Teleparallel Dark Energy with Purely Non-minimal Coupling to Gravity

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Based on:

CQ Geng, CC Lee, E. N. Saridakis, YP Wu Phys. Lett. B704, 384 (2011)
CQ Geng, CC Lee, E. N. Saridakis JCAP 1201, 002 (2012)
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Outline

- Teleparallel Gravity
- Teleparallel Dark Energy model
- Observational Constraints
- Without Potential Case
- Summary

What is the feature of teleparallel gravity?

- An alternative theory of gravity, which is equivalent to General Relativity.
- This is a curvatureless gravity theory, and the gravitational effect comes from torsion instead of curvature.

Teleparallel Gravity
A brief introduction

- The dynamical variable of teleparallel gravity is the vierbein fields $\mathbf{e}_A(x^\mu)$, which form an orthonormal basis for the tangent space at each point x^μ of the manifold: $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$, where $\eta_{AB} = diag(1, -1, -1, -1)$.
- Notation:
  Greek indices $\mu, \nu, \ldots$: coordinate space-time.
  Latin indices $A, B, \ldots$: tangent space-time.
- The relationship between metric and vierbein fields is

$$g_{\mu\nu}(x) = \eta_{AB}\, e^A_\mu(x)\, e^B_\nu(x).$$

- Weitzenböck connection: a curvatureless connection

$$\overset{\mathbf{w}}{\Gamma}{}^{\lambda}_{\nu\mu} \equiv e^{\lambda}_{A}\, \partial_{\mu} e^{A}_{\nu}$$

- The torsion tensor is defined as

$$T^{\lambda}_{\mu\nu} \equiv \overset{\mathbf{w}}{\Gamma}{}^{\lambda}_{\nu\mu} - \overset{\mathbf{w}}{\Gamma}{}^{\lambda}_{\mu\nu} = e^{\lambda}_{A}\,(\partial_{\mu} e^{A}_{\nu} - \partial_{\nu} e^{A}_{\mu}).$$

- Under Weitzenböck connection, the Riemann tensor vanishes:

$$R^{\rho}_{\mu\sigma\nu} = \overset{\mathbf{w}}{\Gamma}{}^{\rho}_{\mu\nu,\sigma} - \overset{\mathbf{w}}{\Gamma}{}^{\rho}_{\mu\sigma,\nu} + \overset{\mathbf{w}}{\Gamma}{}^{\rho}_{\delta\sigma} \overset{\mathbf{w}}{\Gamma}{}^{\delta}_{\mu\nu} - \overset{\mathbf{w}}{\Gamma}{}^{\rho}_{\delta\nu} \overset{\mathbf{w}}{\Gamma}{}^{\delta}_{\mu\sigma} = 0.$$

Teleparallel Gravity
A brief introduction

- We can construct the "teleparallel Lagrangian" by using the torsion tensor,
  $\mathcal{L}_T = T = a_1 T^{\rho\mu\nu} T_{\rho\mu\nu} + a_2 T^{\rho\mu\nu} T_{\nu\mu\rho} + a_3 T_{\rho\mu}{}^{\rho} T_{\nu}{}^{\mu\nu}$.
- It is a good approach of General Relativity when we choose the suitable parameters $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{2}$ and $a_3 = -1$:

$$\tilde{R} = -T - 2\nabla^{\mu} T^{\nu}{}_{\mu\nu},$$

where $\tilde{R}$ is constructed by Levi-Civita connection.

Teleparallel Gravity
A brief introduction

- The action of teleparallel gravity is

$$S = \int d^4xe \left[\frac{T}{2\kappa^2} + \mathcal{L}_M \right],$$

where $e = det\left(e^A_{\ \mu}\right) = \sqrt{-g}$.

- Varying this action respect to the vierbein fields gives the field equation

$$e^{-1}\partial_\mu(ee^\rho_A S_\rho{}^{\mu\nu}) - e^\lambda_A T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} - \frac{1}{4}e^\nu_A T = \frac{\kappa^2}{2} e^\rho_A \overset{\mathbf{em}}{T}{}_\rho{}^\nu,$$

where $\overset{\mathbf{em}}{T}{}_\rho{}^\nu$ stands for the energy-momentum tensor and $S_\rho{}^{\mu\nu} = \frac{1}{4}\left(T^{\nu\mu}{}_\rho - T^{\mu\nu}{}_\rho + T_\rho{}^{\mu\nu}\right) + \frac{1}{2}\left(\delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha\right)$.

- Teleparallel dark energy model is a dark energy model, which can explain the late time accelerating universe.
- This model combines quintessence model with teleparallel gravity.
- This model differs from quintessence model when we turn on the non-minimal coupling term.

- Quintessence is one of the most popular dark energy model.
- The generalized quintessence model action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + \xi R\phi^2\right) - V(\phi) + \mathcal{L}_M\right]$$

- Under the flat Friedmann-Robertson-Walker (FRW) background $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, the gravitational field equations can be derived as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\left(\rho_\phi + \rho_m + \rho_r\right),$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\rho_\phi + p_\phi + \rho_m + 4\rho_r/3\right),$$

where ρ_ϕ and p_ϕ are the effective energy and pressure density

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 6\xi H\phi\dot{\phi} + 3\xi H^2\phi^2,$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + \xi\left(2\dot{H} + 3H^2\right)\phi^2 + 4\xi H\phi\dot{\phi} + 2\xi\phi\ddot{\phi} + 2\xi\dot{\phi}^2$$

Teleparallel Dark Energy

- Similar to quintessence model, we can construct teleparallel dark energy model, and the action is given by

$$S = \int d^4x e \left[\frac{T}{2\kappa^2} + \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 \right) - V(\phi) + \mathcal{L}_M \right].$$

- Variation of action with respect to the vierbein fields yields the field equation

$$\left(\frac{2}{\kappa^2} + 2\xi\phi^2 \right) \left[e^{-1} \partial_\mu (e e_A^\rho S_\rho{}^{\mu\nu}) - e_A^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} - \frac{1}{4} e_A^\nu T \right]$$
$$- e_A^\nu \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + e_A^\mu \partial^\nu \phi \partial_\mu \phi$$
$$+ 4\xi e_A^\rho S_\rho{}^{\mu\nu} \phi \left(\partial_\mu \phi \right) = e_A^\rho \overset{\mathbf{em}}{T}{}_\rho{}^\nu .$$

Teleparallel Dark Energy

- Again, the effective energy and pressure density under FRW metric $(e_\mu^A = \mathrm{diag}(1,a,a,a))$ are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 3\xi H^2\phi^2,$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + 4\xi H\phi\dot{\phi} + \xi\left(3H^2 + 2\dot{H}\right)\phi^2.$$

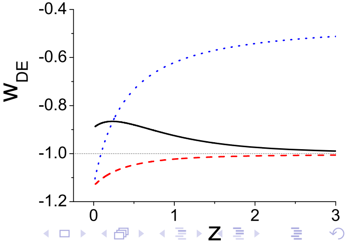
- Variation of action with respect to the scalar field gives us the equation of motion of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + 6\xi H^2\phi + V'(\phi) = 0.$$

- These equations lead to the continuity equation $\dot{\rho}_\phi + 3H(1+w_\phi)\rho_\phi = 0$, where w_ϕ is the equation of state of the scalar field, which is defined as $w_\phi \equiv \frac{p_\phi}{\rho_\phi}$.

Teleparallel Dark Energy

- In the minimal coupling case ($\xi = 0$), the teleparallel dark energy is equivalent to quintessence model
- However, these two models are different theories when we turn on the non-minimal coupling constant ($\xi \neq 0$)
- Teleparallel dark energy model can cross the phantom-divide easily.

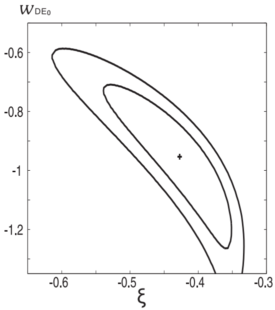


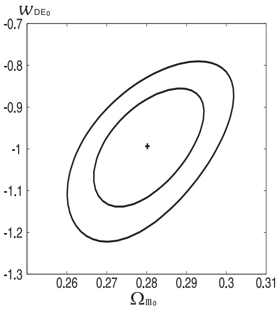
Teleparallel Dark Energy: Observational Constraints

- We would like to test teleparallel dark energy model by using the SNIa, BAO and CMB data. These observational data can tell us whether this is a suitable model for dark energy or not
- We consider three kinds of potential cases:
  Power-Law potential: $V(\phi) = V_0\phi^4$
  Exponential potential: $V(\phi) = V_0 e^{-\kappa\phi}$
  Inverse hyperbolic cosine potential: $V(\phi) = \frac{V_0}{cosh(\kappa\phi)}$

Teleparallel Dark Energy: Observational Constraints

- Potential: $V(\phi) = V_0\phi^4$
- Left: fixing $\Omega_m = 27\%$, the best fit locates at $h \simeq 0.7$, $\xi \simeq -0.42$, $w_\phi \simeq -0.96$ and $\chi^2 \simeq 543.9$
  Right: fixing $\xi = -0.41$, the best fit locates at $h \simeq 0.7$, $\Omega_m \simeq 28.0\%$, $w_\phi \simeq -0.99$ and $\chi^2 \simeq 544.5$

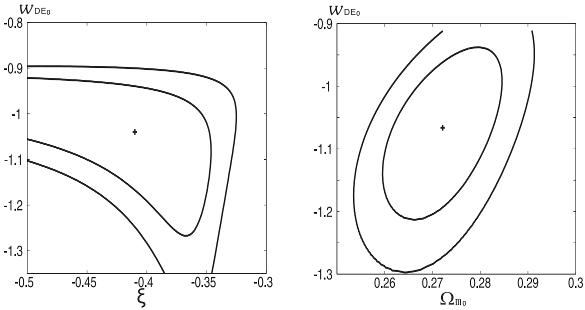




Teleparallel Dark Energy: Observational Constraints

- Potential: $V(\phi) = V_0 e^{-\kappa\phi}$
- Left: fixing $\Omega_m = 27\%$, the best fit locates at $h \simeq 0.7$, $\xi \simeq -0.41$, $w_\phi \simeq -1.04$ and $\chi^2 \simeq 544.3$

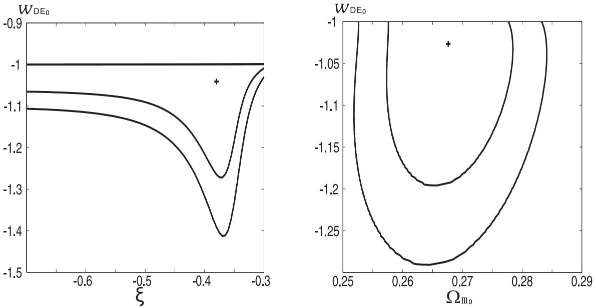
  Right: fixing $\xi = -0.41$, the best fit locates at $h \simeq 0.7$, $\Omega_m \simeq 27.1\%$, $w_\phi \simeq -1.07$ and $\chi^2 \simeq 544.6$



Teleparallel Dark Energy: Observational Constraints

- Potential: $V(\phi) = \frac{V_0}{cosh(\kappa\phi)}$
- Left: fixing $\Omega_m = 27\%$, the best fit locates at $h \simeq 0.7$, $\xi \simeq -0.38$, $w_\phi \simeq -1.05$ and $\chi^2 \simeq 544.8$

  Right: fixing $\xi = -0.41$, the best fit locates at $h \simeq 0.7$, $\Omega_m \simeq 26.7\%$, $w_\phi \simeq -1.03$ and $\chi^2 \simeq 545.1$



$V(\phi) = 0$ Case
The Basic Idea

- Here, we consider the simplest model $V(\phi) = 0$.
- The gravity action is rewritten as

$$S = \int d^4xe \left[\frac{T}{2\kappa^2} + \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 \right) + \mathcal{L}_M \right].$$

$V(\phi) = 0$ Case
Modified Friedmann Equation

- Under FRW metric ($e_\mu^A = \mathrm{diag}(1, a, a, a)$), the scalar field and the gravitational field equations read

$$\ddot{\phi} + 3H\dot{\phi} + 6\xi H^2 \phi = 0\,,$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\left(\rho_\phi + \rho_m + \rho_r\right)\,,$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\rho_\phi + p_\phi + \rho_m + 4\rho_r/3\right)\,,$$

  where the matter energy density $\rho_m \propto a^{-3}$, the radiation energy density $\rho_r \propto a^{-4}$
- The energy density and pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 - 3\xi H^2 \phi^2,$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 + 4\xi H \phi \dot{\phi} + \xi\left(3H^2 + 2\dot{H}\right)\phi^2.$$

$V(\phi) = 0$ Case
Exact Solution in RD and MD

- When the universe is undergoing a radiation-dominated (RD) and matter-dominated (MD) eras, the Hubble parameter has the form: $H = \frac{A}{t}$, where $A = \frac{2}{3(1+w_{eff})}$ and w_{eff} is the total EoS.
- Substituting that into the scalar field equation, one can obtain the exact solution:

$$\begin{aligned} \phi(t) &= C_1 t^{l_1} + C_2 t^{l_2}, \\ l_{1,2} &= \frac{1}{2}\left[\pm\sqrt{(3A-1)^2 - 24\xi A^2} - (3A-1)\right], \end{aligned}$$

where $C_{1,2}$ are constants.

- There exist an increasing mode and a decreasing mode.

$V(\phi) = 0$ Case
Exact Solution in RD and MD

- Consequently, one obtains

$$\text{RD:} \qquad w_\phi = \frac{1}{3}\left(2 - \sqrt{1-24\xi}\right), \quad \rho_\phi \propto a^{-5+\sqrt{1-24\xi}}\,;$$

$$\text{MD:} \qquad w_\phi = \frac{1}{2}\left(1 - \sqrt{1-32\xi/3}\right), \quad \rho_\phi \propto a^{(-9+\sqrt{9-96\xi})/2}\,.$$

Table: Explicit solutions of the scalar field in RD and MD with different values of ξ.

ξ	(i) 0^-		(ii) $-1/8$		(iii) $-3/4$		(iv) -1	
Era	RD	MD	RD	MD	RD	MD	RD	MD
l_1	-3ξ	$\frac{-8\xi}{3}$	$\frac{1}{4}$	$+0.26$	0.84	$+1$	$+1$	$+1.21$
w_ϕ	$\frac{1}{3} + 4\xi$	$\frac{-8\xi}{3}$	0	-0.26	-0.79	-1	-1	-1.21
$\rho_\phi \propto$	$a^{-4-12\xi}$	$a^{-3-8\xi}$	a^{-3}	$a^{-2.21}$	$a^{-0.64}$	Const.	Const.	$a^{0.62}$

$V(\phi) = 0$ Case
Exact Solution in SD

- When the universe is dominated by dark energy, the scalar-dominated era, we deduce the evolution of the scalar field from the second modified Friedmann equation (gravitational field equation),

$$\frac{1}{F(\phi)}\left(\frac{d\phi}{d\ln a}\right)^2 = \frac{6}{\kappa^2} \Rightarrow \phi(a) = \pm\frac{\sin\theta}{\sqrt{-\kappa^2\xi}},$$

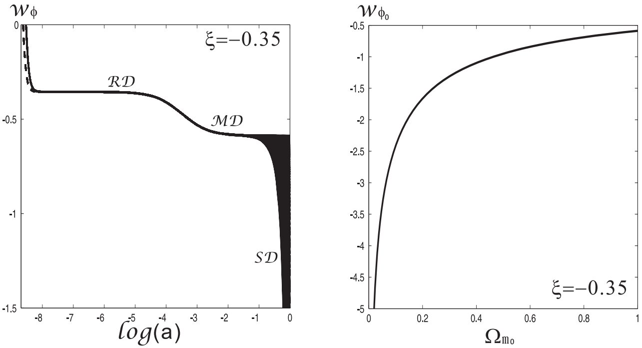
where $F(\phi) \equiv 1 + \kappa^2\xi\phi^2$ and $\theta(a) = \sqrt{-6\xi}\ln a + C_3$.

- The equation of state can also be derived as

$$w_\phi = -1 - \sqrt{-32\xi/3}\tan\theta.$$

$V(\phi) = 0$ Case
Numerical Demonstration

- The EoS has the fix points at RD and MD eras.
- The EoS is monotonically decreasing and reaches the singularity $w_\phi \to \infty$.
- The different initial condition corresponds to the same final state of (Ω_m, w_ϕ) figure in current stage.



Classification of Finite-time Singularities

- There are four classes of finite-time singularity [1]
  - Type I (Big Rip): For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$.
  - Type II (sudden): For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$.
  - Type III: For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$.
  - Type IV: For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge.

[1] S. 'i. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005)

$V(\phi) = 0$ Case
"Type III" Singularity

- For the universe dominated by matter and dark energy, we deduce an integrable equation from the gravitational field equation:

$$\frac{d}{dt}\left[F(\phi)a^3H\right] = \frac{\kappa^2}{2}\rho_m a^3 = \frac{3}{2}H_0^2\Omega_m^{(0)} ,$$
$$F(\phi) \equiv 1 + \kappa^2\xi\phi^2,$$

- Through the equation

$$Fa^3H = C_4 \rightarrow H = \frac{C_4}{a^3F} = \frac{C_4}{a^3cos^2\theta(a)},$$

we can estimate that H is always positive and goes to infinite at $\theta = \pi/2$ in ϕ-dominated era.

$V(\phi) = 0$ Case
Data Fitting

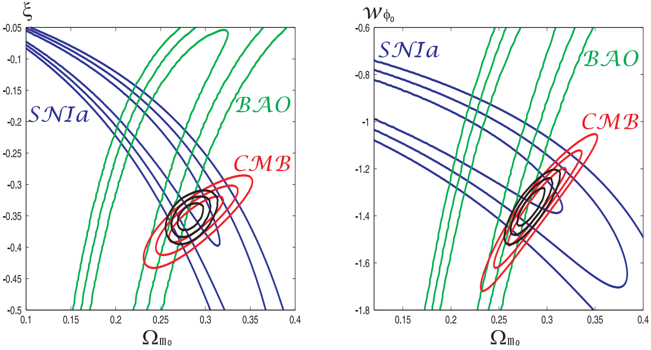


Figure: The 1σ–3σ confidence regions in (Ω_{m0}, ξ) (left panel) and $(\Omega_{m0}, w_{\phi 0})$ (right panel) obtained from the SNIa (blue), BAO (green), CMB (red), and the combined (black) data.

Summary

- Teleparallel gravity is an alternative gravity theory of the universe.
- Teleparallel dark energy model is equivalent to quintessence model happens at the minimal coupling case ($\xi = 0$), but it has a different behavior when we include a non-minimal coupling term ($\xi \neq 0$).
- We show that the equation of state of teleparallel dark energy model can cross the phantom-divide easily.
- The observational constraints show a good result on this model. This model is suitable for the late-time accelerating universe.
- We deduce the exact solution of teleparallel dark energy without potential, and demonstrate this behavior by numerical result.