Crossing of Phantom Divide in $F(R)$ Gravity

Reference:

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I. Introduction

- Current cosmic acceleration
- $F(R)$ gravity
- Crossing of the phantom divide

II. Reconstruction of a $F(R)$ gravity model with realizing the crossing of the phantom divide

III. Summary

* We use the ordinary metric formalism, in which the connection is written by the differentiation of the metric.
I. Introduction

- Recent observations of Supernova (SN) Ia confirmed that the current expansion of the universe is accelerating.

  [Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999)]
  [Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998)]
  [Astier et al. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

- There are two approaches to explain the current cosmic acceleration. [Copeland, Sami and Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006)]

< Gravitational field equation >

\[ G_{\mu \nu} = \kappa^2 T_{\mu \nu} \]

- **Gravity**
- **Matter**

(1) General relativistic approach \( \rightarrow \) Dark Energy

(2) Extension of gravitational theory
(1) General relativistic approach

- Cosmological constant

- Scalar field : X matter, Quintessence

  [Caldwell, Dave and Steinhardt, Phys. Rev. Lett. 80, 1582 (1998)]
  Cf. Pioneering work: [Fujii, Phys. Rev. D 26, 2580 (1982)]

Phantom  
Wrong sign kinetic term

  [Caldwell, Phys. Lett. B 545, 23 (2002)]

K-essence  
Non canonical kinetic term

  [Chiba, Okabe and Yamaguchi, Phys. Rev. D 62, 023511 (2000)]
  [Armendariz-Picon, Mukhanov and Steinhardt, Phys. Rev. Lett. 85, 4438 (2000)]

Tachyon  
String theories

  [Padmanabhan, Phys. Rev. D 66, 021301 (2002)] \( A > 0 \) : Constant

- Chaplygin gas  
\( \rho = - \frac{A}{\rho} \)


\( \rho \) : Energy density
\( p \) : Pressure
(2) Extension of gravitational theory

- **$F(R)$ gravity**
  
  $F(R)$: Arbitrary function of the Ricci scalar $R$
  
  [Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]
  [Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]
  \[ f_i(\phi) \] : Arbitrary function of a scalar field $\phi$

- **Scalar-tensor theories**
  
  \[ f_1(\phi)R \]  
  \[ f_2(\phi)G \]
  
  [Boisseau, Esposito-Farese, Polarski and Starobinsky, Phys. Rev. Lett. 85, 2236 (2000)]
  [Gannouji, Polarski, Ranquet and Starobinsky, JCAP 0609, 016 (2006)]

- **Ghost condensates**
  
  [Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405, 074 (2004)]

- **Higher-order curvature term**
  
  Gauss-Bonnet term with a coupling to a scalar field: $f_2(\phi)G$
  
  \[ G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \]
  
  [Nojiri, Odintsov and Sasaki, Phys. Rev. D 71, 123509 (2005)]

- **DGP braneworld scenario**
  
  [Dvali, Gabadadze and Porrati, Phys. Lett B 485, 208 (2000)]
  [Deffayet, Dvali and Gabadadze, Phys. Rev. D 65, 044023 (2002)]
< Flat Friedmann-Robertson-Walker (FRW) space-time >

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)dx^2 \]

\( a(t) \): Scale factor

< Equation for \( a(t) \) with a perfect fluid >

\[ \frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho + 3p) \]

\( \rho \): Energy density, \( p \): Pressure

\[ \dot{\rho} + 3H(1 + w)\rho = 0 \]

\( \ddot{a} > 0 \): Accelerated expansion

< Equation of state (EoS) \( w \) >

\[ w \equiv \frac{p}{\rho} \]

Condition for accelerated expansion:

\[ w < -\frac{1}{3} \]

Cf. Cosmological constant \( \iff w = -1 \)

- Continuity equation

\[ \dot{\rho} + 3H(1 + w)\rho = 0 \]

\( H = \dot{a}/a \): Hubble parameter

\[ a \propto t^{\frac{2}{3(1+w)}} \]

\[ \rho \propto a^{-3(1+w)} \]
5-year WMAP data on the current value of $w$


$$\Omega_\Lambda \equiv \frac{\kappa^2 \rho_\Lambda^{(0)}}{3H_0^2} = \frac{\rho_\Lambda^{(0)}}{\rho_c^{(0)}}$$

- $\rho_c^{(0)}$: Critical density
- $\rho_\Lambda^{(0)}$: Current energy density of dark energy

HST: Hubble Space Telescope Key Project

- Baryon acoustic oscillation (BAO): Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

For the flat universe, constant $w$: (From WMAP+BAO+SN)

$-0.14 < 1 + w < 0.12$ (95% CL)

Cf. $\Omega_\Lambda = 0.726 \pm 0.015$ (68% CL)
For the flat universe, a variable EoS:

\[ -0.33 < 1 + w_0 < 0.21 \]  
(95\% CL)

\[ w^\prime \equiv \frac{dw}{dz} \bigg|_{z=0} \]

\[ w(a) = \frac{a\tilde{w}(a)}{a + a_{trans}} - \frac{a_{trans}}{a + a_{trans}} \]

\[ \tilde{w}(a) = \tilde{w}_0 + (1 - a)\tilde{w}_a \]

\[ w_0 = w(a = 1) \]

\[ z > z_{trans} : \]
\[ w(z) \text{ approaches to } -1. \]
<Canonical scalar field>

\[ S_\phi = \int d^4 x \sqrt{-g} \left[ - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \]

\[ g = \text{det}(g_{\mu \nu}) \]

\[ \phi : \text{Scalar field} \]

\[ V(\phi) : \text{Potential of } \phi \]

\[ \cdot \text{ For a homogeneous scalar field } \phi = \phi(t) : \]

\[ \rightarrow \rho = \frac{1}{2} \ddot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \ddot{\phi}^2 - V(\phi) \]

\[ \Rightarrow \quad w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\ddot{\phi}^2 - 2V(\phi)}{\ddot{\phi}^2 + 2V(\phi)} \]

\[ \text{If } \ddot{\phi}^2 \ll V(\phi), \quad w_\phi \approx -1. \]

\[ \rightarrow \quad \text{Accelerated expansion can be realized.} \]
\[ \langle F(R) \text{ gravity} \rangle \]
\[ S = \int d^4 x \sqrt{-g} \frac{F(R)}{2\kappa^2} \]

\[ F(R) = R : \text{General Relativity} \]


[Capozziello and Francaviglia, Gen. Rel. Grav. 40, 357 (2008)]

[Sotiriou and Faraoni, arXiv:0805.1726 [gr-qc]]

\[ F(R) = R \]

\[ g = \det(g_{\mu\nu}) \]

• Example : \[ F(R) \propto R^n \ (n \neq 1) \]

\[ \rightarrow \quad a \propto t^q, \quad q = \frac{-2n^2+3n-1}{n-2} \]

\[ w_{\text{eff}} = -\frac{6n^2-7n-1}{6n^2-9n+3} \]

If \( q > 1 \), accelerated expansion can be realized.

(For \( n = 3/2 \) or \( n = -1 \), \( q = 2 \) and \( w_{\text{eff}} = -2/3 \).)
<Conditions for the viability of $F(R)$ gravity>

(1) $F'(R) > 0$

- Positivity of the effective gravitational coupling
  
  $G_{\text{eff}} = G_0/F'(R) > 0$  \quad $G_0$ : Gravitational constant

  (The graviton is not a ghost.)

(2) $F''(R) > 0$


- Stability condition: $M^2 \approx 1/(3F''(R)) > 0$

  $M$ : Mass of a new scalar degree of freedom (called the "scalaron") in the weak-field regime.

  (The scalaron is not a tachyon.)

(3) $F(R) \rightarrow R - 2\Lambda$  \quad for  \quad $R \gg R_0$

  $R_0$ : Current curvature

  $\Lambda$ : Cosmological constant

- Realization of the $\Lambda$ CDM-like behavior in the large curvature regime

  $\uparrow$ Standard cosmology [ $\Lambda + $ Cold dark matter (CDM)]
(4) Solar system constraints

\[ F(R) \text{ gravity} \quad \leftrightarrow \quad \text{Brans-Dicke theory with } \omega_{BD} = 0 \]

\[ \omega_{BD} : \text{Brans-Dicke parameter} \]

\[ \omega_{BD} > 40000 \]

- However, if the mass of the scalar degree of freedom \( M \) is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.

- \( M = M(R) \leftarrow \) Scale-dependence: “Chameleon mechanism”

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.
(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

- Combing local gravity constraints, it is shown that

\[ m \equiv \frac{RF''(R)}{F'(R)} \] has to be several orders of magnitude smaller than unity.

\[ m \] quantifies the deviation from the \( \Lambda \) CDM model.

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

(6) Stability of the de Sitter space

\[
\left( \frac{F'_d}{F_d} \right)^2 - 2F'_d \frac{F''_d}{F'_d} > 0
\]

\[ F_d = F(R_d) \]

\[ R_d : \text{Constant curvature in the de Sitter space} \]

- Linear stability of the inhomogeneous perturbations in the de Sitter space


\[ R_d = \frac{2F_d}{F'_d} \iff m < 1 \]
(7) Free of curvature singularities


• Existence of relativistic stars


Models of $F(R)$ gravity

(a) Hu-Sawicki model  

$$F_{HS}(R) = R - \frac{\bar{M}^2 c_2 (R/\bar{M}^2)^p}{c_3 (R/\bar{M}^2)^p + 1}$$

$c_2, c_3$: Constants  
$ar{M}$: Mass scale  
$p > 0$: Constant

(b) Starobinsky’s model  

$$F_S(R) = R + \lambda \Lambda_0 \left[ \left( 1 + \frac{R^2}{\Lambda_0^2} \right)^{-n} - 1 \right]$$

$\lambda > 0$, $n > 0$: Constants  
$\Lambda_0$: Current cosmological constant

(c) Appleby-Battye model  

$$F_{AB}(R) = \frac{R}{2} + \frac{1}{2a} \log \left[ \cosh(aR) - \tanh(b) \sinh(aR) \right]$$

$a > 0$, $b$: Constants

(d) Tsujikawa’s model  

$$F_T(R) = R - \mu R_c \tanh \left( \frac{R}{R_c} \right)$$

$R_c > 0$: Constants
Regarding the condition (7), under debate.
[Babichev and Langlois, arXiv:0904.1382 [gr-qc]]
[Upadhye and Hu, Phys. Rev. D 80, 064002 (2009)]

\[ F_{MJWQ}(R) = R - \alpha R_* \ln \left( 1 + \frac{R}{R_*} \right) \]

\( \alpha > 0, \ R_* > 0 : \) Constants

[Miranda, Joras, Waga and Quartin, Phys. Rev. Lett. 102, 221101 (2009)]

Cf. [de la Cruz-Dombriz, Dobado and Maroto, Phys. Rev. Lett. 103, 179001 (2009)]
Various observational data (SN, Cosmic microwave background radiation (CMB), BAO) imply that the effective EoS of dark energy $w_{DE}$ may evolve from larger than -1 (non-phantom phase) to less than -1 (phantom phase). Namely, it crosses -1 (the crossing of the phantom divide).

[Alam, Sahni and Starobinsky, JCAP 0406, 008 (2004)]
[Nesseris and Perivolaropoulos, JCAP 0701, 018 (2007)]
[Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)]

(i) $w_{DE} > -1$

Non-phantom phase

(ii) $w_{DE} = -1$

Crossing of the phantom divide

(iii) $w_{DE} < -1$

Phantom phase

$t_c$: Time of the crossing of the phantom divide
< Models to account for the crossing of the phantom divide >

• Two scalar field model, e.g., Quintom
  Canonical scalar field + phantom
  

• Scalar-tensor theories
  
  [Perivolaropoulos, JCAP 0510, 001 (2005)]

• Single scalar field model with nonlinear kinetic terms
  
  [Vikman, Phys. Rev. D 71, 023515 (2005)]

• String-inspired models
  

We reconstruct an explicit model of $F(R)$ gravity with realizing the crossing of the phantom divide.

< Preceding work >


$$F(R) = \left( R^{1/c} - \Lambda \right)^c$$

$c$, $\Lambda$ : Constants

- Example: $c = 1.8$

< Recent work >


- It has been shown that in the Hu-Sawicki model, the transition from the phantom phase to the non-phantom one can also occur.
II. Reconstruction of a $F(R)$ gravity model with realizing the crossing of the phantom divide

< II A. Reconstruction method >

< Action >

$$S = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right]$$

By introducing proper functions $P(\phi)$ and $Q(\phi)$, we find

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ P(\phi)R + Q(\phi) \right] + \mathcal{L}_{\text{matter}} \right\}$$

- Equation of motion for $\phi$: $0 = \frac{dP(\phi)}{d\phi} R + \frac{dQ(\phi)}{d\phi} \rightarrow \phi = \phi(R)$

$\Longrightarrow F(R) = P(\phi(R)) R + Q(\phi(R))$

< Gravitational field equation >

$$\frac{1}{2} g_{\mu\nu} \left[ P(\phi) R + Q(\phi) \right] - R_{\mu\nu} P(\phi) - g_{\mu\nu} \Box P(\phi) + \nabla_\mu \nabla_\nu P(\phi) + \kappa^2 T^{(\text{matter})}_{\mu\nu} = 0$$

$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ : Covariant d'Alembertian

$\nabla_\mu$ : Covariant derivative operator

$T^{(\text{matter})}_{\mu\nu}$ : Energy-momentum tensor of matter
Gravitational field equations in the flat FRW background:

- $(\mu, \nu) = (0, 0)$ component

\[ -6H^2P(\phi(t)) - Q(\phi(t)) - 6H \frac{dP(\phi(t))}{dt} + 2\kappa^2 \rho = 0 \]

- Trace part of $(\mu, \nu) = (i, j)$ $(i, j = 1, \cdots, 3)$ components

\[ 2 \frac{d^2P(\phi(t))}{dt^2} + 4H \frac{dP(\phi(t))}{dt} + \left(4\dot{H} + 6H^2\right) P(\phi(t)) + Q(\phi(t)) + 2\kappa^2 p = 0 \]

$\rho$ and $p$ are the sum of the energy density and pressure of matters with a constant EoS parameter $w_i$, respectively, where $i$ denotes some component of the matters.

\[ \frac{d^2P(\phi(t))}{dt^2} - H \frac{dP(\phi(t))}{dt} + 2\dot{H} P(\phi(t)) + \kappa^2 (\rho + p) = 0 \]

- $\phi$ may be taken as $\phi = t$ because $\phi$ can be redefined properly.

**< Scale factor >**

\[ a(t) = \tilde{a} \exp \left(\tilde{g}(t)\right) \quad \tilde{a} : \text{Constant, } \tilde{g}(t) : \text{Proper function} \]
\[ \frac{d^2 P(\phi)}{d\phi^2} - \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} + 2\frac{d^2 \tilde{g}(\phi)}{d\phi^2} P(\phi) \]

\[ + \kappa^2 \sum_i (1 + w_i) \bar{\rho}_i \bar{a}^{-3(1+w_i)} \exp [-3 (1 + w_i) \tilde{g}(\phi)] = 0 \]

\[ \bar{\rho}_i : \text{Constant}, \quad H = \frac{d\tilde{g}(\phi)}{d\phi} \]

\[ \rightarrow Q(\phi) = -6 \left[ \frac{d\tilde{g}(\phi)}{d\phi} \right]^2 P(\phi) - 6 \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} \]

\[ + 2\kappa^2 \sum_i \bar{\rho}_i \bar{a}^{-3(1+w_i)} \exp [-3 (1 + w_i) \tilde{g}(\phi)] \]

\[ \rightarrow \quad \text{We derive the solutions of } P(\phi) \text{ and } Q(\phi). \]
< II B. Explicit model >

< Solution without matter >

\[ P(\phi) = e^{\tilde{g}(\phi)/2}\tilde{p}(\phi), \quad \tilde{g}(\phi) = -10 \ln \left[ \left( \frac{\phi}{t_0} \right)^{-\gamma} - C \left( \frac{\phi}{t_0} \right)^{\gamma+1} \right] \]

\[ \tilde{p}(\phi) = \tilde{p}_+\phi^\beta+ + \tilde{p}_-\phi^\beta- \]

\[ \beta_\pm = \frac{1 \pm \sqrt{1 + 100\gamma(\gamma + 1)}}{2} \]

\[ \gamma > 0, \quad C > 0 : \text{Constants} \]

\[ t_0 : \text{Present time} \]

\[ \tilde{p}_\pm : \text{Arbitrary constants} \]

- When \( \phi = t_s \equiv t_0C^{-1/(2\gamma+1)} \), \( \tilde{g}(t) \) diverges.

\[ \Rightarrow \quad a(t) = \bar{a} \exp(\tilde{g}(t)) \rightarrow \infty : \text{Big Rip singularity} \]

- We take \( \phi = t \) and only consider the period \( 0 < t < t_s \).
\(< \text{Effective EoS} >\)

- \(\rho_{\text{eff}} = \frac{3H^2}{\kappa^2}, \text{ } p_{\text{eff}} = -\left(2\dot{H} + 3H^2\right) / \kappa^2\)

\(\mathcal{W}_{\text{eff}} \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} \equiv -1 - \frac{2\dot{H}}{3H^2}\)

Cf. \(w_{\text{DE}} \approx \frac{w_{\text{eff}}}{1 - (F'(R)/F'(R_0))\Omega_m}\) \approx w_{\text{eff}}

(i) \(\dot{H} < 0 \iff w_{\text{eff}} > -1\) \hspace{1cm} \text{Non-phantom phase}

(ii) \(\dot{H} = 0 \iff w_{\text{eff}} = -1\) \hspace{1cm} \text{Crossing of the phantom divide}

(iii) \(\dot{H} > 0 \iff w_{\text{eff}} < -1\) \hspace{1cm} \text{Phantom phase}

\(< \text{Hubble rate} >\)

\[H = \left(\frac{10}{t}\right) \left[\frac{\gamma + (\gamma + 1) \left(\frac{t}{t_s}\right)^{2\gamma+1}}{1 - \left(\frac{t}{t_s}\right)^{2\gamma+1}}\right]\]
\[
< \text{Evolution of } \mathcal{W}_{\text{eff}} > \rightarrow \mathcal{W}_{\text{eff}} = -1 + U(t)
\]

\[
U(t) \equiv -\frac{2\dot{H}}{3H^2} = -\frac{-\gamma + 4\gamma (\gamma + 1) \left(\frac{t}{t_s}\right)^{2\gamma+1} + (\gamma + 1) \left(\frac{t}{t_s}\right)^{2(2\gamma+1)}}{15 \left[\gamma + (\gamma + 1) \left(\frac{t}{t_s}\right)^{2\gamma+1}\right]^2}
\]

- \( t < t_c \) : \( U(t) > 0 \) \( U(t) \) decreases monotonously.
- \( \frac{dU(t)}{dt} < 0 \)
  \( \rightarrow \) It evolves from positive to negative.

\( t_c \) : Time of the crossing of the phantom divide

\( t_c = t_s \left(-2\gamma + \sqrt{4\gamma^2 + \frac{\gamma}{\gamma + 1}}\right)^{1/(2\gamma+1)} \)

\( \mathcal{W}_{\text{eff}} \) crosses \(-1\).
< Forms of $P$ and $Q$ >

\[ P(t) = \left[ \frac{\left( \frac{t}{t_0} \right)^\gamma}{1 - \left( \frac{t}{t_s} \right)^{2\gamma+1}} \right]^5 \sum_{j=\pm} \tilde{p}_j t^{\beta_j} \]

\[ Q(t) = -6H \left[ \frac{\left( \frac{t}{t_0} \right)^\gamma}{1 - \left( \frac{t}{t_s} \right)^{2\gamma+1}} \right]^5 \sum_{j=\pm} \left( \frac{3}{2}H + \frac{\beta_j}{t} \right) \tilde{p}_j t^{\beta_j} \]

• Scalar curvature: $R = 6 \left( \dot{H} + 2H^2 \right)$

\[
R = \frac{60 \left[ \gamma (20\gamma - 1) + 44\gamma (\gamma + 1) \left( \frac{t}{t_s} \right)^{2\gamma+1} + (\gamma + 1) (20\gamma + 21) \left( \frac{t}{t_s} \right)^{2(2\gamma+1)} \right]}{t^2 \left[ 1 - \left( \frac{t}{t_s} \right)^{2\gamma+1} \right]^2}
\]

→ If we have the solution $t = t(R)$, we can obtain

\[ F(R) = P(t(R))R + Q(t(R)). \]
Behavior of $t_s^2 F(\tilde{R})$ as a function of $\tilde{R} \equiv t_s^2 R$ for

$\gamma = 1/2, \, \tilde{p}_+ = -1/t_s^{\beta_+}, \, \tilde{p}_- = 0, \, \beta_+ = (1 + 2\sqrt{19}) / 2,$

and $t_s = 2t_0 \, [C = (t_0/t_s)^{2\gamma+1} = 1/4 ]$. $R_0$ : Current curvature

$\cdot \tilde{R} = t_s^2 R = 4R/R_0$, $t_0 \approx H_0^{-1}$ $H_0$ : Present Hubble parameter
< Analytic form of $F(R)$ >

(1) $t \to 0 \ (t/t_s \ll 1) : \quad t \sim \sqrt{\frac{60\gamma (20\gamma - 1)}{R}}$

\[
F(R) \approx A_1 R^{-5\gamma/2 + 1} \sum_{j=\pm} C_j R^{-\beta_j/2}
\]

\[A_1 = \left[ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)} \right]^{5\gamma}, \quad \beta_{\pm} = \frac{1 \pm \sqrt{1 + 100\gamma(\gamma + 1)}}{2}\]

\[C_{\pm} = \left( \frac{5\gamma - \beta_{\pm} - 1}{20\gamma - 1} \right)^{\tilde{p}_{\pm}} [60\gamma (20\gamma - 1)]^{\beta_{\pm}/2}\]

(2) $t \to t_s : \quad t \sim t_s - 3 \sqrt{\frac{140}{R}}$

\[
F(R) \approx A_2 R^{7/2}
\]

\[A_2 = \frac{2}{7} \left[ \frac{1}{3\sqrt{140} (2\gamma + 1)} \left( \frac{t_s}{t_0} \right)^{\gamma} \right]^{\tilde{\gamma}} \left( \sum_{j=\pm} \tilde{p}_j t_s^{\beta_j} \right) t_s^5
\]

III. Summary

- A scenario to explain the current accelerated expansion of the universe is to study a modified gravitational theory, such as $F(R)$ gravity.

- Various observational data imply that the effective EoS of dark energy may evolve from larger than -1 (non-phantom phase) to less than -1 (phantom phase). Namely, it crosses -1 (the crossing of the phantom divide).

- We have reconstructed an explicit model of $F(R)$ gravity with realizing the crossing of the phantom divide.

  → The Big Rip singularity appears.

  → Around the Big Rip singularity, $F(R) \propto R^{7/2}$. 
Backup Slides
\[ \frac{1}{H_0^2} \frac{\ddot{a}}{a} = - \frac{\Omega_m}{2} (1 + z)^3 + \Omega_\Lambda \]

\[ \Omega_m \equiv \frac{\kappa^2 \rho(t_0)}{3H_0^2} \] : Density parameter for matter

\[ \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2} \] : Density parameter for \( \Lambda \)

From [Astier et al. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

\[ 1 + z = \frac{a_0}{a}, \quad z : \text{Red shift} \]

“0” denotes quantities at the present time \( t_0 \) .
< Residuals for the best fit to a flat $\Lambda$ cosmology >

$\Delta (m - M)$

From [Astier et al. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]
\[ S = \int d^4 x \sqrt{-g} \frac{F(R)}{2\kappa^2} \quad \text{F(R) gravity} \]

\[ F(R) = R : \text{General Relativity} \]

[Capozziello and Francaviglia, Gen. Rel. Grav. 40, 357 (2008)]
[Sotiriou and Faraoni, arXiv:0805.1726 [gr-qc]]

\[ F'(R) = dF(R)/dR \]

\[ F'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(R) + g_{\mu\nu}\Box F'(R) - \nabla_{\mu}\nabla_{\nu}F'(R) = 0 \]

\[ \Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} : \text{Covariant d'Alembertian} \]
\[ \nabla_{\mu} : \text{Covariant derivative operator} \]
- In the flat FRW background, gravitational field equations read

\[
H^2 = \frac{\kappa^2}{3} \rho_{\text{eff}}, \quad \dot{H} = -\frac{\kappa^2}{2} \left( \rho_{\text{eff}} + p_{\text{eff}} \right) \quad \rho_{\text{eff}}, \; p_{\text{eff}}: \text{Effective energy density and pressure from the term } F(R) - R
\]

\[
\rho_{\text{eff}} = \frac{1}{\kappa^2 F'(R)} \left[ \frac{1}{2} \left( -F(R) + RF'(R) \right) - 3H \dot{R} F''(R) \right]
\]

\[
p_{\text{eff}} = \frac{1}{\kappa^2 F'(R)} \left[ \frac{1}{2} \left( F(R) - RF'(R) \right) + \left( 2H \dot{R} + \ddot{R} \right) F''(R) + \dot{R}^2 F'''(R) \right]
\]

\[
\Rightarrow w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\left( F(R) - RF'(R) \right) / 2 + \left( 2H \dot{R} + \ddot{R} \right) F''(R) + \dot{R}^2 F'''(R)}{\left( -F(R) + RF'(R) \right) / 2 - 3H \dot{R} F''(R)}
\]

- Example: \( F(R) \propto R^n \) \( (n \neq 1) \)

\[
\rightarrow a \propto t^q, \quad q = \frac{-2n^2 + 3n - 1}{n-2}
\]

\[
w_{\text{eff}} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}
\]

If \( q > 1 \), accelerated expansion can be realized.

(For \( n = 3/2 \) or \( n = -1 \), \( q = 2 \) and \( w_{\text{eff}} = -2/3 \).)
We reconstruct an explicit model of $F(R)$ gravity with realizing the crossing of the phantom divide.

\[ F(R) = (R^{1/c} - \Lambda)^c \]

$c, \Lambda$ : Constants

- Example: $c = 1.8$
< 5-year WMAP data on $\omega$ >


- For the flat universe, constant $\omega$ : (From WMAP+BAO+SN)
  - $0.14 < 1 + \omega < 0.12$ (95% CL)

  Baryon acoustic oscillation (BAO) : Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

- For a variable EoS :
  - $0.33 < 1 + \omega_0 < 0.21$ (95% CL) $\leftarrow z_{\text{trans}} = 10$

  \[
  w(a) = \frac{a\tilde{w}(a)}{a + a_{\text{trans}}} - \frac{a_{\text{trans}}}{a + a_{\text{trans}}} \quad a < a_{\text{trans}} : \text{Dark energy density tends to a constant value}
  \]

  \[
  \tilde{w}(a) = \tilde{w}_0 + (1 - a)\tilde{\omega}_a \quad \omega_0 = w(a = 1)
  \]

  Cf. Dark Energy : $\Omega_\Lambda = 0.726 \pm 0.015$
  Dark Matter : $\Omega_c = 0.228 \pm 0.013$
  Baryon : $\Omega_b = 0.0456 \pm 0.0015$ (68% CL)

  \[
  \Omega_i \equiv \frac{\kappa^2 \rho_i^{(0)}}{3H_0^2} = \frac{\rho_i^{(0)}}{\rho_c^{(0)}} \quad i = \Lambda, c, b
  \]

  $\rho_c^{(0)}$ : Critical density
\[ <F(R)\, \text{gravity}> \]

\[ S = \int d^4x \sqrt{-g} \frac{F(R)}{2\kappa^2} \]

\[ F(R) \, \text{gravity} \]

[Capozziello and Francaviglia, Gen. Rel. Grav. 40, 357 (2008)]
[Sotiriou and Faraoni, arXiv:0805.1726 [gr-qc]]

Cf. \( F(R) = R \): General Relativity

- \( F(R) = R - \frac{\mu^{2(n+1)}}{R^n} \)

\( \mu \): Mass scale, \( n \): Constant

\[ \Rightarrow a \propto t^q, \quad q = \frac{(2n+1)(n+1)}{n+2} \]

\[ w_{\text{eff}} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)} \]

For \( n = 1, \ q = 2 \) and \( w_{\text{eff}} = -2/3 \). \( \rightarrow \) Accelerated expansion

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

Second term become important as \( R \) decreases.
(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

→ Analysis of $m(r)$ curve on the $(r, m)$ plane

\[ m \equiv RF''(R)/F'(R), \quad r \equiv - RF'(R)/F(R) \]

- Presence of a matter-dominated stage

\[ m(r) \approx + 0 \text{ and } \frac{dm}{dr} > - 1 \text{ at } r \approx - 1 \]

- Presence of a late-time acceleration

(i) \[ m(r) = - r - 1, \quad \frac{\sqrt{3}-1}{2} < m \leq 1 \quad \text{and} \quad \frac{dm}{dr} < - 1 \]

(ii) \[ 0 < m \leq 1 \quad \text{at} \quad r = - 2 \]

- Combing local gravity constraints, we obtain

\[ m(r) = C(- r - 1)^p \quad \text{with} \quad p > 1 \quad \text{as} \quad r \to - 1. \]

\[ C > 0, \quad p : \text{Constants} \]

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]
\[ w(z) = w_0 + w_1 \frac{z}{1+z} \]

From [Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)]

- **SN gold data set**
  [Riess et al. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004)]

- **SNLS data set**
  [Astier et al. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

- **Cosmic microwave background radiation (CMB) data**
  [Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007)]

- **SDSS baryon acoustic peak (BAO) data**
  [Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]

Shaded region shows 1σ error.
• For most observational probes (except the SNLS data), a low $\Omega_{0m}$ prior ($0.2 < \Omega_{0m} < 0.25$) leads to an increased probability (mild trend) for the crossing of the phantom divide.

$\Omega_{0m}$ : Current density parameter of matter

[Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)]
< Data fitting of $w(z)$ (2) >

\[
w(x) = \frac{(2x/3) \ln H/\ln x - 1}{1 - (H_0/H)^2 \Omega_{0m} x^3}
\]

\[
x = 1 + z
\]

From [Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)]

SN gold data set+CMB+BAO

| $\Omega_{0m}$ | 0.28 ± 0.03 |

SNLS data set+CMB+BAO

| $2\sigma$ confidence level. |
Baryon acoustic oscillation (BAO)

\[ \Omega_{bh}^2 = 0.24 \]

\[ \Omega_{mh}^2 = 0.12, 0.13, 0.14, 0.105 \] (From top to bottom)

\[ \Omega_{bh}^2 = 0.024 \]

From [Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]
< Note on the reconstruction method >

• If we redefine $\phi = \Phi(\varphi)$ and define $\tilde{P}(\varphi) \equiv P(\Phi(\varphi))$, we obtain

$$S = \int d^4x \sqrt{-g} \left[ \frac{\tilde{F}(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right]$$

$$\tilde{F}(R) \equiv \tilde{P}(\varphi) R + \tilde{Q}(\varphi) \quad \text{This is equivalent to the original action.}$$

$$\varphi = \varphi(R) = \Phi^{-1}(\phi(R))$$

$$\tilde{F}(R) = F(R)$$

• We have the choices in $\Phi$ like a gauge symmetry and thus we can identify $\Phi$ with time $t$, i.e., $\phi = t$, which can be interpreted as a gauge condition corresponding to the reparameterization of $\phi = \phi(\varphi)$. 

$\Phi$: Proper function
< Reconstruction of an explicit model >

- Equation for $P(\phi)$ without matter:

$$0 = \frac{d^2 P(\phi)}{d\phi^2} - \frac{d\tilde{g}(\phi)}{d\phi} \frac{dP(\phi)}{d\phi} + 2 \frac{d^2 \tilde{g}(\phi)}{d\phi^2} P(\phi)$$

\[\downarrow\]

By redefining $P(\phi)$ as $P(\phi) = e^{\tilde{g}(\phi)/2} \tilde{p}(\phi)$

$$\frac{1}{\tilde{p}(\phi)} \frac{d^2 \tilde{p}(\phi)}{d\phi^2} = 25e^{\tilde{g}(\phi)/10} \frac{d^2 (e^{-\tilde{g}(\phi)/10})}{d\phi^2}$$

\[\downarrow\]

The model $\tilde{g}(\phi) = -10 \ln \left[ \left( \frac{\phi}{t_0} \right)^{-\gamma} - C \left( \frac{\phi}{t_0} \right)^{\gamma+1} \right]$}

$$\frac{1}{\tilde{p}(\phi)} \frac{d^2 \tilde{p}(\phi)}{d\phi^2} = 25 \frac{\gamma(\gamma + 1)}{\phi^2}$$

< Solutions >

$$\tilde{p}(\phi) = \tilde{p}_+ \phi^{\beta_+} + \tilde{p}_- \phi^{\beta_-}, \quad \beta_\pm = \frac{1 \pm \sqrt{1 + 100 \gamma(\gamma + 1)}}{2}$$
\[ H = \left( \frac{10}{t} \right) \left\{ \frac{\gamma + (\gamma + 1) \left( \frac{t}{t_s} \right)^{2\gamma + 1}}{1 - \left( \frac{t}{t_s} \right)^{2\gamma + 1}} \right\} \]

(i) \( t \to 0 \)  \hspace{1cm} \text{Non-phantom phase}

\[ H(t) \sim \frac{10\gamma}{t}, \quad w_{\text{eff}} = -1 + \frac{1}{15\gamma} > -1 \]

(ii) \( t_c = t_s \left( -2\gamma + \sqrt{4\gamma^2 + \frac{\gamma}{\gamma + 1}} \right)^{1/(2\gamma + 1)} \)  \hspace{1cm} \text{Crossing of the phantom divide}

\[ \dot{H} = 0 , \quad w_{\text{eff}} = -1 \]

(iii) \( t \to t_s \)  \hspace{1cm} \text{Phantom phase}

\[ H(t) \sim \frac{10}{t_s - t} , \quad w_{\text{eff}} = -1 - \frac{1}{15} = -\frac{16}{15} < -1 \]
\[
\begin{align*}
R &= \frac{60 \left[ \gamma (20\gamma - 1) + 44\gamma (\gamma + 1) \left( \frac{t}{t_s} \right)^{2\gamma+1} + (\gamma + 1) (20\gamma + 21) \left( \frac{t}{t_s} \right)^{2(\gamma+1)} \right]}{t^2 \left[ 1 - \left( \frac{t}{t_s} \right)^{2\gamma+1} \right]^2} \\
\end{align*}
\]

< Scalar curvature >

< Behavior of \( U(t) \) >

- \( U(t) > 0 \) for \( t < t_c \), because \( \gamma > 0 \).

\[
\frac{dU(t)}{dt} = -\frac{2\gamma (\gamma + 1) (2\gamma + 1)^2}{15 \left[ \gamma + (\gamma + 1) \left( \frac{t}{t_s} \right)^{2\gamma+1} \right]^3} \left( \frac{1}{t_s} \right) \left( \frac{t}{t_s} \right)^{2\gamma} \left[ 1 - \left( \frac{t}{t_s} \right)^{2\gamma+1} \right]
\]

\[\rightarrow \frac{dU(t)}{dt} < 0 \text{ for } 0 < t < t_s\]

\( U(t) \) decreases monotonously. It evolves from positive to negative. Consequently, \( w_{\text{eff}} \) crosses -1.
< Analytic form of $F(R)$ >

(1) $t \to 0 : \quad t \sim \sqrt{\frac{60\gamma (20\gamma - 1)}{R}}$

$$F(R) \sim \left\{ \frac{\left[ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)R^{-1/2}} \right]^\gamma}{1 - \left[ \frac{1}{t_s} \sqrt{60\gamma (20\gamma - 1)R^{-1/2}} \right]^{2\gamma + 1}} \right\}^5 R$$

$$\times \sum_{j=\pm} \left\{ \left( \frac{5\gamma - 1 - \beta_j}{20\gamma - 1} \right) \tilde{p}_j [60\gamma (20\gamma - 1)]^{\beta_j/2} R^{-\beta_j/2} \right\}$$

• In the limit $t/t_s \ll 1$ , [KB and Geng, Phys. Lett. B 679, 282 (2009)]

$$F(R) \approx A_1 R^{-5\gamma/2 + 1} \sum_{j=\pm} C_j R^{-\beta_j/2}$$

$$A_1 = \left[ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)} \right]^{5\gamma} , \quad \beta_{\pm} = \frac{1 \pm \sqrt{1 + 100\gamma(\gamma + 1)}}{2}$$

$$C_{\pm} = \left( \frac{5\gamma - \beta_{\pm} - 1}{20\gamma - 1} \right) \tilde{p}_{\pm} [60\gamma (20\gamma - 1)]^{\beta_{\pm}/2}$$
(2) $t \rightarrow t_s : \quad t \sim t_s - 3 \sqrt{\frac{140}{R}}$

$$F(R) \sim \left( \frac{\left\{ \frac{1}{t_0} \left[ t_s - 3 \sqrt{140 R^{-1/2}} \right] \right\}^{\gamma}}{1 - \left[ 1 - \frac{3 \sqrt{140}}{t_s} R^{-1/2} \right]^{2\gamma + 1}} \right)^{5} R \sum_{j=\pm} \tilde{p}_j \left[ t_s - 3 \sqrt{140 R^{-1/2}} \right]^{\beta_j}$$

$$\times \left\{ 1 - \sqrt{\frac{20}{7}} \left[ \sqrt{\frac{15}{84}} t_s + (\beta_j - 15) R^{-1/2} \right] \frac{1}{t_s - 3 \sqrt{140 R^{-1/2}}} \right\}$$

- For large $R$, namely, $t_s^2 R \gg 1$,

$$F(R) \approx A_2 R^{7/2}$$

$$A_2 = \frac{2}{7} \left[ \frac{1}{3 \sqrt{140} (2\gamma + 1)} \left( \frac{t_s}{t_0} \right)^\gamma \right]^5 \left( \sum_{j=\pm} \tilde{p}_j t_s^\beta_j \right) t_s^5$$
II C. Property of the singularity in the corresponding scalar field theory

Action of $F(R)$ gravity

\[
S = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right]
\]

We introduce two scalar fields: $\zeta$ and $\xi$.

Action in the Jordan frame

There is a non-minimal coupling between $\xi$ and $R$.

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ \xi (R - \zeta) + F(\zeta) \right] + \mathcal{L}_{\text{matter}} \right\}
\]

- Equation of motion for $\xi$: $\zeta = R$
- Equation of motion for $\zeta$: $\xi = F'(\zeta)$ \hspace{1cm} $F'(\zeta) = dF(\zeta)/d\zeta$

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( F'(\zeta) R + F(\zeta) - F'(\zeta) \zeta \right) + \mathcal{L}_{\text{matter}} \right]
\]
Conformal transformation: 
\[ g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = e^\sigma g_{\mu\nu}, \quad e^\sigma = F'(\zeta) \]

\( \sigma \): Scalar field

Action in the Einstein frame

There is no non-minimal coupling between a scalar field and \( \hat{R} \).

\[ S_E = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \left( \frac{\hat{R}}{2} - \frac{3}{2} \hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right) + e^{-2\sigma} \mathcal{L}_\text{matter} \right] \]

\[ V(\sigma) = e^{-\sigma} \zeta(\sigma) - e^{-2\sigma} F' \left( \zeta(\sigma) \right) = \frac{\zeta}{F''(\zeta)} - \frac{F(\zeta)}{(F'(\zeta))^2} \]

A hat denotes quantities in the Einstein frame.

We redefine \( \varphi \) as \( \varphi \equiv \sqrt{3/2\sigma/\kappa} \).

Canonical scalar field theory

\[ S_{ST} = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) + e^{-2\sqrt{2/3\kappa}\varphi} \mathcal{L}_\text{matter} \right] \]
\[ F(R) = c_1 M^2 \left( \frac{R}{M^2} \right)^{-n} \]

\[ c_1, n : \text{Constants}, \quad M : \text{Mass scale} \]

→ Around the Big Rip singularity, the reconstructed model of \( F(R) \) gravity behaves as \( n = -7/2 \).

\[ a(t) = \bar{a} (t_s - t)^{(n+1)(2n+1)/(n+2)} \]

\[ R = \frac{6n(n + 1)(2n + 1)(4n + 5)}{(n + 2)^2} \cdot \frac{1}{(t_s - t)^2} \]

< Relation between \( dt \) and \( d\hat{t} \) >

\[ d\hat{t} = \pm e^{\sigma/2} dt \]

\[ e^{\sigma/2} = \sqrt{-nc_1} \left[ \frac{(n + 2)^2}{6n(n + 1)(2n + 1)(4n + 5)} \right]^{(n+1)/2} M^{n+1} (t_s - t)^{n+1} \]
< Relation between $t$ and $\hat{t}$ >

\[ \hat{t} = \mp \frac{\sqrt{-nc_1}}{n + 2} \left[ \frac{(n + 2)^2}{6n(n + 1)(2n + 1)(4n + 5)} \right]^{(n+1)/2} M^{n+1} (t_s - t)^{n+2} \]

- If $n < -2$, the limit of $t \rightarrow t_s$ corresponds to that of $\hat{t} \rightarrow \mp \infty$.
- In the reconstructed model of $F(R)$ gravity, around the Big Rip singularity, $n = -7/2$.

→ **‘Finite-time’ Big Rip singularity in $F(R)$ gravity (the Jordan frame)**

↓

[Conformal transformation]

‘**Infinite-time**’ singularity in the corresponding scalar field theory (the Einstein frame)
< Scale factor \( \hat{a} \left( \hat{t} \right) \) in the Einstein frame >

\[
ds^2 = e^\sigma ds^2 = -d\hat{t}^2 + \hat{a} \left( \hat{t} \right) d\vec{x}^2
\]

\[\rightarrow \quad \hat{a} \left( \hat{t} \right) = \hat{a} \hat{t}^{3 \left[ (n+1)/(n+2) \right]^2}\]

\[
\hat{a} = \bar{a} \left( \mp \frac{1}{n+2} \right)^{-3 \left[ (n+1)/(n+2) \right]^2}
\]

\[
\times \left\{ \sqrt{-n c_1} \left[ \frac{(n+2)^2}{6n(n+1)(2n+1)(4n+5)} \right]^{(n+1)/2} M^{n+1} \right\}^{-(2n^2+2n-1)/(n+2)^2}
\]
We have shown that the (finite-time) Big Rip singularity in the reconstructed model of $F(R)$ gravity becomes the infinite-time singularity in the corresponding scalar field theory obtained through the conformal transformation.
It has been demonstrated that the scalar field theories describing the non-phantom phase (phantom one with the Big Rip singularity) can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation.

We have examined that the quantum correction of massless conformally-invariant fields could be small when the crossing of the phantom divide occurs and the obtained solutions of the crossing of the phantom divide could be stable under the quantum correction, although it becomes important near the Big Rip singularity.
Appendix A

Relation between scalar field theory and $F(R)$ gravity
< Relation between scalar field theory and $F(R)$ gravity >

**Action in the Einstein-frame**

$$S_{\chi} = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\hat{R}}{2\kappa^2} + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{W}(\chi) \right]$$

(1) **Non-phantom (canonical) field**

- Real conformal transformation:

$$\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = e^{\pm \sqrt{2/3\kappa} \chi} \hat{g}_{\mu\nu}$$

**Action in the Jordan-frame**

$$S_{\text{NP}} = \int d^4x \sqrt{-g} \frac{F_{\text{NP}}(R)}{2\kappa^2}$$

$$F_{\text{NP}}(R) \equiv e^{\pm \sqrt{2/3\kappa} \chi(R)} R - 2\kappa^2 e^{\pm 2\sqrt{2/3\kappa} \chi(R)} \tilde{W}(\chi(R))$$

- $\chi$ can be expressed as $\chi = \chi(R)$ by solving the equation of motion for $\chi$:

$$R = e^{\pm \sqrt{2/3\kappa} \chi} \left(4\kappa^2 \tilde{W}(\chi) \pm \sqrt{6\kappa} \frac{d\tilde{W}(\chi)}{d\chi} \right)$$

- $\chi$ : Real scalar field
- $-$ : Non-phantom case
- $+$ : Phantom case
- $\tilde{W}(\chi)$ : Potential of $\chi$

[Capozziello, Nojiri and Odintsov, Phys. Lett. B 634, 93 (2006)]
(2) Phantom field


- Complex conformal transformation:

\[ \hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = e^{\pm i \sqrt{2/3\kappa\chi}} \hat{g}_{\mu\nu} \]

Action in the Jordan-frame

\[ S_P = \int d^4x \sqrt{-g} \frac{F_P(R)}{2\kappa^2} \]

\[ F_P(R) \equiv e^{\pm i \sqrt{2/3\kappa\chi(R)}} R - 2\kappa^2 e^{\pm i2\sqrt{2/3\kappa\chi(R)}} \tilde{W} (\chi(R)) \]

- We can obtain the relation \( \chi = \chi(R) \) by solving the equation of motion for \( \chi \):

\[ R = e^{\pm i \sqrt{2/3\kappa\chi}} \left( 4\kappa^2 \tilde{W}(\chi) \mp i\sqrt{6}\kappa \frac{d\tilde{W}(\chi)}{d\chi} \right) \]
Condition for $R$ to be real:

$$e^{i\sqrt{2/3}\kappa \chi} \left(4\kappa^2 \tilde{W}(\chi) - i\sqrt{6} \kappa \frac{d\tilde{W}(\chi)}{d\chi} \right) = e^{-i\sqrt{2/3}\kappa \chi} \left(4\kappa^2 \tilde{W}(\chi) + i\sqrt{6} \kappa \frac{d\tilde{W}(\chi)}{d\chi} \right)$$

$$\frac{1}{\tilde{W}(\chi)} \frac{d\tilde{W}(\chi)}{d\chi} = 2\sqrt{\frac{2}{3}} \kappa \tan \sqrt{\frac{2}{3}} \kappa \chi$$

With the except of $\tilde{W}(\chi)$ satisfying this relation, $R$ is complex.

< Summary >

Non-phantom (canonical) field theory $\rightarrow$ Real $F(R)$ gravity

Real conformal transformation

Phantom field theory $\rightarrow$ Complex $F(R)$ gravity

Complex conformal transformation (except the special case)
< Summary >

- It has been demonstrated that the scalar field theories describing the non-phantom phase (phantom one with the Big Rip singularity) can be represented as the theories of real (complex) $F(R)$ gravity through the inverse (complex) conformal transformation.
Appendix B

I. Stability under a quantum correction

II. Model of $F(R)$ gravity with the transition from the de Sitter universe to the phantom phase

III. Summary
I. Stability under a quantum correction

- Quantum effects produce the conformal anomaly:

\[ T_A = b \left( F + \frac{2}{3} \Box R \right) + b' G + b'' \Box R \]

\[ F = \frac{1}{3} R^2 - 2 R_{ij} R^{ij} + R_{ijkl} R^{ijkl} \quad : \text{Square of 4d Weyl tensor} \]

\[ G = R^2 - 4 R_{ij} R^{ij} + R_{ijkl} R^{ijkl} \quad : \text{Gauss-Bonnet invariant} \]

- In the flat FRW background, we find \( F = 0 \), \( G = 24 \left( \dot{H} H^2 + H^4 \right) \).

\[
\begin{align*}
b &= \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2} \\
b' &= -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\text{HD}}}{360(4\pi)^2}
\end{align*}
\]

\( b'' \) can be arbitrary, e.g., we can choose \( b'' = -2b/3 \) or \( b'' = 0 \).
Assuming $T_A = -\rho_A + 3p_A$ and

the conservation law:

$$\dot{\rho}_A + 3H (\rho_A + p_A) = 0,$$

we find

$$\rho_A = -\frac{1}{a^4} \int dt a^4 H T_A, \quad p_A = -\frac{1}{3a^4} \int dt a^4 H T_A + \frac{T_A}{3}.$$

Effective energy density $\rho_F$ and pressure $p_F$ from $f(R) = F(R) - R$

$$\kappa^2 \rho_F = -\frac{1}{2} (F(R) - R) + 3 \left( H^2 + \dot{H} \right) (F'(R) - 1) - 18 \left( 4H^2 \dot{H} + H \ddot{H} \right) F''(R)$$

$$\kappa^2 p_F = \frac{1}{2} (F(R) - R) - \left( 3H^2 + \dot{H} \right) (F'(R) - 1)$$

$$+ 6 \left( 8H^2 \dot{H} + 4\dot{H}^2 + 6H \ddot{H} + \dddot{H} \right) F''(R) + 36 \left( 4H \dot{H} + \dddot{H} \right)^2 F'''(R)$$

We assume the Hubble rate at the phantom crossing is given by

$$H \sim H_0 \sim 10^{-33} \text{ eV}$$
- $\rho_A \sim p_A \sim T_A$

$$\rightarrow \rho_A \sim p_A \sim C H_0^4, \quad C \sim 10^{2-3} \quad \text{: Dimensionless constant}$$

- $\rho_F \sim p_F \sim f(R)/\kappa^2$

$$f(R) \sim H_0^2 \quad \text{f}(R) \text{ plays the role of the effective cosmological constant}$$

$$\rightarrow \rho_F \sim p_F \sim \frac{H_0^2}{\kappa^2}, \quad 1/\kappa \sim 10^{28} \text{ eV}$$

$$\rightarrow |\rho_F| \gg |\rho_A|, \quad |p_F| \gg |p_A|$$

- The quantum correction could be small when the crossing of the phantom divide occurs and the obtained solutions of the crossing of the phantom divide could be stable under the quantum correction. (The quantum correction becomes important near the Big Rip singularity.)
II. Model of $F(R)$ gravity with the transition from the de Sitter universe to the phantom phase

- Hubble rate: \[ H = g_0 + \frac{g_1}{t_s - t} \quad g_0 > 0, \quad g_1 > 0, \quad t_s > 0 \]
  
  (1) $t \rightarrow -\infty$ : $H \rightarrow g_0$ : Constant \[ \text{Asymptotically de Sitter space} \]

  (2) $t \rightarrow t_s$ : $H \sim \frac{g_1}{(t_s - t)}$

  $\dot{H} \sim \frac{g_1}{(t_s - t)^2} > 0$

  $\rightarrow w_{\text{eff}} < -1$ \[ \text{Phantom phase} \]

  $t = t_s$ : **Big Rip singularity**

- $R = 6 \left[ 2g_0^2 + \frac{4g_0g_1}{t_s - t} + \frac{g_1(2g_1 + 1)}{(t_s - t)^2} \right]$
(1) Case with neglecting matter

\[
\frac{d\tilde{g}(\phi)}{d\phi} = g_0 + \frac{g_1}{t_s - \phi}
\]

\[
\rightarrow 0 = \frac{d^2P(\phi)}{d\phi^2} - \left( g_0 + \frac{g_1}{t_s - \phi} \right) \frac{dP(\phi)}{d\phi} + \frac{2g_1}{(t_s - \phi)^2} P(\phi)
\]

\textless Solutions \textgreater

\[P(z) = C_+ z^\alpha F_K (\alpha, \tilde{\gamma}; z) + C_- z^{1-\tilde{\gamma}} F_K (\alpha - \tilde{\gamma} + 1, 2 - \tilde{\gamma}; z)\]

\[z \equiv g_0 (\phi - t_s), \quad \alpha \equiv \frac{1 - g_1 \pm \sqrt{g_1^2 - 10g_1 + 1}}{4} \quad C_\pm : \text{Dimensionless constants}\]

\[\tilde{\gamma} \equiv 1 \pm \frac{\sqrt{g_1^2 - 10g_1 + 1}}{2}\]

\[F_K (\alpha, \tilde{\gamma}; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{\tilde{\gamma}(\tilde{\gamma} + 1) \cdots (\tilde{\gamma} + n - 1)} \frac{z^n}{n!}\]

\[F_K : \text{Kummer functions (confluent hypergeometric function)}\]
We have taken $\phi = t$ and therefore $z = g_0 (t - t_s)$.

1. $t \rightarrow -\infty : \quad R \sim 12g_0^2 : \text{Constant} \quad \textbf{\rightarrow} \quad \text{de Sitter phase}$

2. $t \rightarrow t_s : \quad R \sim 6g_1 (2g_1 + 1) / (t_s - t)^2$

\[ z \sim -g_0 \sqrt{\frac{6g_1 (2g_1 + 1)}{R}} \]

- In this limit, $|z| \ll 1$ because $R$ diverges.
Expanding the Kummer functions in and taking the first leading order in $z$, we obtain

$$F(R) \approx \frac{R}{(\tilde{\gamma} - 1) (2g_1 + 1)} \left\{ C_+ (\alpha - 1) (\alpha + g_1 + 1) \left[ -g_0 \sqrt{6g_1 (2g_1 + 1)} \right]^\alpha R^{-\alpha/2} - C_- (\alpha - \tilde{\gamma}) (2 - \tilde{\gamma} + g_1) \left[ -g_0 \sqrt{6g_1 (2g_1 + 1)} \right]^{1-\tilde{\gamma}} R^{-(1-\tilde{\gamma})/2} \right\}$$
(2) Case with the cold dark matter

- We take into account the cold dark matter with its EoS $w = 0$.
- We numerically solve the equation for $P(\phi)$.

\[ \frac{F(\tilde{R})}{(2\kappa^2)} \] as a function of $\tilde{R} \equiv t_s^2 R$ for $t_s g_0 = g_1$, $t_s = 2t_0$ and $\bar{\rho} = 0.233 \rho_c$.

\[ \bar{\rho} : \text{Current energy density of the cold dark matter} \]

\[ \rho_c = \frac{3H_0^2}{8\pi G} \]

\[ = 3.97 \times 10^{-47} \text{GeV}^4 \]

\[ : \text{Critical density} \]
Behavior of $P(\tilde{R})$ and $t_s^2 Q(\tilde{R})$ as a function of $\tilde{R}$ for $t_s g_0 = g_1$, $t_s = 2t_0$ and $\bar{\rho} = 0.233\rho_c$. 
<Summary>

- We have examined that the quantum correction of massless conformally-invariant fields could be small when the crossing of the phantom divide occurs and the obtained solutions of the crossing of the phantom divide could be stable under the quantum correction, although it becomes important near the Big Rip singularity.

- We have reconstructed a model of $F(R)$ gravity in which the transition from the de Sitter universe to the phantom phase can occur.
Appendix C

Scalar field theory with realizing the constructed $H(t)$
< Scalar field theory with realizing the constructed $H(t)$ >

\[ S_\Phi = \int d^4 x \sqrt{-\hat{g}} \left[ \frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \omega(\Phi) \hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - W(\Phi) \right] \]

\( \Phi \) : Scalar field, \( \omega(\Phi) \) : Function of \( \Phi \), \( W(\Phi) \) : Potential of \( \Phi \)

- In the flat FRW background, the Einstein equations are given by
  \[
  \frac{3}{\kappa^2} H^2 = \rho_\Phi , \quad -\frac{2}{\kappa^2} \dot{H} = p_\Phi + \rho_\Phi \\
  \rho_\Phi = \frac{1}{2} \omega(\Phi) \dot{\Phi}^2 + W(\Phi) , \quad p_\Phi = \frac{1}{2} \omega(\Phi) \dot{\Phi}^2 - W(\Phi) \\
  \rightarrow \omega(\Phi) \dot{\Phi}^2 = -\frac{2}{\kappa^2} \dot{H} , \quad W(\Phi) = \frac{1}{\kappa^2} \left( 3H^2 + \dot{H} \right)
  \]

- We define \( \omega(\Phi) \) and \( W(\Phi) \) in terms of a single function \( I(\Phi) \).
  \[
  \omega(\Phi) = -\frac{2}{\kappa^2} \frac{dI(\Phi)}{d\Phi} , \quad W(\Phi) = \frac{1}{\kappa^2} \left( 3I^2(\Phi) + \frac{dI(\Phi)}{d\Phi} \right) \\
  \text{Solutions} : \Phi = t , \quad H = I(t)
  \]
- Defining $\chi$ as $\chi \equiv \int d\Phi \sqrt{\omega(\Phi)}$, we obtain

$$S_\chi = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\hat{R}}{2\kappa^2} + \frac{1}{2}\hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{W}(\chi) \right]$$

$\omega(\Phi) > 0 \quad \rightarrow \quad -\text{sign}$

$\omega(\Phi) < 0 \quad \rightarrow \quad +\text{sign}$

$\tilde{W}(\chi) = W(\Phi(\chi))$  

- $H(t)$ with realizing the phantom crossing in $F(R)$ gravity

$$I(\Phi) = \left(\frac{10}{\Phi}\right) \left[ \frac{\gamma + (\gamma + 1) \left(\frac{\Phi}{t_s}\right)^{2\gamma + 1}}{1 - \left(\frac{\Phi}{t_s}\right)^{2\gamma + 1}} \right]$$

$\rightarrow$  

$$H = \left(\frac{10}{t}\right) \left[ \frac{\gamma + (\gamma + 1) \left(\frac{t}{t_s}\right)^{2\gamma + 1}}{1 - \left(\frac{t}{t_s}\right)^{2\gamma + 1}} \right]$$
\[ \omega (\Phi) = \frac{20}{\kappa^2 \Phi^2 \left[ 1 - \left( \frac{\Phi}{t_s} \right)^{2\gamma+1} \right]^2} \left[ \gamma - 4\gamma (\gamma + 1) \left( \frac{\Phi}{t_s} \right)^{2\gamma+1} - (\gamma + 1) \left( \frac{\Phi}{t_s} \right)^{2(2\gamma+1)} \right] \]

\[ \rightarrow \chi = \frac{\sqrt{20}}{\kappa} \int d\Phi \sqrt{\frac{\gamma - 4\gamma (\gamma + 1) \left( \frac{\Phi}{t_s} \right)^{2\gamma+1} - (\gamma + 1) \left( \frac{\Phi}{t_s} \right)^{2(2\gamma+1)}}{\Phi \left[ 1 - \left( \frac{\Phi}{t_s} \right)^{2\gamma+1} \right]}} \]

\[ \tilde{W} (\chi) = \frac{10}{\kappa^2 \Phi^2 (\chi) \left[ 1 - \left( \frac{\Phi(\chi)}{t_s} \right)^{2\gamma+1} \right]^2} \times \left[ \gamma (30\gamma - 1) + 64\gamma (\gamma + 1) \left( \frac{\Phi(\chi)}{t_s} \right)^{2\gamma+1} + (\gamma + 1) (30\gamma + 31) \left( \frac{\Phi(\chi)}{t_s} \right)^{2(2\gamma+1)} \right] \]
(i) \( t < t_c : \dot{H} < 0 \implies \omega(\Phi) > 0 \) → − sign

Non-phantom phase

(ii) \( t = t_c : \dot{H} = 0 \implies \omega(\Phi) = 0 \)

Crossing of the phantom divide

(iii) \( t > t_c : \dot{H} > 0 \implies \omega(\Phi) < 0 \) → + sign

Phantom phase


- It has been shown that for such a scalar field theory in the presence of the background cold dark matter, the crossing of the phantom divide can occur.
< Summary >

- We have studied the scalar field theory with realizing the constructed $H(t)$. 