Quantum Computation for Pedestrians

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"Information is Physical"

There are no unavoidable energy consumption requirements per step in a computer. Related analysis has provided insight into the measurement process and the communications channel, and has prompted speculations about the nature of physical laws.

Rolf Landauer, Physics Today, May 1991, Page 23

Can We Process the Information Quantum Mechanically?

Outline

- Basic Ideas of Quantum Computation
    - What is Quantum bit (Qubit)
    - Physical realization and Bloch sphere representation
    - Multiple qubits and entanglement
    - Quantum gate or quantum operation
    - Quantum circuit diagram and Quantum algorithm
    - Universal quantum computation
    - De-coherence

- Quantum Dot Based Quantum Computation Scheme
    - Semiconductor Quantum Dots
    - Quantum Optical Control of Excitons in QDs
    - Quantum Optical Control of Spins in QDs

What is a Quantum Bit, or Qubit?

- Classical Bit: $\psi=0$ or $\psi=1$.

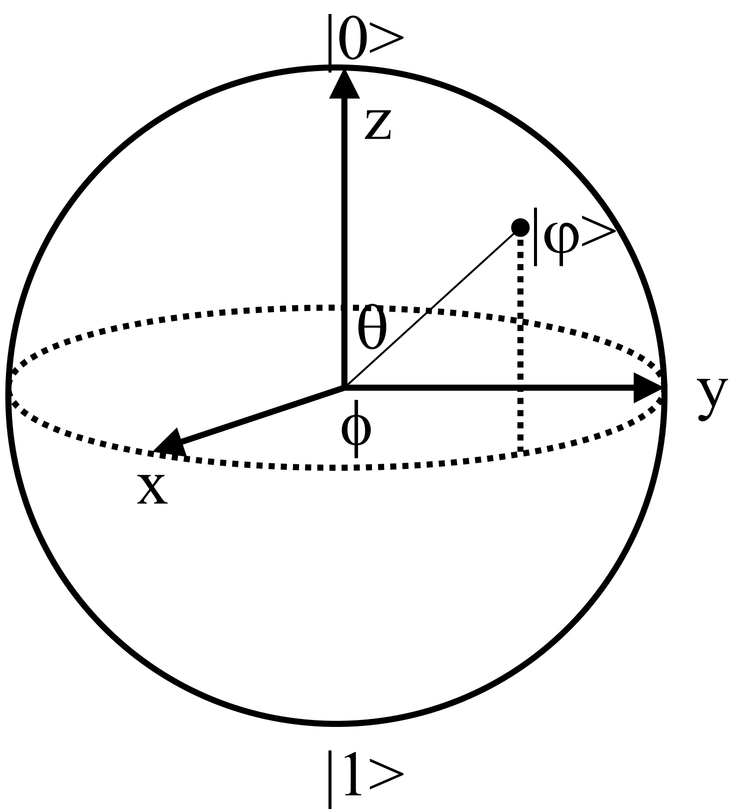
- Quantum Bit: $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$, $\alpha^2+\beta^2=1$
  - Abstract *mathematical objects*.
    - But isn't information *physical*?
    - Yes, but abstraction allows us to construct *general* theories.

  - $|0\rangle$ and $|1\rangle$ are known as *computational basis states*.

- Measurements? Measurements !
  - Classical bit => easy to determine if $\psi=0$ or $\psi=1$.
  - Quantum bit => Probabilistic measurement results.
    => Can be measured in a different basis.

Physical Realizations

- $|0\rangle$=ground state, $|1\rangle$=first excited state
- $|0\rangle$=spin up, $|1\rangle$=spin down
- Many other possibilities



Bloch sphere representation

$$|\varphi\rangle = e^{i\gamma}\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\}$$

Multiple qubits and Entanglement

- Single qubit : $|\psi\rangle_1 = \alpha_0|0\rangle + \alpha_1|1\rangle \quad \in H_2$
- Two qubits : $|\psi\rangle_2 = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad \in (H_2)^2$
- Multiple qubits: $|\psi\rangle_n = \Sigma\alpha_{i1i2\ldots in}|i_1 i_2 \ldots i_n\rangle \quad \in (H_2)^n$

- Hilbert space is a HUGE space !
  - For n-qubits, need to specify 2^n amplitudes.
  - For n=500, 2^{500}> estimated number of atoms in the universe!

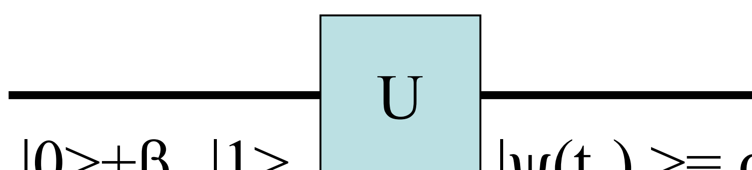
- Entanglement: Trivial multiple qubits v.s. Non-trivial multiple qubits
  - Trivial: $|\psi\rangle_2 = |\psi_a\rangle_1 |\psi_b\rangle_1$
  - Non-Trivial: $|\psi\rangle_2 \neq |\psi_a\rangle_1 |\psi_b\rangle_1$ ==> Entangled

Quantum gate or Quantum operation I

- QM : $|\psi(t_2)\rangle = U(t_2,t_1)\,|\psi(t_1)\rangle$ $\qquad U = e^{-\frac{i}{\hbar}\int H(t)dt}$

- The unitary operator U can be expressed as a matrix in computational basis

- Single qubit gate $\qquad |\psi(t_2)\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = U\,|\psi(t_1)\rangle$

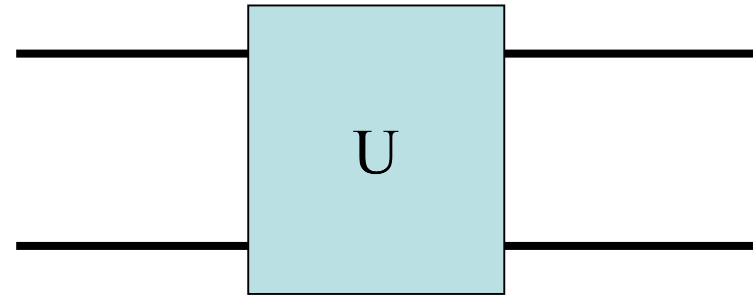


$|\psi(t_2)\rangle = \alpha_2|0\rangle + \beta_2\,|1\rangle \qquad |\psi(t_1)\rangle = \alpha_1|0\rangle + \beta_1\,|1\rangle$

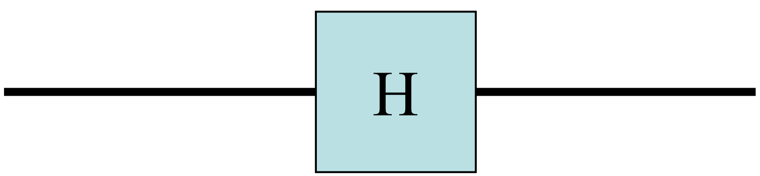
Time

Quantum gate or Quantum operation II

- Two qubits gate

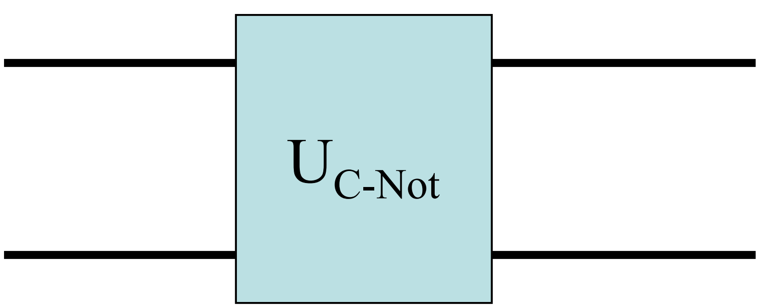


- Examples:
- Hadamard Gate



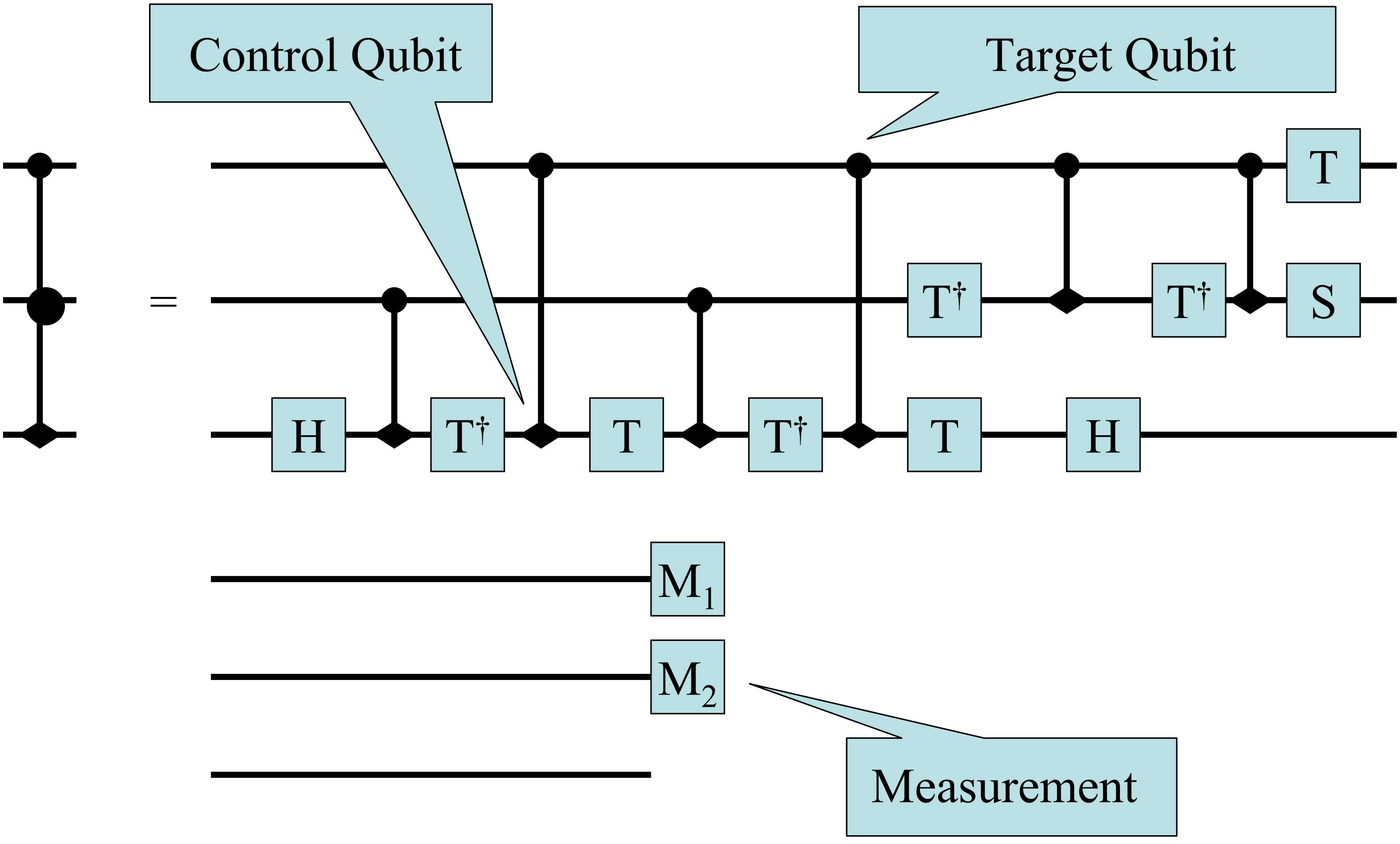
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Controlled-Not Gate (Non-trivial)

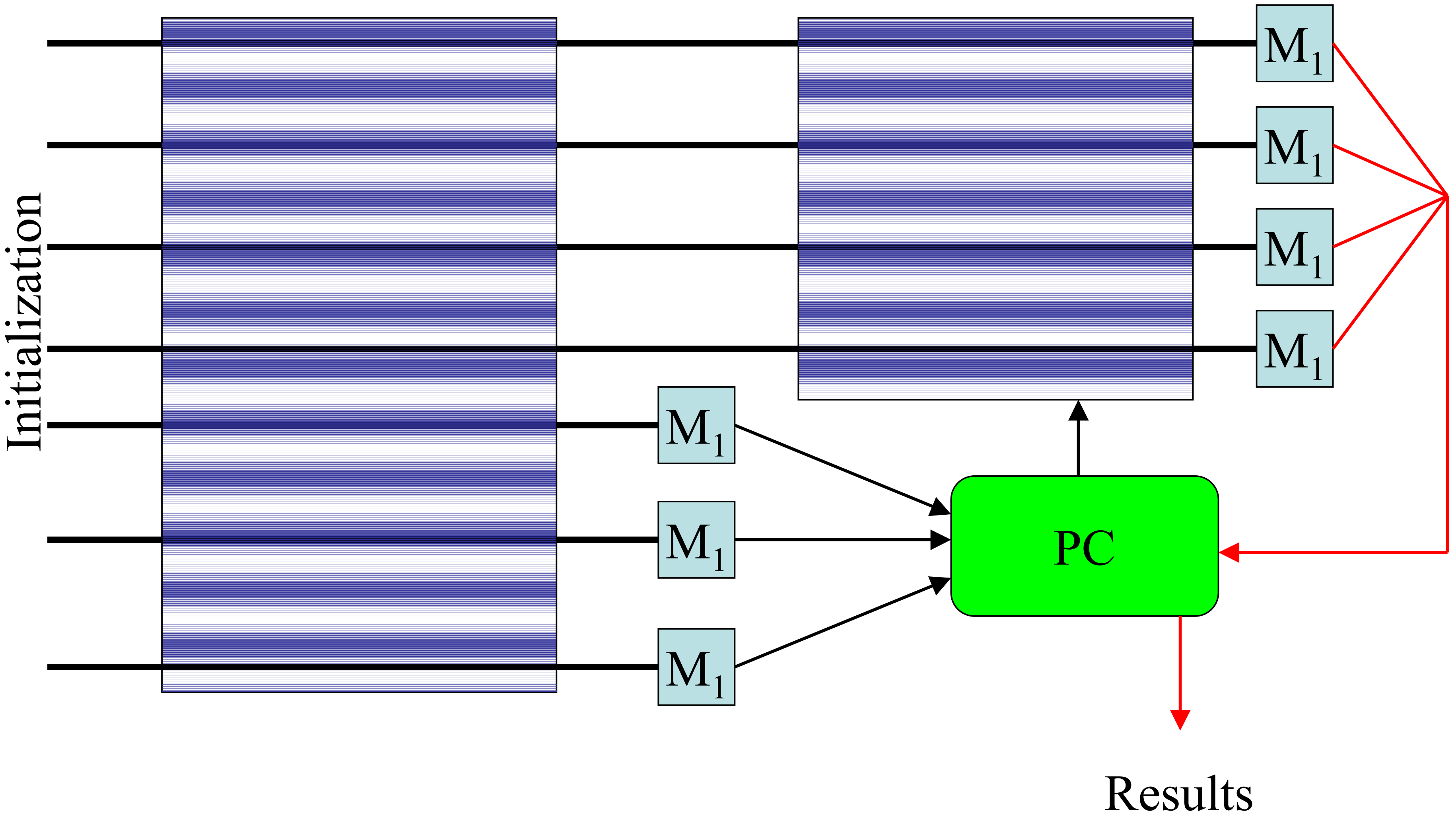


$$U_{C-NOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

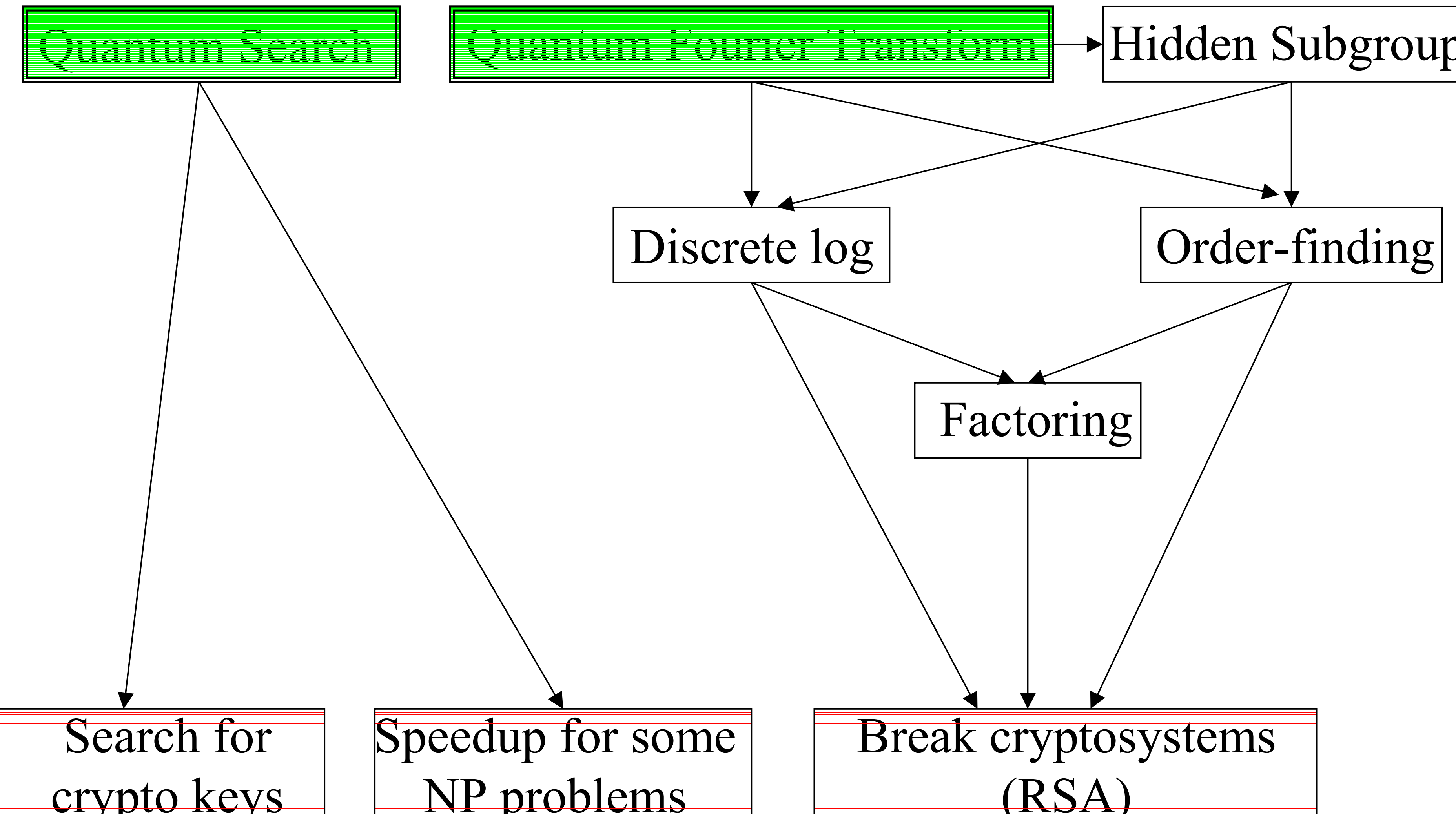
Quantum Circuit Diagram



Quantum Algorithms I



Quantum Algorithms II



Universal Quantum Computation

- *Two-level* unitary gates are universal
  - Act non-trivially only on two-or-fewer vector components
  - Any unitary matrix U which acts on a d-dimensional Hilbert space can be decomposed into a product of two-level unitary matrices.
  - $U=V_1 \ldots V_k$, $k \leq d(d-1)/2$

- (All) *Single qubit* and *CNOT* are universal
  - Single qubit and CNOT can implement an arbitrary two-level unitary operation acting on n-qubit space

- A *discrete set* of universal operations
  - Example: Hadamard+phase+CNOT+$\pi/8$ gates
  - Approximate unitary operators efficiently

- Warning: *Not Optimal*

Decoherence

Quantum computer is so great, why can't I buy one yet ?
Decoherence, decoherence, and decoherence !!!

•Ideal qubit : Pure state => Pure density matrix
•Non-ideal qubit: Mix state => Mixed density matrix

$$|\psi>=\frac{1}{\sqrt{2}}\left(|0>+|1>\right)\Rightarrow\rho=|\psi><\psi|=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\xrightarrow{decoherence}\rho^{'}=|\psi><\psi|=\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

•Why the decoherence destroies the quantum computation?

•Considering measurement operators $M_1=|+><+|$, $M_2=|-><-|$
  •Ideal qubit: p(1)=1, p(2)=0
  •Non-ideal qubit: p(1)=1/2, p(2)=1/2

Where does the decoherence come from?

- $H=H_{sys} \otimes I+I \otimes H_{env}+H_{int}$

- Without the environment:
  - $\rho(t)=\exp(-iH_{sys}t)\ \rho(0)\exp(+iH_{sys}t)$ Pure

- With the environment:
  - $R(t)=\exp(-iHt)\ (\rho(0) \otimes \rho_{env}(0))\ \exp(+iHt)$ Pure

- With the environment but we forget about it:
  - $\rho(t)=Tr_{env}[\exp(-iHt)\ (\rho(0) \otimes \rho_{env}(0))\ \exp(+iHt)]$ *Not p*

- Markov approximation:
  - $d\rho(t)/dt=-I/h[H, \rho]+\Sigma_l(2L_j\rho L^+_j-\{L^+_jL_j, \rho\})$

How to fight the decoherence?

- Active:
  - Quantum error correction
  - Quantum feedback control
  - …….
- Passive:
  - Decoherence-free subspace
  - Bang-bang decoupling
  - …….

- Quantum Error Correction
  - Knowing the error without knowing the state
  - Encoded qubit, Ex: $|0>_L=|000>$, $|1>_L=|111>$
  - Error detection
  - Error correction

- Threshold for resilient quantum computation
  - $P_{th} \approx 10^{-5}$

Quantum Dot Based Quantum Computation Scheme

Design Hamiltonians for Quantum Manipulation

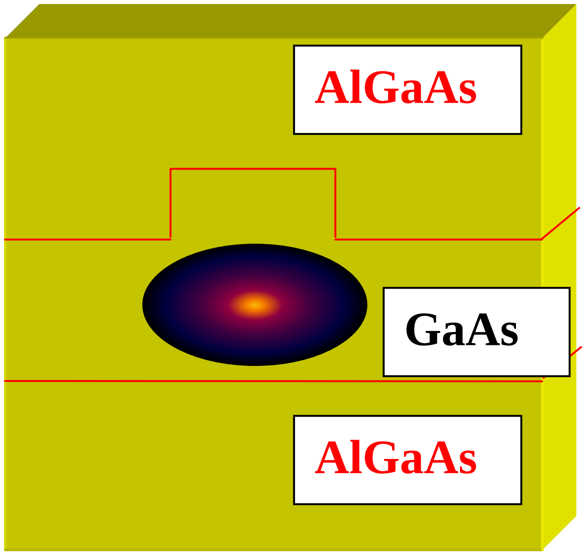
$$\left|\Psi\right\rangle = \alpha\left|\psi_0\right\rangle + \beta\left|\psi_1\right\rangle + \gamma\left|\psi_2\right\rangle + \delta\left|\psi_3\right\rangle + \ldots$$

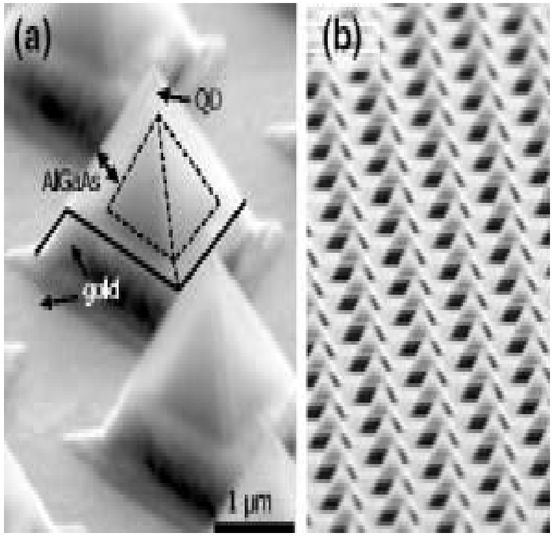
$$H = H_0 + H_{control}(t, \sigma_1, \sigma_2, \ldots)$$

- Semiconductor Quantum Dots
- Quantum Optical Control of Excitons in QDs
- Quantum Optical Control of Spins in QDs

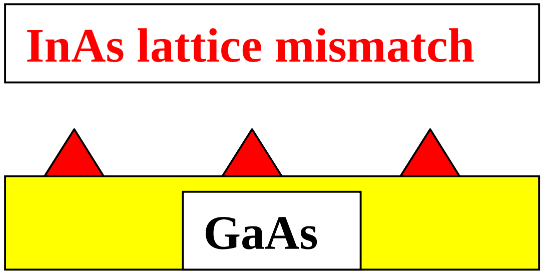
Semiconductor Quantum Dots

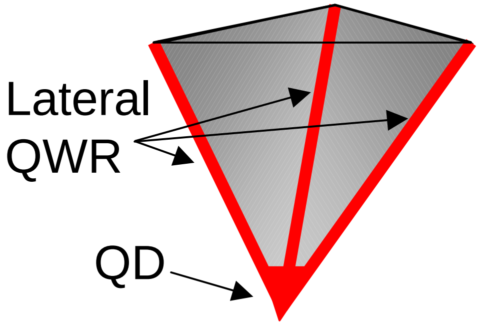
Interface fluctuation





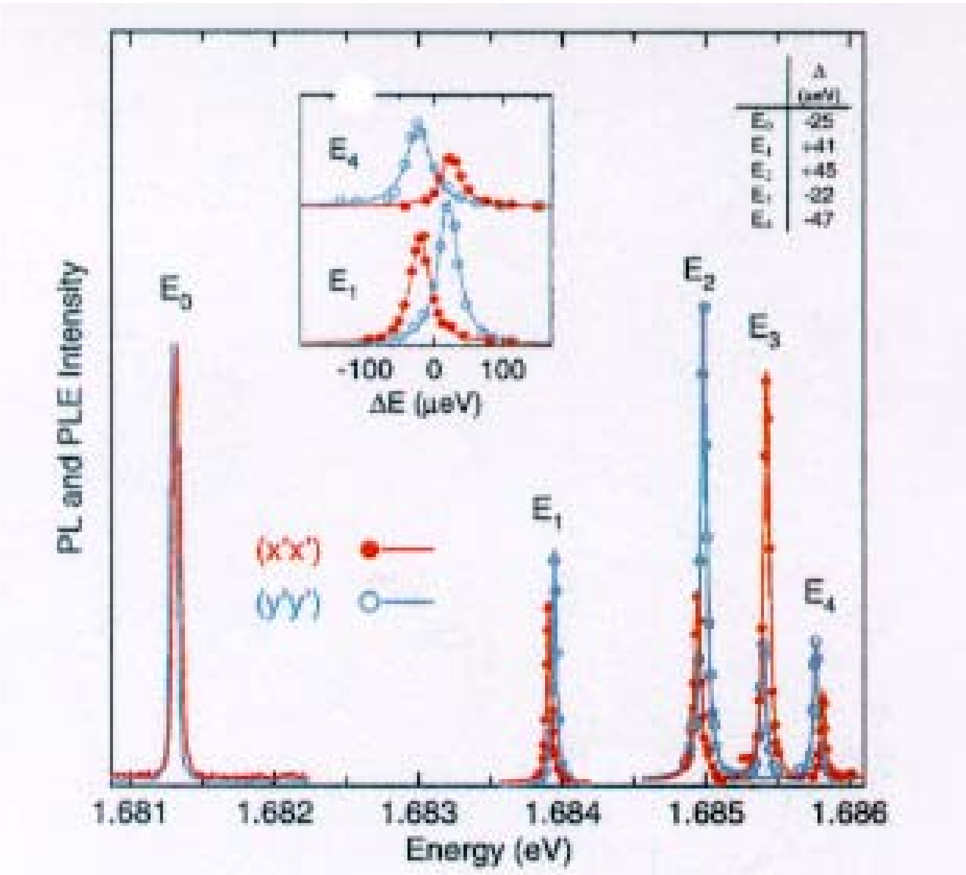
Self assembled



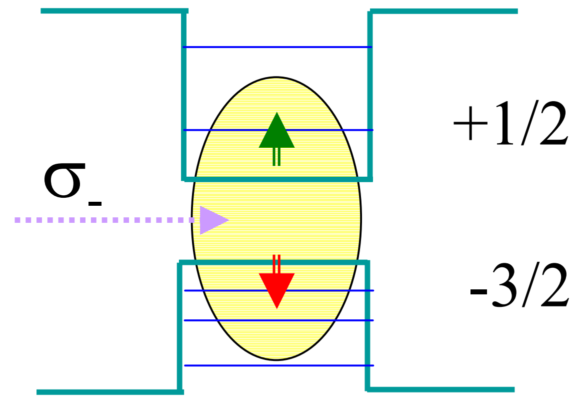


Excitons in a Single QD

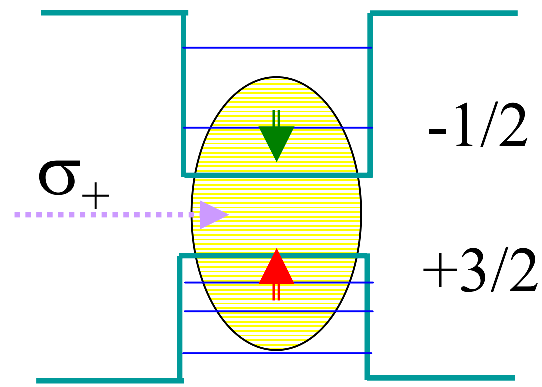
Interface fluctuation QD



NRL Group, PRL **76**,3005 (1996)



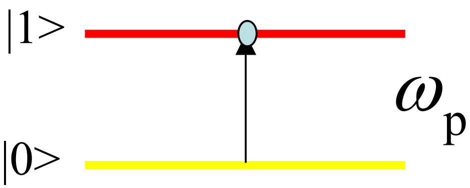
$X_{-}=e^{+}_{\uparrow}h^{+}_{\downarrow}|G\rangle$



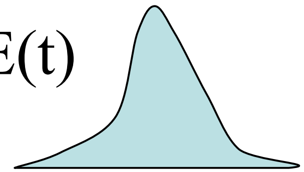
$X_{+}=e^{+}_{\downarrow}\,h^{+}_{\uparrow}\,|G\rangle$

Rabi Oscillation

Two Level System

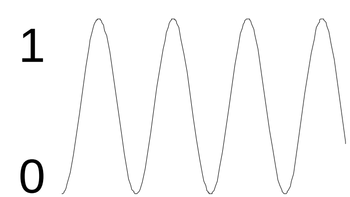


E(t)



$$H(t) = \begin{bmatrix} 0 & \Omega_R(t) \\ \Omega_R(t) & 0 \end{bmatrix} \quad \Omega_R(t) = \frac{d \cdot |E|}{\hbar}$$

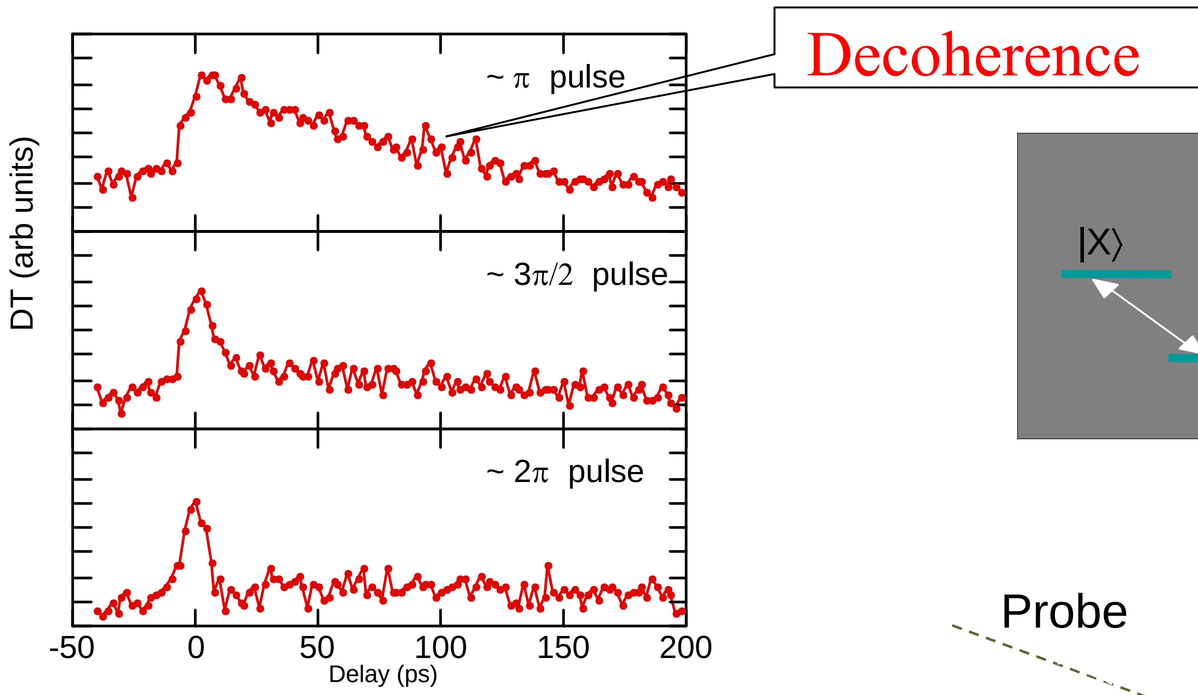
$$U = e^{-\frac{i}{\hbar}\int dt H(t)} = \begin{bmatrix} \cos(\frac{\beta}{2}) & -e^{i\alpha}\sin(\frac{\beta}{2}) \\ -e^{i\alpha}\sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2}) \end{bmatrix}$$

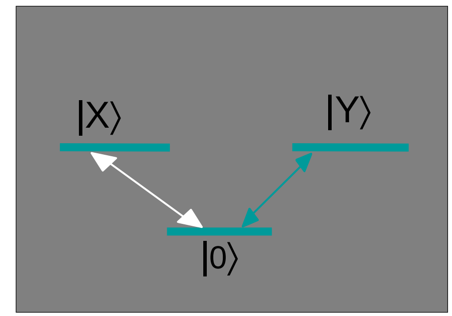


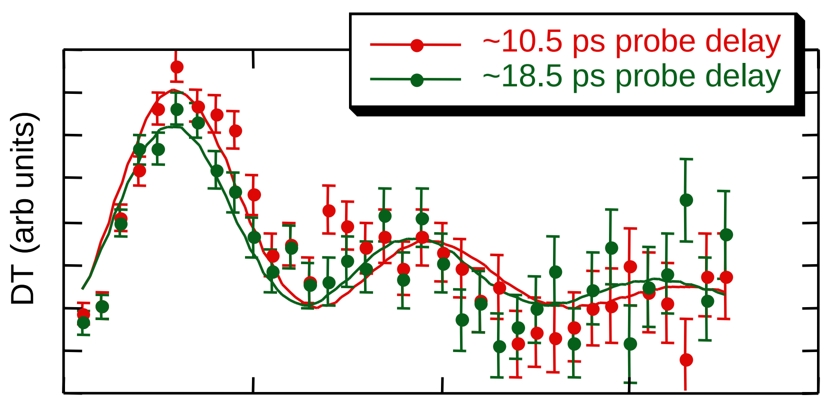
$$\beta = \int dt \Omega_R(t)$$

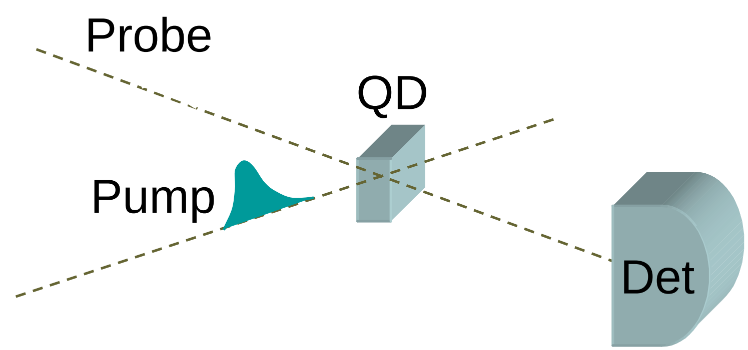
Perfect Rabi oscillation enables us to perform *arbitrary* single qubit gate

Rabi Oscillation of Excitons



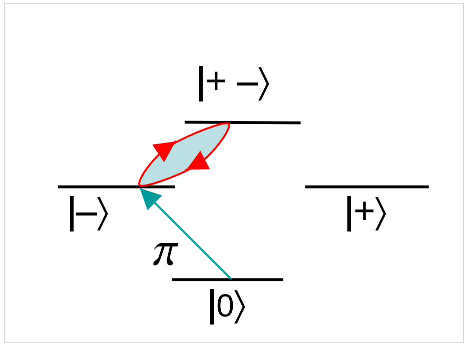






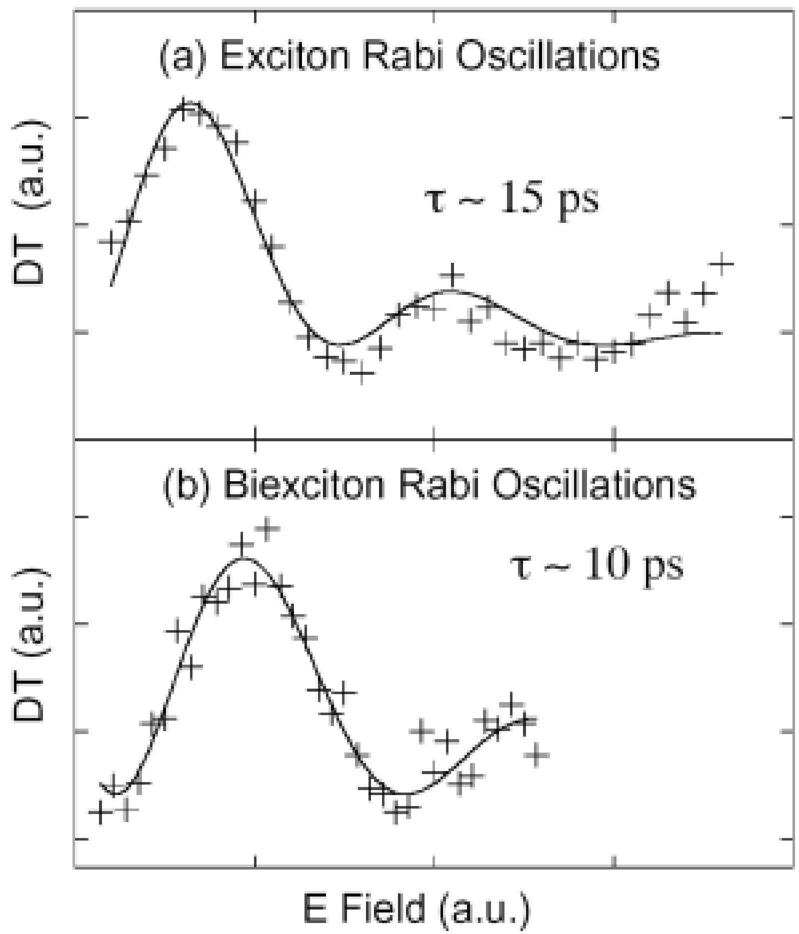
Michigan Group, PRL (2001)

Optical Control of Excitons and Biexcitons



Full quantum control two excitons in a single QD

Ready to be used as a two-qubit quantum computer



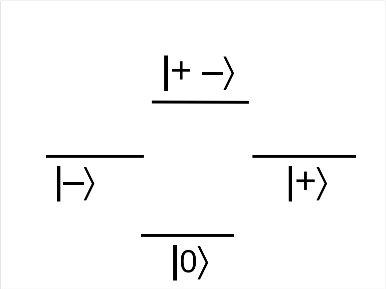
Michigan Group (2001)

Map Excitons to Qubits

Qubit #1	1	$\|-\rangle$ Exciton	Single qubit #1 rotation are provided by σ_- pulses
	0	$\|0\rangle$	
Qubit #2	1	$\|+\rangle$ Exciton	Single qubit #2 rotation are provided by σ_+ pulses
	0	$\|0\rangle$	

$$H_1 \otimes H_2$$

$$\begin{array}{cc} |11\rangle & |+-\rangle \\ |10\rangle & |+\rangle \\ |01\rangle & |-\rangle \\ |00\rangle & |0\rangle \end{array}$$



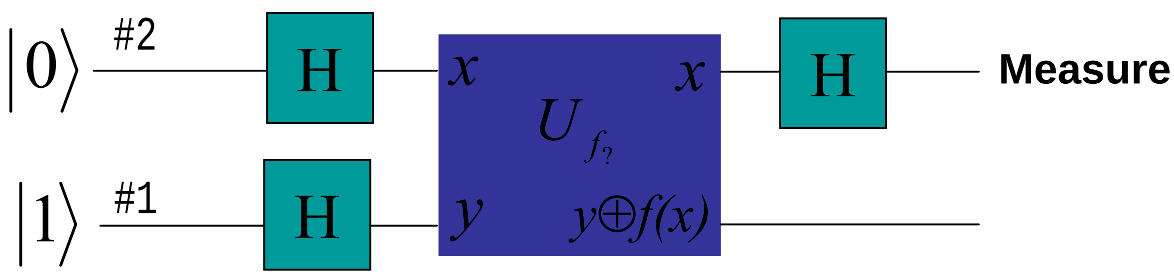
Deutsch Algorithm

Classical computing: I have to evaluate both

$f_?(0)$ $f_?(1)$ and compare the results.

$f_?(x)$ constant or balanced?

Quantum computing: build a Unitary transformation associated to $f_?$ acting on two qubits



➡ 1 Balanced

➡ 0 Constant

We get the answer in one shot !

Universal Quantum Computation is *Not Optimal*

- Universal approach
  - Given a control Hamiltonian, try to implement the well known discrete set of universal gates.

- (Global) Optimization approach
  - Given a control Hamiltonian, try to achieve the same result in (sub)-optimal way. Modify the algorithm if necessary.

H

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{i}{4}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & i\\ i & 1\end{bmatrix} \Longleftrightarrow R_2\left(-\frac{\pi}{2}\right)$$

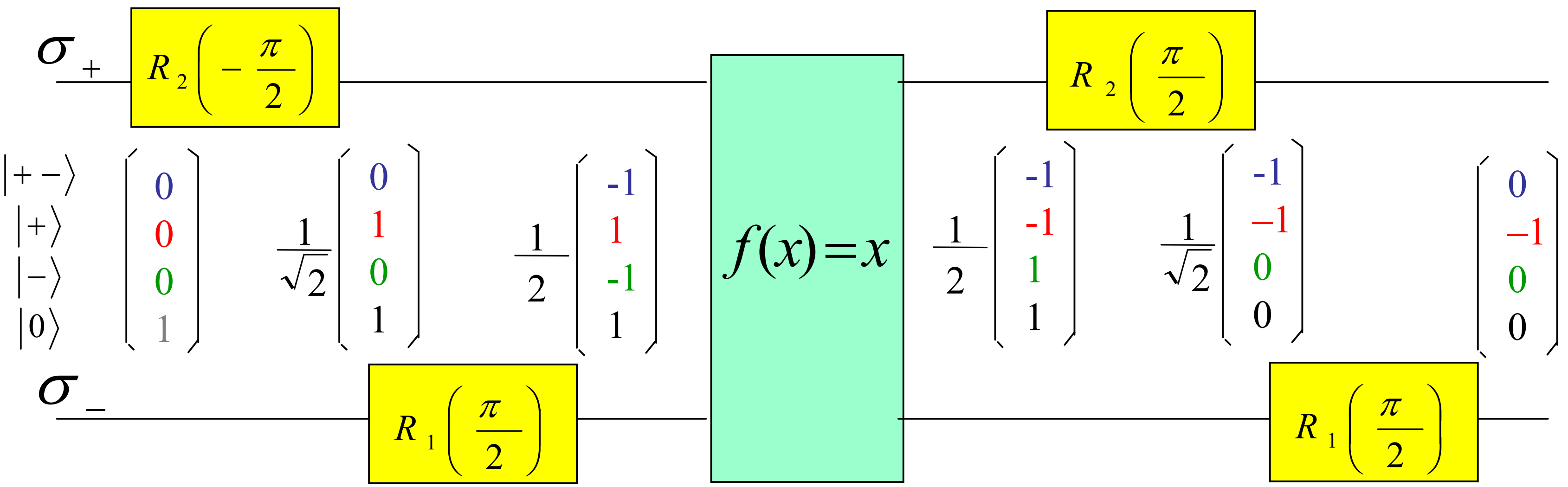
$U_{f(x)=x}$

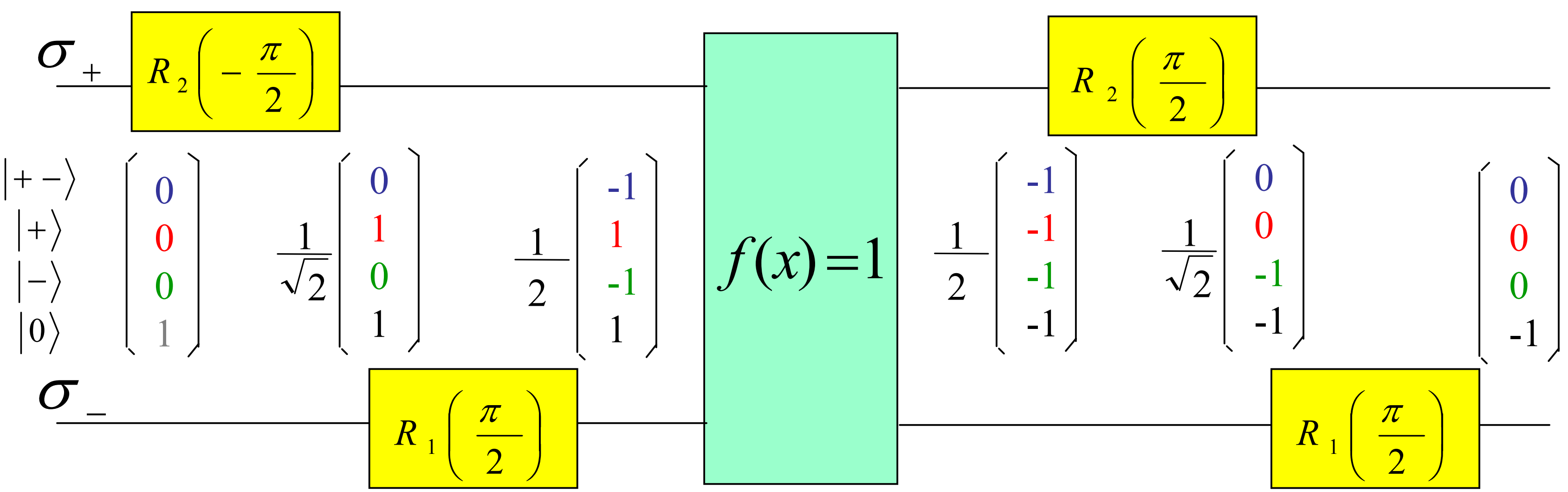
$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \Longleftrightarrow CROT_{2,1} \quad \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$U_{f_?}\left|\overset{2}{m},\overset{1}{n}\right\rangle = R_1^{f_?(m)}(\pi)\,|m,n\rangle$$

$R_1^1(\pi)$ Rotates +π

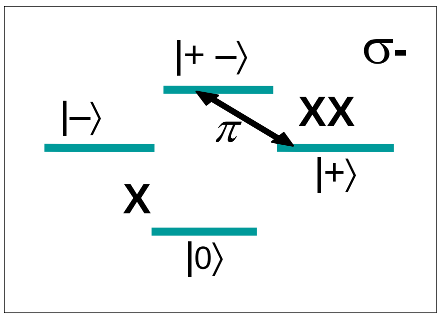
$R_1^0(\pi)$ Rotates -π

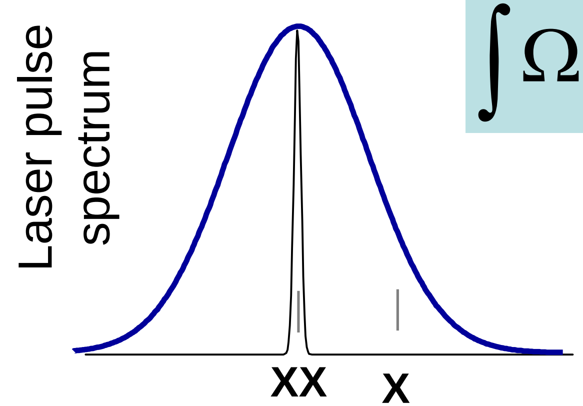




(Local) Optimization of the design: Pulse Shaping

Double two-level system





$$\int \Omega_R d\tau = \pi$$

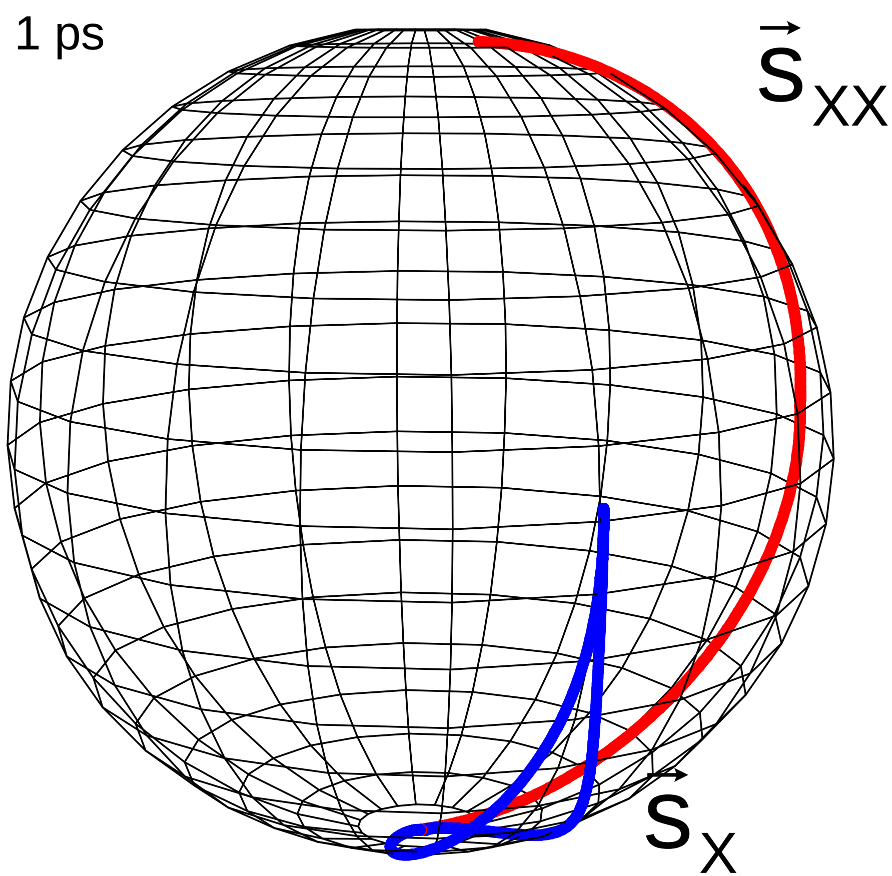
Resonant Excitation ⟺ Long Pulses

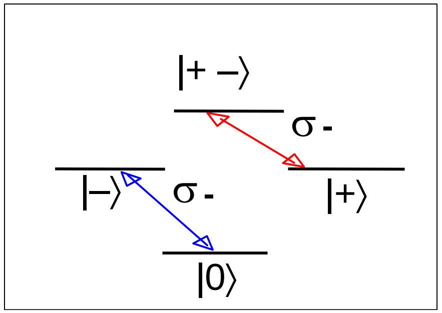
Dephasing (~ 40 ps)

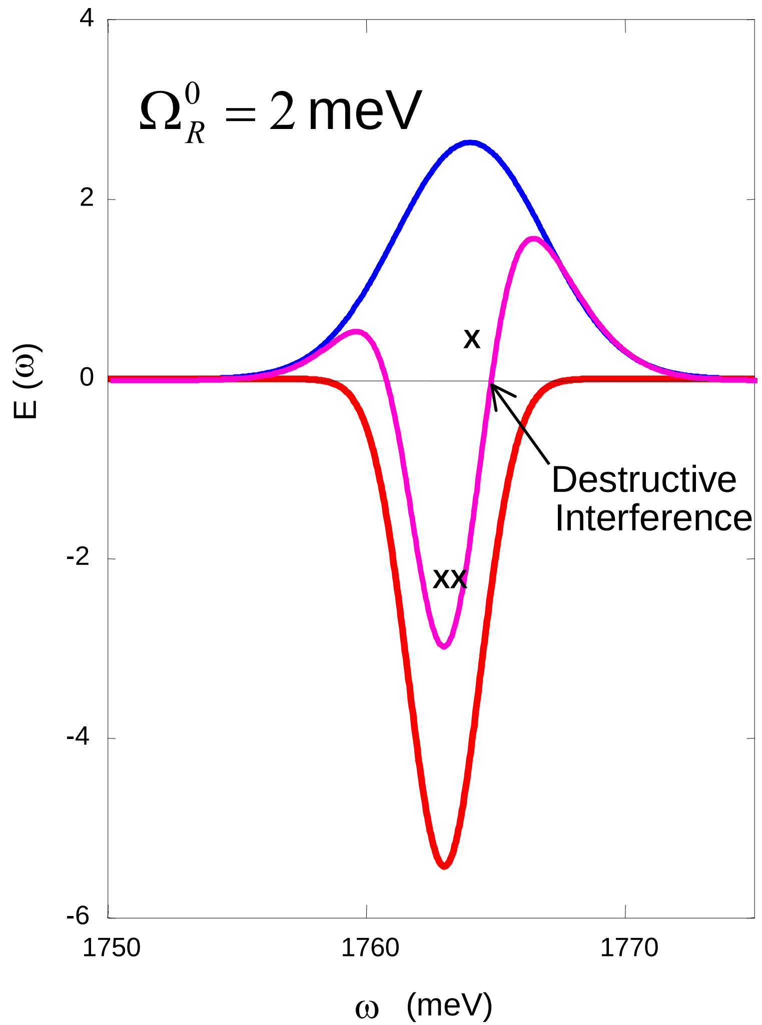
Short time for the operation

Two phase-locked pulses:

$$\Omega_R(t) = \Omega_R^0 \left(e^{-(t/s)^2} e^{-i\omega_X t} + e^{-(t/s1)^2} e^{-i\omega_{XX} t - i\pi} \right)$$







Fidelity: $$\overline{\left|\left\langle \psi_{in} \,|\, \widetilde{U}^{+} U_{ideal} \,|\, \psi_{in} \right\rangle\right|^2}$$

F=0.535 without shaping

F=0.995 with shaping

Analytical Methods

$$H_{control}(t, s, s_1, \Omega_0)$$ Numerical maximization of the Fidelity

Magnus expansion

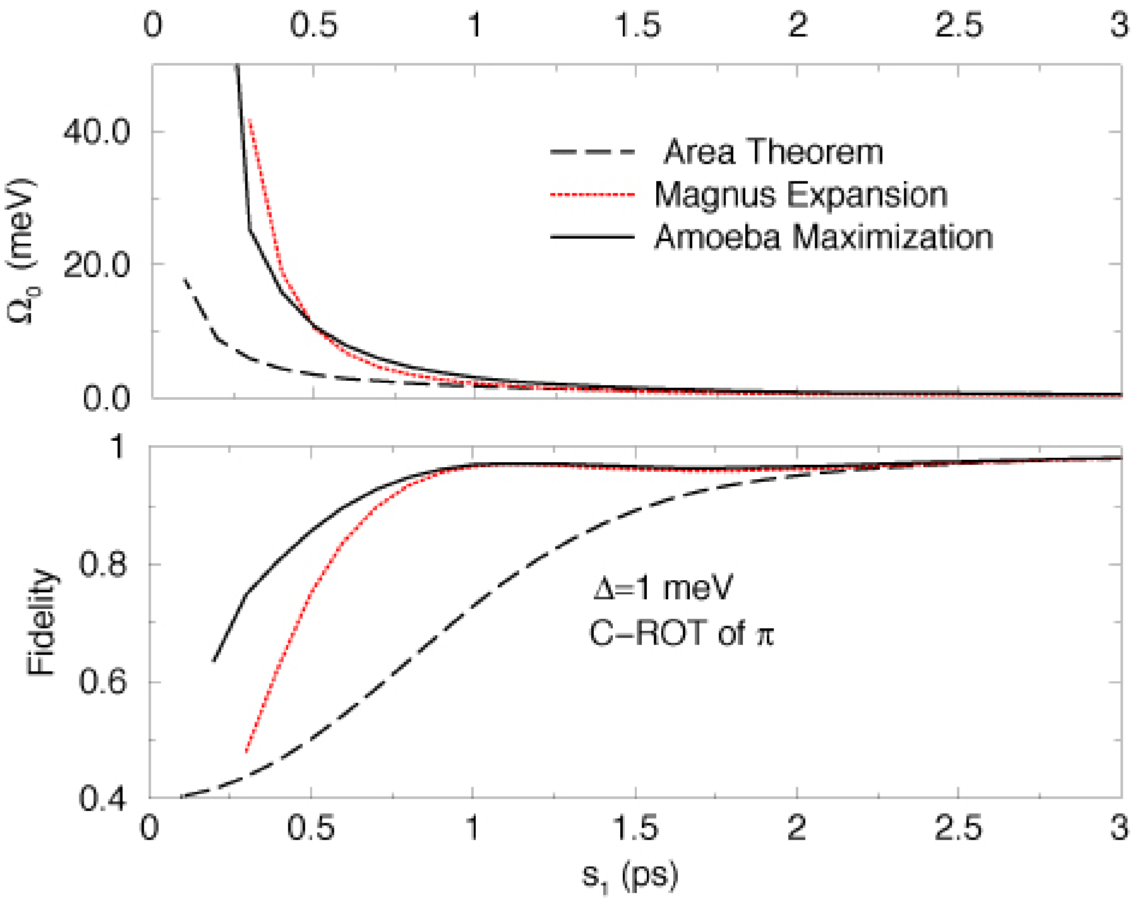
$$U = Te^{-\frac{i}{\hbar}\int_0^\infty H_C(t)dt} = e^{-\frac{i}{\hbar}(S^1{}_C + S^2{}_C + S^3{}_C + ...)}$$

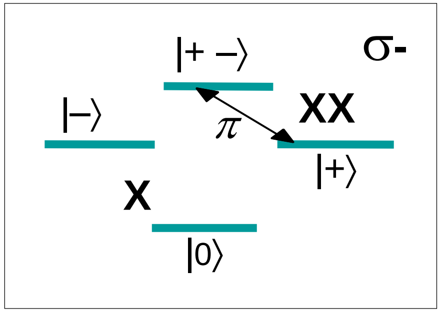
$$S_C^1 = \frac{1}{2}\int_0^\infty H_C(t)dt \qquad S_C^2 = \frac{-i}{8}\int_0^\infty dt \int_0^t dt' \left[H_C(t), H_C(t')\right]$$

For a given U is possible to find an analytical expressions for the control parameters

C. P. , P. Chen, and L. J. Sham
PRB (2002)

Fidelity of C-ROT



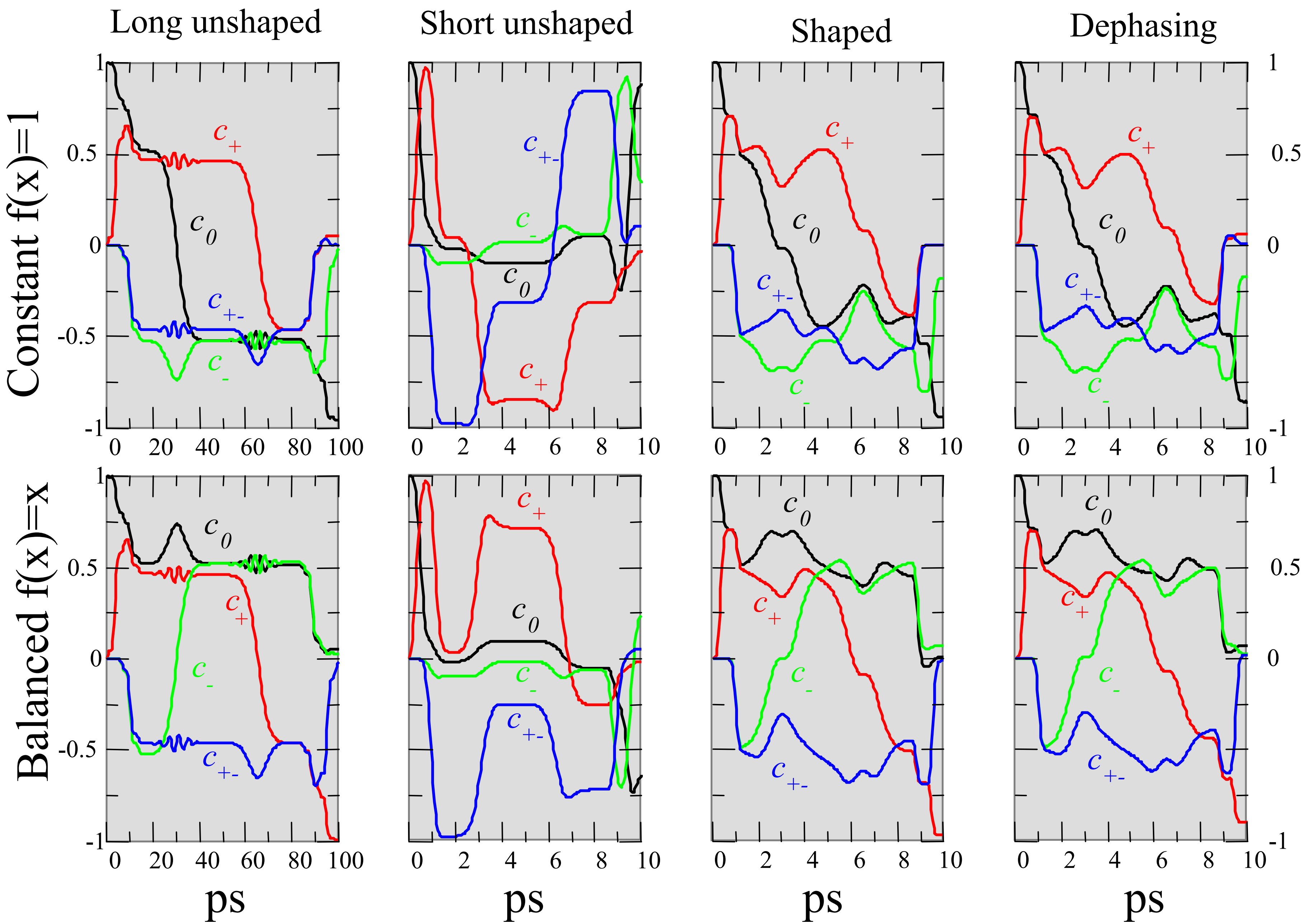


Magnus expansion

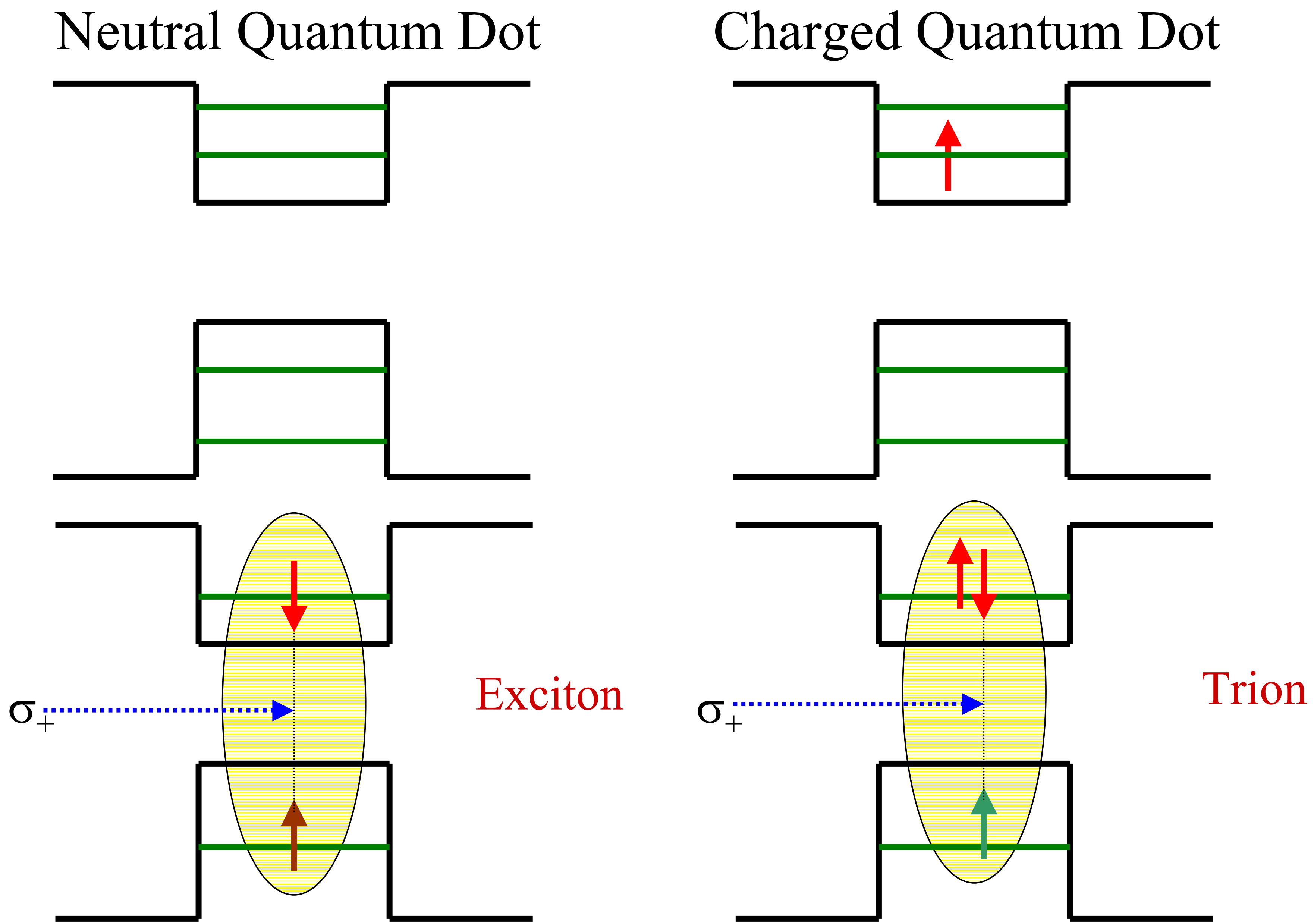
$$s = s_1 e^{-(\Delta s_1/2)^2}$$

$$\Omega_0 = \sqrt{\pi}\left(s_1 - s e^{-(\Delta s/2)^2}\right)$$

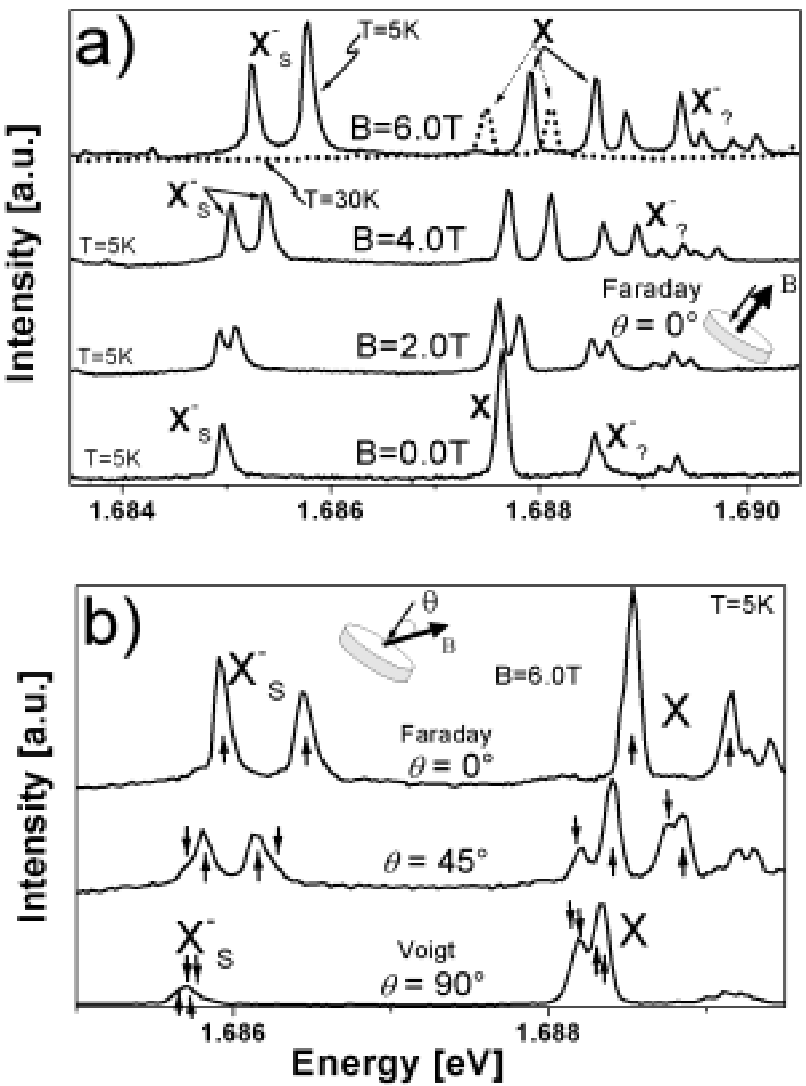
Time Evolution of Two Qubits

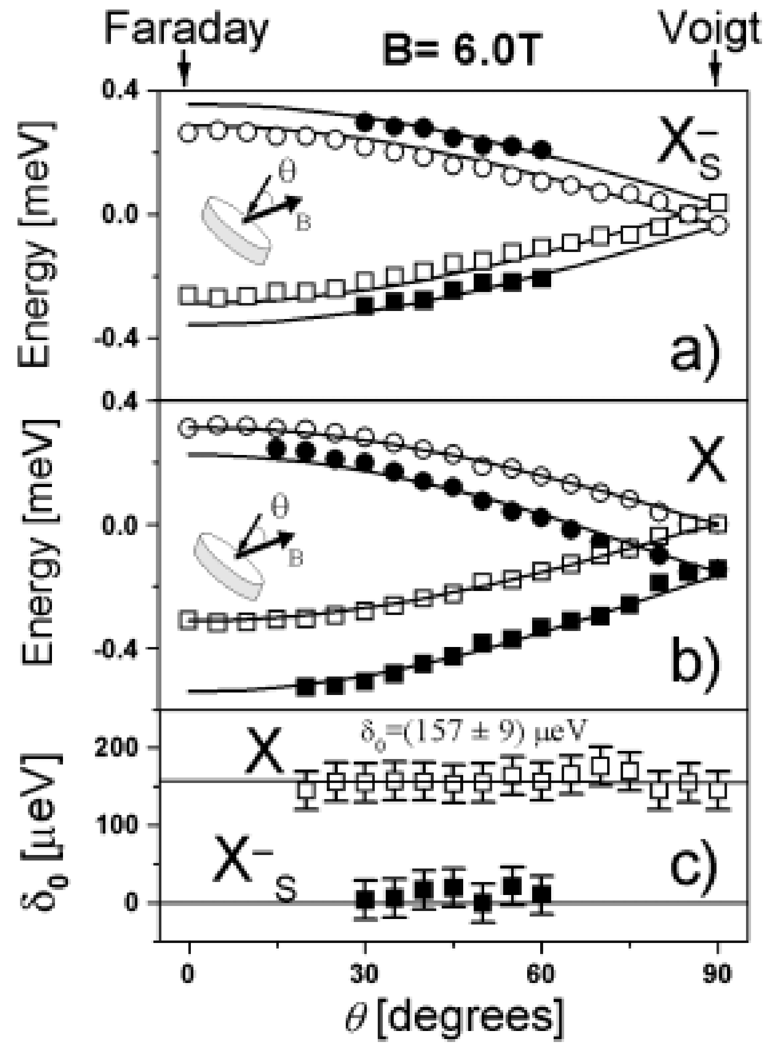


Spins in Quantum Dot as Qubit

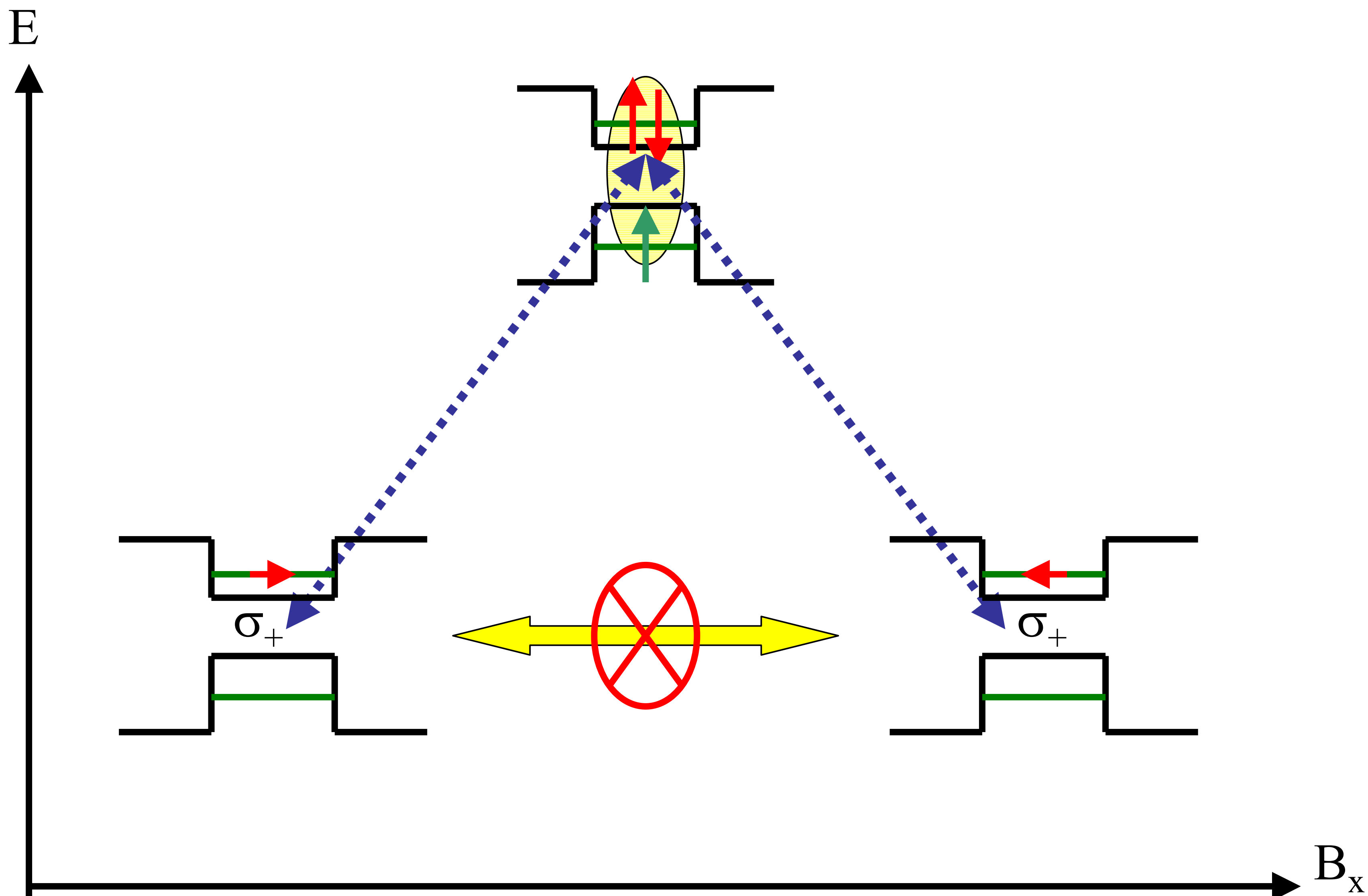


Status on the Experiment

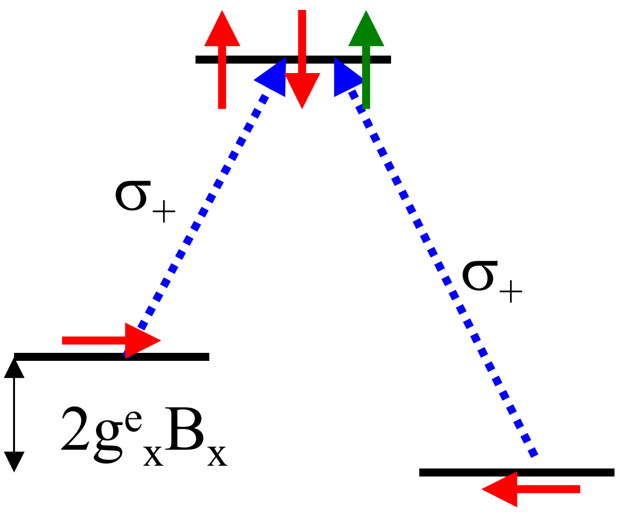


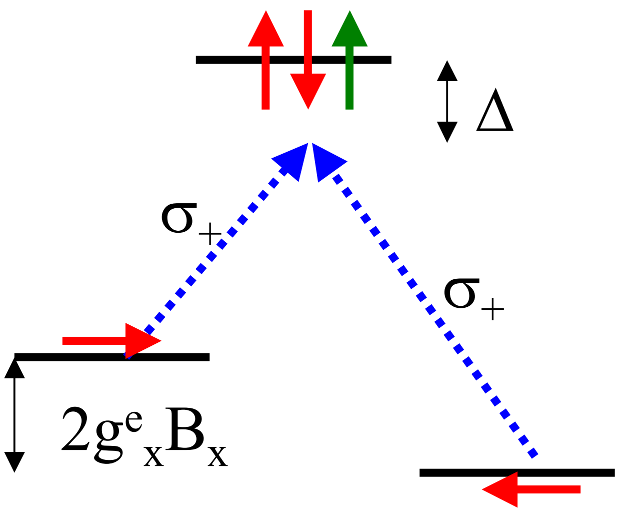


Single Qubit Operations : Λ System



Rabi Flopping v.s. Adiabatic Raman Transition





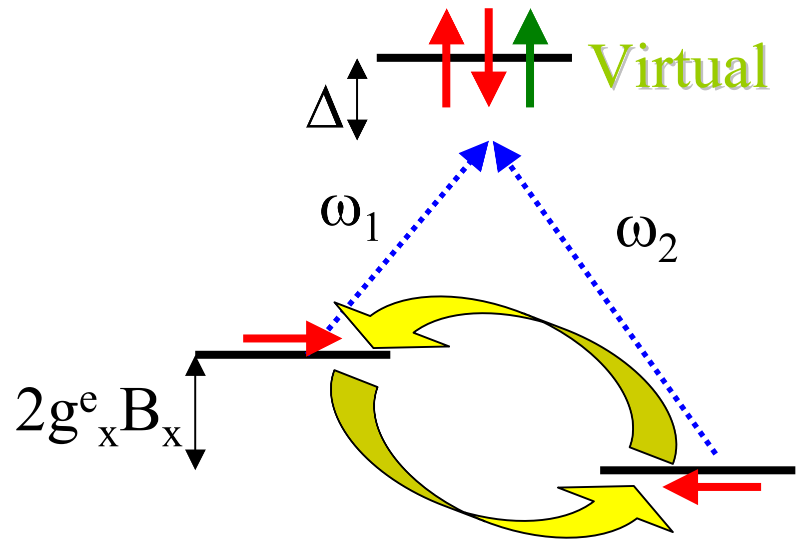
- ❑The decay of the trion state
- ❑Sequence of pulses

- ❑Robust against the decoherence
- ❑Single composite pulse

Adiabatic Raman Transition
Single Λ system with Perturbation Method

Rotating frame

$$
\begin{array}{cccc}
 & |\rightarrow\rangle & |\leftarrow\rangle & |\uparrow\downarrow\Uparrow\rangle \\
H = & \begin{bmatrix} 0 & 0 & \Omega(t)e^{i\alpha} \\ 0 & 0 & \Omega(t)e^{i\beta} \\ \Omega(t)e^{-i\alpha} & \Omega(t)e^{-i\beta} & \Delta \end{bmatrix}
\end{array}
$$



Second order perturbation

$$
\begin{array}{cc}
 & |\rightarrow\rangle \quad |\leftarrow\rangle \\
H_{eff} = & \begin{bmatrix} 0 & \frac{\Omega^2(t)}{\Delta} \\ \frac{\Omega^2(t)}{\Delta} & 0 \end{bmatrix}
\end{array}
\quad \rightarrow \quad
U_{eff} = e^{i\frac{\lambda}{2}} \begin{bmatrix} \cos(\frac{\lambda}{2}) & ie^{i(\alpha-\beta)}\sin(\frac{\lambda}{2}) \\ ie^{-i(\alpha-\beta)}\sin(\frac{\lambda}{2}) & \cos(\frac{\lambda}{2}) \end{bmatrix}
$$

Effective Hamiltonian in spin space

Effective rotation in spin space

Adiabatic Raman Transition
Single Λ system with Analytic Method

Rotating frame

$$H = \begin{bmatrix} 0 & 0 & \Omega(t)e^{i\alpha} \\ 0 & 0 & \Omega(t)e^{i\beta} \\ \Omega(t)e^{-i\alpha} & \Omega(t)e^{-i\beta} & \Delta \end{bmatrix}$$

Instantaneous eigenvectors

$$W(t) = [|\lambda_1(t)\rangle, |\lambda_2(t)\rangle, |\lambda_3(t)\rangle]$$

$$H_{ada}(t) = W^+(t)H(t)W(t) - iW^+(t)\frac{dW(t)}{dt}$$

$$H_{ada} = \begin{bmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{bmatrix}$$

adiabatic condition

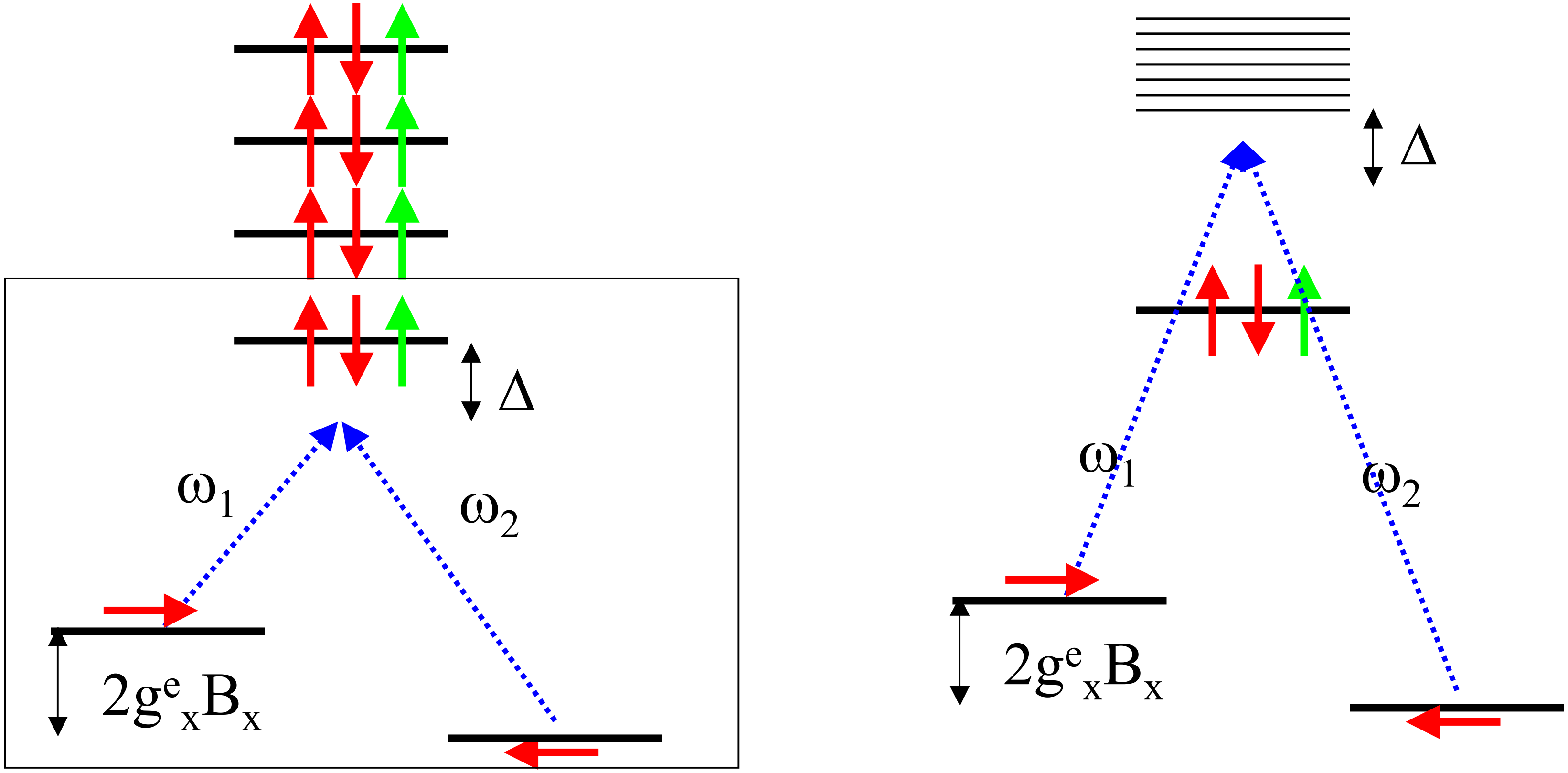
$$-iW^+(t)\frac{dW(t)}{dt} \Rightarrow \frac{d\Omega(t)}{dt} \le \Delta^2$$

$$U = W(\infty)U_{ada}W^+(-\infty)$$

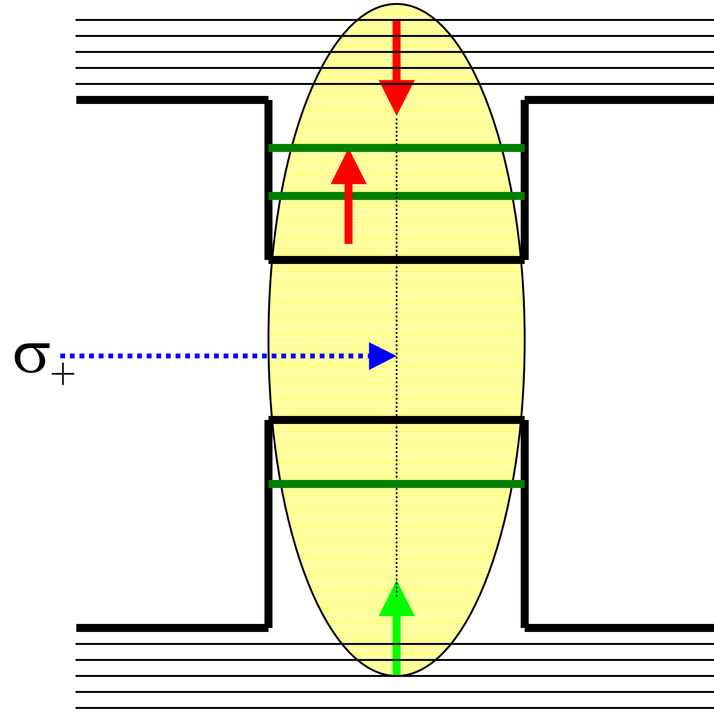
$$U_{ada} = \begin{bmatrix} e^{-i\Lambda_1} & 0 & 0 \\ 0 & e^{-i\Lambda_2} & 0 \\ 0 & 0 & e^{-i\Lambda_3} \end{bmatrix}$$

$$U_{eff} = e^{i\frac{\Lambda}{2}} \begin{bmatrix} \cos(\frac{\Lambda}{2}) & ie^{i(\alpha-\beta)}\sin(\frac{\Lambda}{2}) \\ ie^{-i(\alpha-\beta)}\sin(\frac{\Lambda}{2}) & \cos(\frac{\Lambda}{2}) \end{bmatrix}$$

Multiple Λ system and Continuum Λ system



Adiabatic Raman Transition: Continuum Λ system



Rotating frame

$$H = \begin{bmatrix} 0 & 0 & f_1\Omega(t)e^{i\alpha} & \cdots & f_k\Omega(t)e^{i\alpha} \\ 0 & 0 & f_1\Omega(t)e^{i\beta} & \cdots & f_k\Omega(t)e^{i\beta} \\ f_1\Omega(t)e^{-i\alpha} & f_1\Omega(t)e^{-i\beta} & \Delta_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ f_k\Omega(t)e^{-i\alpha} & f_k\Omega(t)e^{-i\beta} & 0 & 0 & \Delta_k \end{bmatrix}$$

$$E_{-x}(t) = \int d\varepsilon g(\varepsilon) \frac{|f_\varepsilon \Omega(t)|^2}{E_{-x}(t) - \Delta_\varepsilon}$$

Effective rotation in spin space

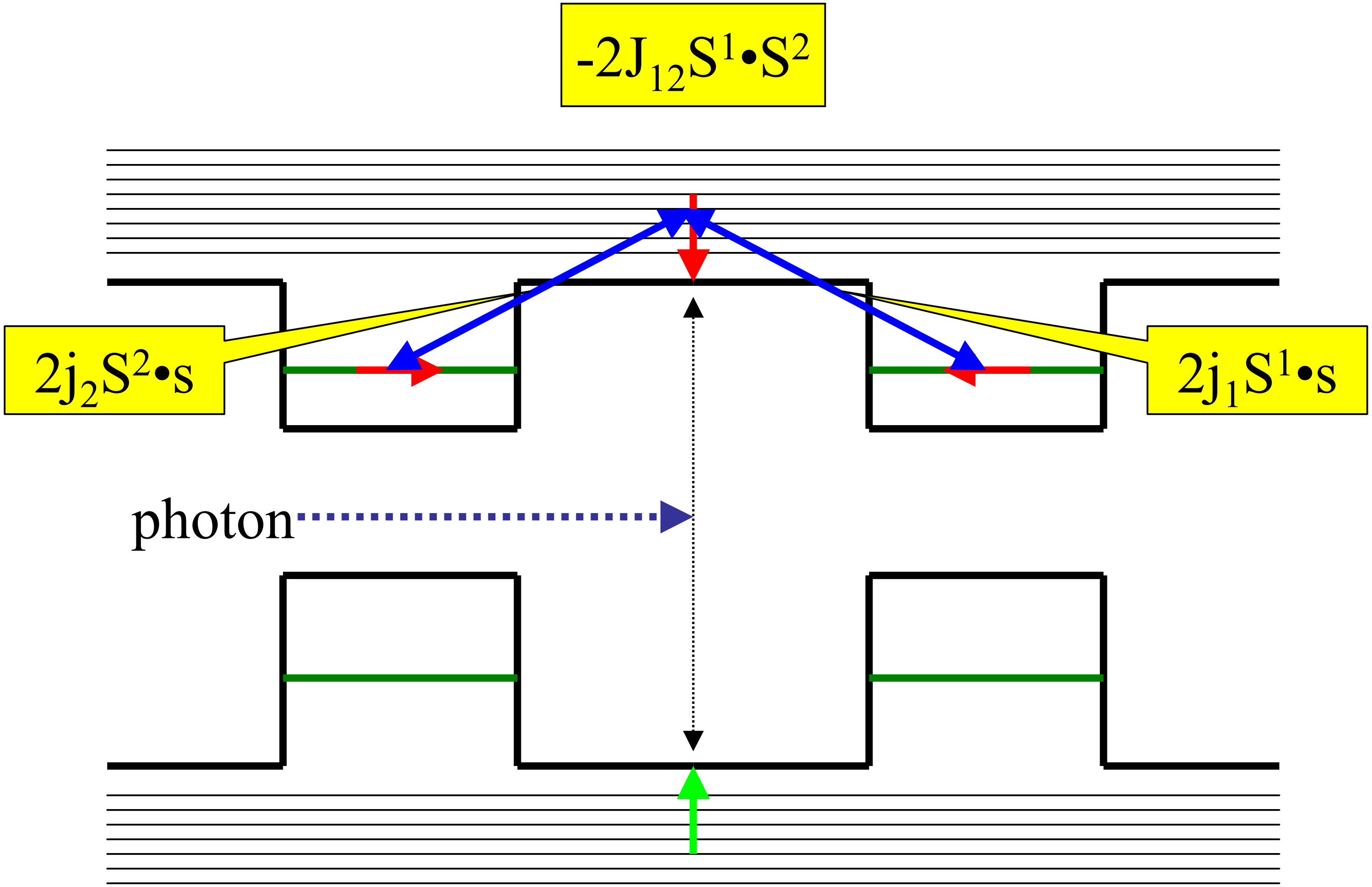
$$U = e^{i\frac{\lambda}{2}} \begin{bmatrix} \cos(\frac{\lambda}{2}) & ie^{i(\alpha-\beta)}\sin(\frac{\lambda}{2}) \\ ie^{-i(\alpha-\beta)}\sin(\frac{\lambda}{2}) & \cos(\frac{\lambda}{2}) \end{bmatrix}$$

$$\lambda = \int E_{-x}(t)dt$$

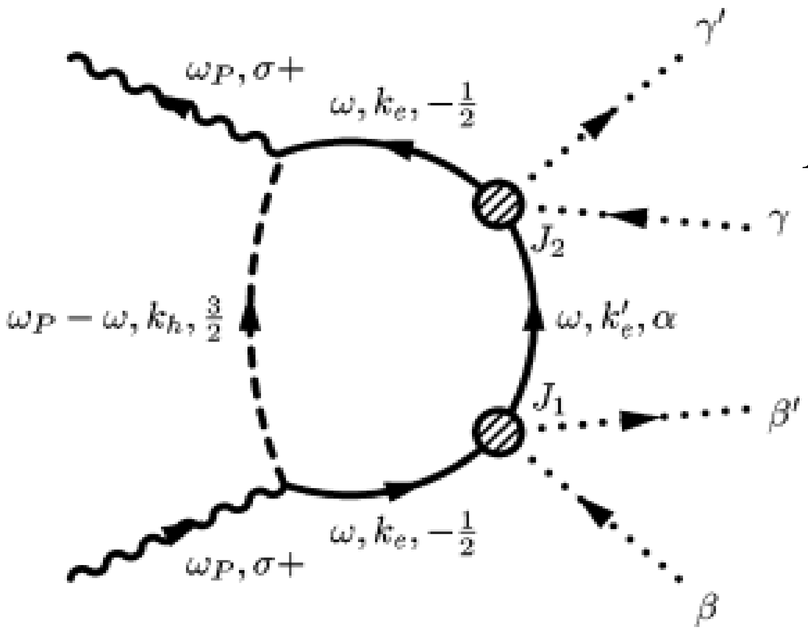
Adiabatic condition

$$\frac{d\Omega(t)}{dt} \le \Delta^2$$

Optical RKKY Interaction Between Spins in Neighboring Quantum Dots

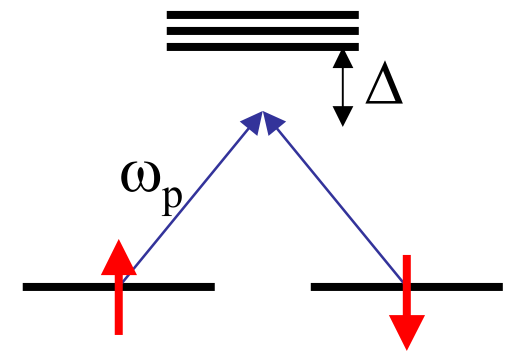


Optical RKKY Interaction via Delocalized *e-h* pairs



$$H = -4J_{12}(\mathbf{S}^1 \cdot \mathbf{s})(\mathbf{S}^2 \cdot \mathbf{s}) - 4J_{12}(\mathbf{S}^2 \cdot \mathbf{s})(\mathbf{S}^1 \cdot \mathbf{s})$$

$$= -2J_{12}(\mathbf{S}^1 \cdot \mathbf{S}^2)$$



$$J_{12}(R) = \frac{|\Omega(t)|^2}{16} j_1^d j_2^d \iint \frac{d^d\mathbf{k}_e}{(2\pi)^d} \frac{d^d\mathbf{k}_e'}{(2\pi)^d} e^{-i(\mathbf{k}_e - \mathbf{k}_e')\cdot\mathbf{R}} \left(\Delta + \frac{k_e^{\ 2}}{2m_h} + \frac{k_e^{\ 2}}{2m_e}\right)^{-2} \left(\Delta + \frac{k_e^{\ 2}}{2m_h} + \frac{k_e'^2}{2m_e}\right)^{-}$$

$$j_i^{\ d} = \iint d^d\mathbf{r} d^d\mathbf{r}' \Psi^*(\mathbf{r}') V(\mathbf{r}' - \mathbf{r}) \Psi(\mathbf{r}) \approx IRy^* a_B^* \xi^{d-1}$$

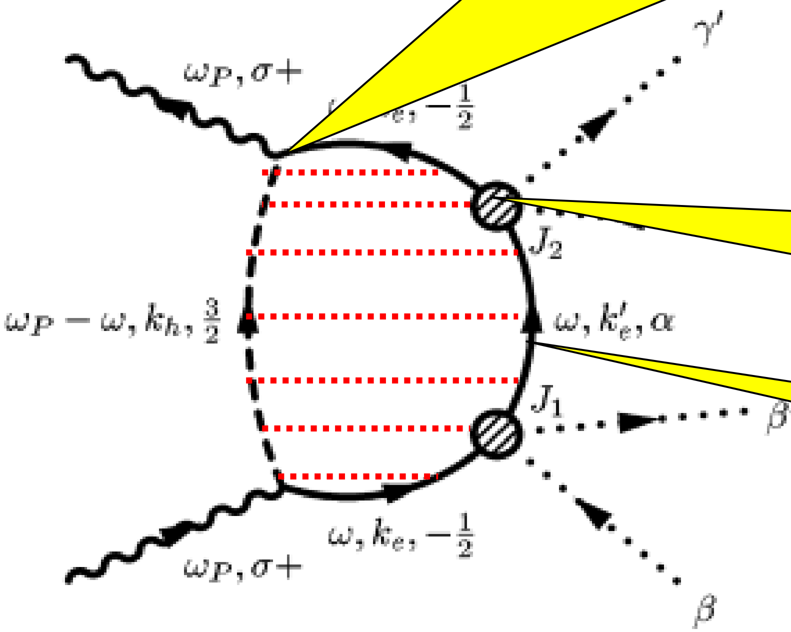
Optical RKKY Interaction via Delocalized Excitons

Exciton enhancement $|\Psi(0)|^2$

Form factor: F(q)

$$F_{n1,n2}(\mathbf{q}) = \int dr e^{-i\frac{m_h}{M}\mathbf{q}\cdot\mathbf{r}} \Psi^*_{n1}(r)\Psi_{n2}(r)$$

Exciton propagator



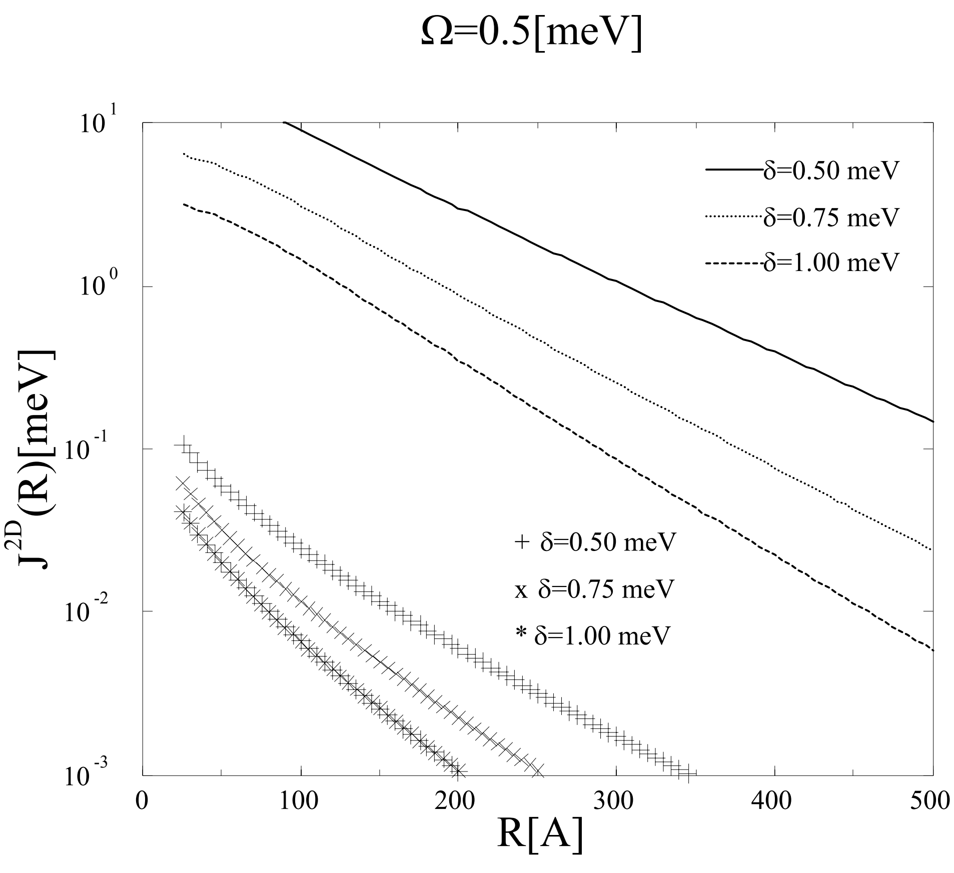
$$J^d{}_{12}(R) = \frac{|\Omega(t)|^2}{16} j_1^d j_2^d \sum_{n1,n2,n3} \Psi_{n1}(0)\frac{1}{\Delta+E_{n1}} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{F_{n1,n2}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{R}}F_{n2,n3}(\mathbf{q})}{\Delta+E_{n2}+\frac{q^2}{2M}} \frac{1}{\Delta+E_{n3}}\Psi_{n3}(0)$$

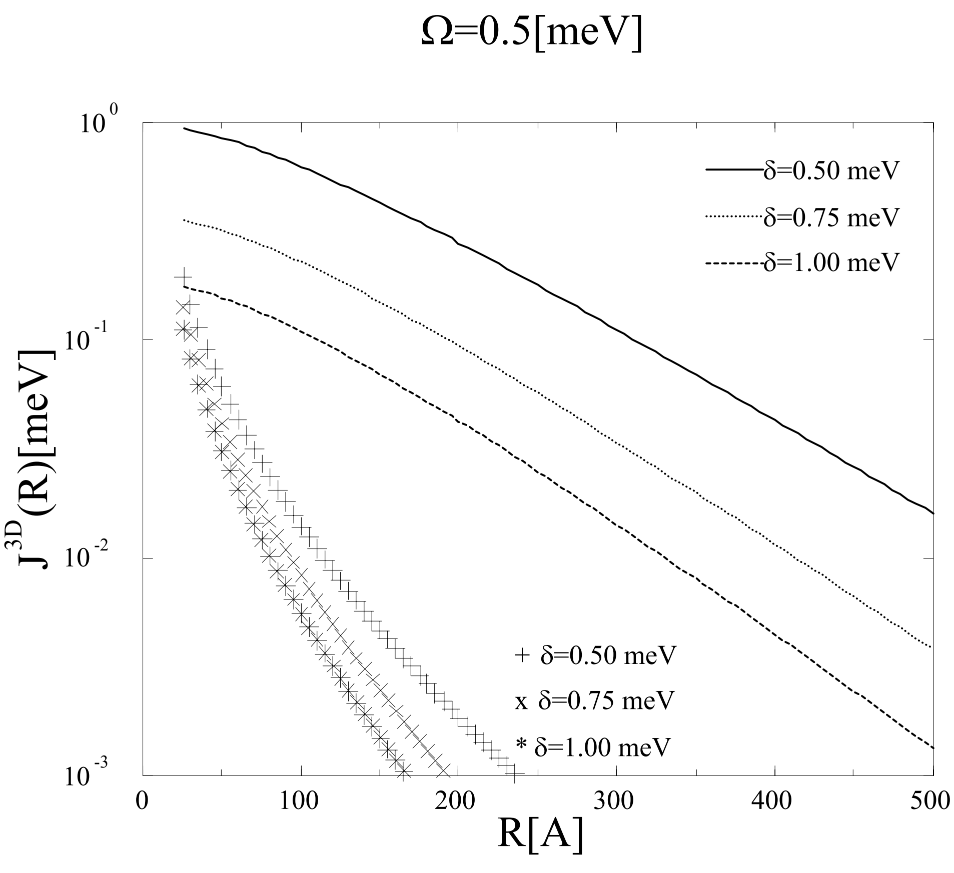
$$\approx \frac{|\Omega(t)|^2}{16} j_1^d j_2^d \frac{1}{\Delta^3}\left|\Psi_{1s}(0)\right|^2 I^d(R), \qquad I^d(R) = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{e^{-i\mathbf{q}\cdot\mathbf{R}}}{1+(\lambda_M q)^2}\left|F_{1s,1s}(q)\right|^2$$

Numerical Results on the Exchange Constant *J*

m^*_e=0.07m, m^*_h=0.5m, ξ=300 Å

2D:	3D:
•Ry^*=10 meV	•Ry^*=5 meV
•a_B^*= 100 Å	•a_B^*= 150 Å





Asymptotic Form of the Optical RKKY Interaction

Important length scales: $a_B, \lambda_M = 1/\sqrt{2M\Delta}, \lambda_\mu = 1/\sqrt{2\mu\Delta}$

$\lambda_M << a_B$

$$I^{3d}(R) \approx \frac{1}{2\pi R \lambda_M^2} e^{-\frac{R}{\lambda_M}}$$

$$I^{1d}(R) \approx \frac{1}{2\pi \lambda_M^2} \sqrt{\frac{\pi}{2R/\lambda_M}} e^{-\frac{R}{\lambda_M}}$$

$\lambda_M >> a_B$

$$I^{3d}(R) \approx \frac{1}{24\pi R a_B^2} e^{-\frac{R}{a_B}} P^{3d}(\frac{R}{a_B})$$

$$I^{1d}(R) \approx \frac{16}{\pi a_B^2} \sqrt{\frac{\pi}{8R/a_B}} e^{-\frac{R}{a_B}} P^{2d}(\frac{R}{a_B})$$

Summary

- Information is physical
- We can process the information quantum mechanically
- Decoherence is our major enemy
- It is possible to suppress the decoherence and/or to correct the error
- You should design your own Hamiltonian for quantum manipultaion