Cosmological Perturbations in F(R) Gravity

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Introduction

- Observations of type Ia supernovae give us the information about background evolution of the Universe.

↙ ↘

Homogeneous Universe   Inhomogeneous Universe

↓

Accelerated expansion of the Universe

→ Modification of the Einstein equation is needed.

$$R_{\mu\nu} - g_{\mu\nu}R/2 = \kappa^2 T_{\mu\nu}$$

Modified Gravity	Dark Energy
F(R) gravity	Cosmological Constant
Scalar-Tensor theory	Quintessence
Galileon	Ghost Condensate
…	…

Introduction II

- How to clarify what model describes the nature?

- Background evolution of the Universe

(SN Ia, CMB, BAO, ...)

However, the method, `` reconstruction", have been developed in dark energy and modified gravity models. Therefore, we can not find the differences between models in the background evolution of the Universe.

- The matter density perturbation

(LSS, Microlensing of the galaxies and quasars)

FL equations in F(R) gravity

Action: $$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[R + f(R)] + S_{\mathrm{matter}}.$$ $\kappa^2 = 8\pi G$

Metric: $$ds^2 = a^2(\eta)(d\eta^2 - \textstyle\sum_{i=1}^{3} dx^i dx^i)$$

Friedmann-Lemaitre equations

$$\frac{3\mathcal{H}'}{a^2}(1+f_R) - \frac{1}{2}(R+f) - \frac{3\mathcal{H}}{a^2}f_R' = -\kappa^2\rho$$

$$\frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1+f_R) - \frac{1}{2}(R+f) - \frac{1}{a^2}(\mathcal{H}f_R' + f_R'') = \kappa^2 w\rho$$

$w = p/\rho$ $\quad \mathcal{H} \equiv a'/a \quad R = 6a^{-2}(\mathcal{H}' + \mathcal{H}^2) \quad f_R \equiv df(R)/dR$

The equation of continuity

$$\rho' + 3(1+w)\mathcal{H}\rho = 0.$$

Perturbative equations

Newtonian gauge $ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - (1+2\Psi)\sum_{i=1}^{3} dx^i dx^i]$

Einstein equations

(0,0)
$$(1+f_R)\{-k^2(\Phi+\Psi) - 3\mathcal{H}(\Phi'+\Psi') + (3\mathcal{H}' - 6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi\} + f'_R(-9\mathcal{H}\Phi + 3\mathcal{H}\Psi - 3\Psi') = \kappa^2 \rho a^2 \delta$$

(i,i)
$$(1+f_R)\{\Phi''+\Psi'' + 3\mathcal{H}(\Phi'+\Psi') + 3\mathcal{H}'\Phi + (\mathcal{H}' + 2\mathcal{H}^2)\Psi\} + f'_R(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Phi') + f''_R(3\Phi - \Psi) = c_s^2 \kappa^2 \rho a^2 \delta$$

(0,i)
$$(1+f_R)\{\Phi' + \Psi' + \mathcal{H}(\Phi+\Psi)\} + f'_R(2\Phi - \Psi) = -\kappa^2 \rho a^2 (1+w) v$$

(i,j)
$$\Phi - \Psi - \frac{2 f_{RR}}{a^2(1+f_R)}\{3\Psi'' + 6(\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}(\Phi' + 3\Psi') - k^2(\Phi - 2\Psi)\} = 0$$

The equations of continuity

(0)
$$3\Psi'(1+w) - \delta' + 3\mathcal{H}(w - c_s^2)\delta + k^2(1+w)v = 0,$$

(i)
$$\Phi + \frac{c_s^2}{1+w}\delta + v' + \mathcal{H}v(1-3w) = 0.$$

$\delta = \delta\rho/\rho$

$c_s^2 \equiv \delta p/\delta\rho$

Subhorizon approximation

Quasi-static approximation

$\phi' \sim H\phi,\ \psi' \sim H\psi,\ \delta' \sim H\delta,\ H' \sim H^2$

Small scale approximation

$H << k$

Einstein equations

(0,0)

$$(1+f_R)\{-k^2(\Phi+\Psi)-3\mathcal{H}(\Phi'+\Psi')+(3\mathcal{H}'-6\mathcal{H}^2)\Phi-3\mathcal{H}'\Psi\} + f'_R(-9\mathcal{H}\Phi+3\mathcal{H}\Psi-3\Psi') = \kappa^2\rho a^2\delta$$

(i,i)

$$(1+f_R)\{\Phi''+\Psi''+3\mathcal{H}(\Phi'+\Psi')+3\mathcal{H}'\Phi+(\mathcal{H}'+2\mathcal{H}^2)\Psi\} + f'_R(3\mathcal{H}\Phi-\mathcal{H}\Psi+3\Phi')+f''_R(3\Phi-\Psi) = c_s^2\kappa^2\rho a^2\delta$$

(0,i)

$$(1+f_R)\{\Phi'+\Psi'+\mathcal{H}(\Phi+\Psi)\}+f'_R(2\Phi-\Psi) = -\kappa^2\rho a^2(1+w)v$$

(i,j)

$$\Phi-\Psi-\frac{2f_{RR}}{a^2(1+f_R)}\{3\Psi''+6(\mathcal{H}'+\mathcal{H}^2)\Phi+3\mathcal{H}(\Phi'+3\Psi')-k^2(\Phi-2\Psi)\}=0$$

The equations of continuity

(0)

$$3\Psi'(1+w)-\delta'+3\mathcal{H}(w-c_s^2)\delta+k^2(1+w)v = 0,$$

(i)

$$\Phi+\frac{c_s^2}{1+w}\delta+v'+\mathcal{H}v(1-3w) = 0.$$

$\delta = \delta\rho/\rho$

$c_s^2 \equiv \delta p/\delta\rho$

Subhorizon approximation

Quasi-static approximation

$\phi' \sim H\phi,\ \psi' \sim H\psi,\ \delta' \sim H\delta,\ H' \sim H^2$

Small scale approximation

$H << \mathrm{k}$

Einstein equations

(0,0)

$$(1+f_R)\{-k^2(\Phi+\Psi) \quad \}$$
$$+ f'_R(-9\mathcal{H}\Phi + 3\mathcal{H}\Psi - 3\Psi') = \kappa^2 \rho a^2 \delta$$

(i,i)

$$(1+f_R)\{\Phi''+\Psi''+3\mathcal{H}(\Phi'+\Psi')+3\mathcal{H}'\Phi+(\mathcal{H}'+2\mathcal{H}^2)\Psi\}$$
$$+f'_R(3\mathcal{H}\Phi-\mathcal{H}\Psi+3\Phi')+f''_R(3\Phi-\Psi) = c_s^2\kappa^2\rho a^2\delta$$

(0,i)

$$(1+f_R)\{\Phi'+\Psi'+\mathcal{H}(\Phi+\Psi)\}+f'_R(2\Phi-\Psi) = -\kappa^2\rho a^2(1+w)v$$

(i,j)

$$\Phi - \Psi - \frac{2f_{RR}}{a^2(1+f_R)}\{ \quad k^2(\Phi-2\Psi)\} = 0$$

The equations of continuity

(0)

$$3\Psi'(1+w) - \delta' + 3\mathcal{H}(w-c_s^2)\delta + k^2(1+w)v = 0,$$

(i)

$$\Phi + \frac{c_s^2}{1+w}\delta + v' + \mathcal{H}v(1-3w) = 0.$$

$\delta = \delta\rho/\rho$

$c_s^2 \equiv \delta p/\delta\rho$

Approximations adequate to a case

(0,0)
$$(1+f_R)\{-k^2(\Phi+\Psi) \quad \}$$
$$+f'_R(-9\mathcal{H}\Phi+3\mathcal{H}\Psi-3\Psi') = \kappa^2\rho a^2\delta$$

(i,j)
$$\Phi-\Psi-\frac{2f_{RR}}{a^2(1+f_R)}\{ \quad k^2(\Phi-2\Psi)\}=0$$

The equations of continuity

(0)
$$3\Psi'(1+w)-\delta'+3\mathcal{H}(w-c_s^2)\delta+k^2(1+w)v = 0,$$

$\delta = \delta\rho/\rho$

(i)
$$\Phi+\frac{c_s^2}{1+w}\delta+v'+\mathcal{H}v(1-3w) = 0.$$

$c_s^2 \equiv \delta p/\delta\rho$

	$\lvert f_{RR}\rvert k^2/a^2 \ll 1$	$\lvert f_{RR}\rvert k^2/a^2 \gg 1$
(i,j) →	$\phi = \psi$	$\phi = 2\psi$

$H^4\lvert f_{RR}\rvert \ll a^2k^2$

(0,0) → $-k^2(1+f_R)(\phi+\psi) = \kappa^2\rho a^2\delta$

$w, c_s = 0$

(0), (i) → $\delta'' + H\delta' - 3\psi'' - 3H\psi' + k^2\phi = 0$

Quasi-static equation of δ

$|f_{RR}|k^2/a^2 << 1$

$$\delta'' + \mathcal{H}\delta' - \frac{3a^2\Omega_{\rm m}}{2(1+f_R)}\delta = 0.$$

$|f_{RR}|k^2/a^2 >> 1$

$$\delta'' + \mathcal{H}\delta' - \frac{2a^2\Omega_{\rm m}}{(1+f_R)}\delta = 0,$$

$\Omega m = \kappa^2\rho/(3a^2 H^2)$

↓

Consistent with the equation in ΛCDM model:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_{\rm m}H^2\delta \approx 0.$$

$H = a\,\mathcal{H}$

Note: There are several ways to derive the differential equation of matter density perturbation when we use approximations. However, some procedure deduce the wrong result caused by the cancellation of leading terms.

Other solutions of δ

$\partial/\partial\eta \sim H,\ H << \mathrm{k}$ → $\partial/\partial\eta \sim \mathrm{k},\ H << \mathrm{k}$

(i,i)

$$(1+f_R)\{\Phi''+\Psi''+3\mathcal{H}(\Phi'+\Psi')+3\mathcal{H}'\Phi+(\mathcal{H}'+2\mathcal{H}^2)\Psi\}$$
$$+f_R'(3\mathcal{H}\Phi-\mathcal{H}\Psi+3\Phi')+f_R''(3\Phi-\Psi)=c_s^2\kappa^2\rho a^2\delta$$

$H^4|f_{RR}| << a^2k^2$, w, $c_s = 0$ → $\Phi \approx -\Psi$

(i,j) → $$\Psi+\frac{3f_{RR}}{a^2(1+f_R)}(\Psi''+k^2\Psi)\approx 0.$$

$|f_{RR}|k^2/a^2 << 1$ — $|f_{RR}|k^2/a^2 >> 1$

$\psi = 0$ — $\Psi = C_1 e^{-ik\eta} + C_2 e^{ik\eta},$

(0), (i) → $\delta'' \approx 2\Psi''$ → $\delta = 2C_1 e^{-ik\eta} + 2C_2 e^{ik\eta} + C_3\eta + C_4,$

There are also oscillating solutions, but the effective growth rate of δ is not determined.

Differential equation without approximations

A. de la Cruz-Dombriz, A. Dobado, and A. L. Maroto, Phys. Rev. D **77**, 123515 (2008)

$|f_{RR}|k^2/a^2 >> 1$

$$\delta'''' + \mathcal{H}\left(3 + \frac{f_R'}{\mathcal{H}(1+f_R)} + O(\mathcal{H}^2/k^2)\right)\delta''' + \mathcal{H}^2\left(\frac{k^2}{\mathcal{H}^2} + O(1)\right)\delta''$$
$$+ \mathcal{H}^3\left(\frac{k^2}{\mathcal{H}^2} + O(1)\right)\delta' - \mathcal{H}^4\left(\frac{2\kappa^2 k^2 \rho a^2}{3\mathcal{H}^4(1+f_R)} + O(1)\right)\delta = 0,$$

$\partial/\partial\eta \sim \mathcal{H}$, $\mathcal{H} << k$ →

$$\frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\rm eff}\right)\frac{d\delta}{dN} - \frac{2}{1+f_R}\Omega_{\rm m}\delta = 0.$$

$N = \ln a(t)$ $\quad w_{\rm eff} = -2\dot{H}/(3H^2) - 1$, $\quad H \equiv \dot{a}(t)/a(t)$

$\partial/\partial\eta \sim k$, $\mathcal{H} << k$

→ $$\delta(\eta) = C_1(\eta)\mathrm{e}^{ik\int d\eta} + C_2(\eta)\mathrm{e}^{-ik\int d\eta}$$
$$= const. \times \frac{1}{a\sqrt{1+f_R}}\mathrm{e}^{ik\int d\eta} + const. \times \frac{1}{a\sqrt{1+f_R}}\mathrm{e}^{-ik\int d\eta}$$

Differential equation without approximations II

$|f_R| k^2/H^2$, $|f_{RR}| k^2/a^2 << 1$

$$\delta'''' + \left\{ \frac{12\mathcal{H}^2(-2+\mathcal{H}''/\mathcal{H}^3) f_{RRR}}{a^2 f_{RR}} + \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right\} \mathcal{H}\delta'''$$
$$+\chi^2 \left\{ (1+O(\mathcal{H}^2/\chi^2))\,\delta'' + \mathcal{H}\,(1+O(\mathcal{H}^2/\chi^2))\,\delta' + \mathcal{H}^2 \left(2\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right) \delta \right\} = 0,$$

$$\chi \equiv \sqrt{\frac{a^2}{3 f_{RR}} \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}}.$$

$\partial/\partial\eta \sim \mathcal{H}$, $\quad \mathcal{H} << k, \chi$

$$\longrightarrow \quad \frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2} w_{\text{eff}} \right) \frac{d\delta}{dN} + \left(2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \right) \delta = 0.$$

$$2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} = -\frac{3\Omega_{\text{m}}}{2(1+f_R)} - \frac{f_R'}{\mathcal{H}(1+f_R)} - \frac{f_R''}{\mathcal{H}^2(1+f_R)} + \frac{f_R'''}{2\mathcal{H}^3(1+f_R)}.$$

Differential equation without approximations III

$|f_R|k^2/H^2$, $|f_{RR}|k^2/a^2 << 1$

$$\delta'''' + \left\{ \frac{12\mathcal{H}^2(-2+\mathcal{H}''/\mathcal{H}^3)f_{RRR}}{a^2 f_{RR}} + \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right\} \mathcal{H}\delta'''$$
$$+\chi^2 \left\{ \left(1+O(\mathcal{H}^2/\chi^2)\right)\delta'' + \mathcal{H}\left(1+O(\mathcal{H}^2/\chi^2)\right)\delta' + \mathcal{H}^2\left(2\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2)\right)\delta \right\} = 0,$$

$$\chi \equiv \sqrt{\frac{a^2}{3f_{RR}} \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}}.$$

$\partial/\partial\eta \sim \chi$, $\mathcal{H} << k, \chi$

→

$$\delta(\eta) = C_1 e^{\int f_{\mathrm{eff}} dN + i\int \chi d\eta} + C_2 e^{\int f_{\mathrm{eff}} dN - i\int \chi d\eta}.$$

$$f_{\mathrm{eff}} = -\frac{5}{2}\frac{d}{dN}\ln|\chi| - \frac{d}{dN}\ln|a^3 f_{RR}^2| - \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}.$$

Actual models of F(R) gravity

W. Hu and I. Sawicki, Phys. Rev. D **76**,064004 (2007).

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

A. A. Starobinsky, JTEP Lett. **86**, 157 (2007).

$$f(R) = \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right)$$

E. V. Linder, Phys. Rev. D **80**, 123528 (2009); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **77**, 046009 (2008).

$$f(R) = -cr(1 - \mathrm{e}^{-R/r}),$$

The parameters in these models are tuned to be $|f_R| << 1$ to satisfy local gravity constraints, and $|f_{RR}|k^2/a^2 << 1$ would be satisfied as a result of it.

Behaviors of the solutions

Quasi-static modes

$$\frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\rm eff}\right)\frac{d\delta}{dN} + \left(2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}\right)\delta = 0.$$

$$2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} = -\frac{3\Omega_{\rm m}}{2(1+f_R)} - \frac{f_R'}{\mathcal{H}(1+f_R)} - \frac{f_R''}{\mathcal{H}^2(1+f_R)} + \frac{f_R'''}{2\mathcal{H}^3(1+f_R)}.$$

$$\longrightarrow \quad \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_{\rm m}H^2\delta \approx 0$$

Same as ΛCDM model's

f = dlnδ/dN = 1

Fast variational modes

$$\delta(\eta) = C_1 e^{\int f_{\rm eff}dN + i\int \chi d\eta} + C_2 e^{\int f_{\rm eff}dN - i\int \chi d\eta}.$$

$$f_{\rm eff} = -\frac{5}{2}\frac{d}{dN}\ln|\chi| - \frac{d}{dN}\ln|a^3 f_{RR}^2| - \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}.$$

$$\chi \equiv \sqrt{\frac{a^2}{3f_{RR}}\frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}}.$$

$$\frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3} \sim -1$$

Behaviors of the solutions II

Best fit models to the Sloan Digital Sky Survey (SDSS) and the seven years data of the Wilkinson Microwave Anisotropy Prove (WMAP7)

V. F. Cardone, S. Camera, and A. Diaferio, [arXiv:1201.3272 [astro-ph.CO]]

$n = 1.53$, $c_1 = 10^{3.47}$ and $c_2 = 10^{2.28}$, $$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

$n = 1.34$, $\lambda = 10^{1.50}$ and $R_\star/R_0 = 10^{-1.73}$, $$f(R) = \lambda R_\star \left(\left(1 + \frac{R^2}{R_\star^2} \right)^{-n} - 1 \right)$$

$$f(R) = -cr(1 - \mathrm{e}^{-R/r}),$$

→ $f_{RR} < 0$

$$\delta(\eta) = C_1 \mathrm{e}^{\int f_{\mathrm{eff}} dN + i \int \chi d\eta} + C_2 \mathrm{e}^{\int f_{\mathrm{eff}} dN - i \int \chi d\eta}. \quad \chi \equiv \sqrt{\frac{a^2}{3 f_{RR}} \frac{1 - \mathcal{H}'/\mathcal{H}^2}{-2 + \mathcal{H}''/\mathcal{H}^3}}.$$

The growth rate of the matter density perturbation, $f = d\ln\delta/dN = \chi/\mathcal{H}$, is too large compared to the observed value.

Summary

- Cosmological perturbations in F(R) gravity have been considered.

- Not only quasi-static modes of the matter density perturbation but also fast variational modes of that should be considered.

- Models thought of viable ones have an instability when we fit the background evolution to that of ΛCDM model.

- Exponential type model is completely rejected.