Precessional Memory Effect

A New Cosmological Probe

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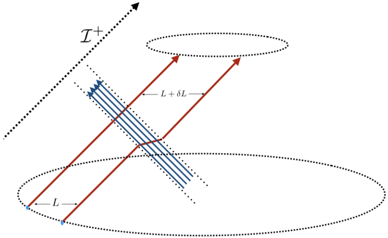
Outline

Motivation

Gravitational Wave Memory Effect

The permanent relative displacement of a pair of test mass particles upon the passage of gravitational wave is called **gravitational memory effect.** (Zel'dovich, Braginski, Thorne, Christodoulou...)



Bigger Picture!

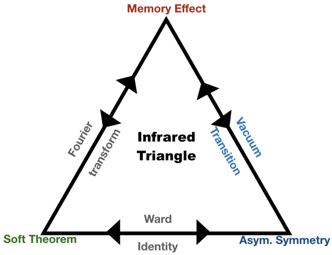


Figure: Strominger et. al.

Different kinds

- Displacement Christodoulou91,Thorne92
- Electromagnetic memory Bieri 10
- Velocity Grishchuk 89
- Spin-memory Pasterski 15

Fundamentals of Displacement Memory

Geodesic Deviation Equation(GDE) of particles of trajectories $x^\mu(\tau)$ and $x^\mu(\tau)+\xi^\mu(\tau)$

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}\xi^\lambda\frac{dx^\beta}{d\tau}. \tag{1}$$

Weak field approximation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$.

The transverse traceless(TT) guage condition:

$$h_{0\mu} = 0, \qquad h_{ij,j} = 0, \qquad h^\mu_\mu = 0 \tag{2}$$

In TT gauge: $\frac{D}{d\tau} \to \frac{d}{dt}$ and $\frac{dx^\alpha}{d\tau} = (1,0,0,0)$

The Riemann curvature tensor:

$$R_{i0j0} = -\frac{1}{2}h^{TT}_{ij,00} \tag{3}$$

For particles in laboratory frame of reference:

$$\frac{d^2\xi^i}{d\tau^2} = \frac{1}{2}h^{TT}_{ij,00}\xi^j. \tag{4}$$

Displacement memory in the leading order:

$$\Delta\xi^i(t\to+\infty, t\to-\infty) = \frac{1}{2}\Delta h^{TT}_{ij}\xi^j(t_i) = \frac{1}{2}h_{+,\times}(t\to+\infty) - h_{+,\times}(t\to-\infty)\xi^j(t_i). \tag{5}$$

Formulation of Gravitational Thomas Precession

Thomas Precession in a Relativist's Perspective

Thomas Precession is due to the non-commutative nature of Lorentz groups.

$$\left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right)_{\text{Non-Rot}} = \left(\frac{\mathrm{d}\vec{S}}{\mathrm{d}t}\right)_{\text{R}} + \vec{\omega_{\text{T}}} \times \vec{S}, \tag{6}$$

Where ω_{T} is the frequency of precession, it is given by

$$\omega_{\text{T}} = \frac{1}{2}\frac{\vec{a}\times\vec{v}}{c^2}$$

For an electron, the energy associated with Thomas Precession is

$$U_{\text{Non-Rot}} = U_{\text{R}} + \vec{S}\cdot\vec{\omega}_{\text{T}}$$

For a Bohr atom with $n = 1$, $r = r_0$,

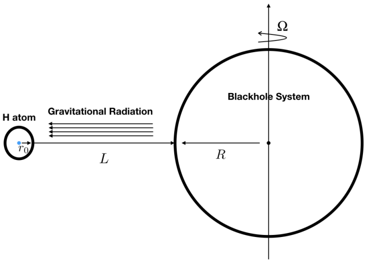
$$\frac{|\vec{v}|}{c} \equiv \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \qquad r_0 = \frac{\hbar}{m\alpha c}$$

$$|\vec{a}| = \frac{e^2}{4\pi\epsilon_0 r_0^2 m} = \frac{\hbar\alpha c}{m r_0^2}, \qquad E_0 = \frac{1}{2}mc^2\alpha^2$$

TP energy for Hydrogen atom is due to the Spin-Orbit coupling. This causes fine structure of hydrogen energy levels.

$$\begin{aligned} U_{\text{TP}} &= \vec{S}\cdot\frac{1}{2}\frac{\vec{a}}{c}\times\frac{\vec{v}}{c} \\ &\simeq \frac{|\vec{S}|}{\hbar}\alpha^2 E_0. \end{aligned} \tag{7}$$

Gravitational Thomas Precession for a Toy Model



L: distance from the source to observer
Ω is the angular frequency of the orbit.

Case: **Binary system stars of mass M, revolving in a circular orbital of radius R**

Quadrupole moment: $I \sim MR^2$

Upon the passage of Gravitational waves there exists an accelaration:

$$a^i = \frac{d^2\xi^i}{d\tau^2} = \frac{1}{2}h^{TT}_{ij,00}\xi^j.$$

Using quadrupole approximation :

$$a^i = \frac{G}{c^4}\Omega^4\frac{I^{TT}_{ij}}{L}\xi^j.$$

This acceleration causes Thomas precession in Bohr atom. The energy of spin-orbit coupling is:

$$U_{\mathbf{GTP}} \simeq \frac{|\vec{S}|}{2\hbar}\cdot\alpha^2 E_0\left[\frac{\phi_s R}{L}\left(\frac{v_s}{c}\right)^4\left(\frac{r_0}{R}\right)^2\cdot\frac{1}{\alpha^4}\right]. \quad (8)$$

ϕ_s**: dimensionless gravitational potential**
v_s**:surface velocity of the source**

$$\phi_s = \frac{GM}{c^2R} \quad , \quad v_s = \Omega R.$$

Gravitational Fine Structure and Memory Effect

- Gravitational waves cause Thomas Precession which is encoded in the fine structure of hydrogen. Compared with standard spin-orbit interaction energy:

$$\frac{U_{\text{GTP}}}{U_{\text{TP}}} \sim \phi_s \left(\frac{v_s}{c}\right)^4 \left(\frac{R}{L}\right)\left(\frac{r_0}{R}\right)^2 \cdot \frac{1}{\alpha^4}.$$

- Event's signature can be found on the hydrogen fine structure spectra, which is unique, hence termed it as **Precession Memory Effect(PME).**

If ν is the frequency of the radiation without memory and $\nu(M)$ is with PME, then

$$\frac{\delta\nu(M)}{\nu} \simeq \left[\phi_s \left(\frac{v_s}{c}\right)^4 \left(\frac{r_0^2}{3 \times 10^3 \text{m} L}\right)\left(\frac{M_\odot}{M}\right) \cdot \frac{1}{\alpha^4}\right] \tag{9}$$

Few simple scenarios:

- **Pulsars:**

Mass $M_\odot$
Radius $\sim$ 10km

Distance between hydrogen atom and pulsar $L \sim 100R$

$$\frac{U_{\rm GTP}}{U_{\rm TP}} \sim 10^{-31} \tag{10}$$

Too small to be observed!

- **Blackholes with $M \geq M_\odot$**

For a Schwarzchild BH, the radius of the source is equal to Schwarzchild radius:

$$R = \frac{2GM}{c^2} = 3\text{km} \times \left(\frac{M}{M_\odot}\right). \tag{11}$$

Too small to be observed since $\left(\frac{r_0}{R}\right)^2$ is still suppressing the energy contribution!

Perfect candidates: Primordial Black Holes(PBHs)

Masses of PBHs $\geq M_\odot$ are not suitable.

Consider lower mass PBHs, say $\left(\frac{M}{M_\odot}\right) \sim 10^{-n}$. Then the radius of the source is

$$R \sim 3\text{km} \times 10^{-n} \sim 10^{-(n-3)}\text{m}$$

The suppressing factor: $\left(\frac{r_0}{R}\right)^2 \sim 10^{2n-27}$. For a source of $\phi_s \approx 1$ and the distance between source to atom is $L = 10^3 R$ the energy of GTP is

$$\frac{U_{\text{GTP}}}{U_{\text{TP}}} \sim 10^{-2} \times 10^{2(n-14)}. \qquad (12)$$

To estimate the spectral splitting, quantum mechanical perturbation is used. To apply it, the mass range of PBH source is $10^{-14} \leq \left(\frac{M}{M_\odot}\right) < 10^{-13}$ **.**

For sources in this range, the spectral line observed should have the frequency of the order of $\delta\nu \sim 10^7$Hz(Radio Waves). As we go farther away the spectral line becomes redder.

As a Cosmological probe

Perfect candidates: Primordial Black Holes(PBHs)

Advantage: They come in wide range of masses from Planck mass to supermassive! Zeldovich67,Hawking74...

Why are they interesting?

- Prime dark matter candidates
- Candidates of Supermassive BHs in galaxy centers and AGNs
- Possible contribution to GRBs and cosmic rays
- Possible contribution to BBN synthesis

Interesting Recent Developements

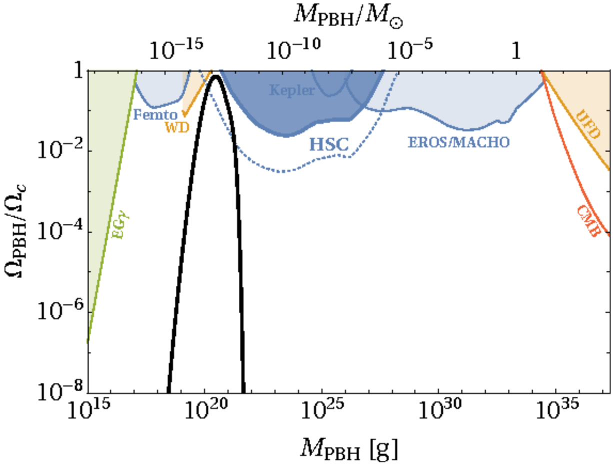


Figure: Inflationary PBHs as all dark matter (Inomata et. al. 2017)

Preliminaries: Mass and Abundances of PBHs

For simplicity, radiation dominant era is considered.
During the time of formation, the mass of PBH is proportional to Horizon mass.

$$\begin{aligned} M_{\text{PBH}} &= \gamma \frac{4\pi\rho}{3} H^{-3}\Big|_{\text{Formation}} = \frac{\gamma}{2G} H^{-1}\Big|_{\text{Formation}} \\ &\simeq 10^{20}\text{g}\left(\frac{\gamma}{0.2}\right)\left(\frac{g_*}{106.75}\right)^{-\frac{1}{6}}\left(\frac{k}{7\times 10^{20}\text{MPC}^{-1}}\right)^{-2} \end{aligned} \tag{13}$$

The abundance of PBHs are estimated using mass fraction defined as $\beta(M) := \frac{\rho_{\text{PBH}}(M)}{\rho}$.
Mass Fraction depends on the profile of density perturbations.

$$\beta = \gamma \int_{\delta_C}^{1} P(\delta) d\delta \tag{14}$$

For a gaussian Distribution:

$$\begin{aligned} \beta(M) &= \gamma \int_{\delta_C}^{1} \frac{d\delta}{\sqrt{2\pi\sigma^2_{\text{PBH}}(M)}} e^{-\frac{\delta^2}{2\sigma^2_{\text{PBH}}(M)}} \\ &\simeq \frac{3\gamma}{\sqrt{2\pi}} \sigma_{\text{PBH}}(M) e^{-\frac{1}{18\sigma^2_{\text{PBH}}(M)}}. \end{aligned} \tag{15}$$

PBH Background in Early Universe

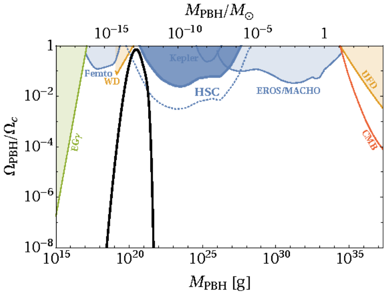


Figure: Inflationary PBHs as all dark matter (Inomata et. al. 2017

If P is the probability of mergers per unit time, event rate of PBH mergers of a certain mass at fixed z per unit time is

$$\mathrm{d}E = \mathrm{d}N(M)P$$

Total number of events:

$$E = \int \frac{\beta(M)\rho(t_i)}{M} P \mathrm{d}lnM$$

Number density as monochromatic mass function: $n_{\text{PBH}}(M) = \frac{\beta(M)\rho_{\text{total}}}{M}$

Cumulative number density

$$\mathrm{d}N(M) = n_{\text{PBH}}(M)\mathrm{d}lnM$$

- For $30M_\odot$, $E \rightarrow 2\text{Gpc}^{-3}\text{yr}^{-1}$ (Sasaki16)
- $10M_\odot$, $E \rightarrow 5\text{Gpc}^{-3}\text{yr}^{-1}$
- $2M_\odot$, $E \rightarrow 2000\text{Gpc}^{-3}\text{yr}^{-1}$ (Chen16)

Our Proposal: As a new Cosmological Probe

Our Proposal

- After the end of radiation dominant era, light elements were formed. Lightest and the most abundant element during that epoch was hydrogen.
- If there was a large abundance of PBHs in the early universe, then the number of merger events should be large.
- Gravitational waves from these merger events would be stronger near the source, therefore it should leave strong Gravitational Thomas Precession Signatures in the hydrogen cloud in the vicinity.
- These local GTP effect should have a distinct $\frac{1}{L}$ profile.
- Assuming the abundance of PBHs of desired mass range and also with considerable event rate, we can map GTP fluctuations in the entire sky.
- Event rate could be used to constrain mass fraction and PBH contribution to dark matter.

THANK YOU.

A bit about calculations

Gravitational perturbations for the above case can be written as:

$$h^{TT}_{ij,00} = \frac{G}{2c^4}\Omega^4 \frac{MR^2}{L}\xi^j = \phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\frac{c^2}{R^2}. \quad (16)$$

ϕ_s: dimensionless gravitational potential
v_s:surface velocity of the source

$$\phi_s = \frac{GM}{c^2 R} \quad , \quad v_s = \Omega R. \quad (17)$$

Acceleration Approximation

$$a \sim |h^{TT}_{ij,00}| r_0$$

Estimation of Thomas Precession Energy:

$$\frac{a}{c} \sim \phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\left(\frac{r_0 c}{R}\right)\cdot\frac{E_0 r_0}{\alpha c\hbar} = \frac{\phi_s R}{L}\left(\frac{v_s}{c}\right)^4\left(\frac{r_0}{R}\right)^2\cdot\frac{E_0}{\alpha\hbar}.$$

Energy:

$$U_{\mathrm{GTP}} \sim \frac{|\vec{S}|}{2\hbar}\phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\left(\frac{r_0}{R}\right)^2\cdot\frac{E_0}{\alpha^2} = \frac{|\vec{S}|}{2\hbar}\cdot\alpha^2 E_0\left[\frac{\phi_s R}{L}\left(\frac{v_s}{c}\right)^4\left(\frac{r_0}{R}\right)^2\cdot\frac{1}{\alpha^4}\right].$$

Gravitational Thomas Precession for a Toy Model (contd...)

Energy:

$$\begin{aligned} U_{\mathbf{GTP}} &\sim \frac{|\vec{S}|}{2\hbar}\phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\left(\frac{r_0}{R}\right)^2\cdot\frac{E_0}{\alpha^2} \\ &= \frac{|\vec{S}|}{2\hbar}\cdot\alpha^2 E_0\left[\frac{\phi_s R}{L}\left(\frac{v_s}{c}\right)^4\left(\frac{r_0}{R}\right)^2\cdot\frac{1}{\alpha^4}\right]. \end{aligned}$$

Dimensionless energy parameter:

$$\frac{U_{\mathrm{GTP}}}{U_{\mathrm{TP}}}\sim\phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\left(\frac{r_0}{R}\right)^2\cdot\frac{1}{\alpha^4}. \quad (18)$$

Memory Signature on atomic spectra

•After the end of radiation dominant era, light elements were formed. Lightest and the most abundant element during that epoch was hydrogen.

•In case of a merger of PBH, the produced gravitational waves would leave a signature on Hydrogen atomic cloud around them.

•Therefore, it is a unique signature of the event on the hydrogen fine structure spectra, hence called as **Precession Memory Effect(PME).**

If ν is the frequency of the radiation without memory and $\nu'(M)$ is with PME, then

$$\delta\nu'(M) = \nu\left[\phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{R}{L}\right)\left(\frac{r_0}{R}\right)^2\cdot\frac{1}{\alpha^4}\right]$$

$$\implies \frac{\delta\nu'(M)}{\nu} \simeq \left[\phi_s\left(\frac{v_s}{c}\right)^4\left(\frac{r_0^2}{3\times 10^3\mathrm{m}L}\right)\left(\frac{M_\odot}{M}\right)\cdot\frac{1}{\alpha^4}\right] \tag{19}$$