$F(R)$ bigravity

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"bigravity"
$=$ system of massive spin 2 field (massive graviton)
$+$ gravity (includes massless spin 2 field $=$ graviton)

$F(R)$ extension of bigravity

Application to Cosmology

Contents

1 Introduction

Mainly based on

S. Nojiri and S. D. Odintsov, "Ghost-free $F(R)$ bigravity and accelerating cosmology," Phys. Lett. B **716**, 377 (2012) [arXiv:1207.5106 [hep-th]].

and

S. Nojiri, S. D. Odintsov, and N. Shirai, "Variety of cosmic acceleration models from massive $F(R)$ bigravity," arXiv:1212.2079 [hep-th].

Review on Massive Gravity

K. Hinterbichler, "Theoretical Aspects of Massive Gravity," Rev. Mod. Phys. **84** (2012) 671 [arXiv:1105.3735 [hep-th]].

♠ Motivation or status of the massive gravity:

- Consistent interacting theory of massive spin-2 field?
  cf. String Theory, Kaluza-Klein Theory.

- Fierz-Pauli action (linearized or free theory)
  M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond. A **173** (1939) 211.

  The Lagrangian of the massless spin-two field (graviton) $h_{\mu\nu}$ is given by $\left(h \equiv h^\mu{}_\mu\right)$.

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\mu\nu} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \,.$$

  Massive graviton: 5 degrees of freedom.

The Lagrangian of the massive graviton with mass m is given by

$$\mathcal{L}_m = \mathcal{L}_0 - \frac{m^2}{2} \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) \, .$$

- Boulware-Deser ghost.

  D. G. Boulware and S. Deser, "Classical General Relativity Derived from Quantum Gravity," Annals Phys. **89** (1975) 193.

  In non-linear (interacting) theory, 6th degree of freedom appears as a ghost.

- vDVZ(van Dam, Veltman, and Zakharov) discontinuity

  H. van Dam and M. J. G. Veltman, “Massive and massless Yang-Mills and gravitational fields,” Nucl. Phys. B **22** (1970) 397.

  V. I. Zakharov, “Linearized gravitation theory and the graviton mass,” JETP Lett. **12** (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. **12** (1970) 447].

  Discontinuity of $m \to 0$ limit in the free massive gravity with the Einstein gravity due to the extra degrees of freedom in the limit.

$\Rightarrow$ the Vainstein mechanism

A. I. Vainshtein, Phys. Lett. B **39** (1972) 393.

Non-linearity screens the extra degrees of freedom (non-linearity becomes strong when m is small).

Massive gravity without ghost

C. de Rham and G. Gabadadze, “Generalization of the Fierz-Pauli Action,” Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

C. de Rham, G. Gabadadze and A. J. Tolley, “Resummation of Massive Gravity,” Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]].

S. F. Hassan and R. A. Rosen, “Resolving the Ghost Problem in non-Linear Massive Gravity,” Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].

Non-dynamical metric $f_{\mu\nu}\,(\sim \eta_{\mu\nu})$

$$\sqrt{g^{-1}f}\,:\ \sqrt{g^{-1}f}\sqrt{g^{-1}f} = g^{\mu\lambda} f_{\lambda\nu}\,.$$

Minimal extension of Fierz-Pauli action:

$$S = M_p^2 \int d^4x \sqrt{-g}\,\left[R - 2m^2 \left(\mathrm{tr}\,\sqrt{g^{-1}f} - 3\right)\right]\,.$$

$\Rightarrow$ vDVZ discontinuity $\Rightarrow$

$$S = M_p^2 \int d^4x \sqrt{-g} \left[R + 2m^2 \sum_{n=0}^{3} \beta_n \, e_n(\sqrt{g^{-1}f}) \right] ,$$

$$e_0(\mathbb{X}) = 1 , \quad e_1(\mathbb{X}) = [\mathbb{X}] , \quad e_2(\mathbb{X}) = \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]) ,$$

$$e_3(\mathbb{X}) = \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]) ,$$

$$e_4(\mathbb{X}) = \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]) ,$$

$$e_k(\mathbb{X}) = 0 \ \text{ for } k > 4 ,$$

$$\mathbb{X} = (X^\mu{}_\nu) , \quad [\mathbb{X}] \equiv X^\mu{}_\mu ,$$

$\sim$ Galileon $\Rightarrow$ Vainshtein mechanism

Bimetric gravity

S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].

Dynamical $f_{\mu\nu}$ (background independent).

$$
\begin{aligned}
S \;=\; & M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
& + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n(\sqrt{g^{-1} f})\,, \\
& 1/M_{\mathrm{eff}}^2 \equiv 1/M_g^2 + 1/M_f^2\,.
\end{aligned}
$$

$R^{(g)}$: scalar curvature for $g_{\mu\nu}$,
$R^{(f)}$: scalar curvature for $f_{\mu\nu}$.

Spectrum of the linearized theory

Minimal case:

$$\beta_0 = 3\,, \quad \beta_1 = -1\,, \quad \beta_2 = 0\,, \quad \beta_3 = 0\,, \quad \beta_4 = 1\,.$$

Linearize

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}\,.$$

$\Rightarrow$

$$S = \int d^4x \left(h_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}h_{\alpha\beta} + l_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}l_{\alpha\beta}\right)$$
$$-\frac{m^2 M_{\rm eff}^2}{4}\int d^4x \left[\left(\frac{h^\mu_{\ \nu}}{M_g} - \frac{l^\mu_{\ \nu}}{M_f}\right)^2 - \left(\frac{h^\mu_{\ \mu}}{M_g} - \frac{l^\mu_{\ \mu}}{M_f}\right)^2\right].$$

$\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$: usual Einstein-Hilbert kinetic operator.

Change of variables

$$\frac{1}{M_{\mathrm{eff}}} u_{\mu\nu} = \frac{1}{M_f} h_{\mu\nu} + \frac{1}{M_g} l_{\mu\nu} ,$$

$$\frac{1}{M_{\mathrm{eff}}} v_{\mu\nu} = \frac{1}{M_g} h_{\mu\nu} - \frac{1}{M_f} l_{\mu\nu} .$$

$\Rightarrow$

$$
\begin{aligned}
S &= \int d^4x \, (u_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}u_{\alpha\beta} + v_{\mu\nu}\hat{\mathcal{E}}^{\mu\nu\alpha\beta}v_{\alpha\beta}) \\
& \quad -\frac{m^2}{4}\int d^4x \, \left(v^{\mu\nu}v_{\mu\nu} - v^{\mu}_{\ \mu}v^{\nu}_{\ \nu}\right) \, .
\end{aligned}
$$

One massless spin-2 particle $u_{\mu\nu}$ and one massive spin-2 particle $v_{\mu\nu}$ with mass m.

Standard $F(R)$ gravity $\Leftrightarrow$ scalar tensor theory

$$S_{F(R)} = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter}\right) .$$

Introducing the auxiliary field A,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right) + F(A)\right\} .$$

Variation of $A \Rightarrow A = R$: original action

Rescaling of metric

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu} \, , \quad \sigma = - \ln F'(A) \, .$$

$\Rightarrow$ Einstein frame action:

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) \, ,$$

$$V(\sigma) = \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \, .$$

$$A = g\left(\mathrm{e}^{-\sigma}\right) \Leftarrow \sigma = - \ln\left(1 + f'(A)\right) = - \ln F'(A)$$

Coupling of σ with matters appears
by the rescaling $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$.

Adding the following actions to the bigravity action

$$S_\varphi = - M_g^2 \int d^4 x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\}$$

$$+ \int d^4 x \mathcal{L}_{\rm matter} \left(\mathrm{e}^\varphi g_{\mu\nu}, \Phi_i \right) \ ,$$

$$S_\xi = - M_f^2 \int d^4 x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\} \ .$$

Conformal transformations

$$g_{\mu\nu} \to \mathrm{e}^{-\varphi} g_{\mu\nu} \ , \quad f_{\mu\nu} \to \mathrm{e}^{-\xi} f_{\mu\nu} \ ,$$

$$
\begin{aligned}
S_F = {} & M_f^2 \int d^4x \sqrt{-\det f}\ \left\{ \mathrm{e}^{-\xi} R^{(f)} + \mathrm{e}^{-2\xi} U(\xi) \right\} \\
& + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{-1} f} \right) \\
& + M_g^2 \int d^4x \sqrt{-\det g} \left\{ \mathrm{e}^{-\varphi} R^{(g)} + \mathrm{e}^{-2\varphi} V(\varphi) \right\} \\
& + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g_{\mu\nu}, \Phi_i \right) \ .
\end{aligned}
$$

Kinetic terms of φ and ξ vanish.

Coupling of φ with matters also disappears.

Variation of φ and ξ $\Rightarrow$

$$
\begin{aligned}
0 =& 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2\right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{-1} f}\right) \\
& + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{(g)} - 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} , \\
0 =& - 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{-1} f}\right) \\
& + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{(f)} - 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} .
\end{aligned}
$$

In principle, can be solved algebraically with respect to φ and ξ

$$\begin{aligned}\varphi =& \varphi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right) \ , \\ \xi =& \xi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right) \ .\end{aligned}$$

$\Rightarrow$ analogue of $F(R)$ gravity:

$$S_F = M_f^2 \int d^4x \sqrt{-\det f}\, F^{(f)} \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right)$$
$$+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g}$$
$$\times \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2} - 2 \right) \varphi \left(R^{(g)}, e_n \left(\sqrt{g^{-1} f} \right) \right)} e_n \left(\sqrt{g^{-1} f} \right)$$
$$+ M_g^2 \int d^4x \sqrt{-\det g}\, F^{(g)} \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right)$$
$$+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g_{\mu\nu}, \Phi_i \right) \ ,$$

Here

$$F^{(g)}\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right) \equiv \left\{ \mathrm{e}^{-\varphi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)} R^{(g)} \right.$$

$$\left. + \mathrm{e}^{-2\varphi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)} V\left(\varphi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)\right) \right\} ,$$

$$F^{(f)}\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right) \equiv \left\{ \mathrm{e}^{-\xi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)} R^{(f)} \right.$$

$$\left. + \mathrm{e}^{-2\xi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)} U\left(\xi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1}f}\right)\right)\right) \right\} .$$

It is difficult to explicitly solve equations with respect to φ and ξ and it might be better to define the model by introducing the auxiliary scalar fields φ and ξ.

Cosmological Reconstruction

Minimal case:

$$S_{\rm bi} = M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\ + 2m^2 M_{\rm eff}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f}\right) .$$

Evaluation of $\delta\sqrt{g^{-1}f}$:

Two matrices M, N: $M^2 = N$.

$$\delta M M + M \delta M = \delta N \ \Rightarrow\ \mathrm{tr}\, \delta M = \frac{1}{2} \mathrm{tr}\, \left(M^{-1} \delta N \right) .$$

Start from the Einstein frame action. Neglect matter.

$\delta g_{\mu\nu} \ \Rightarrow$

$$0 = M_g^2 \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right) + m^2 M_{\text{eff}}^2 \left\{ g_{\mu\nu} \left(3 - \text{tr} \sqrt{g^{-1} f} \right) + f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\quad\ \nu} \right\} + M_g^2 \left[\frac{1}{2} \left(\frac{3}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + V(\varphi) \right) g_{\mu\nu} - \frac{3}{2} \partial_\mu \varphi \partial_\nu \varphi \right] \,.$$

$\delta f_{\mu\nu} \ \Rightarrow$

$$
\begin{aligned}
0 =\ & M_f^2 \left(\frac{1}{2} f_{\mu\nu} R^{(f)} - R^{(f)}_{\mu\nu} \right) \\
& + m^2 M_{\mathrm{eff}}^2 \left\{ f_{\mu\nu} \left(3 - \mathrm{tr}\, \sqrt{g^{-1} f} \right) - f_{\mu\sigma} \left(\sqrt{g^{-1} f} \right)^{-1\,\sigma}_{\quad\ \rho} g^{\rho\tau} f_{\tau\nu} \right\} \\
& + M_f^2 \left[\frac{1}{2} \left(\frac{3}{2} f^{\rho\sigma} \partial_\rho \xi \partial_\sigma \xi + U(\xi) \right) f_{\mu\nu} - \frac{3}{2} \partial_\mu \xi \partial_\nu \xi \right] .
\end{aligned}
$$

Assume FRW universes by using the conformal time t

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) ,$$

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

$\Rightarrow$

$$\delta g_{tt} \ : \ 0 = -\, 3M_g^2 H^2 - 3m^2 M_{\text{eff}}^2 \left(a^2 - ab\right) + \left(-\frac{3}{4}\dot{\varphi}^2 - \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 \,,$$

$$\delta g_{ij} \ : \ 0 = M_g^2 \left(2\dot{H} + H^2\right) + m^2 M_{\text{eff}}^2 \left(3a^2 - 2ab - ac\right) + \left(-\frac{3}{4}\dot{\varphi}^2 + \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 \,,$$

$$H \equiv \frac{\dot{a}}{a} \,.$$

$$\delta f_{tt} \; : \; 0 = -3M_f^2 K^2 + m^2 M_{\rm eff}^2 c^2 \left(-3 + \frac{2c}{a} + \frac{3b}{a} \right) + \left(-\frac{3}{4}\dot\xi^2 - \frac{1}{2} U(\xi) c(t)^2 \right) M_f^2 \, ,$$

$$\delta f_{ij} \; : \; 0 = M_f^2 \left(2\dot K + 3K^2 - 2LK \right) + m^2 M_{\rm eff}^2 c^2 \left(3 - \frac{c}{a} - \frac{7b}{a} \right) + \left(-\frac{3}{4}\dot\xi^2 + \frac{1}{2} U(\xi) c(t)^2 \right) M_f^2 \, .$$

$$K \equiv \dot b / b \, , \quad L = \dot c / c \, .$$

Redefinition of the scalar fields: $\varphi = \varphi(\eta)$, $\xi = \xi(\zeta)$.
Identify $\eta = \zeta = t$

$$\begin{aligned} \omega(t) M_g^2 =& 4 M_g^2 \left(\dot H - H^2 \right) + m^2 M_{\mathrm{eff}}^2 \left(- ac + ab \right) \, , \\ \tilde V(t) a(t)^2 M_g^2 =& - M_g^2 \left(2 \dot H + 4 H^2 \right) \\ & - m^2 M_{\mathrm{eff}}^2 \left(6 a^2 - 5 ab - ac \right) \, , \\ \sigma(t) M_f^2 =& 4 M_f^2 \left(\dot K - L K \right) + 2 m^2 M_{\mathrm{eff}}^2 c^2 \left(\frac{c}{a} - \frac{b}{a} \right) \, , \\ \tilde U(t) a(t)^2 M_f^2 =& - M_f^2 \left(2 \dot K + 6 K^2 - 2 L K \right) \\ & - m^2 M_{\mathrm{eff}}^2 c^2 \left(6 - \frac{3c}{a} - \frac{7b}{a} \right) \, . \end{aligned}$$

Here

$$\omega(\eta) = 3\varphi'(\eta)^2 \,, \quad \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right) \,,$$
$$\sigma(\zeta) = 3\xi'(\zeta)^2 \,, \quad \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right) \,.$$

For arbitrary $a(t)$ and $b(t)$, if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy the above equations, a model admitting the given $a(t)$ and $b(t)$ evolution can be reconstructed.

Physical metric: the scalar field does not directly coupled with matter.

$$g_{\mu\nu}^{\mathrm{phys}} = \mathrm{e}^{\varphi} g_{\mu\nu} \, .$$

FRW universe:

$$ds^2 = \tilde{a}(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i \right)^2 \right) \, .$$

$\tilde{a}(t)^2 = \frac{l^2}{t^2}$: de Sitter universe.
$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$ with $n \neq 1$ case:

Redefinition of time coordinate:

$d\tilde{t} = \pm \frac{l^n}{t^n} dt \ \left(\tilde{t} = \pm \frac{l^n}{n-1} t^{1-n} \right)$

$$\Rightarrow \ ds^2 = -d\tilde{t}^2 + \left(\pm (n-1) \frac{\tilde{t}}{l} \right)^{-\frac{2n}{1-n}} \sum_{i=1}^{3} \left(dx^i \right)^2 .$$

$0 < n < 1$: phantom universe,

$n > 1$: quintessence universe,

$n < 0$: decelerating universe

$$e^{\varphi(t)} a(t)^2 = \tilde{a}(t)^2 \quad \left(g^{\text{phys}}_{\mu\nu} = e^{\varphi} g_{\mu\nu}\right) \quad \Rightarrow$$

$$\omega(t) = 12\left(H - \tilde{H}\right)^2 \ \left(\omega(\eta) = 3\varphi'(\eta)^2\right) , \quad \tilde{H} \equiv \frac{1}{\tilde{a}}\frac{d\tilde{a}}{dt}$$

Universe with $a(t) = c(t) = 1$

If we choose $a(t) = 1$ (flat Minkowski space) and $c(t) = 1$,

$$
\begin{aligned}
12 M_g^2 \tilde{H}^2 &= - m^2 M_{\text{eff}}^2 (1-b) \ , \\
\tilde{V}(t) M_g^2 &= 5 m^2 M_{\text{eff}}^2 (1-b) = 60 M_g^2 \tilde{H}^2 \, , \\
\sigma(t) M_f^2 &= 4 M_f^2 \dot{K} + 2 m^2 M_{\text{eff}}^2 (1-b) \\
&= 4 M_f^2 \dot{K} - 24 M_g^2 \tilde{H}^2 \, , \\
\tilde{U}(t) M_f^2 &= - M_f^2 \left(2\dot{K} + 6K^2 \right) - m^2 M_{\text{eff}}^2 (3 - 7b) \\
&= - M_f^2 \left(2\dot{K} + 6K^2 \right) + 4 m^2 M_{\text{eff}}^2 + 84 M_g^2 \tilde{H}^2 \, .
\end{aligned}
$$

$\Rightarrow$

$$\sigma(t) M_f^2 = \frac{\frac{96 M_f^2 M_g^2}{m^2 M_{\rm eff}^2} \left\{ \tilde H \ddot{\tilde H} + \dot{\tilde H}^2 + \frac{12 M_g^2}{m^2 M_{\rm eff}^2} \left(\tilde H^3 \ddot{\tilde H} - \tilde H^2 \dot{\tilde H}^2 \right) \right\}}{\left(1 + \frac{12 M_g^2}{m^2 M_{\rm eff}^2} \tilde H^2 \right)^2} - 24 M_g^2 \tilde H^2 \,,$$

$$\tilde U(t) M_f^2 = - \frac{\frac{48 M_f^2 M_g^2}{m^2 M_{\rm eff}^2} \left\{ \tilde H \ddot{\tilde H} + \dot{\tilde H}^2 + \frac{12 M_g^2}{m^2 M_{\rm eff}^2} \left(\tilde H^3 \ddot{\tilde H} + 5 \tilde H^2 \dot{\tilde H}^2 \right) \right\}}{\left(1 + \frac{12 M_g^2}{m^2 M_{\rm eff}^2} \tilde H^2 \right)^2} + 4 m^2 M_{\rm eff}^2 + 84 M_g^2 \tilde H^2 \,.$$

$\sigma(t)$ is not always positive:
inconsistent with $\sigma(\zeta) = 3 \xi'(\zeta)^2$, ghost.

In case $\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$,

$$\sigma(t) M_f^2 = \frac{\frac{288 M_f^2 M_g^2 n^2}{m^2 M_{\rm eff}^2 t^4} \left(1 + \frac{4 M_g^2 n^2}{m^2 M_{\rm eff}^2 t^2} \right)}{\left(1 + \frac{12 M_g^2 n^2}{m^2 M_{\rm eff}^2 t^2} \right)^2} - \frac{24 M_g^2 n^2}{t^2} ,$$

$$\tilde{U}(t) M_f^2 = - \frac{\frac{144 M_f^2 M_g^2 n^2}{m^2 M_{\rm eff} t^4} \left(1 + \frac{28 M_g^2 n^2}{m^2 M_{\rm eff}^2 t^2} \right)}{\left(1 + \frac{12 M_g^2 n^2}{m^2 M_{\rm eff}^2 t^2} \right)^2} + 4 m^2 M_{\rm eff}^2 \left(1 + 21 \frac{M_g^2 n^2}{m^2 M_{\rm eff}^2 t^2} \right) .$$

When t is small,

$$\sigma(t) \sim \left(8 - 24 \frac{M_g^2}{M_f^2} n^2 \right) \frac{1}{t^2} \, .$$

In order to avoid the ghost,

$$\frac{M_f^2}{M_g^2} > 3n^2 \, .$$

When t is large,

$$\sigma(t) \sim - \frac{24 M_g^2 n^2}{M_f^2 t^2} \, .$$

Ghost or inconsistent!

Universe with $a(t) = b(t) = c(t)$

We may choose $a(t) = b(t) = c(t)$, $H = K = L \Rightarrow$

$$3\left(H - \tilde{H}\right)^2 = \dot{H} - H^2 \,,$$

$$\tilde{V}(t)a(t)^2 M_g^2 = - M_g^2 \left(2\dot{H} + 4H^2\right) \,,$$

$$\sigma(t) M_f^2 = 4M_f^2 \left(\dot{H} - H^2\right) \,,$$

$$\tilde{U}(t)a(t)^2 M_f^2 = - M_f^2 \left(2\dot{H} + 4H^2\right) + 4m^2 M_{\mathrm{eff}}^2 a^2 \,,$$

$$\Rightarrow \sigma(t) = \omega(t) = 12\left(H - \tilde{H}\right)^2 \,,$$

No ghost

To determine a, solve the differential equation

$$3\left(H-\tilde{H}\right)^2=\left(\dot{H}-H^2\right)\,.$$

In case that there is no solution in the differential equation, we cannot reconstruct such a model.

In case $\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$,

$$H = \frac{h_0}{t} ,$$
$$h_0 = \frac{-6n - 1 \pm \sqrt{-12n^2 + 12n + 1}}{8} .$$

In order that h_0 to be real,

$$\frac{3 - 2\sqrt{3}}{6} < n < \frac{3 + 2\sqrt{3}}{6} .$$

$0 > n > \frac{3-2\sqrt{3}}{6} = -0.07735 \cdots \Leftrightarrow$ decelerating

$0 < n < 1 \Leftrightarrow$ phantom

$1 < n < \frac{3+2\sqrt{3}}{6} = 1.07735 \cdots \Leftrightarrow$ quintessence

$\Rightarrow$

$$\omega(\eta) = \frac{12\left(h_0 + n\right)^2}{\eta^2}\,,$$

$$\sigma(\zeta) = \frac{12\left(h_0 + n\right)^2}{\zeta^2}\,,$$

$$\tilde{V}(\eta) = \frac{-4h_0^2 + 2h_0}{a_0^2 \eta^{2+2h_0}}\,,$$

$$\tilde{U}(\zeta) = \frac{-4h_0^2 + 2h_0}{a_0^2 \zeta^{2+2h_0}} + \frac{4m^2 M_{\mathrm{eff}}^2}{M_f^2}\,.$$

4 Summary

- Construction $F(R)$ bigravity via auxiliary fields. Stability of solution?
- Correction to the Newton law?

In general, the cosmological singularities in g metric are manifested as cosmological singularities in f metric. However, there are models where cosmological singularity does not occur in g metric while it occurs in f metric and vice-versa.

Example:

$n = 1$ case (de Sitter, no singularity) when $a(t) = c(t) = 1$.

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2$$

$$\sim - dt^2 + \left(\frac{12 M_g^2}{m^2 M_{\mathrm{eff}}^2 t^2} \right)^2 \left(dx^i\right)^2 .$$

$\Rightarrow$

$$\left(ds_f^{\mathrm{J}}\right)^2 = \sum_{\mu,\nu=0}^{3} f^{\mathrm{J}}_{\mu\nu} dx^\mu dx^\nu = \mathrm{e}^\xi ds_f^2$$

$$\propto \frac{1}{t^{\mp 2\alpha}} \left(-dt^2 + \left(\frac{12 M_g^2}{m^2 M_{\mathrm{eff}}^2 t^2} \right)^2 \left(dx^i\right)^2 \right)$$

$$\propto - dt_f^2 + \frac{144 M_g^4}{m^4 M_{\mathrm{eff}}^4 \left|1 \pm \alpha\right|^{\frac{4\mp 2\alpha}{1\pm\alpha}} t_f^{\frac{4\mp 2\alpha}{1\pm\alpha}}} \left(dx^i\right)^2 ,$$

$$t_f \equiv \frac{t^{1\pm\alpha}}{\left|1\pm\alpha\right|} .$$

If $\frac{4\mp 2\alpha}{1\pm\alpha} > 0$, $t_f = 0$ corresponds to Big Rip singularity.