Time variation of particle and anti-particle number in curved space time.

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Introduction

1. Time variation of the particle and anti-particle number asymmetry is a very important issue in Baryogenesis and Leptogenesis.

2. They are important when we study how the asymmetry of Baryon number or Lepton number evolves in the curved space time.

3. **In the first part of my talk, we study the time variation of particle number in a simple quantum mechanical model. The toy model shows the meaning of the particle and anti-particle asymmetry in the scence of the statistical average with the density matrix. It also shows the importance of the role of the chemical potential.**

4. In the second part of my talk, the model is extended to the field theoretical model. This is mainly due to that we are applying the path integral. The (wave) functional representation of the density matrix with non-zero chemical potential is derived.

5. In the final part, I explain the field theoretical model in curved space time.

1 Time variation of particle number in quantum mechanics

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega^2(x^2 + y^2) - \epsilon\omega^2 xy$$

Classical trajectory

$$x(t) = \frac{x_0}{2}(\cos\omega_+ t + \cos\omega_- t),$$
$$y(t) = \frac{x_0}{2}(\cos\omega_+ t - \cos\omega_- t).$$
$$\omega_\pm = \omega\sqrt{1 + \epsilon}$$

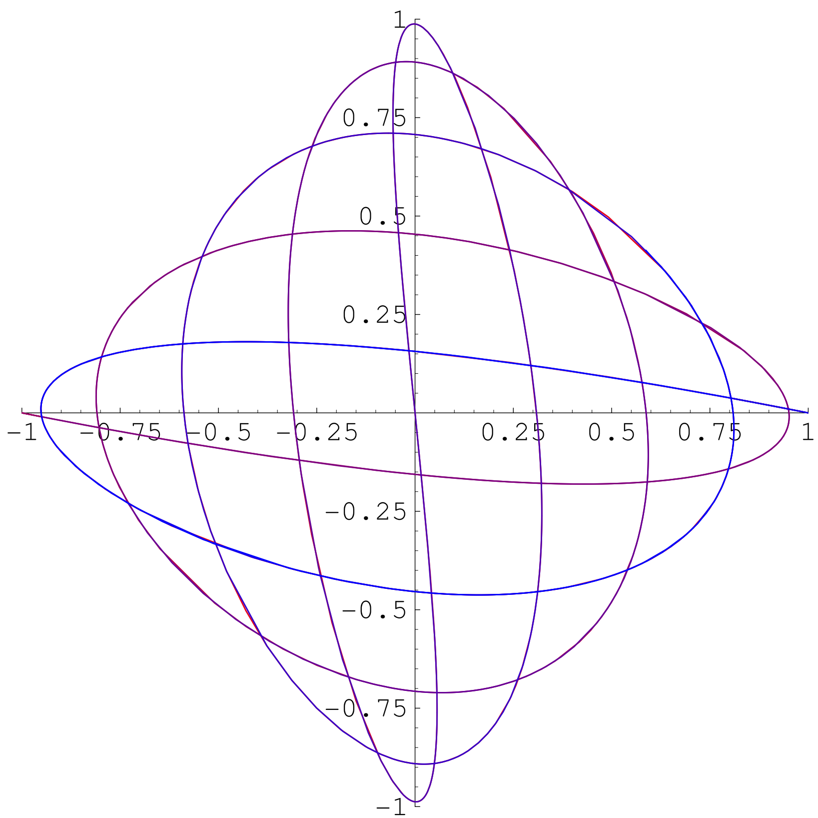

Figure 1: Lissajous figure:(Trajectory)

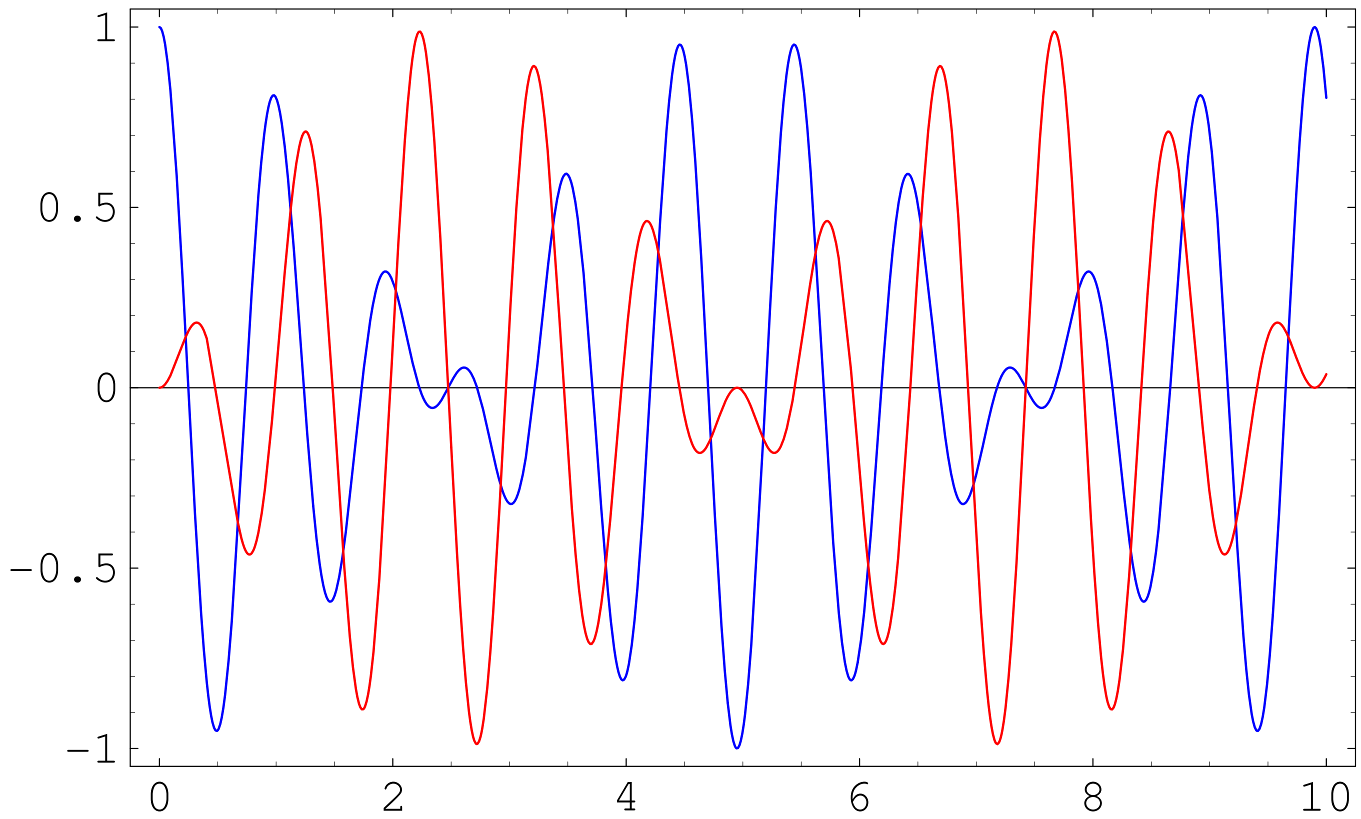

Figure 2: motion of x, y

Quantum mechanics of the oscillator model

the complex coordinate.

$$z = \frac{x + iy}{\sqrt{2}}$$

Rotation on (x, y) plane corresponds to U(1) transformation on z.

$$z \longrightarrow z' = e^{i\theta} z$$

$$V(z) = \omega^2 |z|^2 - i\frac{\epsilon\omega^2}{2}(z^2 - z^{*2})$$

Charge N corresponding to U(1) transformation $\leftrightarrow$ angular momentum

$$N = i(z^* \frac{dz}{dt} - \frac{dz^*}{dt} z) = -L_z.$$

$$z = \frac{a + b^\dagger}{\sqrt{2\omega}}.$$

$$N = a^\dagger a - b^\dagger b$$

Hamiltonian and Particle number Particle Number $N \equiv N_a - N_b$.

$$H = H_0 + H^{|\Delta N|=2},$$

$$H_0 = \omega(a^\dagger a + b^\dagger b) = \omega(N_a + N_b)$$

$$H^{|\Delta N|=2} = -i\frac{\epsilon\omega}{4}(a^2 + b^{\dagger 2} - a^{\dagger 2} - b^2).$$

$$[H, N] \neq 0, [H_0, N] = 0$$

Initial density matrix and the particle number in statistical average
Grand Canonical ensemble

$$\rho(t=0) = e^{-\beta(H_0-\mu N)}/\mathrm{Tr}e^{-\beta(H_0-\mu N)}$$

$$\beta = 1/T$$

Chemical potential μ. Temperature T.

$H_0 - \mu N$ is similar to the Hamitonian for Landau level.

$$H_0 - \mu N = (\omega - \mu) a^\dagger a + (\omega + \mu) b^\dagger b$$

$$\omega = \mu \rightarrow \omega_B = \frac{|eB|}{2m}$$

$$H_{Landau} = 2\omega_B (b^\dagger b),$$

$$-\mu(a^\dagger a - b^\dagger b) = \omega_B L_z.$$

Energy shift due to the Magnetic moment of the orbital angular momentum.

statistical average of the particle number

$$\langle N(0)\rangle = \mathrm{Tr}N\rho(0)$$
$$= \frac{1}{e^{\beta(\omega-\mu)}-1} - \frac{1}{e^{\beta(\omega+\mu)}-1}$$
$$= \frac{\sinh\beta\mu}{\cosh\beta\omega - \cosh\beta\mu}$$

When $\omega > |\mu|$, the sign of the chemical potential controls the excess of particle a quanta or anti-particle b.

Time development of particle number operator: $N(t) = a^\dagger(t)a(t) - b^\dagger(t)b(t)$.

The time development is exactly solvable:

$$X = \frac{x_1 + x_2}{\sqrt{2}}, Y = \frac{x_1 - x_2}{\sqrt{2}}$$

$$L = \frac{\dot{X}^2 + \dot{Y}^2}{2} - \frac{\omega_+^2}{2}X^2 - \frac{\omega_-^2}{2}Y^2$$

$$\omega_+ = \omega\sqrt{1+\epsilon}, \omega_- = \omega\sqrt{1-\epsilon}$$

$$a(t) = (f_+(t) - iH_+(t))a - h_-(t)a^\dagger$$
$$+i(f_-(t) - iH_-(t))b + ih_+(t)b^\dagger$$
$$b(t) = -i(f_-(t) - iH_-(t))a + ih_+(t)a^\dagger$$
$$+(f_+(t) - iH_+(t))b + h_-(t)b^\dagger$$
$$f_\pm(t) = \frac{\cos\omega_+ t \pm \cos\omega_- t}{2},$$
$$H_\pm(t) = \frac{R_+ \sin\omega_+ t \pm R_- \sin\omega_- t}{2}, R_\pm = \frac{1}{2}(\frac{\omega}{\omega_\pm} + \frac{\omega_\pm}{\omega})$$
$$h_\pm(t) = \frac{r_+ \sin\omega_+ t \pm r_- \sin\omega_- t}{2}, r_\pm = \frac{1}{2}(\frac{\omega}{\omega_\pm} - \frac{\omega_\pm}{\omega})$$

$$
\begin{aligned}
N(t) &= a^\dagger(t)a(t) - b^\dagger(t)b(t) \\
&= (a^\dagger a - b^\dagger b)(|p|^2 - |q|^2 - h_+^2 + h_-^2) \\
&- (a^\dagger a^\dagger + b^\dagger b^\dagger)(h_- p^* - i h_+ q^*) + h.c. \\
&+ 2Re.(qp^*)(a^\dagger b + b^\dagger a)
\end{aligned}
$$

$p = f_+ - iH_+, q = f_- - iH_-.$

$|p|^2 - |q|^2 - h_+^2 + h_-^2 = f_+^2 - f_-^2 + H_+^2 - H_-^2 - h_+^2 + h_-^2.$

$$\frac{\langle N(t)\rangle}{\langle N(0)\rangle} = \frac{\mathrm{Tr}\,N(t)\rho(0)}{\mathrm{Tr}\,N(0)\rho(0)}$$

$$= \cos(\omega_+ - \omega_-)t +$$

$$\left\{\frac{1}{2}\left(\frac{\omega_+}{\omega_-} + \frac{\omega_-}{\omega_+}\right) - 1\right\}\sin\omega_+ t\sin\omega_- t.$$

$$\simeq \cos(\omega\epsilon t) + \frac{\epsilon^2}{2}(\cos\omega\epsilon t - \cos 2\omega t).$$

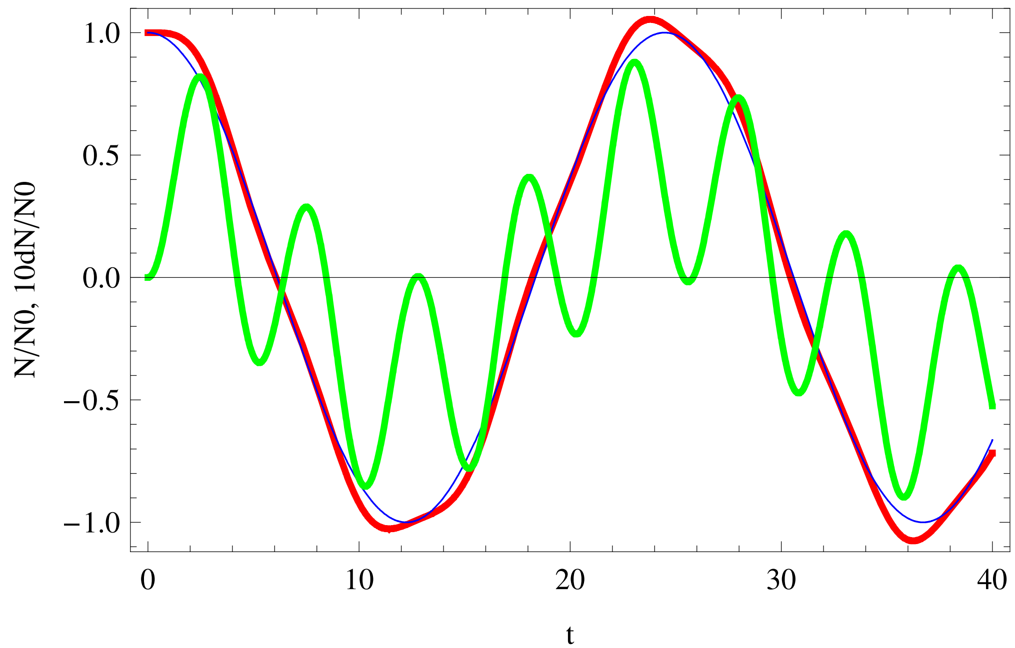

Figure 3: $\frac{\langle N(t)\rangle}{\langle N(0)\rangle}$ ($\epsilon = 0.4$). **Green curve:** $10 \times \epsilon^2$ **suppressed term. Blue curve:** $\cos \omega\epsilon t$.

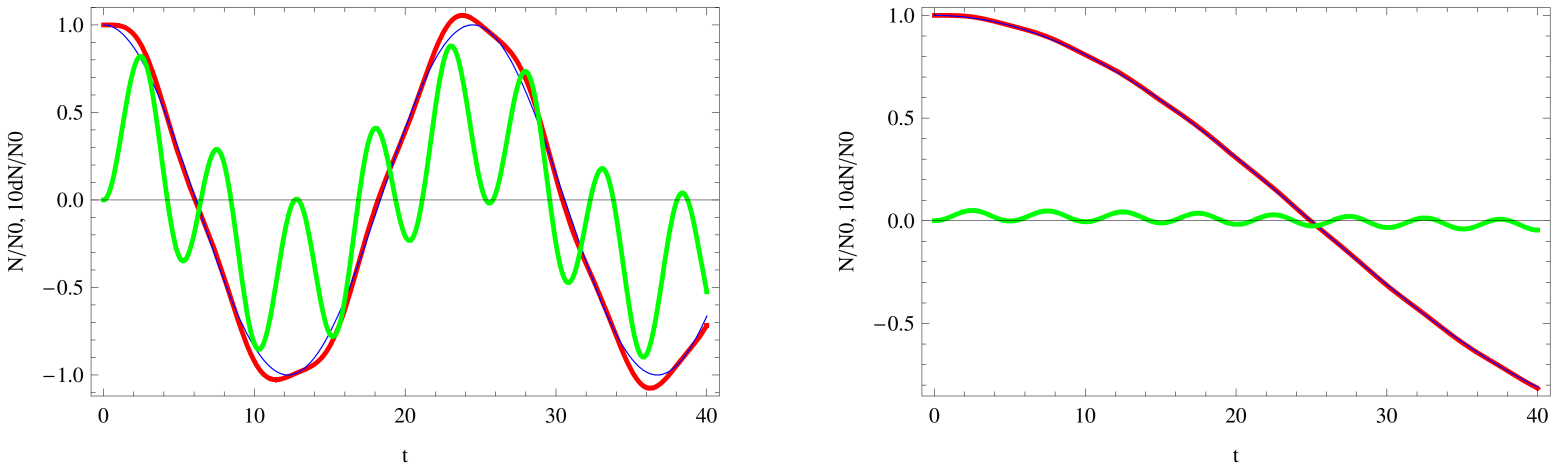

Figure 4: $\frac{\langle N(t) \rangle}{\langle N(0) \rangle}$ **(left** $\epsilon = 0.4$**, right** $\epsilon = 0.1$**) Green curve:** $10 \times \epsilon^2$ **suppressed term. Blue curve:** $\cos \omega \epsilon t$**.**

Lesson from quantum mechanical model

- The large time behaviour of the particle number is determined by the period related to the difference of the oscillation frequencies $\omega_1 - \omega_2 = \omega\epsilon$.
- The amplitude of the rapid oscillation is suppressed by ϵ^2.

Particle number in field theoretical Model

Complex scalar field with U(1) breaking

mass term:

$$L = \partial_\mu \phi^\dagger \partial^\mu \phi - m_\phi^2 \phi^\dagger \phi + B^2(\phi^2 + \phi^{\dagger 2})$$

B^2: **U(1) breaking term.** $\phi \rightarrow e^{i\theta}\phi$.

In terms of the real scalars $\phi = \dfrac{\phi_1 + i\phi_2}{\sqrt{2}}$,

$$L = \frac{1}{2}(\partial_\mu \phi_i \partial^\mu \phi_i) - \frac{m_i^2}{2}\phi_i^2$$

$$m_1^2 = m_\phi^2 - B^2 \neq m_2^2 = m_\phi^2 + B^2$$

Under the presence of U(1) breaking term B^2, the mass splitting of the real scalars is non-zero and U(1) current does not conserve.

Computing the current for U(1) charge (particle number).

$$
\begin{aligned}
j_\mu &= i(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi) \\
&= \phi_2 \partial_\mu \phi_1 - \partial_\mu \phi_2 \phi_1
\end{aligned}
$$

One wants to compute the expectation value of the U(1) current with some initial condition.

The initial condition is specifed by the density matrix.

$$\langle j_\mu(X)\rangle = \mathrm{Tr} j_\mu(X)\rho(0),$$
$$= \left(\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu}\right) G_{12}^{12}(x,y)\Big|_{x=y=X}$$

$$G_{12}^{12}(x,y) = Tr[\phi_2(y)\phi_1(x)\rho(0)]$$

The expectation value can be written by the Green function of Keldysh.

Notation:

upper index (a,b,...$= 1, 2$) denotes if the operator sits on the time ordered contour or anti-time ordered contour of closed time path.

$$\mathrm{Tr}[T(\phi_i(x_1)\phi_j(x_2)..)\rho(0)\tilde{T}(\phi_l(y_1)\phi_m(y_2))].$$
$$= G^{11...22}_{ij..lm}(x_1, x_2, .., y_1, y_2).$$

The operators at the left(right) of ρ is (anti-) time ordered. The lower indices of $i(j, l, m) = 1, 2$ denote the scalar field ϕ_i with the mass m_i.

The functional approach (Path Integral approach) for Keldysh Green function from 2 PI effective action.

Introduce the non-local source $K(x,y)$:

$$e^{iW[K]} = \int d\phi_i e^{i(S+\frac{1}{2}\int\int \phi_i^a(x) K_{ij}^{ab}(x,y)\phi_j^b(y))}$$

$$\Gamma[G] = W[K] - \frac{1}{2} Tr[KG]$$

$$\frac{\delta\Gamma}{\delta G_{ij}^{ab}} = -\frac{1}{2}K_{ij}^{ab}$$

$$\Gamma[G_{ij}^{ab}] = -i\frac{1}{2}\mathrm{TrLn}G_{ij}^{ab}$$

$$-\int d^4x \frac{c^{ab}}{2}(\Box + m_i^2)G_{ij}^{ab}(x,x)$$

$c^{11} = -c^{22} = 1, c^{12} = 0.$

The non local source K is related to the density matrix at $t = 0$.

$$\langle \phi_i^1 | \rho(0) | \phi_j^2 \rangle =$$
$$e^{-\frac{1}{2}\int d^3x \int d^3y \phi_i^a(\mathrm{x}) \kappa_{\mathrm{ij}}^{\mathrm{ab}}(\mathrm{x}-\mathrm{y}) \phi_{\mathrm{j}}^{\mathrm{b}}(\mathrm{y})}$$

$$K(x, y) = -i\kappa(\mathrm{x} - \mathrm{y})\delta(\mathrm{x}_0)\delta(\mathrm{y}_0)$$

$$(\Box_x + m_i^2)G_{ij}^{ab}(x,y) = -ic^{ab}\delta_{ij}\delta(x-y)$$
$$+\int d^4z(cK(x,z)G(z,y))_{ij}^{ab}$$
$$(\Box_y + m_j^2)G_{ij}^{ab}(x,y) = -ic^{ab}\delta_{ij}\delta(x-y)$$
$$+\int d^4z(G(x,z)K(z,y)c)_{ij}^{ab}.$$

One can solve the Fourier transform of the Green function: $G_{12}^{12}(x^0, y^0, k)$.

$$G_{12}^{12}(x^0, y^0, k) = \frac{\sinh \beta\mu}{\cosh \omega - \cosh \beta\mu}$$

$$\times \left(\frac{\cos \omega_1 y^0 \sin \omega_2 x^0}{2\omega_1} - \frac{\sin \omega_1 y^0 \cos \omega_2 x^0}{2\omega_2} \right)$$

$$\omega(k) = \sqrt{k^2 + m_\phi^2},$$

$$\omega_1(k) = \sqrt{k^2 + m_1^2}, \omega_2(k) = \sqrt{k^2 + m_2^2}.$$

$$m_1^2 = m_\phi^2 - B^2, m_2^2 = m_\phi^2 + B^2$$

The particle number density.

$$\langle j^0(t)\rangle = \mathrm{Tr} j^0(t)\rho(0) = \int \frac{d^3k}{(2\pi)^3}$$

$$(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial y^0})G^{12}_{12}(x^0, y^0, k)|_{x^0=y^0=t}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{\sinh\beta\mu}{\cosh\omega - \cosh\beta\mu}(\cos(\omega_1 - \omega_2)t$$

$$+\{\frac{1}{2}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) - 1\}\sin\omega_1 t \sin\omega_2 t).$$

It is similar to the quantum oscillator model

$\epsilon \longrightarrow \dfrac{B^2}{\omega^2}$.

$$\omega_1 = \omega\sqrt{1 - \frac{B^2}{\omega^2}}, \omega_2 = \omega\sqrt{1 + \frac{B^2}{\omega^2}}$$

Now we have contribution from many modes corresponding to k.

Extention to the curved space time.

$ds^2 = dt^2 - a(t)^2 dx^i dx^i$ S.A. Ramsey and B. L. Hu (PRD36,1997)

$$e^{iW[K,g]} = \int d\phi e^{iS}$$

$$e^{\frac{i}{2}\int d^4x d^4y \sqrt{-g(x)}\sqrt{-g(y)}\phi(x)K(x,y)\phi(y)}$$

$$S = \int d^4x \sqrt{-g}(D_\mu \phi_i D^\mu \phi_i - (m_i^2 + \xi R)\phi_i \phi$$

$$\Gamma[G,g] = W[K,g] - \frac{1}{2}\int d^4x d^4y G(x,y) K(x,y) \sqrt{-g(x)}\sqrt{-g(y)}$$

$$\Gamma = -\frac{i}{2} Tr Ln G - \frac{1}{2}\int d^4x \sqrt{-g}(D_\mu D^\mu + m^2 + \xi R) G(x,x)$$

$$\frac{d\Gamma}{dG} = -\frac{1}{2}\sqrt{-g(x)} K(x,y) \sqrt{-g(y)}$$

Schwinger Dyson Eq.

$$-iG^{-1} - \sqrt{-g}(D_\mu D^m u + m^2 + \xi R)\delta(x - y$$

$$= -\sqrt{-g(x)}K(x,y)\sqrt{-g(y)}$$

Summary

- Time variation of particle number is investigated.
- Quantum mechanical model for particle and anti-particle oscillation is proposed. The expectation value of the particle number is studied.
- The quantum field theory extention of the quantum mechnical oscillator

model is studied.

- The outline to the extention to the model in curved space time is shown.
- The model with interaction is given in R. Hotta's talk.