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Modified gravity theories
&
alternatives to the Dirac algebra

Chopin Soo 許祖斌

Department of Physics,
National Cheng Kung University

成功大學
National Cheng Kung University

*Einstein's General Relativity in 4-dimensions:

Not renormalizable as a perturbative QFT (Goroff, Sagnotti; t' Hooft Veltman; van der Ven …)

*GR with higher derivatives as perturbative QFTs :

Renormalizable; BUT not unitary (Stelle; Julve,Tonin; Fradkin,Tseytlin; Avramidi,Barvinsky;…)

$$\int d^4x\sqrt{g}\left[\frac{1}{16\pi G}(2\Lambda - R) + \frac{1}{2\lambda}C^2 - \frac{\omega}{3\lambda}R^2 + \frac{\theta}{\lambda}E\right]$$

$C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ is the square of the Weyl tensor.

$E \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss-Bonnet term.

taming of divergences due to higher derivatives

(General covariance => no. of time and space derivatives are equal)

=> problem with unitarity $G_0(k) \propto P^{(2)}\frac{1}{m_2^2}\left(\frac{1}{k^2} - \frac{1}{k^2+m_2^2}\right).$

*Horava's proposal:

improve convergence with higher spatial derivatives but keep time derivatives to 2nd order only.

(=> Give up (!) spacetime covariance at the "fundamental" level)

Space and time are not on equal footing!

PHYSICAL REVIEW D **79**, 084008 (2009)

Quantum gravity at a Lifshitz point

Petr Hořava

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z} = \frac{1}{\omega^2 - \mathbf{k}^2} + \frac{1}{\omega^2 - \mathbf{k}^2}G(\mathbf{k}^2)^z \times \frac{1}{\omega - \mathbf{k}^2} + \cdots.$$

Alternatives:

String Theory
Loop Quantum Gravity (LQG)
Higher Derivative Gravity Theories
Dynamical Triangulation
Euclidean Path Integral Quantum Gravity
Twistor Models
Stochastic Gravity
Acoustic metric and other models of analog gravity
Entropic Gravity and models inspired by Thermodynamics
....

*Reduce 4-dimensional diffeomorphism (general coordinate) symmetry
-> 3-dimensional spatial diffeomorphism invariance
(?+? time reparametrization invariance)

*Assume Arnowitt-Deser-Misner (ADM) decomposition of spacetime metric

$$ds^2 = -N^2(dt)^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Horava's proposed action (in canonical form):

$$S = \int \pi^{ij}\dot{q}_{ij}\, d^3x\, dt - \int (NH + N^i H_i)\, d^3x\, dt.$$

*Guiding principle: maintain 3-dim. diffeomorphism invariance

**To eliminate many many possible terms: Impose "detailed balance"

$$H = \frac{\kappa^2}{2}\frac{G_{ijkl}}{\sqrt{q}}\left[\pi^{ij}\pi^{kl} + \frac{\delta W_T}{\delta q_{ij}}\frac{\delta W_T}{\delta q_{kl}}\right]$$

$$H_i = 2q_{ij}\nabla_k\pi^{kj}$$

Supermetric: $G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda - 1}q_{ij}q_{kl}$

Deformation parameter λ
(=1 for DeWitt supermetric)

$W_T = W_{CS} + W_{EH\Lambda} = \frac{1}{4w^2}\int \tilde{\epsilon}^{ikj}(\Gamma^l_{im}\partial_j\Gamma^m_{kl} + \frac{2}{3}\Gamma^l_{im}\Gamma^m_{jn}\Gamma^n_{kl})\, d^3x + \frac{\mu}{2}\int \sqrt{q}(R - 2\Lambda_W)\, d^3x$. The Cotton tensor density can be expressed as $\tilde{C}^{ij} = w^2\frac{\delta W_{CS}}{\delta q_{ij}} = w^2\tilde{\epsilon}^{ikl}\nabla_k(R_l{}^j - \frac{1}{4}R\delta_l{}^j)$.

$$S = \int dt d^3\mathbf{x}\sqrt{g}N\Big\{\frac{2}{\kappa^2}K_{ij}G^{ijk\ell}K_{k\ell} - \frac{\kappa^2}{2}\Big[\frac{1}{w^2}C^{ij} - \frac{\mu}{2}\Big(R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda_W g^{ij}\Big)\Big] \times \mathcal{G}_{ijk\ell}\Big[\frac{1}{w^2}C^{k\ell} - \frac{\mu}{2}\Big(R^{k\ell} - \frac{1}{2}Rg^{k\ell} + \Lambda_W g^{k\ell}\Big)\Big]\Big\}.$$

$$S = \int dt d^3\mathbf{x}\sqrt{g}N\Big\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}C_{ij}C^{ij} + \frac{\kappa^2\mu}{2w^2}\varepsilon^{ijk}R_{i\ell}\nabla_j R^\ell_k - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\Big(\frac{1-4\lambda}{4}R^2 + \Lambda_W R - 3\Lambda_W^2\Big)\Big\}.$$

$$G = \frac{\kappa^2 c^3}{32\pi}$$

$$\Lambda = \tfrac{3}{2}\Lambda_W$$

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}.$$

*Short distance behavior: interacting fundamentally non-rel. gravitons

*Power-counting renormalizable in 3+1 dimensions.

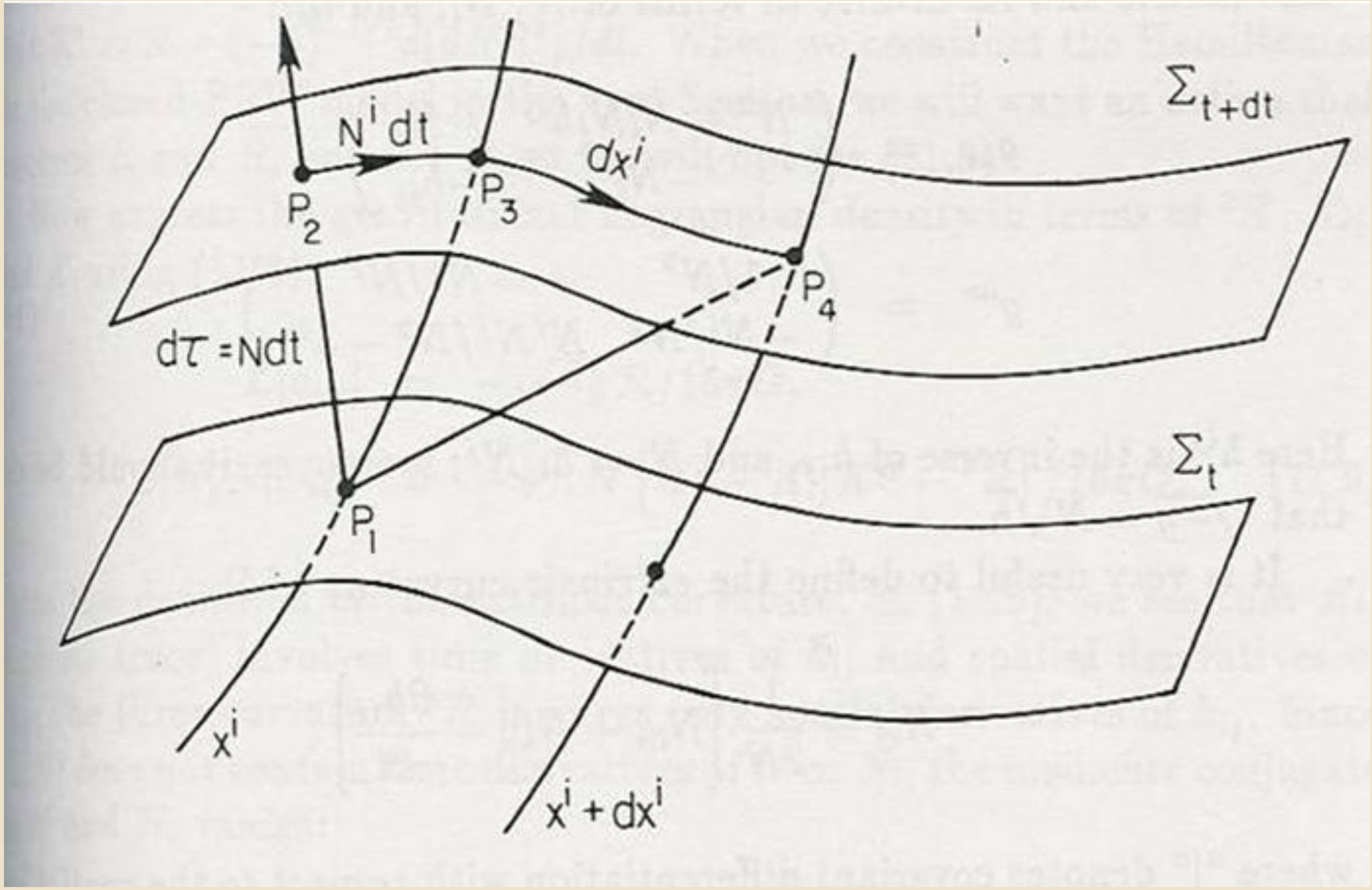
=> If successful as perturbative QFT:

coupling parameters obey renormalization group flow;

*c, G emerge from non-relativistic fundamental theory.

*Long distance behaviour: flows to Einstein's theory (hopefully(!)) $\lambda = 1$)

*4-dim. spacetime covariance recovered at low energies/curvatures.



$$ds^2 = -N^2(cdt)^2 + q_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

*Horava Gravity : *comes in (at least) 2 versions*

$$S = \int \pi^{ij} \dot{q}_{ij} \, d^3x \, dt - \int (NH + N^i H_i) \, d^3x \, dt.$$

1)"Projectable" (lapse function: N(t only))

⇒ Global (integrated) Hamiltonian constraint [$\int d^3x \, H(x)$] = 0

⇒ (Pathological) *extra d.o.f.

*2) *"Non-projectable" (lapse function N(t,x))

=>*Local constraint H(x) = 0

*Subdivision: with and without detailed balance

$$H = \frac{\kappa^2}{2} \frac{G_{ijkl}}{\sqrt{q}} \left[\pi^{ij} \pi^{kl} + \frac{\delta W_T}{\delta q_{ij}} \frac{\delta W_T}{\delta q_{kl}} \right]$$

$$H = \frac{\kappa^2}{2\sqrt{q}} \left(\tilde{\pi}_{ij} \tilde{\pi}^{ij} - \frac{\lambda}{3\lambda - 1} \tilde{\pi}^2 \right) + \sqrt{q} V(q)$$

Einstein's General Relativity :

$$G_{ijkl} = (1/2) g^{-1/2} (g_{ik} g_{jl} + g_{il} g_{kj} - g_{ij} g_{kl})$$

$$\lambda = 1$$

$$\mathcal{H} = 2\kappa G_{ijkl} \pi^{ij} \pi^{kl} - (2\kappa)^{-1} g^{1/2} (R - 2\Lambda)$$

$$\mathcal{H}_i = -2\pi_i{}^j{}_{/j}$$

Constraints obeys the *Dirac algebra* :

$$[\mathcal{H}_i(x), \mathcal{H}_j(x')] = \mathcal{H}_i(x')\delta_{,j}(x,x') + \mathcal{H}_j(x)\delta_{,i}(x,x')$$

$$[\mathcal{H}_i(x), \mathcal{H}(x')] = \mathcal{H}(x)\delta_{,i}(x,x')$$

$$[\mathcal{H}(x), \mathcal{H}(x')] = (g^{ij}(x)\mathcal{H}_i(x) + g^{ij}(x')\mathcal{H}_i(x'))\, \delta_{,j}(x,x')$$

*Hallmark of spacetime covariance, and of the embeddability of hypersurface deformations (Hojman-Kuchar-Teitelboim(Bunster))

* Departures from General Relativity e.g. Horava gravity:

Q: *What takes the place of the Dirac algebra ?

*Conversely, Dirac Algebra :

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')] = -\varepsilon[g^{rs}(x)\mathcal{H}_s(x) + g^{rs}(x')\mathcal{H}_s(x')]\delta_{,r}(x, x')$$

$$[\mathcal{H}_r(x), \mathcal{H}_\perp(x')] = \mathcal{H}_\perp(x)\delta_{,r}(x, x')$$

$$[\mathcal{H}_r(x), \mathcal{H}_s(x')] = \mathcal{H}_r(x')\delta_{,s}(x, x') + \mathcal{H}_s(x)\delta_{,r}(x, x')$$

with

$$\mathcal{H}_\perp^{\text{grav}} = \tfrac{1}{2}M_{ijkl}\pi^{ij}\pi^{kl} + V[g_{ij}]$$

=>

$$M_{ijkl} = 2\kappa g^{-1/2}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right)$$

(=> $\lambda = 1$!)

AND

$$V = \varepsilon(2\kappa)^{-1}g^{1/2}(R - 2\Lambda)$$

$$G_{ijkl} = \tfrac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$$

"*Geometrodynamics Regained" program:

S. Hojman, K. Kuchar and C. Teitelboim, Nature Phys. Sci. **245**, 97 (1973); Ann. Phys.**96**, 88 (1976).

C. Teitelboim, *The Hamiltonian structure of spacetime*, in General Relativity and Gravitation Vol. 1, edited by A. Held (Plenum, New York, 1980).

$\lambda \neq 1$

*Case 2a) $H = \frac{2\kappa'}{\sqrt{q}} G_{ijkl}\pi^{ij}\pi^{kl}$ BUT ultralocal theory (V=0)

$$\{\int NH\, d^3x, \int MH\, d^3y\}_{\text{P.B.}} = 0$$

=> Modification of Dirac algebra;

but arbitrary hypersurface deformations (N, **N**) still allowed

*Case 2b) $H = \left(\frac{2\kappa'}{\sqrt{q}} G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{q}}{2\kappa'} R\right)$

$$G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda-1} q_{ij}q_{kl}$$

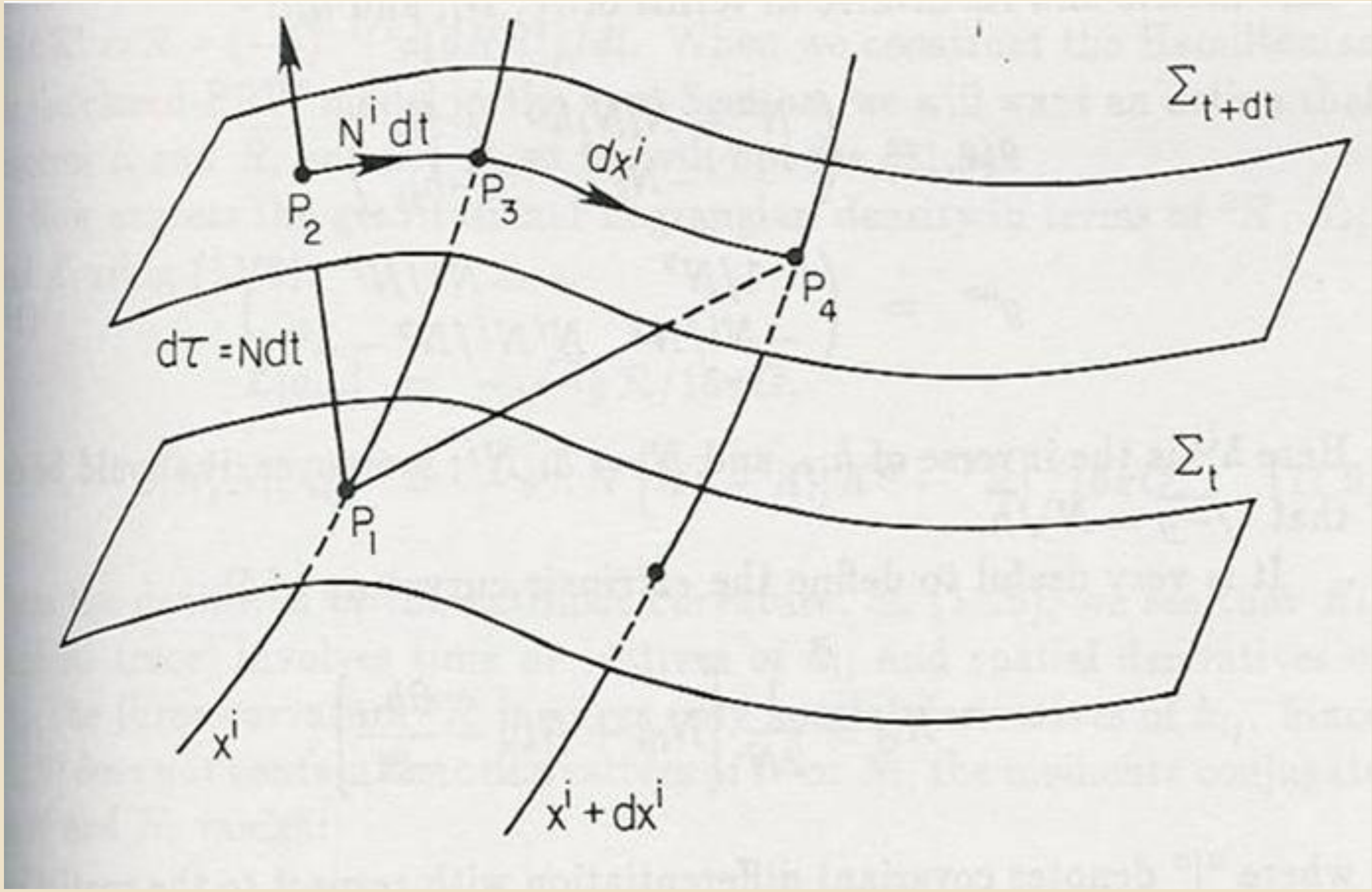
$$\{\int NH\, d^3x, \int MH\, d^3y\}_{\text{P.B.}}$$

$$= \int (N\nabla^i M - M\nabla^i N) H_i\, d^3x - \frac{2(1-\lambda)}{3\lambda-1} \int (N\nabla^i M - M\nabla^i N)\nabla_i \pi\, d^3x$$

Secondary constraint $Z_i := \nabla_i \pi = 0$

$$\{\int \xi^i Z_i\, d^3x, \int \chi^j Z_j\, d^3y\}_{\text{P.B.}} = \int \frac{3}{2}(\chi^i \nabla_j \xi^j - \xi^i \nabla_j \chi^j) Z_i\, d^3x$$

$$\{\int N^i H_i\, d^3x, \int \xi^i Z_i\, d^3y\}_{\text{P.B.}} = \int (\mathcal{L}_{\vec{N}} \xi^i) Z_i\, d^3x$$



$$ds^2 = -N^2(cdt)^2 + q_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

$$\{Z_i[\xi^i], Z_j[\chi^j]\} = Z_i\left[\frac{3}{2}(\chi^i \nabla_j \xi^j - \xi^i \nabla_j \chi^j)\right],$$

$$\{H_i[N^i], Z_i[\xi^i]\} = Z_i\left[\mathscr{L}_{\vec{N}} \xi^i\right],$$

$$\{Z_i[\xi^i], H[N]\} = Z_i\left[-\frac{2\kappa'}{(3\lambda-1)\sqrt{q}} N\pi\xi^i\right] - H\left[\frac{3}{2} N \nabla_i \xi^i\right]$$
$$-\frac{1}{\kappa'}\int d^3x \sqrt{q}\,(\nabla_j \xi^j) W, \qquad (5)$$

with $W := \left[-\nabla^2 + R + \frac{2\kappa'^2\pi^2}{(3\lambda-1)q}\right] N$. Thus $W = 0$ is required

$$Z_i := \nabla_i \pi = 0 \Leftrightarrow \pi = K(t)\sqrt{q}$$

$$R = \frac{4\kappa'^2}{q}\left(\overline{\pi}_{ij}\overline{\pi}^{ij} - \frac{1}{3(3\lambda-1)}\pi^2\right)$$ (from H=0)

$$\overline{\pi}^{ij} := \pi^{ij} - \frac{1}{3} q^{ij} \pi$$

$$W = \left[-\nabla^2 + \frac{4\kappa'^2}{q} \overline{\pi}_{ij} \overline{\pi}^{ij} + \frac{2\kappa'^2 K^2}{3(3\lambda - 1)} \right] N = 0.$$

Note: $-\nabla^2$ and $\frac{4\kappa'^2}{q}\overline{\pi}_{ij}\overline{\pi}^{ij}$ are both positive semi-definite

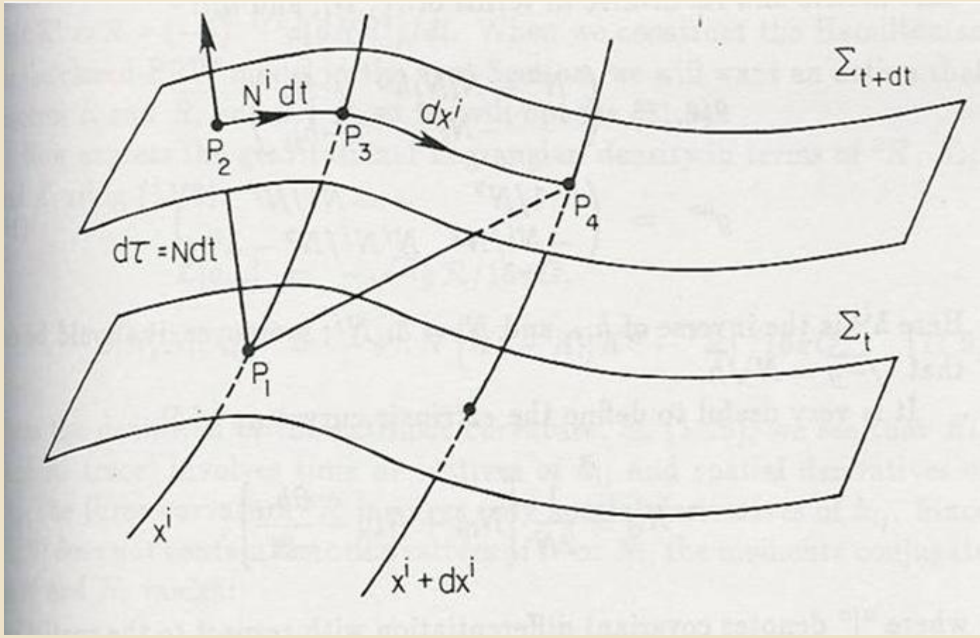
$W = 0$ can have non-vanishing solution [4] for N only if $\lambda < \frac{1}{3}$

For $\lambda > \frac{1}{3}$, we are lead to the fact $N = 0$

[4] N can be expanded in terms of eigenfunctions of the Hermitian operator $-\nabla^2 + \frac{4\kappa'^2}{q}\overline{\pi}_{ij}\overline{\pi}^{ij}$. Then N is non-trivial iff $-\frac{2\kappa'^2 K^2}{3(3\lambda-1)}$ coincides with at least one of its (positive semi-definite) eigenvalues. This can be achieved *only* for $\lambda < \frac{1}{3}$.

$\lambda < \frac{1}{3}$ restricted set of hypersurface deformations

with very specific N satisfying W=0



For $\lambda > \frac{1}{3}$, we are lead to the fact $N = 0$

And *degenerate* Arnowitt-Deser-Misner metric

$$ds^2 = -N^2(cdt)^2 + q_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

$$S = \int \pi^{ij} \dot{q}_{ij} \, d^3x \, dt - \int (N H + N^i H_i) \, d^3x \, dt$$

Case 2c) Horava Gravity:

$$H = \frac{\kappa^2}{2} \frac{G_{ijkl}}{\sqrt{q}} \left[\pi^{ij} \pi^{kl} + \frac{\delta W_{\mathrm{T}}}{\delta q_{ij}} \frac{\delta W_{\mathrm{T}}}{\delta q_{kl}} \right]$$

$$H_i = 2 q_{ij} \nabla_k \pi^{kj}$$

$$G_{ijkl} = \frac{1}{2}(q_{ik} q_{jl} + q_{il} q_{jk}) - \frac{\lambda}{3\lambda - 1} q_{ij} q_{kl}$$

$W_{\mathrm{T}} = W_{\mathrm{CS}} + W_{\mathrm{EH}\Lambda} = \frac{1}{4w^2} \int \tilde{\epsilon}^{ikj} (\Gamma^l_{im} \partial_j \Gamma^m_{kl} + \frac{2}{3} \Gamma^l_{im} \Gamma^m_{jn} \Gamma^n_{kl}) \, d^3x + \frac{\mu}{2} \int \sqrt{q}(R - 2\Lambda_W) \, d^3x$. The Cotton tensor density can be expressed as $\tilde{C}^{ij} = w^2 \frac{\delta W_{\mathrm{CS}}}{\delta q_{ij}} = w^2 \tilde{\epsilon}^{ikl} \nabla_k (R_l{}^j - \frac{1}{4} R \delta_l{}^j)$.

*Neither H nor is G_{ijkl} is of the form in "geometrodynamics regained"

*Non-Projectable Horava gravity with local super-hamiltonian constraint

Inconsistencies in the canonical formulation:

M. Li and Y. Pang, "A Trouble with Hořava-Lifshitz Gravity," JHEP **0908**, 015 (2009) [arXiv:0905.2751 [hep-th]].

"Troubles" in the constraint algebra of Horava Gravity:

$$\{\int d^3\mathrm{x}\zeta_1^i\mathcal{H}_i,\ \int d^3\mathrm{y}\zeta_2^j\mathcal{H}_j\}_{\mathrm{Pb}} = \int d^3\mathrm{x}(\zeta_1^i\partial_i\zeta_2^k - \zeta_2^i\partial_i\zeta_1^k)\mathcal{H}_k,$$

$$\{\int d^3\mathrm{x}\zeta^i\mathcal{H}_i,\ \int d^3\mathrm{y}\eta\mathcal{H}\}_{\mathrm{Pb}} = \int d^3\mathrm{x}\zeta^i\partial_i\eta\mathcal{H},$$

$$\{\mathcal{H}(\mathrm{x}),\ \int d^3\mathrm{y}\eta\mathcal{H}\}_{\mathrm{Pb}} = \underbrace{-2\sqrt{g}\frac{1}{k_W^4}(\alpha^{ijk}\nabla_k\nabla_j\nabla_i\eta + \beta^{ij}\nabla_j\nabla_i\eta + \gamma^i\nabla_i\eta + \omega\eta)}_{=:\ \Delta\eta}$$

Stability of local constraint under evolution

$$\triangle = -2\sqrt{g}\frac{1}{k_W^4}(\alpha^{ijk}\nabla_k\nabla_j\nabla_i + \beta^{ij}\nabla_j\nabla_i + \gamma^i\nabla_i + \omega)$$

$$\{\mathcal{H}(\mathrm{x}), \int d^3\mathrm{y}\eta\mathcal{H}\}_{\mathrm{Pb}} = -2\sqrt{g}\frac{1}{k_W^4}\underbrace{(\alpha^{ijk}\nabla_k\nabla_j\nabla_i\eta + \beta^{ij}\nabla_j\nabla_i\eta + \gamma^i\nabla_i\eta + \omega\eta)}_{=:\ \Delta\eta}$$

$$\alpha^{ijk} = (\widetilde{C}^{ilm}g^{jk} + \widetilde{C}^{klm}g^{ij} - \widetilde{C}^{ilk}g^{jm} - \widetilde{C}^{kli}g^{jm})K_{lm},$$

where $\widetilde{C}^{ijk}$ is defined as $\epsilon^{ijl}C_l{}^k$, in which $C_l{}^k = g_{lm}C^{mk}$

$$\beta^{ij}\nabla_j\nabla_i\eta = \nabla_{(j}\nabla_i\eta\nabla_{k)_c}(K_{lm}\widetilde{C}^{ilm}g^{jk} - K_{lm}\widetilde{C}^{ilk}g^{jm})$$

$$\begin{aligned}\gamma^i\nabla_i\eta = {} & t^{mlkji}\nabla_{(i}\eta\nabla_m\nabla_{k)_c}K_{jl} + K_{jl}\nabla_{(i}\eta\nabla_k\nabla_{m)_c}t^{mlkji} \\ & +2s^{lmijk}\nabla_i\eta\nabla_{[l}\nabla_{k]}K_{jm} + 2K_{jm}\nabla_i\eta\nabla_{[k}\nabla_{l]}s^{lmijk} \\ & +2(\widetilde{C}^{klj}R_l^i + \widetilde{C}^{lki}R_l^j + \widetilde{C}^{lij}R_l^k)K_{jk}\nabla_i\eta \\ & +(\frac{1}{2}R^i_{jkl}\widetilde{C}^{klj} - R^i_{jkl}\widetilde{C}^{jkl})K\nabla_i\eta\end{aligned}$$

$$\begin{aligned}\omega = {} & \nabla_i(\widetilde{C}^{jkl}R_k^iK_{jl} + \widetilde{C}^{jik}R_j^lK_{kl} + \widetilde{C}^{kji}R_k^lK_{jl}) \\ & +\widetilde{C}^{ijk}(\nabla_i\nabla_l\nabla_kK^l{}_j + \nabla_i\nabla_l\nabla_jK^l{}_k - \nabla_i\nabla^l\nabla_lK_{jk} - \nabla_i\nabla_k\nabla_jK) \\ & +(K^l{}_j\nabla_k\nabla_l\nabla_i + K^l{}_k\nabla_j\nabla_l\nabla_i - K_{jk}\nabla^l\nabla_l\nabla_i - K\nabla_j\nabla_k\nabla_i)\widetilde{C}^{ijk}.\end{aligned}$$

M. Henneaux, A. Kleinschmidt and G. L. Gómez, "A dynamical inconsistency of Hořava gravity," Phys. Rev. D 81, 064002 (2010) [arXiv:0912.0399 [hep-th]].

For Horava gravity with local Hamiltonian constraint :
Only consistent solution for stability of constraint under evolution is
$N = 0$

***Dirac algorithm resulting in $N = 0$ =>
*theory inconsistent ?
*suggests H constraint generates on-shell trivial time-reparametrization invariance ?

Note: Only spatial diffeomorphisms are physically relevant gauge symmetries of the theory.

***Consistent Canonical Formulation** (C. S., H.L. Yu, Jinsong Yang (PLB2011))

*Horava's "intended" theory:

$$S = \int dt \int d^3x\, [\tilde{\pi}^{ij}\dot{q}_{ij} + 2N_j \nabla_i \tilde{\pi}^{ij}] - \int dt \boxed{\int d^3x\, N H}$$

REPLACE by *Master Constraint Version:

$$S = \int dt \int d^3x\, [\tilde{\pi}^{ij}\dot{q}_{ij} + 2N_j \nabla_i \tilde{\pi}^{ij}] - \int dt \underbrace{N(t)}_{\varepsilon_0} \boxed{\int d^3x \frac{H^2(x)}{\sqrt{q}}}$$

=: M

$$H = \frac{\kappa^2}{2} \frac{G_{ijkl}}{\sqrt{q}} [\pi^{ij}\pi^{kl} + \frac{\delta W_{\mathrm{T}}}{\delta q_{ij}} \frac{\delta W_{\mathrm{T}}}{\delta q_{kl}}]$$

$$G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda - 1} q_{ij}q_{kl}$$

$W_{\mathrm{T}} = W_{\mathrm{CS}} + W_{\mathrm{EH}\Lambda} = \frac{1}{4w^2} \int \tilde{\epsilon}^{ikj} (\Gamma^l_{im}\partial_j \Gamma^m_{kl} + \frac{2}{3}\Gamma^l_{im}\Gamma^m_{jn}\Gamma^n_{kl})\, d^3x + \frac{\mu}{2} \int \sqrt{q}(R - 2\Lambda_W)\, d^3x$. The Cotton tensor density can be expressed as $\tilde{C}^{ij} = w^2 \frac{\delta W_{\mathrm{CS}}}{\delta q_{ij}} = w^2 \tilde{\epsilon}^{ikl} \nabla_k (R_l{}^j - \frac{1}{4} R \delta_l{}^j)$.

Dirac Algebra

$$\{H_i[N^i], H_j[M^j]\}_{P.B.} = \quad H_i[(\mathcal{L}_{\vec{N}} M)^i]$$
$$\{H_i[N^i], H[M]\}_{P.B.} = \quad H[(\mathcal{L}_{\vec{N}} M]$$
$$\{H[N], H[M]\}_{P.B.} = \quad H_i[(q^{ij}(N\partial_j M - M\partial_j N)]$$

*Structure FUNCTIONS (not infinite dim. Lie Algebra)
*Spatial diffeo. forms subgroup but not ideal.
*Cannot solve constraint in 3-dim. diffeo. invariant subspace (superspace) (H cannot be defined directly therein).

Recently, the master constraint programme for loop quantum gravity (LQG) was proposed as a classically equivalent way to impose the infinite number of Wheeler–DeWitt constraint equations in terms of a single master equation.

T. Thiemann, The Phoenix Project: master constraint programme for loop quantum gravity, Class. Quantum Grav. **23** (2006) 2211.

"M-Theory": Master Constraint Program

Master Constraint Algebra:

$$\{\vec{H}(\vec{N}), \vec{H}(\vec{N}')\} = \vec{H}(\mathscr{L}_{\vec{N}} N')$$
$$\{\vec{H}(\vec{N}), \mathbf{M}\} = 0$$
$$\{\mathbf{M}, \mathbf{M}\} = 0.$$

$$\mathbf{M} := \int_\Sigma \mathrm{d}^3x \frac{[H(x)]^2}{\sqrt{q(x)}}.$$

$\mathbf{M} = 0$ is then equivalent to $H(x) = 0, \forall x \in \Sigma$.

*1st Class Constraints with structure constants

Tested with: finite-dimensional Abelian & non-Abelian algebras
with structure constants & also structure functions,
with contraints polynomial and non-polynomial in momenta,
with electrodynamics and Gauss Law, non-abelian gauge theories,
Free field QFT and interacting theories, linearized gravity.

References

[1] Thiemann T 2003 The Phoenix Project: master constraint programme for loop quantum gravity *Preprint* gr-qc/0305080

[2] Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: I. General framework *Class. Quantum Grav.* **23** 1025–65 (*Preprint* gr-qc/0411138)

[3] Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: II. Finite dimensional systems *Class. Quantum Grav.* **23** 1067–88 (*Preprint* gr-qc/0411139)

[4] Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: III. $SL(2,\mathbb{R})$ models *Class. Quantum Grav.* **23** 1089–120 (*Preprint* gr-qc/0411140)

[5] Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: V. Interacting field theories *Class. Quantum Grav.* **23** 1143–62 (*Preprint* gr-qc/0411142)

**c.f. Einstein's theory

On-shell (modulo constraints +EOM),

constraints do generate 4-d diffeomorphisms

Eventhough Dirac algebra is NOT algebra of 4d diffeomorphisms

$$\delta_{\vec{N}} q_{ab} = \{H_i[N^i], q_{ab}\}_{P.B.} = \mathcal{L}_{\vec{N}} q_{ab}$$

$$\delta_N q_{ab} = \{H[N], q_{ab}\}_{P.B.} = N q^{-\frac{1}{2}}(\pi q_{ab} - \pi_{ab})$$

$$[\text{modulo EOM}] = 2N K_{ab} = \mathcal{L}_{N\vec{n}} q_{ab}$$

$$\delta_{\vec{N}} \pi^{ab} = \{H_i[N^i], \pi^{ab}\}_{P.B.} = \mathcal{L}_{\vec{N}} \pi^{ab}$$

$$\delta_N \pi^{ab} = \{H[N], \pi^{ab}\}_{P.B.}$$

$$= q^{ab} \frac{N}{2} H - N\sqrt{q}(q^{ca}q^{db} - q^{cd}q^{ab}) R^{(4)}_{cd} + \mathcal{L}_{N\vec{n}} \pi^{ab}$$

As structure *functions* are present in the commutator of two Hamiltonian constraints, the Dirac algebra is not the Lie algebra of 4-dim. diffeomorphisms. But H_i and H constraints do generate 4-dimensional diffeomorphisms on-shell(modulo the constraints). In a theory which possesses at the fundamental level only 3-dim. diffeomorphisms as gauge symmetry, we expect the constraints to generate, *on-shell, only spatial diffeomorphisms*. There

*Horava Gravity : explicit realization (representation) of the Master constraint algebra.

*Horava Gravity can't seem to be consistently formulated as a canonical theory otherwise.

$$\{q_{ij}, N(t)\frac{\mathrm{M}}{\epsilon_o} + \int N^k H_k\, d^3x\}|_{\mathrm{M}=0 \Leftrightarrow H=0} \approx$$

$$\{q_{ij}, \int N^k H_k\, d^3x\}_{\mathrm{P.B.}} = \mathcal{L}_{\vec{N}} q_{ij} \text{ (and similarly for } \pi^{ij})$$

Observables O:

$$\{O, \frac{N(t)}{\epsilon_0}\mathrm{M} + \int N^i H_i\, d^3x\}|_{\mathrm{M}=0 \Leftrightarrow H=0} \approx$$

$$\{O, \int N^i H_i\, d^3x\}_{\mathrm{P.B.}} = 0.$$

**Explicitly/concretely realizes on-shell trivial time reparametrization generated by H (and choice of N is on-shell trivial);
physically relevant symmetry is 3-d (spatial) diffeomorphism invariance

***Pecularities of the "detailed balance" condition:

$$H = \frac{\kappa^2}{2}\frac{G_{ijkl}}{\sqrt{q}}\left[\pi^{ij}\pi^{kl} + \frac{\delta W_T}{\delta q_{ij}}\frac{\delta W_T}{\delta q_{kl}}\right]$$

$$G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda - 1}q_{ij}q_{kl}$$

$W_T = W_{CS} + W_{EH\Lambda} = \frac{1}{4w^2}\int \tilde{\epsilon}^{ikj}(\Gamma^l_{im}\partial_j\Gamma^m_{kl} + \frac{2}{3}\Gamma^l_{im}\Gamma^m_{jn}\Gamma^n_{kl})\, d^3x + \frac{\mu}{2}\int \sqrt{q}(R - 2\Lambda_W)\, d^3x$. The Cotton tensor density can be expressed as $\tilde{C}^{ij} = w^2 \frac{\delta W_{CS}}{\delta q_{ij}} = w^2 \tilde{\epsilon}^{ikl}\nabla_k(R_l{}^j - \frac{1}{4}R\delta_l{}^j)$.

$$H = \frac{\kappa^2 G_{ijkl}}{2\sqrt{q}}(\pi^{ij} + i\frac{\delta W_T}{\delta q_{ij}})(\pi^{kl} - i\frac{\delta W_T}{\delta q_{kl}})$$
$$=: \frac{\kappa^2}{2\sqrt{q}} G_{ijkl} Q^{ij}_+ Q^{kl}_- = 0;$$

$$\hat{Q}^{ij}_{\pm} := \hat{\pi}^{ij} \pm i\frac{\delta W_T}{\delta q_{ij}} = e^{\pm \frac{W_T}{\hbar}}\hat{\pi}^{ij} e^{\mp \frac{W_T}{\hbar}}$$

Metric on Superspace:

$$\delta S^2 \equiv G^{ijkl}\delta q_{ij}\delta q_{kl}$$ signature $(\mathrm{sgn}[\frac{1}{3} - \lambda], +, +, +, +, +)$

an "intrinsic time" for $\lambda > \frac{1}{3}$

Emergent speed of light, Newton's constant, cosmological constant:

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}} \qquad G = \frac{\kappa^2 c^3}{32\pi} \qquad \Lambda = \frac{3}{2}\Lambda_W$$

for $\lambda > \frac{1}{3}$ ALL POSITIVE (required phenomenologically)

only if μ is pure imaginary and κ is real. Then H and the action is real only if w^2 is pure imaginary. This set of values renders W_{T} to be pure imaginary, and thus $Q_{\pm}$ become individually hermitian.