Scattering Amplitudes
and
Symmetry in String Theory

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1. Outline

1. Scattering amplitudes

2. High energy limit

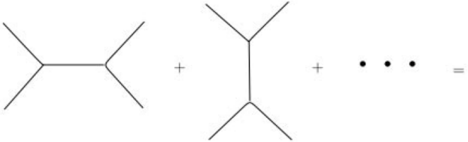
   (a) $E \to \infty$, fixed angle

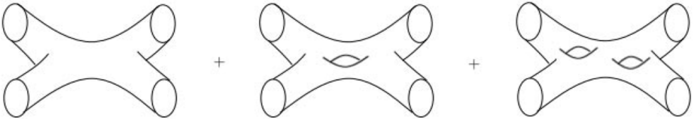
   (b) $E \to \infty$, small angle: Regge limit

3. Summary

2. Scattering Amplitudes

- Amplitudes in field theory and string theory





- Veneziano Amplitude

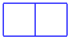
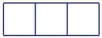
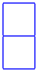
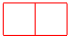
$$\mathcal{T} = \left\langle e^{ik_1 X(z_1)\cdot} e^{ik_2\cdot X(z_2)} e^{ik_3\cdot X(z_3)} e^{ik_4\cdot X(z_4)} \right\rangle$$
$$= B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right)$$

- $E \to \infty$, fixed angle

- Exponential fall-off: $\mathcal{T} \sim e^{-\alpha E}$

- Infinite intermedium states $\to$ symmetry: Gross' conjecture

- String spectrum

Mass level	Positive-norm states	Zero-norm states
$M^2 = -2$	$\bullet$	n/a
$M^2 = 0$		$\bullet$
$M^2 = 2$		 , $\bullet$
$M^2 = 4$	 , 	 , $2 \times$  , $\bullet$

3. High Energy Limits

- $E \to \infty$, fixed angle ϕ

  $\Rightarrow s \sim E^2 \to \infty$, $\dfrac{t}{s} \sim \sin^2 \frac{\phi}{2} \to$ fixed.

  Gross conjecture: symmetry $\to$ linear relations.

- $E \to \infty$, small angle ϕ

  $\Rightarrow s \sim E^2 \to \infty$, $t \sim E^2 \sin^2 \frac{\phi}{2} \to$ fixed

  Regge limit

3.1. $E \to \infty$, Fixed Angle

- 4-point scattering amplitudes

  at mass level $M^2 = 2(N-1)$,

$$\mathcal{T}^{(N,2m,q)} = \langle V_1 V^{(N,2m,q)}(k) V_3 V_4 \rangle,$$

- Linear relations

$$\boxed{\frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} = \left(-\frac{1}{2M}\right)^q \left(\frac{1}{2M^2}\right)^m (2m-1)!!}$$

3.2. $E \to \infty$, Small Angle: Regge Limit

- Scattering amplitude in Regge limit,

$$\mathcal{T}^{(N,k_n,q_m)} = \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1, \frac{t}{2}+2-q_1, \frac{t+M^2+2}{2}\right) \cdot B\left(-1-\frac{s}{2}, -1-\frac{t}{2}\right) \cdot \prod_{n=1}\left[i\sqrt{-t}(n-1)!\right]^{k_n} \cdot \prod_{m=2}\left[i\left(t+M^2+2\right)(m-1)!\left(-\frac{1}{2M_2}\right)\right]^{q_m}$$

- Kummer function of the first kind $U(a,c,x)$

$$xU^{''}(x) + (c-x)U'(x) - aU(x) = 0$$

- General mass level $M^2 = 2(N-1)$,

$$U\left(-2m, \frac{t}{2}+2-2m, \frac{t}{2}\right)$$
$$= 0 \cdot t^{2m} + 0 \cdot t^{2m-1} + \cdots\cdots + 0 \cdot t^{m+1}$$
$$+ \frac{(2m)!}{m!}\left(-\frac{t}{4}\right)^{m} + O\left(t^{m-1}\right)$$

- Prove the identity $\Rightarrow$ Stirling number $s(n,k)$

$$(x+1)\cdots(x+n) = \sum_{k=0}^{n}(-1)^{n-k} s(n,k) x^k$$

4. Summary

- High energy, fixed angle ⇒ linear relations,

$$\lim_{E\to\infty} \frac{\mathcal{T}^{(n,2p,q)}}{\mathcal{T}^{(n,0,0)}} = \left(-\frac{1}{2M}\right)^q \left(\frac{1}{2M^2}\right)^p (2p-1)!!$$

- High energy, small angle (Regge) ⇒ Kummer function

$$\mathcal{T} \sim U\left(-q_1, \frac{t}{2}+2-q_1, \frac{t+M^2+2}{2}\right)$$

- Reproduce the linear relations from Regge limit $t \to \infty$,

$$U\left(-2m, \frac{t}{2}+2-2m, \frac{t}{2}\right) = 0 + \frac{(2m)!}{m!}\left(-\frac{t}{4}\right)^m + \mathcal{O}\left(t^{m-1}\right)$$