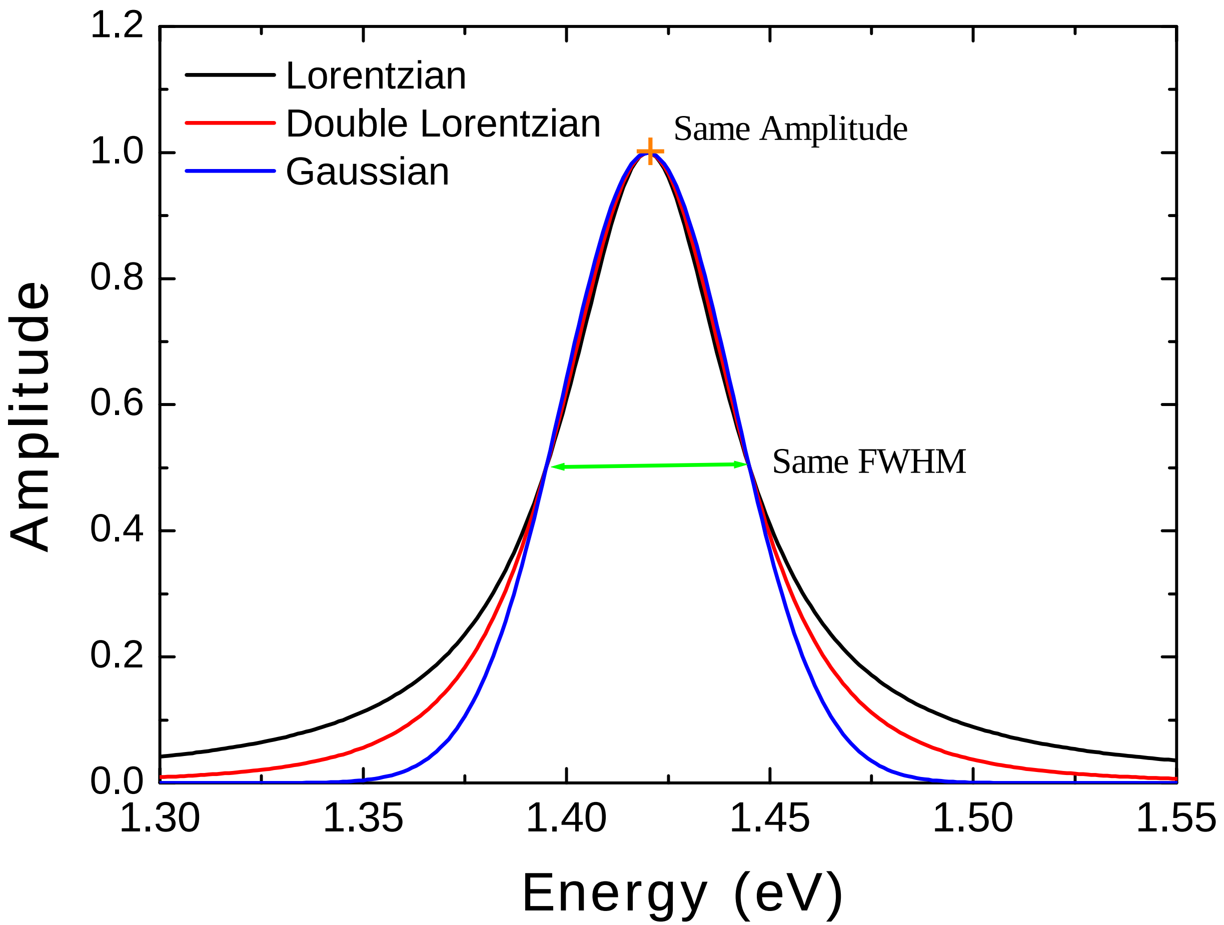
Lorentzian : $L1 = \dfrac{\Gamma_0/(2\pi)}{(\omega_0-\omega)^2+(\Gamma_0/2)^2}$

Double Lorentzian :

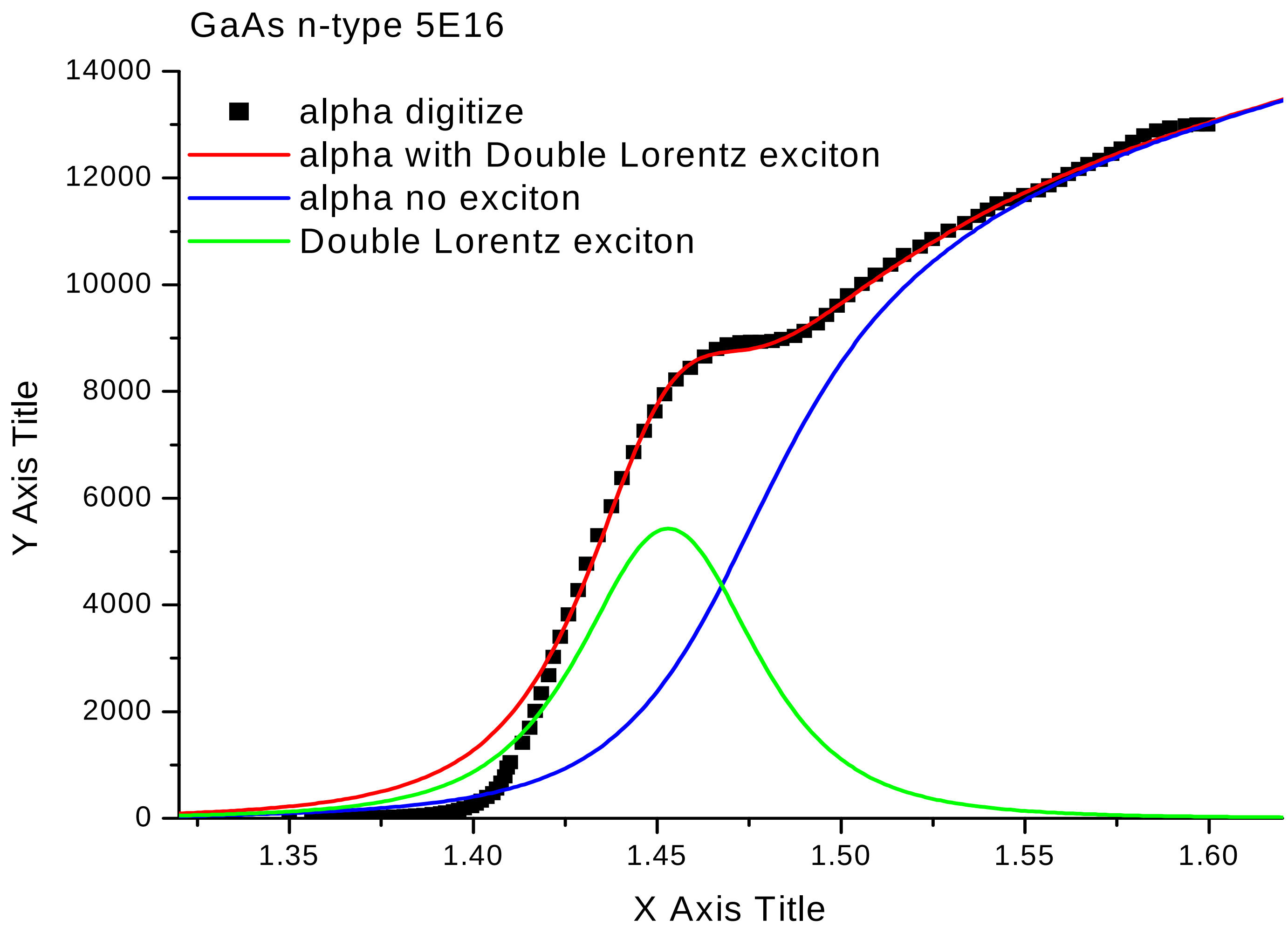
$$L2(\omega) = \frac{\sqrt{6}(\Gamma_0/2)^3/\pi}{\sqrt{3}-\sqrt{2}}\left\{\frac{1}{\left[(\omega_0-\omega)^2+2(\Gamma_0/2)^2\right]\left[(\omega_0-\omega)^2+3(\Gamma_0/2)^2\right]}\right\}$$

$$=\left[\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}\right]\frac{\sqrt{2}(\Gamma_0/2\pi)}{\left[(\omega_0-\omega)^2+2(\Gamma_0/2)^2\right]}-\left[\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]\frac{\sqrt{3}(\Gamma_0/2\pi)}{\left[(\omega_0-\omega)^2+3(\Gamma_0/2)^2\right]}$$

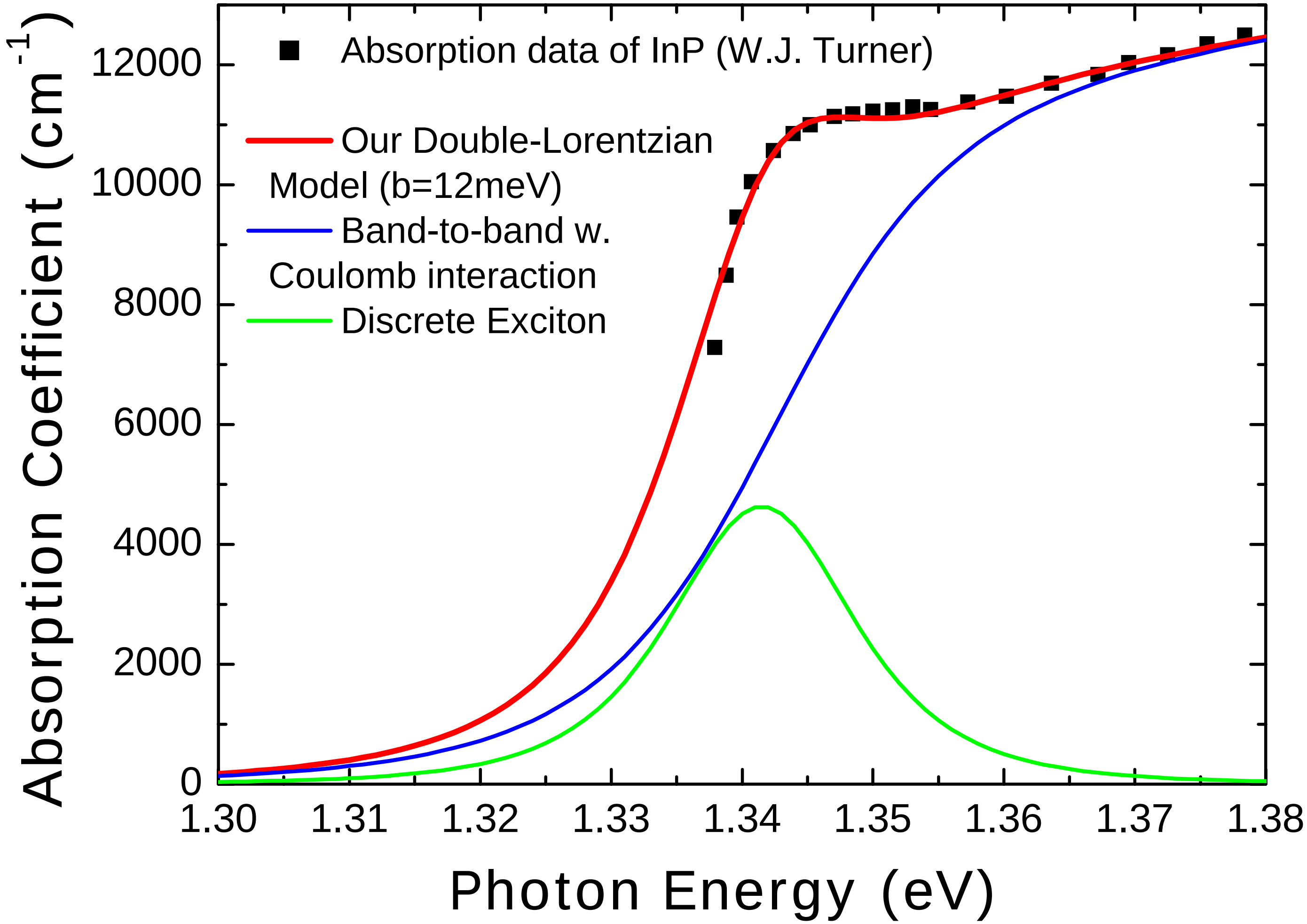
Line Shape Comparison



GaAs Double-Lorentz Fitting



InP Double-Lorentz Fitting



InP High-Energy Spectrum

Further simplification of Sellmeier's equation

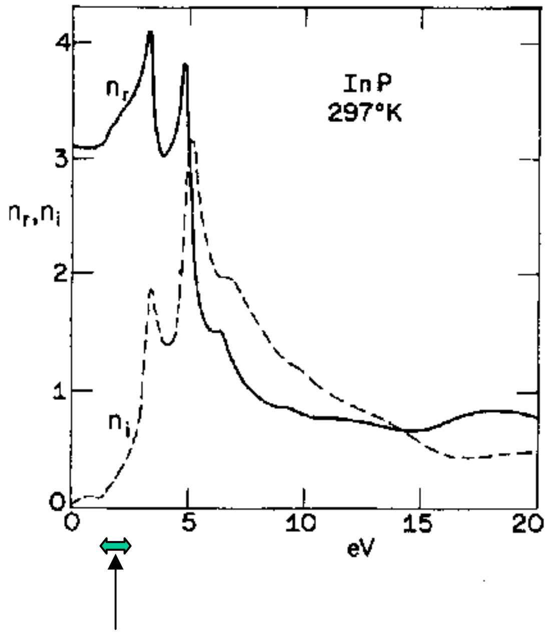
Since $\omega << \omega_j$

$$\chi_{HE}(\omega) \Rightarrow \sum_j \left[\frac{\left(e^2 f_j N_j / \varepsilon_0 m_j\right)}{\left(\omega_j^2 - \omega^2\right)} \right]$$

$$= \sum_j \left[\frac{\left(e^2 f_j N_j / \varepsilon_0 m_j\right)}{\omega_j^2} \right] \left\{ 1 + \frac{\omega^2}{\omega_j^2} + \frac{\omega^4}{\omega_j^4} + \cdots \right\}$$

$$= \boxed{A + \frac{B}{1 - C\omega^2}} = A + B\left[1 + C\omega^2 + C^2\omega^4 + \cdots\right]$$

Single oscillator model

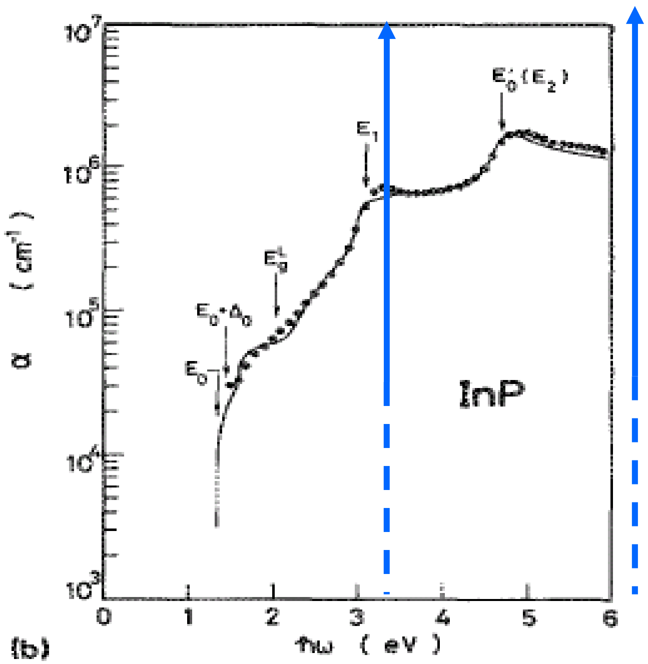


band-to-band

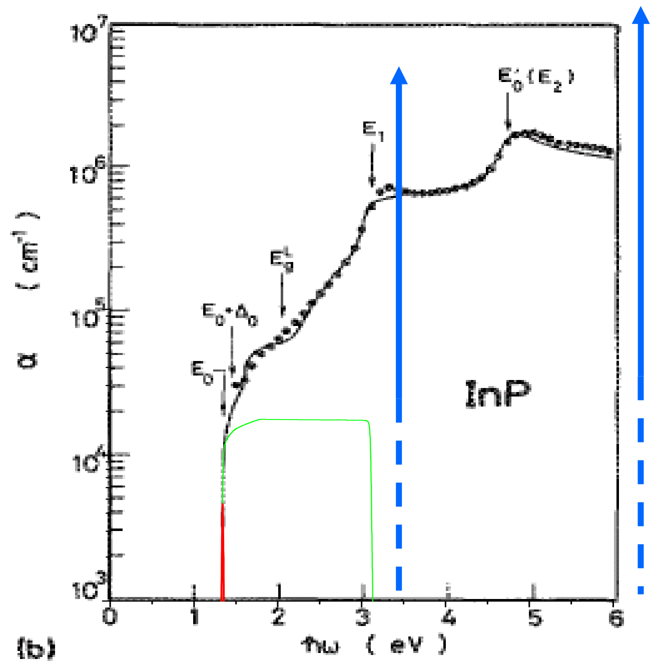
M. Cardona, in "Optical Properties of Solids", Nudelman and Mitra, ed., Plenum (1969)

Composite Model for α

Single oscillator model

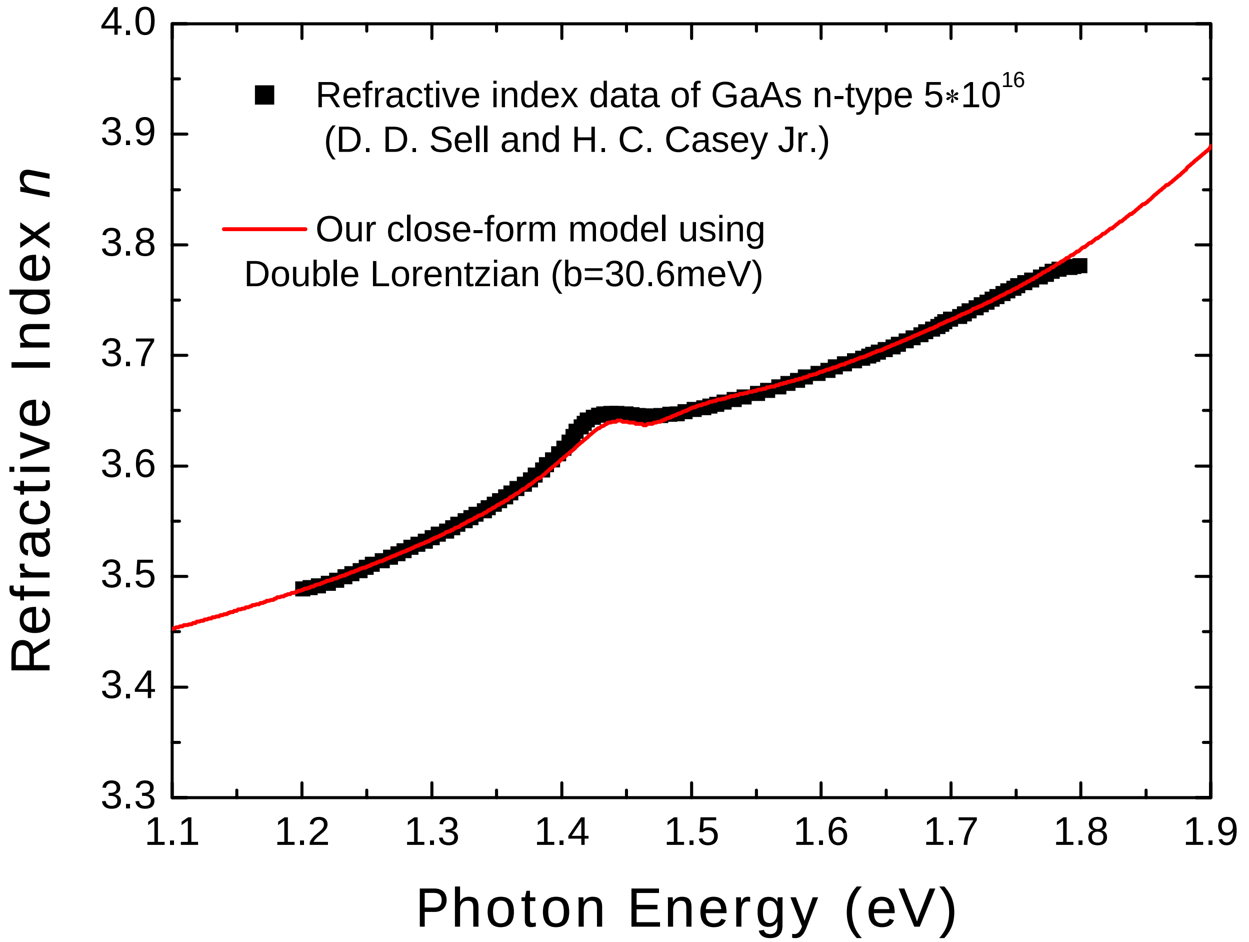


Our model

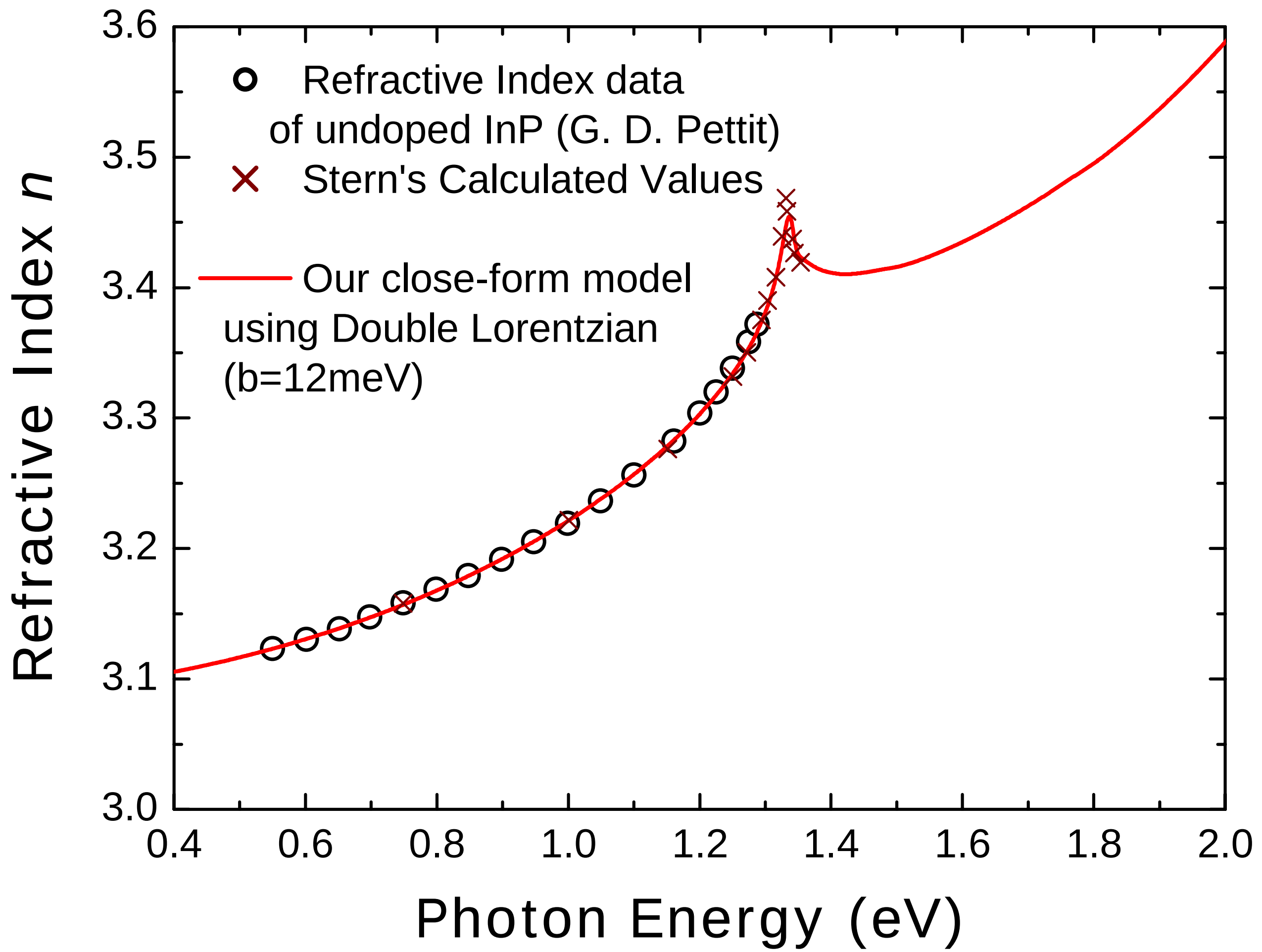


Discrete exciton
Band-to-band
High energy

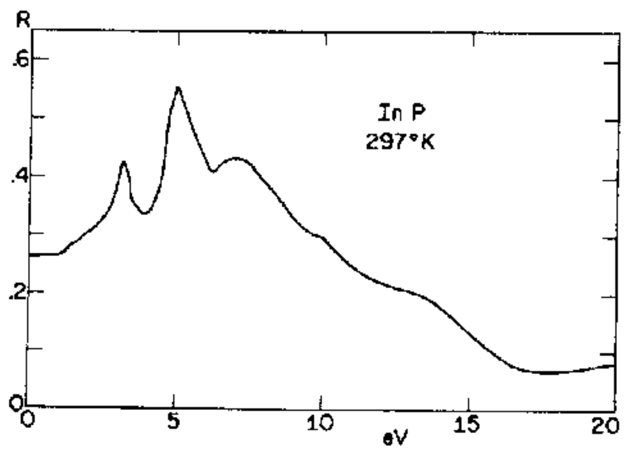
GaAs K-K Transform Result



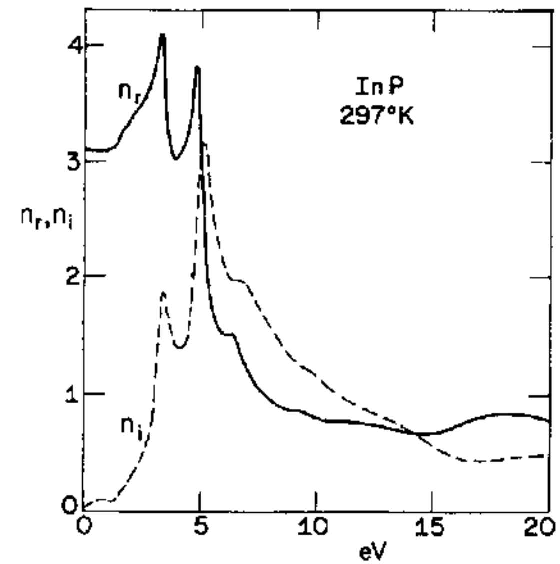
InP K-K Transform Result



InP High-Energy Spectrum



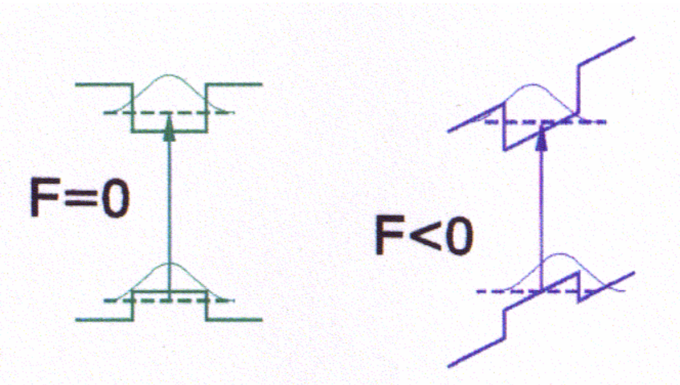
Reflectance ρ

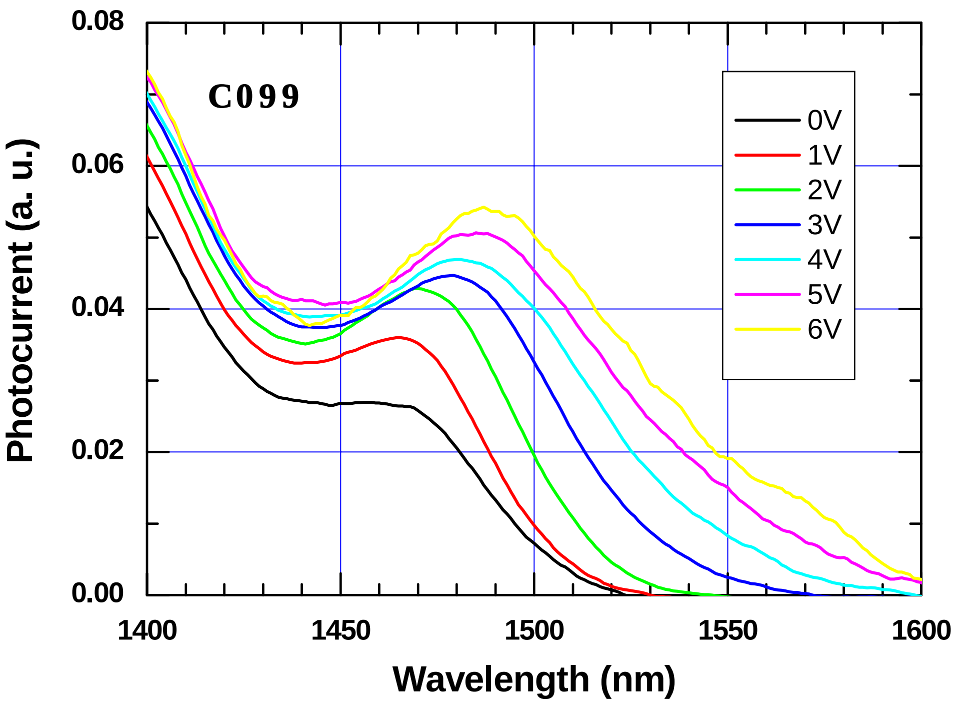


$r = \rho \cdot \exp(i\theta) = (n-n_0)/(n+n_0)$, Equivalent K-K relations for ρ and θ.

M. Cardona, in “Optical Properties of Solids”,
Nudelman and Mitra, ed., Plenum (1969)

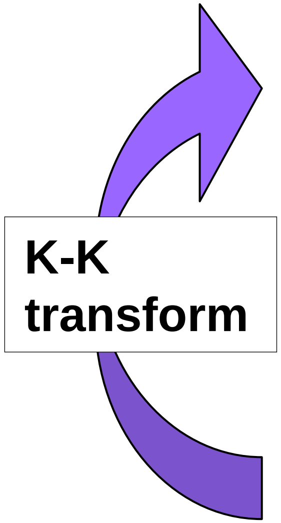
Quantum Confined Stark Effect



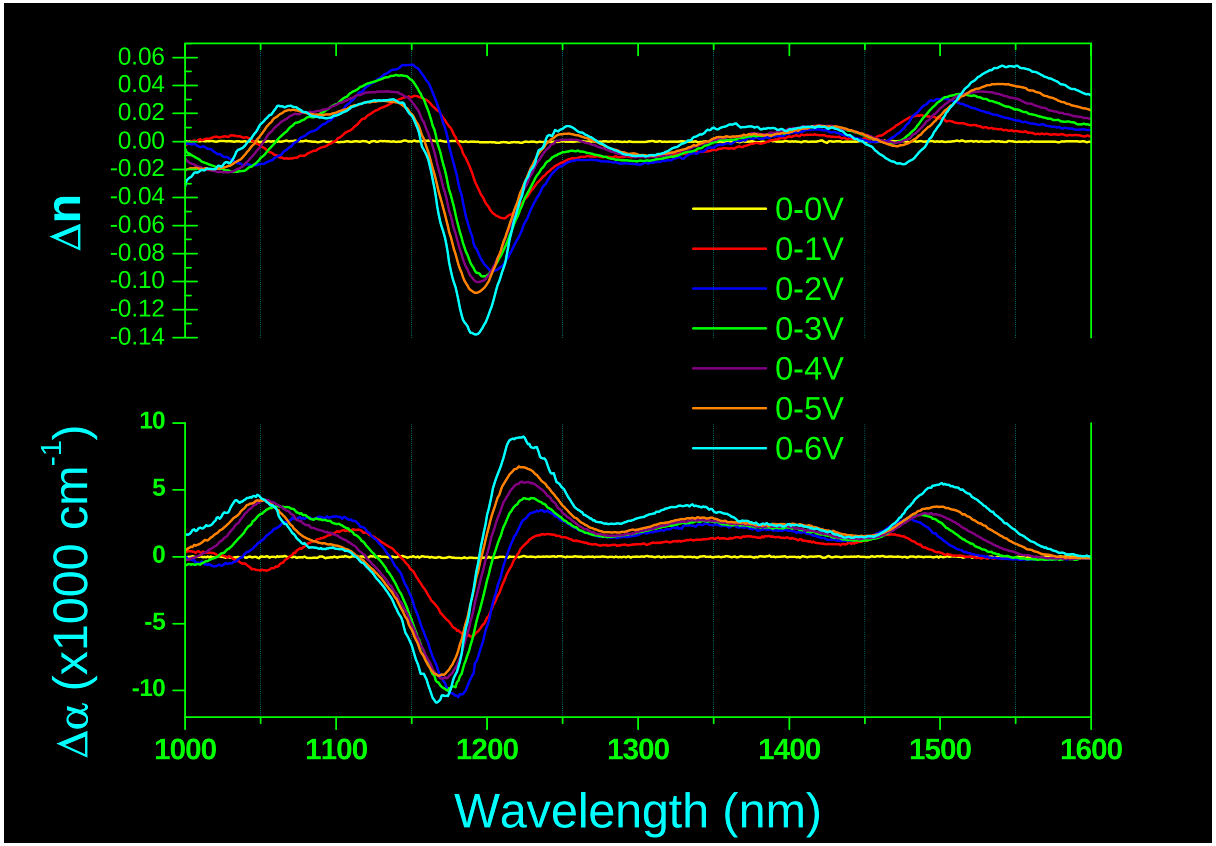


3 Quantum Wells, modulation doped

Electroabsorption (Δα)
Electrorefraction (Δn)



Δα from ΔT/T



K-K Transform: Δα ⇒ Δn

$$\Delta\varepsilon_r + i\Delta\varepsilon_i = 2n_0\Delta n$$

$$= 2n_0\left(\Delta n_r + i\frac{c\Delta\alpha_p}{2\omega}\right) = 2n_0\left(\Delta n_r + i\frac{\lambda\Delta\alpha_p}{4\pi}\right)$$

$$\Delta n_r(\omega) = \frac{c}{\pi}P\int_0^\infty \frac{\Delta\alpha(\omega')}{\omega'^2-\omega^2}\,d\omega'$$

$$\Delta\alpha(\omega) = -\frac{4\omega^2}{\pi c}P\int_0^\infty \frac{\Delta n_r(\omega')}{\omega'^2-\omega^2}\,d\omega'$$

Chirp

Linewidth broadening parameter in lasers:

$$\alpha_H = \frac{dn_r}{dn_i} = \frac{4\pi}{\lambda}\left(\frac{dn_r}{d\alpha}\right) = -\frac{4\pi}{\lambda}\left(\frac{dn_r}{dg}\right)$$

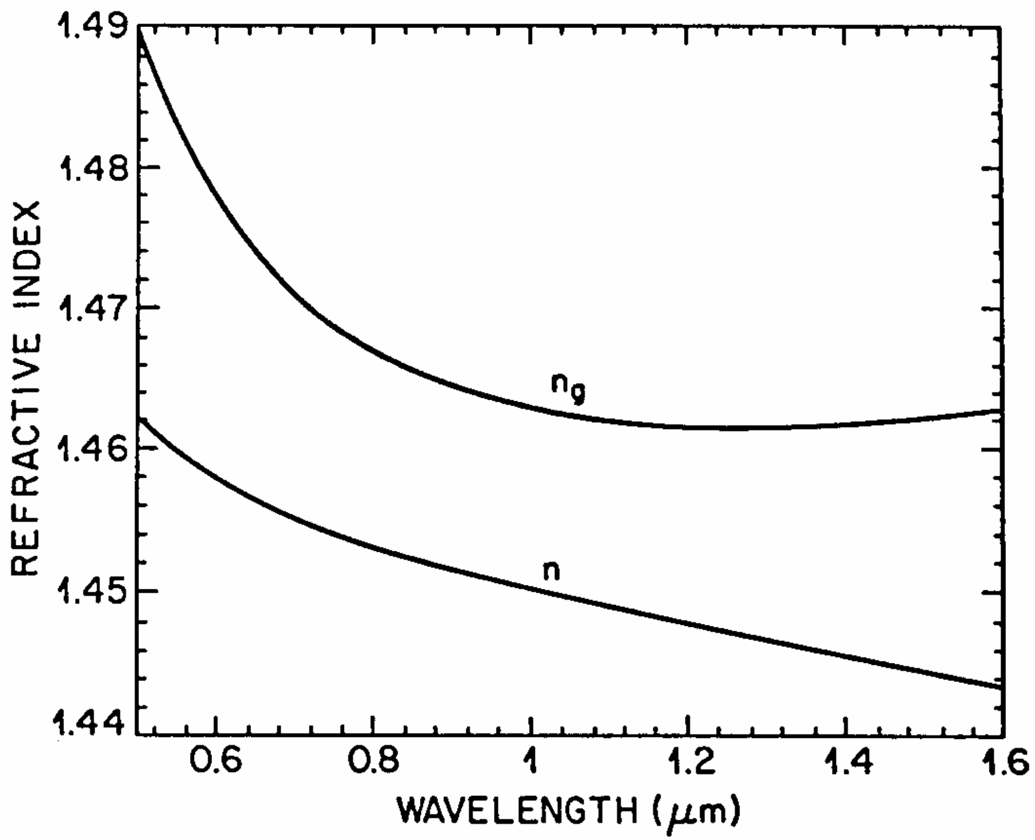
Chirp parameter in lasers and modulators:

$$C \equiv -\frac{2(\delta\omega_o)_{\max}}{\left[\frac{1}{P}\frac{\partial P}{\partial t}\right]} = -\alpha_H$$

Optical Fiber Group Index

$$n_g = \frac{c}{v_g}$$

$$= n - \lambda \frac{dn}{d\lambda}$$

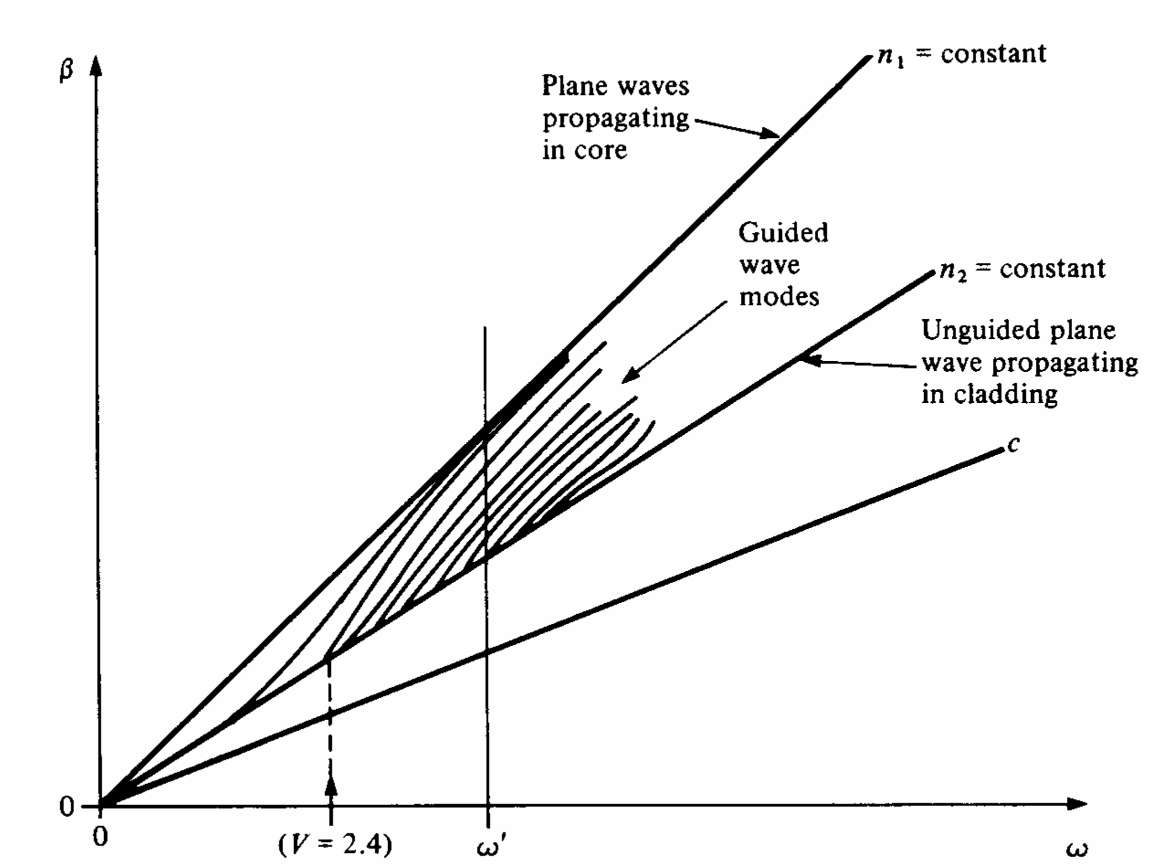


From *Fiber-Optic Communication Systems*, G. P. Agrawal (1997)

Waveguide Dispersion

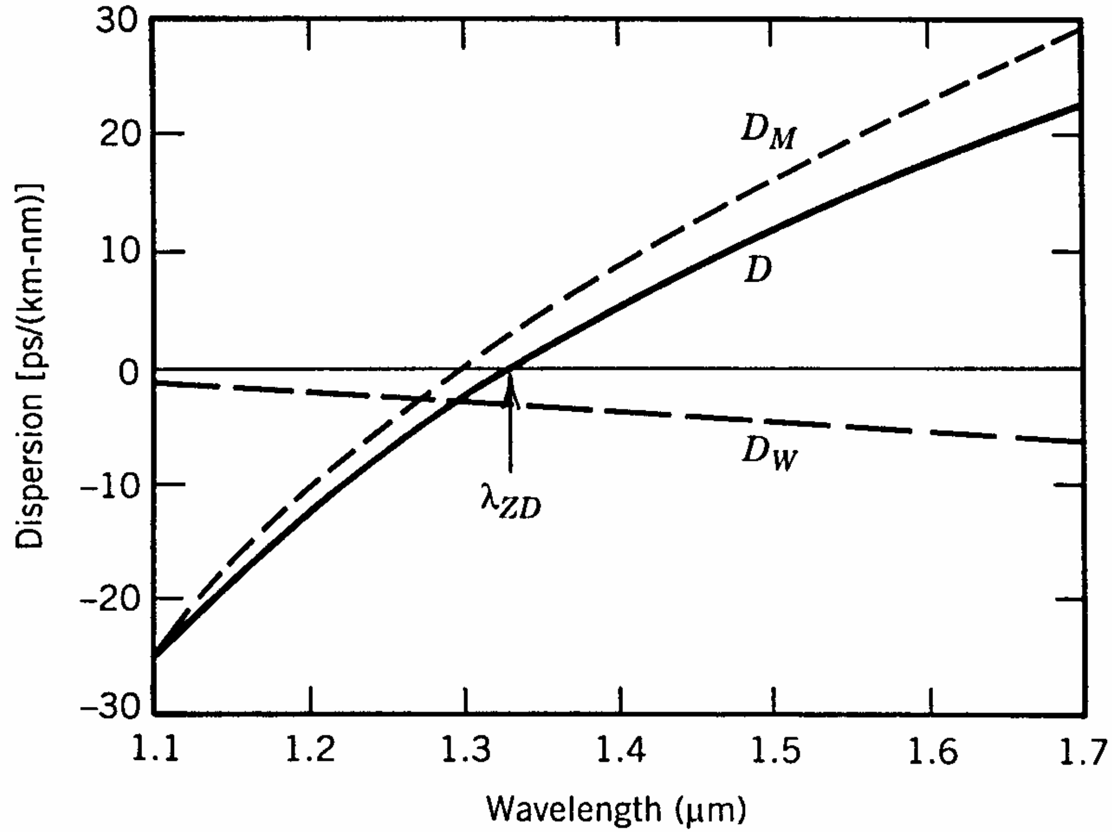
$$D = \frac{1}{c}\left(\frac{dn_g}{d\lambda}\right)$$

$$= -\frac{\omega^2}{2\pi c}\left(\frac{d^2\beta}{d\omega^2}\right)$$



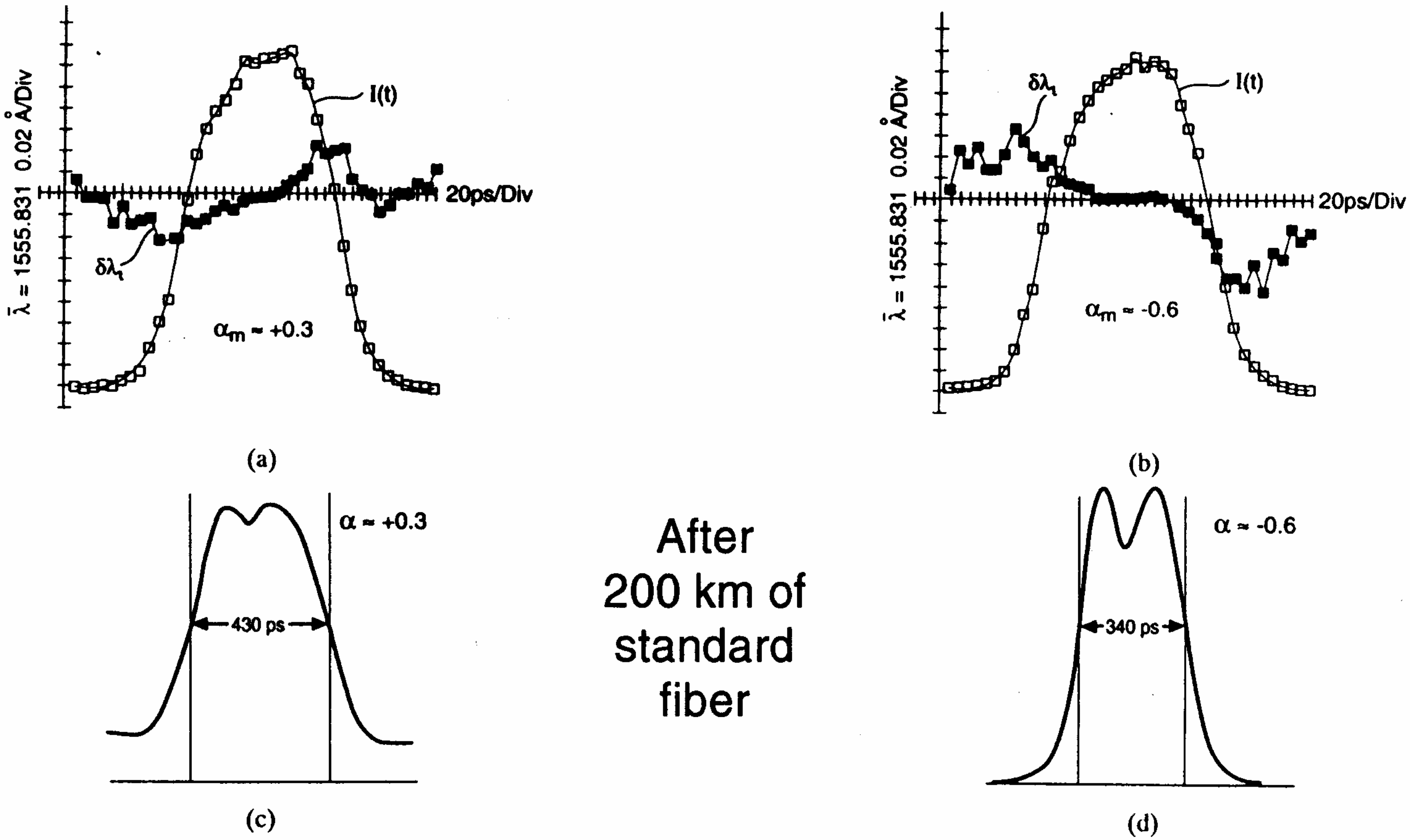
Combined Group Index Dispersion

$$D = \frac{1}{c}\left(\frac{dn_g}{d\lambda}\right)$$



From *Fiber-Optic Communication Systems*, G. P. Agrawal (1997)

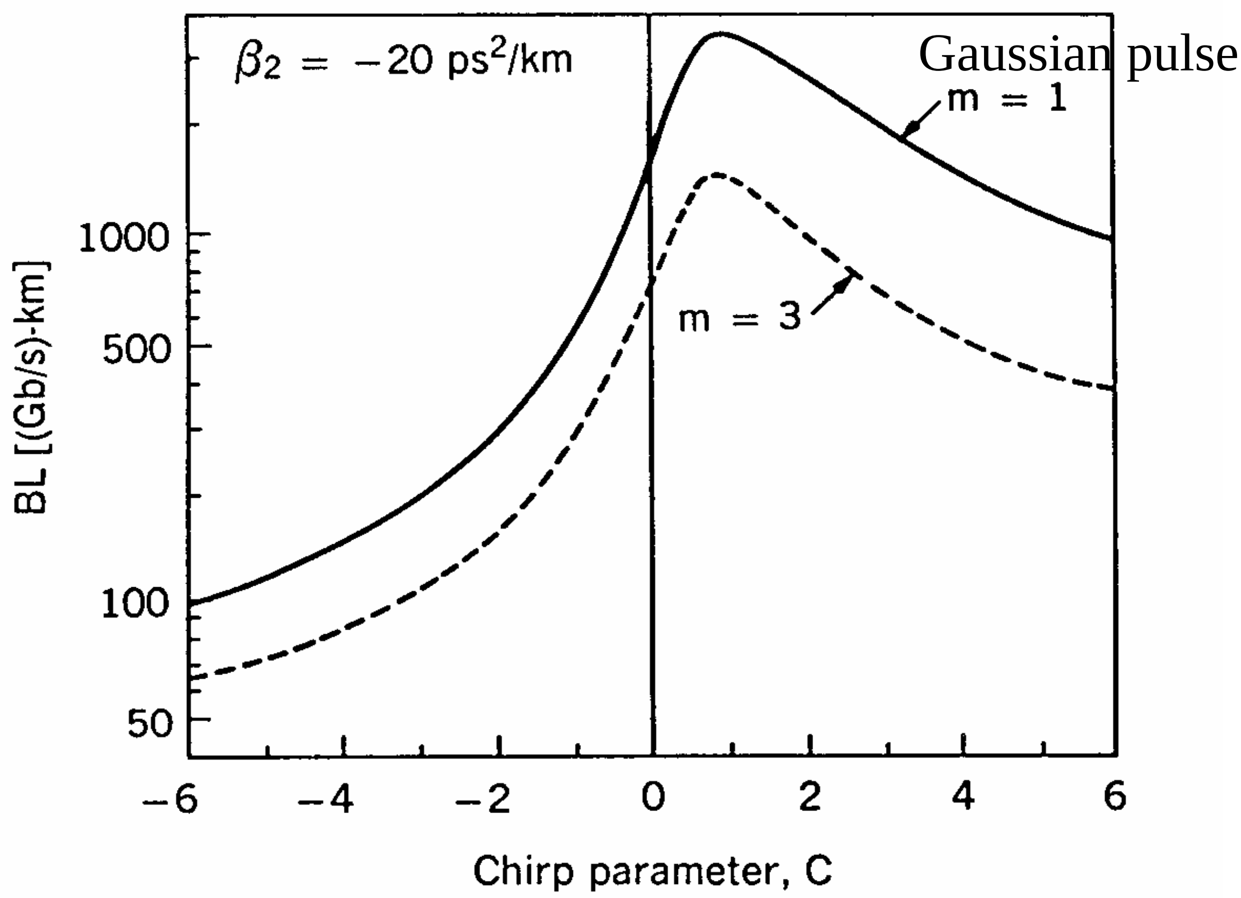
Consequence of Chirp



After 200 km of standard fiber

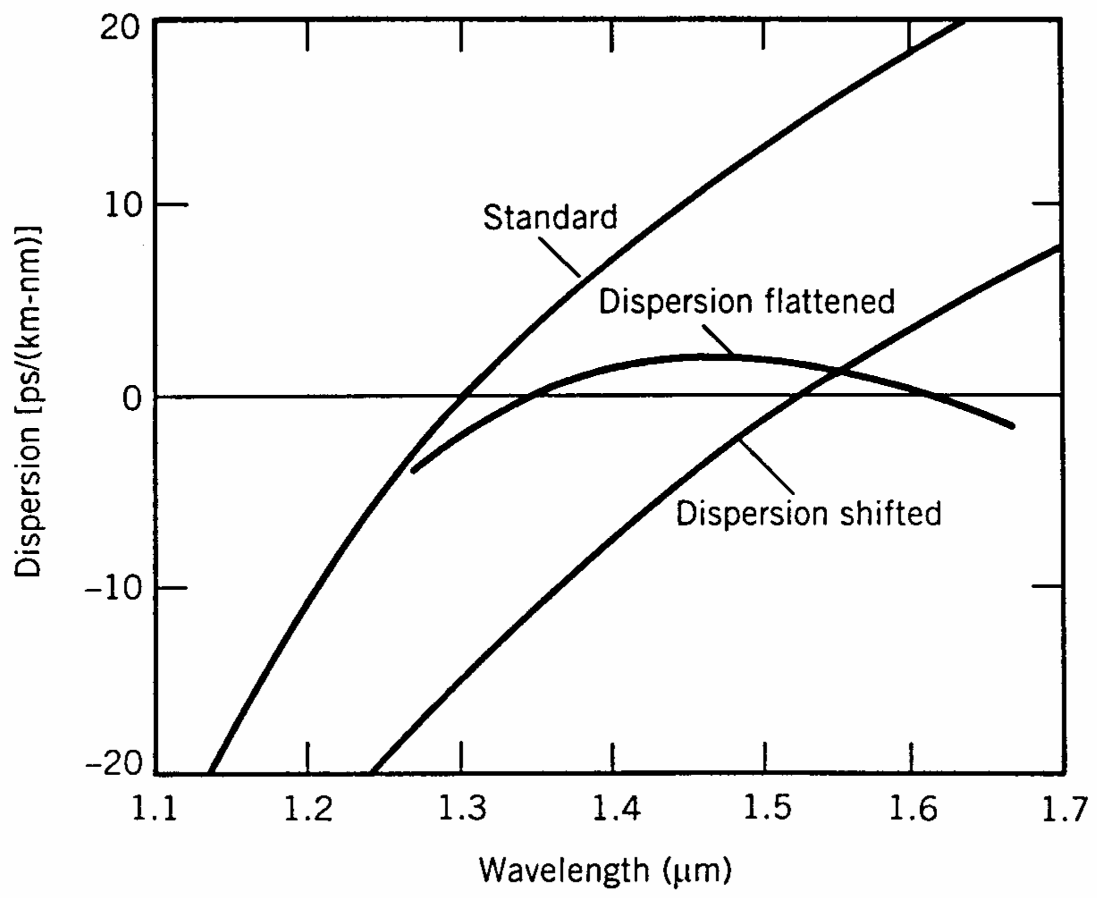
D. A. Fishman, J. Lightwave Technol., **11**, 624 (1993)

Bit-rate-distance Product



From *Fiber-Optic Communication Systems*, G. P. Agrawal (1997)

Dispersion shifting



From *Fiber-Optic Communication Systems*, G. P. Agrawal (1997)

4-wave Mixing Signal

For $\omega_{ijk} = \omega_i + \omega_j - \omega_k$, $\Delta\beta = \beta_i + \beta_j - \beta_k - \beta_{ijk}$

For $L << \pi/\Delta\beta$, $$P_{ijk} \approx \left(\frac{D_{ijk}}{3} \gamma \right)^2 P_i P_j P_k \frac{e^{-\alpha L} \left(1 - e^{-\alpha L} \right)^2}{\alpha^2 + (\Delta\beta)^2}$$

D_{ijk} = 3 for two tone mixing, = 6 for three tone mixing

γ : nonlinear coefficient ($\propto \chi^{(3)}$)

4-Wave Mixing & Dispersion

$$D = \frac{1}{c}\left(\frac{dn_g}{d\lambda}\right)$$

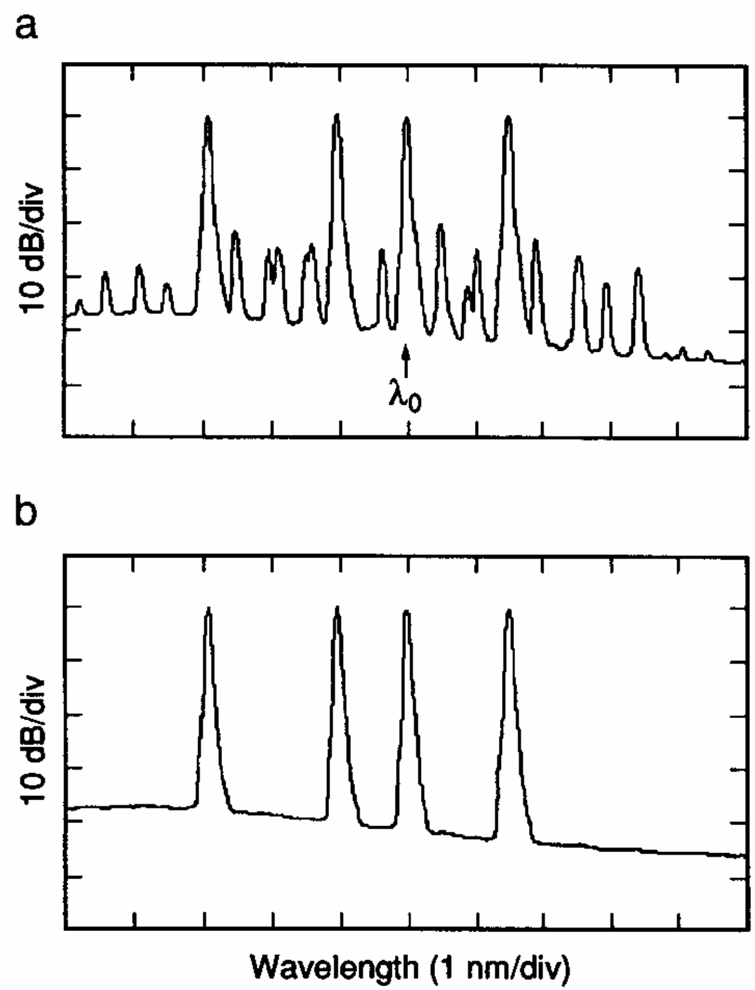
$D \approx 0$

L = 25 km

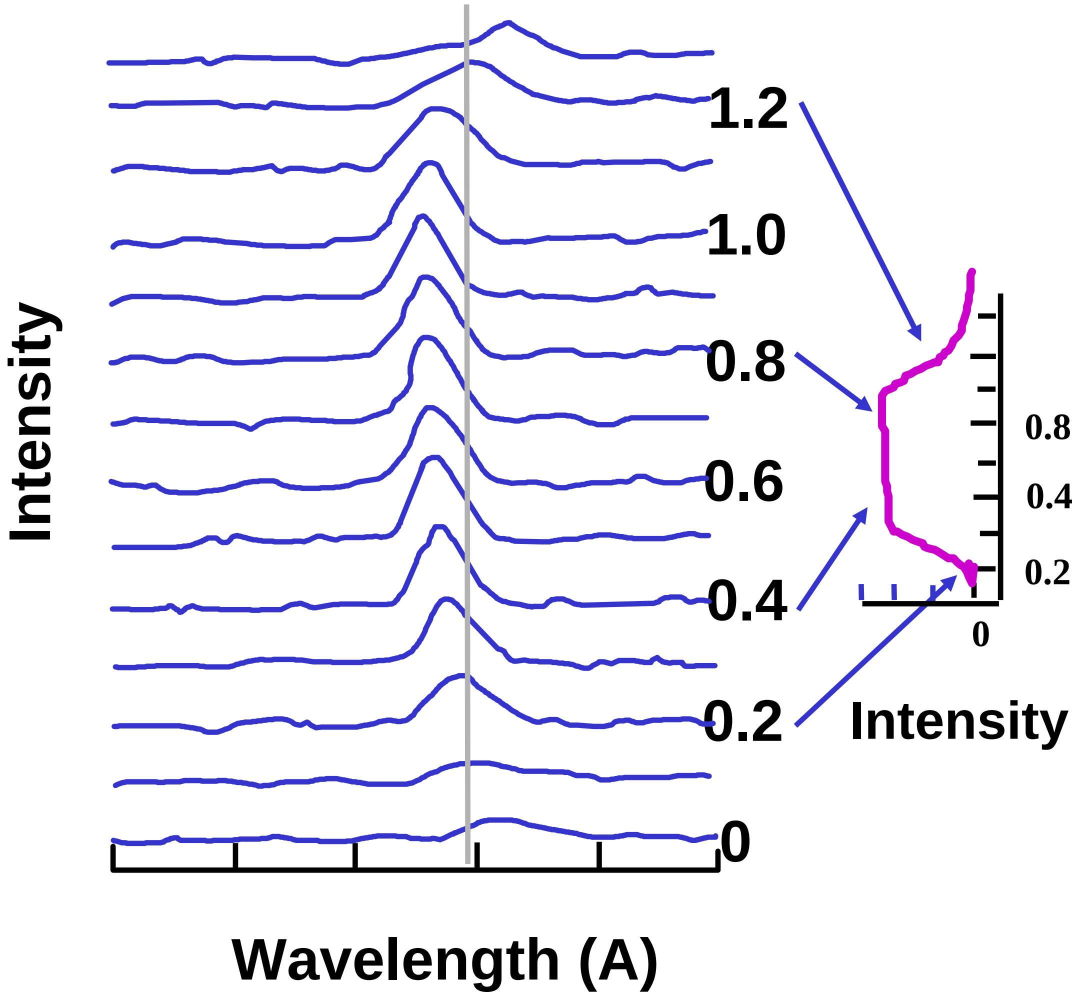
D = 2.5 ps/nm-km

L = 50 km

P_{in} = 3 mW



Chirp in DFB Lasers



Chirp Performance

- For laser diodes
  - $C \approx -6 \rightarrow -3$
- For electroabsorption
  - $C \approx -2 \rightarrow -0.6$
- For Mach-Zehnder's driven in push-pull mode :
  - $C \approx -2 \rightarrow 0 \rightarrow 2$, controllable

Theoretical Basis of K-K Relations-I

- The polarization generated in an optical medium that is made up of equivalent damped harmonic oscillators in response to an electric field is given by

$$\frac{\overline{P}_e}{\varepsilon_0 \overline{E}} = \chi = \sum_j \left[\frac{e^2 f_j N_j / \varepsilon_0 m_j}{\omega_j^2 - \omega^2 - i2\Gamma_j \omega} \right]$$

- Here $\Gamma_j > 0$, and $\chi(\omega)$, analytically continued to the entire complex ω plane, has no pole in the upper half plane. The impulse response of the system is given by the inverse Fourier transform of χ,

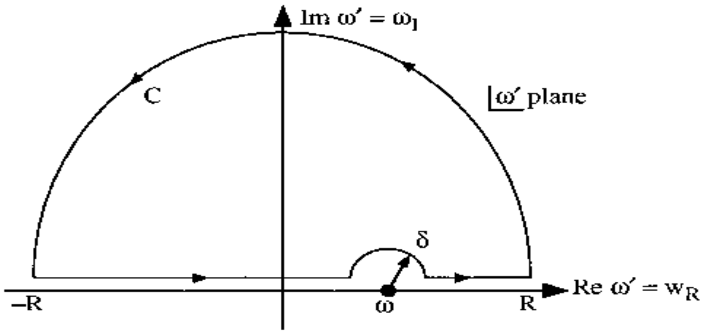
$$T(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \chi(\omega) \exp(-i\omega t)\, d\omega$$

- A contour integral over the upper half ω plane yields *$T(t<0)=0$,* *i.e.* the induce polarization does not exist before the appearance of the electrical impulse.$\Rightarrow$ **Causality** 因果不倒置，時間不倒流

Theoretical Basis of K-K Relations-II

- Analyticity of $\chi(\omega)$ in the upper half plane means

$$\frac{1}{2\pi i}P\int_{-\infty}^{+\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega'-\frac{\chi(\omega)}{2}$$
$$=-\frac{1}{2\pi i}\int_{R}\frac{\chi(\omega')}{\omega'-\omega}d\omega'=0\left(R\to\infty\right)$$



- Equating the real and imaginary parts results in K-K relations.

Passive medium

$\Rightarrow$ $\chi(\omega)$ analytic in the upper half plane

$\Rightarrow$ Causality $\Rightarrow$ K-K relations

Conclusions

- α, n, dispersion, and chirp are key parameters in opto-electronic technologies
- Experimental spectrum of α (or $\Delta\alpha$) allows one to obtain n (or Δn) through K-K transform
- Experimental verification of K-K relations between n and α (in frequency domain) amounts to a confirmation of the concept of causality (in time domain)

哲理的數學模擬

- 陰在內，陽之守也，陽在外，陰之使也。
- 吸收在內，折射之守也，折射在外，吸收之使也。
- 先有前因,再有後果
- 時間不倒流
- 絕對的還是統計的法則?
- 修行,還是遊戲?