Cosmological Evolution in Tsujikawa Model of f(R) Gravity

NCTS Dark Physics Workshop

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J.Y. Cen, S.Y. Chien, C.Q. Geng and C.C. Lee,
Physics of the Dark Universe 26, 100375 (2019)

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Outline

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3. Constraints on Tsujikawa Model
4. Summary

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Introduction

Standard Cosmology:

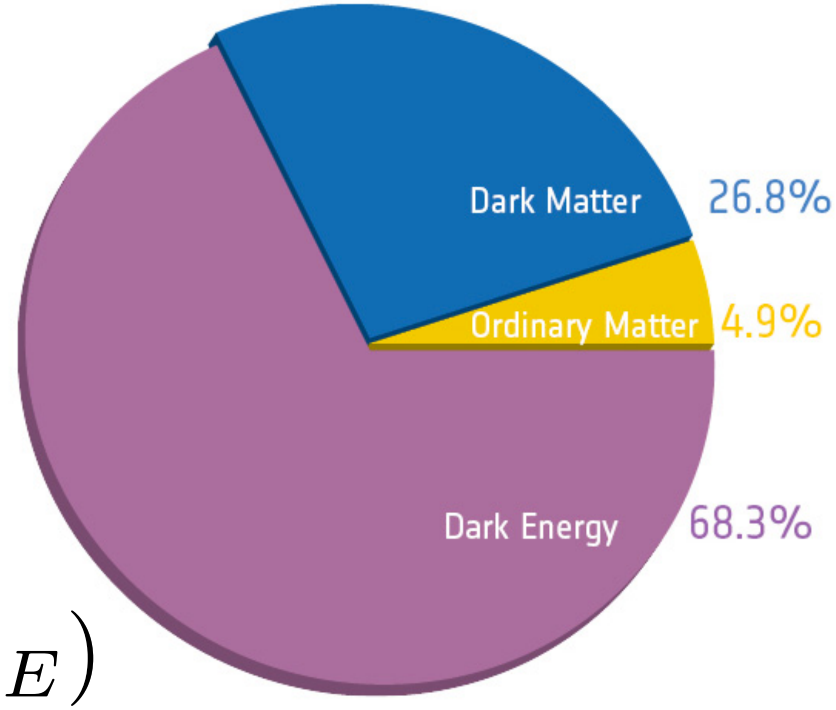
- Cosmological principle:

  Our universe is **homogeneous** and **isotropic** in large scales**.**

  $$\rightarrow \quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2$$

- Einstein field equation:

  $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$



- Friedmann equation

  $$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{DE})$$

Expansion of the universe:

- In 1917, Einstein proposed a static universe by adding cosmological constant in the Einstein field equation.
- In 1920, Friedmann presented the Friedmann equation to show the possibility that the universe can expand.
- The expanding universe was shown with the observational data (galaxies) by Lemaitre (1927) and Hubble (1929).

(Edwin Hubble)

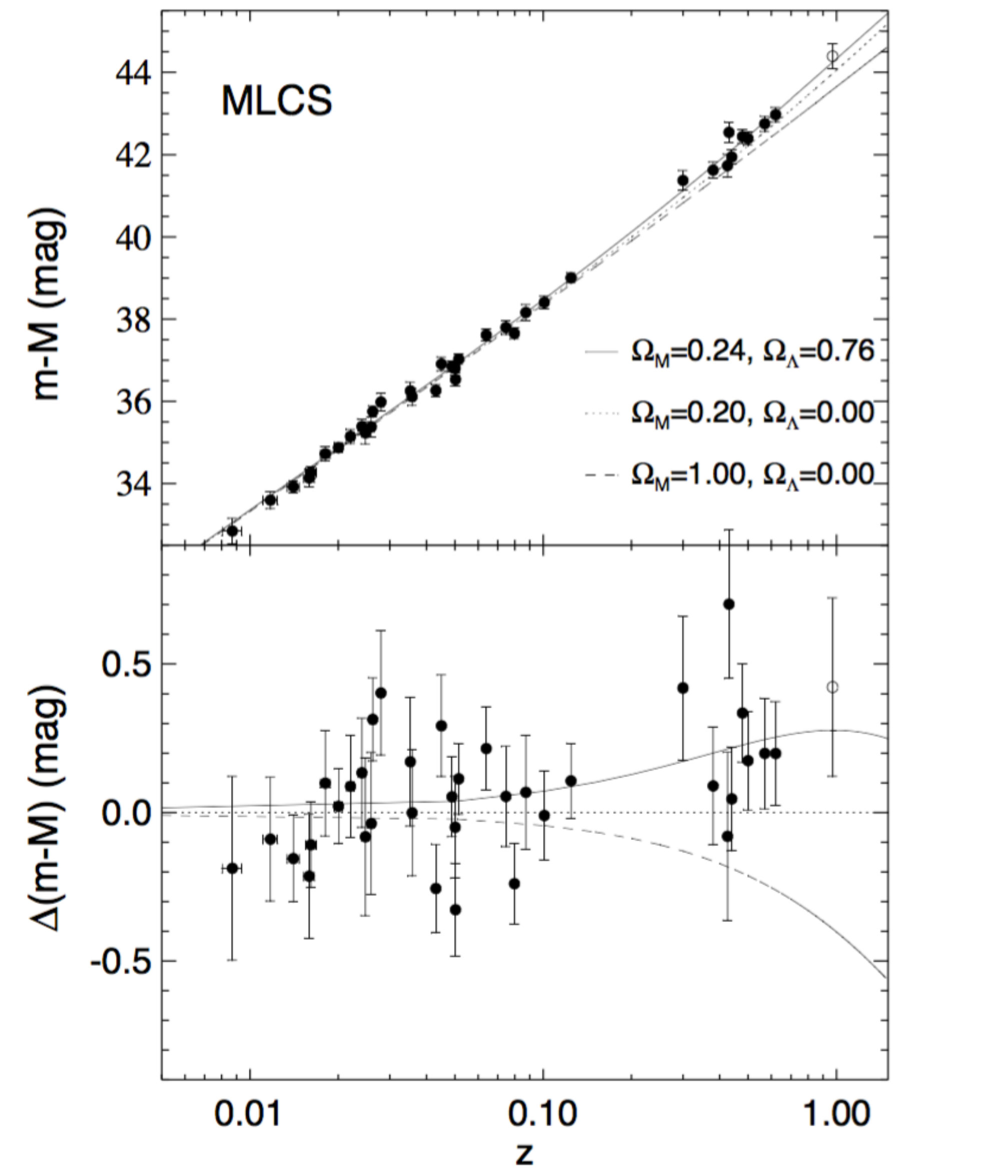
Dark Energy:

- Fact: The universe is accelerated expanding.

- Evidence:

  Supernovae Observation (1998)

  Paper: "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant"
  ~(The Nobel Prize in Physics 2011)



- Acceleration Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) > 0 \Rightarrow w < -\frac{1}{3} \text{ (negative pressure } P < -\frac{\rho}{3}\text{).}$$

Cosmological Constant: $\rho_{DE} = \frac{\Lambda}{8\pi G} = 10^{-47} GeV^4$

- Problems:

  A. Why is the cosmological constant so small if compared with the vacuum energy

$$\rho_{vac} = \frac{1}{2}\int_0^\infty \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2} = \frac{1}{4\pi^2}\int_0^{k_{\max}} dk k^2 \sqrt{k^2+m^2} \approx \frac{k_{\max}^4}{16\pi^2} = 10^{74} GeV^4$$

  B. Coincidence problem?

  The density parameters of dark matter and dark energy are compatible at the current stage.

Alternative approaches:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Modified Gravity		
Name	**Time**	**Authors**
f(R) gravity	1969	Ruzmaĭkina, T. V. Ruzmaĭkin, A. A.
Scalar-tensor theories	1955	Pascual Jordan
Gauss-Bonnet gravity	2005	S. Nojiri, S. D. Odintsov, M. Sasaki
⋮		

Modified Matter		
Name	**Time**	**Authors**
Quintessence	1998	R. R. Caldwell, R. Dave, P. J. Steinhardt
k-essence	2000	T. Chiba, T. Okabe, M. Yamaguchi
Phantom	2002	R. R. Caldwell
⋮		

What is f(R) gravity?

$$S = \int d^4x \frac{\sqrt{-g}}{2\kappa^2}(R - 2\Lambda) + S_M \rightarrow \boxed{S = \int d^4x \frac{\sqrt{-g}}{2\kappa^2} f(R) + S_M \,,}$$

where $\kappa^2 = 8\pi G$ $\quad S_M$: the action of the matter

$f(R)$: an arbitrary function of the Ricci scalar R

g : the determinant of the metric tensor $g_{\mu\nu}$

Viable f(R) gravity model:

The model of the f(R) gravity, which satisfies the viable f(R) conditions, could avoid some unphysical phenomena and instability perturbation happening. (Next page)

Viable f(R) conditions

- A positive effective gravitational coupling, leading to

$$\frac{df(R)}{dR} > 0$$

- The steady cosmological perturbations causing to

$$\frac{d^2 f(R)}{dR^2} > 0$$

- An asymptotic action to the ΛCDM in the larger curvature regime
- A late-time stable de-Sitter like solution.
- Obeying the Weak Equivalence Principle (WEP)

Viable f(R) models

Tsujikawa Model:	$f(R) = R - \lambda R_{ch} \tanh(\frac{R}{R_{ch}})$
Exponential Model:	$f(R) = R - \beta R_{ch}(1 - e^{\frac{-R}{R_{ch}}})$
Starobinsky Model:	$f(R) = R - \lambda R_{ch}\left[1 - \left(1 + \frac{R}{R_{ch}^2}\right)^{-n}\right]$
Hu-Sawicki Model:	$f(R) = R - \frac{c_1 R_{ch}(R/R_{ch})^p}{c_2(R/R_{ch})^p + 1}$

Cosmological evolutions in Tsujikawa Model

In Tsujikawa Model (TM): $f(R) = R - \lambda R_{ch} \tanh(\frac{R}{R_{ch}})$

$$
\begin{aligned}
f_R &= 1 - \lambda \cosh^{-2}\left(\frac{R}{R_{ch}}\right) \\
f_{RR} &= 2\frac{\lambda}{R_{ch}} \cosh^{-2}\left(\frac{R}{R_{ch}}\right) \tanh\left(\frac{R}{R_{ch}}\right) \\
\dot{f}_R &= 2\frac{\lambda}{R_{ch}} \cosh^{-2}\left(\frac{R}{R_{ch}}\right) \tanh\left(\frac{R}{R_{ch}}\right)\dot{R} \\
\dot{f}_{RR} &= 2\frac{\lambda}{R_{ch}^2} \cosh^{-4}\left(\frac{R}{R_{ch}}\right)[-2 + \cosh\left(\frac{2R}{R_{ch}}\right)]\dot{R} \\
\ddot{f}_R &= 2\frac{\lambda}{R_{ch}} \cosh^{-2}\left(\frac{R}{R_{ch}}\right) \tanh\left(\frac{R}{R_{ch}}\right)\ddot{R} \\
&\quad + 2\frac{\lambda}{R_{ch}^2} \cosh^{-4}\left(\frac{R}{R_{ch}}\right)[-2 + \cosh\left(\frac{2R}{R_{ch}}\right)]\dot{R}^2 \\
R &= 6(\dot{H} + 2H^2)
\end{aligned}
$$

$$
f_R : \frac{df(R)}{dR} \qquad f_{RR} : \frac{d^2 f(R)}{dR^2} \qquad \dot{f} : \frac{df(R)}{dt} \qquad f' : \frac{df(R)}{d\ln a}
$$

The relations are needed in the cosmological background and perturbation calculation.

Modified Einstein field Equations:

- Action of f(R) gravity:

$$S = \int d^4x \frac{\sqrt{-g}}{2\kappa^2} f(R) + S_M$$

- From the variation of the Lagrangian with respect to $g_{\mu\nu}$, we get

$$f_R R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f_R = \kappa^2 T^{(M)}_{\mu\nu}$$

where ∇_μ : Covariant Derivative, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$

$$f_R \equiv \frac{df(R)}{dR}, \quad f_{RR} \equiv \frac{d^2 f(R)}{dR^2}$$

Modified Friedmann Equations:

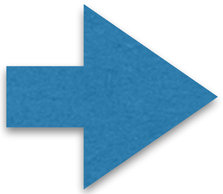
$$3f_R H^2 = \frac{1}{2}\left(f_R R - f\right) - 3H\dot{f}_R + \kappa^2 \rho_M$$

$$2f_R \dot{H} = -\ddot{f}_R + H\dot{f}_R - \kappa^2\left(\rho_M + P_M\right)$$

Effective Density and Pressure of Dark Energy:

$$\rho_{DE} = \kappa^{-2}\left(\frac{1}{2}\left(f_R R - f\left(R\right)\right) - 3H\dot{f}_R + 3\left(1 - f_R\right)H^2\right) ,$$

$$P_{DE} = \kappa^{-2}\left(-\frac{1}{2}\left(f_R R - f(R)\right) + \ddot{f}_R + 2H\dot{f}_R - \left(1 - f_R\right)\left(2\dot{H} + 3H^2\right)\right) .$$



$$w_{DE} = \frac{P_{DE}}{\rho_{DE}}$$

Solve the evolution by 2nd Differential Equation

Using the special parametrization, we rewrite Friedmann equations as the 2nd differential equation:

$$y_H'' + J_1 y_H' + J_2 y_H + J_3 = 0\,,$$

where

$$y_H \equiv \frac{\rho_{DE}}{\rho_m^{(0)}} = \frac{H^2}{m^2} - a^{-3} - \chi a^{-4} \qquad m^2 \equiv \frac{\kappa^2 \rho_m^{(0)}}{3} \qquad \chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}}$$

$$J_1 = 4 + \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{1 - f_R}{6 m^2 f_{RR}}$$

$$J_2 = \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{2 - f_R}{3 m^2 f_{RR}}$$

$$J_3 = -3a^{-3} - \frac{(1 - f_R)\left(a^{-3} + 2\chi a^{-4}\right) + (R - f)/3m^2}{y_H + a^{-3} + \chi a^{-4}} \frac{1}{6 m^2 f_{RR}}\,,$$

Deceleration Parameter:

This quantity describes whether the expansion rate of the universe is accelerating (q < 0) or decelerating (q > 0)

$$
\begin{aligned}
q &\equiv -\left(1+\frac{\dot{H}}{H^2}\right) \\
&= -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} \\
&= \frac{\frac{\kappa^2}{6}\left((1+3w_m)\rho_m + (1+3w_r)\rho_r + (1+3w_{\rm DE})\rho_{\rm DE}\right)}{\frac{\kappa^2}{3}\left(\rho_m+\rho_r+\rho_{\rm DE}\right)} \\
&= \frac{1}{2}\frac{a^{-3}+2\chi a^{-4}+(1+3w_{\rm DE})y_H}{a^{-3}+\chi a^{-4}+y_H}
\end{aligned}
$$

Perturbation theory

Metric perturbation:

$$ds^2 = a^2(\tau)[-(1+2A)d\tau^2 + (B_{,i} + S_j)dx^i d\tau + ((1+2C)\delta_{ij} + E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})dx^i dx^j],$$

SVT Decomposition theory:

The Einstein equations for scalars, vectors and tensors don't mix at linear order and can therefore be treated separately.

Do the scalar perturbation on metric:

$$ds^2 = -\left(1+2\Psi\right)dt^2 + a^2\left(t\right)\left(1-2\Phi\right)d\vec{x}^2 \quad \text{(Newtonian Gauge)}$$

Perturbation theory

Perturbing the modified Einstein Equations,

$$\delta f_R R_{\mu\nu} + f_R \delta R_{\mu\nu} - \frac{\delta f}{2} g_{\mu\nu} - \frac{f}{2} \delta g_{\mu\nu} + ... = \kappa^2 \delta T^{(M)}_{\mu\nu}$$

We obtain the following equations in Fourier space,

$$\frac{k^2}{a^2}\Phi = -3H\left(\dot{\Phi} + H\Psi\right) - \frac{1}{2F}\left[-3H\delta\dot{F} + \left(3H^2 + 3\dot{H} - \frac{k^2}{a^2}\right)\delta F\right] - \frac{1}{2F}\left[3\dot{F}\left(\dot{\Phi} + H\Psi\right) + 3H\dot{F}\Psi + \kappa^2\delta\rho_M\right],$$

$$\Phi = \Psi + \frac{\delta F}{F}$$

Discussing perturbation theory in two limits:

(The equations are too complicated so we need some approximation.)

Regime	limit	Description
Super-horizon	$k \ll aH$	The scale on which the physical wavelength $\lambda = (2\pi/k)a$ of perturbations is much larger than the Hubble radius ($1/H$).
Sub-horizon	$k \gg aH$	The scale on which the physical wavelength $\lambda = (2\pi/k)a$ of perturbations is much less than the Hubble radius ($1/H$).

Under the Sub-Horizon limit (k ≫ aH):

We can drop out the higher order terms:

$$\left\{\frac{k^2}{a^2}|\Phi|, \frac{k^2}{a^2}|\Psi|, \frac{k^2}{a^2}|\delta F|, M^2|\delta F|\right\} \gg \{H^2|\Phi|, H^2|\Psi|, H^2|B|, H^2|\delta F|\} ,$$

$$|\dot{X}| \lesssim |HX| , \quad \text{where} \quad X = \Phi, \Psi, F, \dot{F}, \delta F, \delta\dot{F} .$$

Finally, we get $\frac{k^2}{a^2}\Psi = -4\pi G\mu(k,a)\,\rho_M\Delta_M ,$ where

$$\mu(k,a) = \frac{1}{f_R}\frac{1+4\frac{k^2}{a^2}\frac{f_{RR}}{f_R}}{1+3\frac{k^2}{a^2}\frac{f_{RR}}{f_R}}$$

↓

$(G_{eff} = \mu G)$

and

$$\frac{\Phi}{\Psi} = \gamma(k,a) = \frac{1+2\frac{k^2}{a^2}\frac{f_{RR}}{f_R}}{1+4\frac{k^2}{a^2}\frac{f_{RR}}{f_R}} ,$$

↓

Post-Newtonian Parameter

$$\Delta_M \equiv \delta_M + 3H(1+\omega_M)v_M/k \quad \text{(Comoving density perturbation)}$$

$$\delta_M \equiv \frac{\rho_M - \bar{\rho_M}}{\bar{\rho_M}} \quad \text{(Density contrast)}$$

Evolution equation of density contrast under the Sub-Horizon:

Ignoring the peculiar velocity of the matter, we get

f(R) model: $$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\mu(k,a)\rho_m\delta_m = 0$$

GR: $$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\rho_m\delta_m = 0$$

Performance in different regimes:

$k^2 \ll f_{RR}/(a^2 f_R)$ $$\mu = f_R^{-1}$$

$k^2 \gg f_{RR}/(a^2 f_R)$ $$\mu = \frac{1}{f_R}\frac{1+4k^2 f_{RR}/(a^2 f_R)}{1+3k^2 f_{RR}/(a^2 f_R)}$$

Under the Super-Horizon limit (k ≪ aH):

We can choose to drop out the higher order terms:

$$\left\{\frac{k^2}{a^2}|\Phi|, \frac{k^2}{a^2}|\Psi|, \frac{k^2}{a^2}|\delta F|, M^2|\delta F|\right\} \ll \left\{H^2|\Phi|, H^2|\Psi|, H^2|B|, H^2|\delta F|\right\} ,$$

or starting from the "Curvature fluctuation conservation" and the "momentum conservation". (Wayne Hu 2006)
We finally get the the following relation in the limit (k→0)

$$\Phi'' + \left(1 - \frac{H''}{H'} + \frac{B'}{1-B} + B\frac{H'}{H}\right)\Phi' + \left(\frac{H'}{H} - \frac{H''}{H'} + \frac{B'}{1-B}\right)\Phi = 0\,, (k \to 0).$$

$$2\Phi_{eff} + \left(\frac{B}{2}\frac{E'}{E}\frac{E'}{4E'+E''}\right)S = \left(\frac{-1}{f_R}\frac{\kappa^2 a^2 \rho_M}{k^2}\right)\Delta_M, \text{ (Modified Poisson equation)}$$

where

$$B = \frac{f_{RR}}{f_R}R'\frac{H}{H'}\,, \quad E = \frac{H^2}{H_0^2}, \quad \Phi_{eff} = \frac{1}{2}(\Phi + \Psi)\,, \quad S = -2\Phi + \Psi\,.$$

Using Code: MGCAMB + MGCosmoMC

Code	Author	Description
CAMB	Antony Lewis	A code for computing **CMB**, **matter power spectra** and **cosmological background evolutions**, etc.
CosmoMC		Using the **Markov-Chain Monte-Carlo** (MCMC) method to fit the cosmological parameters with observational data.
MGCAMB	Gong-Bo Zhao Alex Zucca, Levon Pogosian, Alessandra Silvestri	Besides GR, it can also be used to calculate the **dynamical dark energy model** (which depends in time) with different parameterizations.
MGCosmoMC		

MG-CosmoMC

MCMC: Markov-Chain Monte-Carlo

Parameters	Priors
Model parameter	$10^{-4} < \lambda^{-1} < 1$
Density parameter of baryon	$5 \times 10^{-3} < \Omega_b h^2 < 0.1$
Density parameter of CDM	$10^{-3} < \Omega_c h^2 < 0.99$
Sum of the neutrino mass	$0 < \Sigma m_\nu < 1$ eV
Spectral index	$0.9 < n_s < 1.2$
Scalar power spectrum amplitude	$2 < \ln(10^{10} A_s) < 4$
Optical depth to reionization	$0.01 < \tau < 0.8$
Sound horizon /ang. distance	$0.5 < 100\ \theta_{\rm MC} < 10$
Hubble parameter (km/s · Mpc)	$20 < H_0 < 100$

→ MGCAMB

↓

Calculate χ^2 with the data

1. Baryon Acoustic Oscillations
2. Planck 2015 likelihoods
3. Supernova Legacy Survey

← (back to parameters)

Fitting results, Contour plots …

MGCAMB

Modified part

CAMB part:

Solve $\rho_{DE}(z),\ w_{DE}(z)$

by evolution equation

$$y''_H + J_1 y'_H + J_2 y_H + J_3 = 0$$

Input parameters:

$\Omega_b,\ \Omega_c,\ \Omega_\nu, H_0, A_s, n_s, \lambda, \ldots$

Solve the Cosmological Background, such as $H(z), \rho_m(z), \rho_r(z)$

1. Solve $\Phi(k,z), \Psi(k,z)$ by

$$\frac{k^2}{a^2}\Psi = -4\pi G\mu\,(k,a)\,\rho_M \Delta_M\,,$$
$$\frac{\Phi}{\Psi} = \gamma\,(k,a)$$

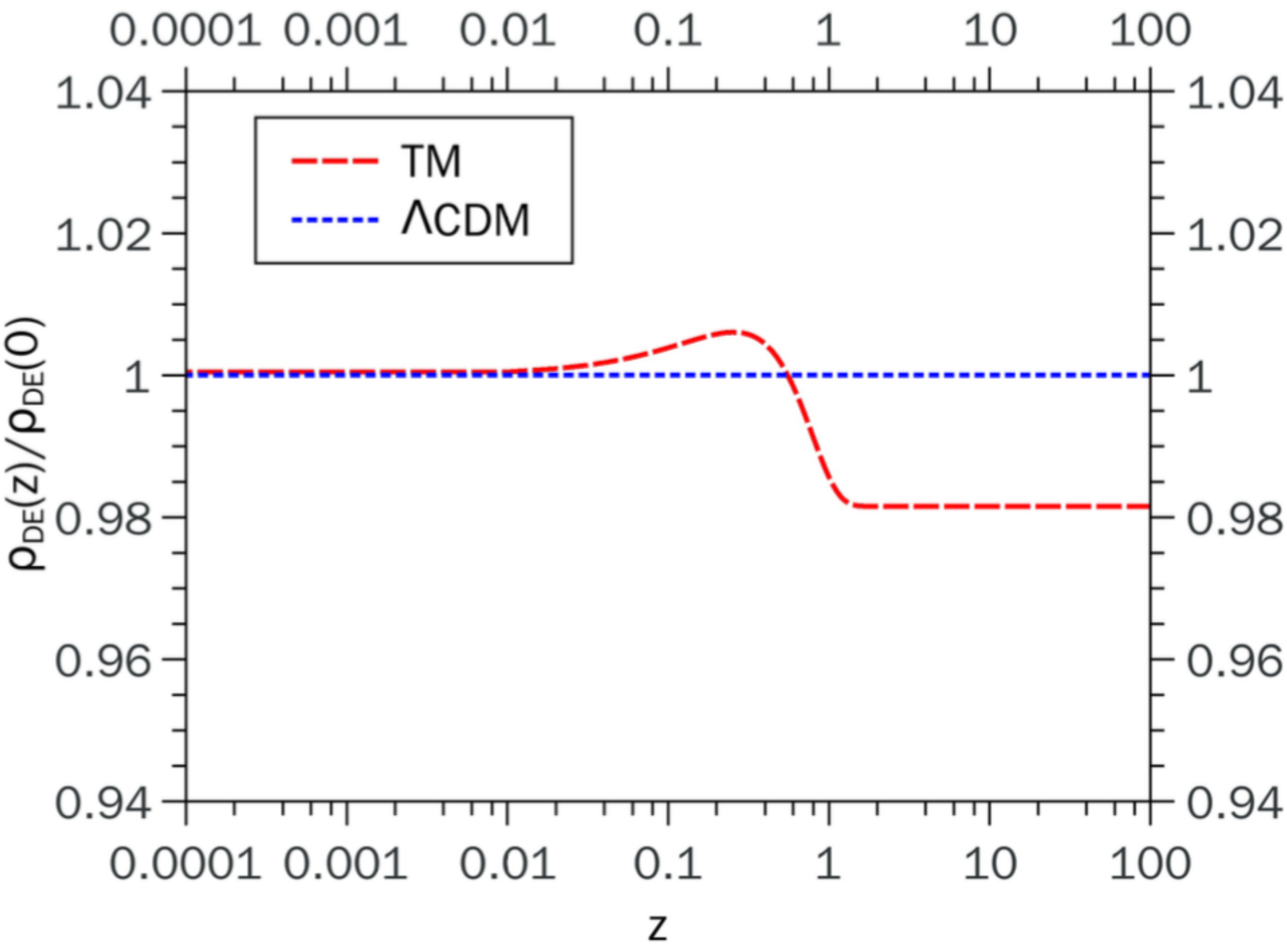
2. Gauge transformation

$\Phi, \Psi \Rightarrow \alpha, \eta$

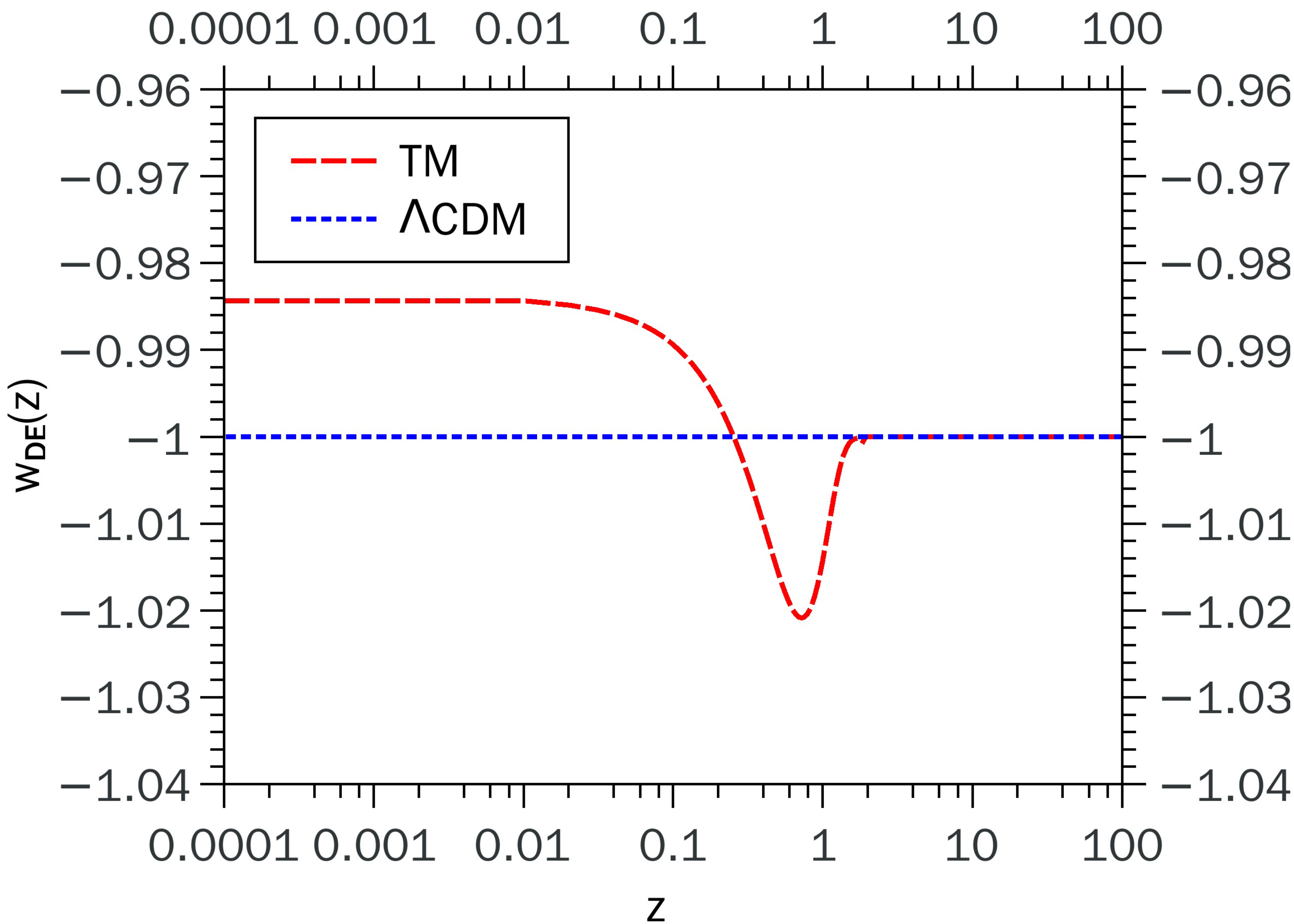
Calculate the metric perturbation (synchronous gauge) and matter perturbation, such as $\alpha, \eta, \delta_m, \delta_r, q$.

Output : CMB power spectrum, Matter power spectrum…

Effective Dark Energy Evolution:

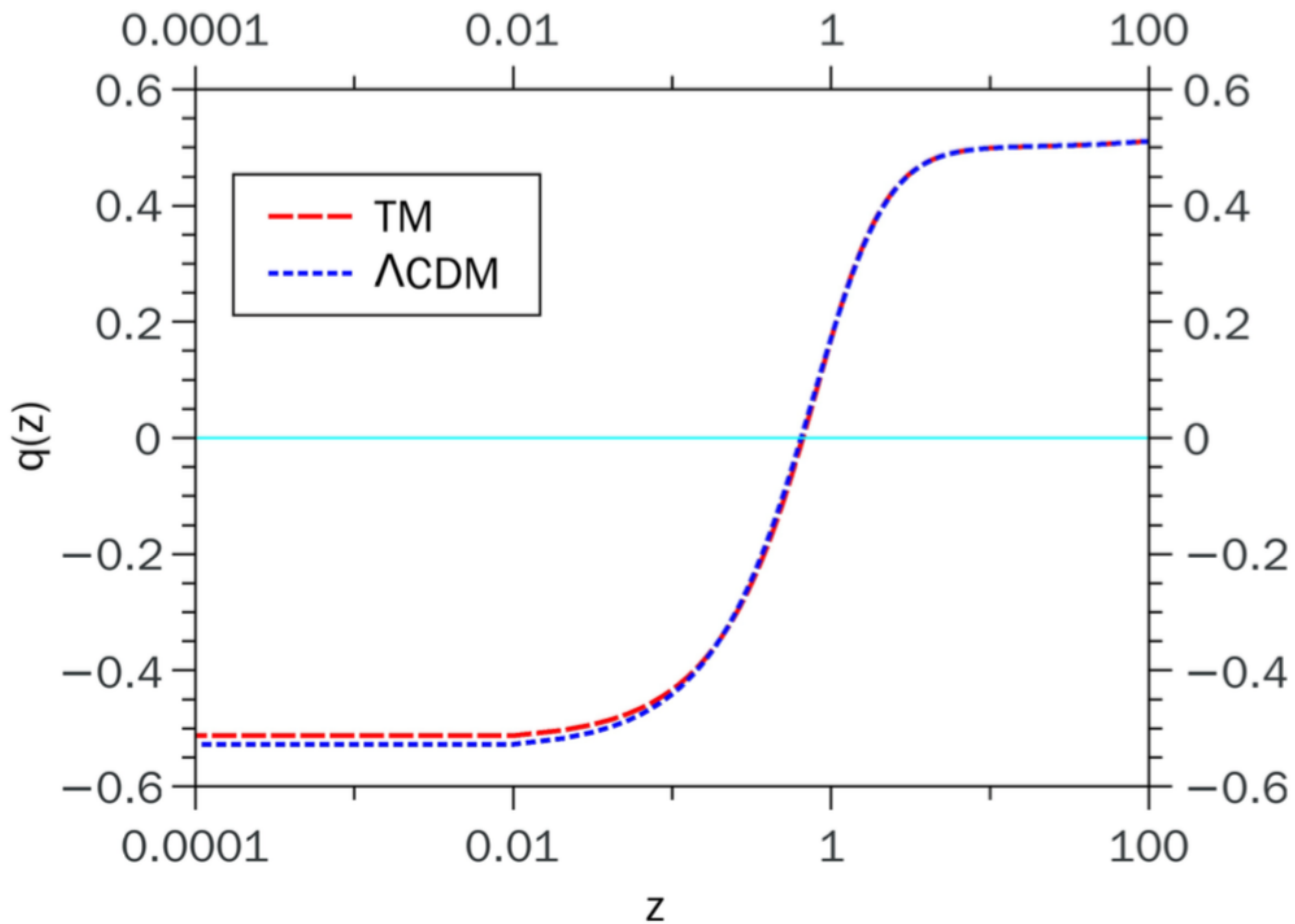


Equation of state of Dark Energy:

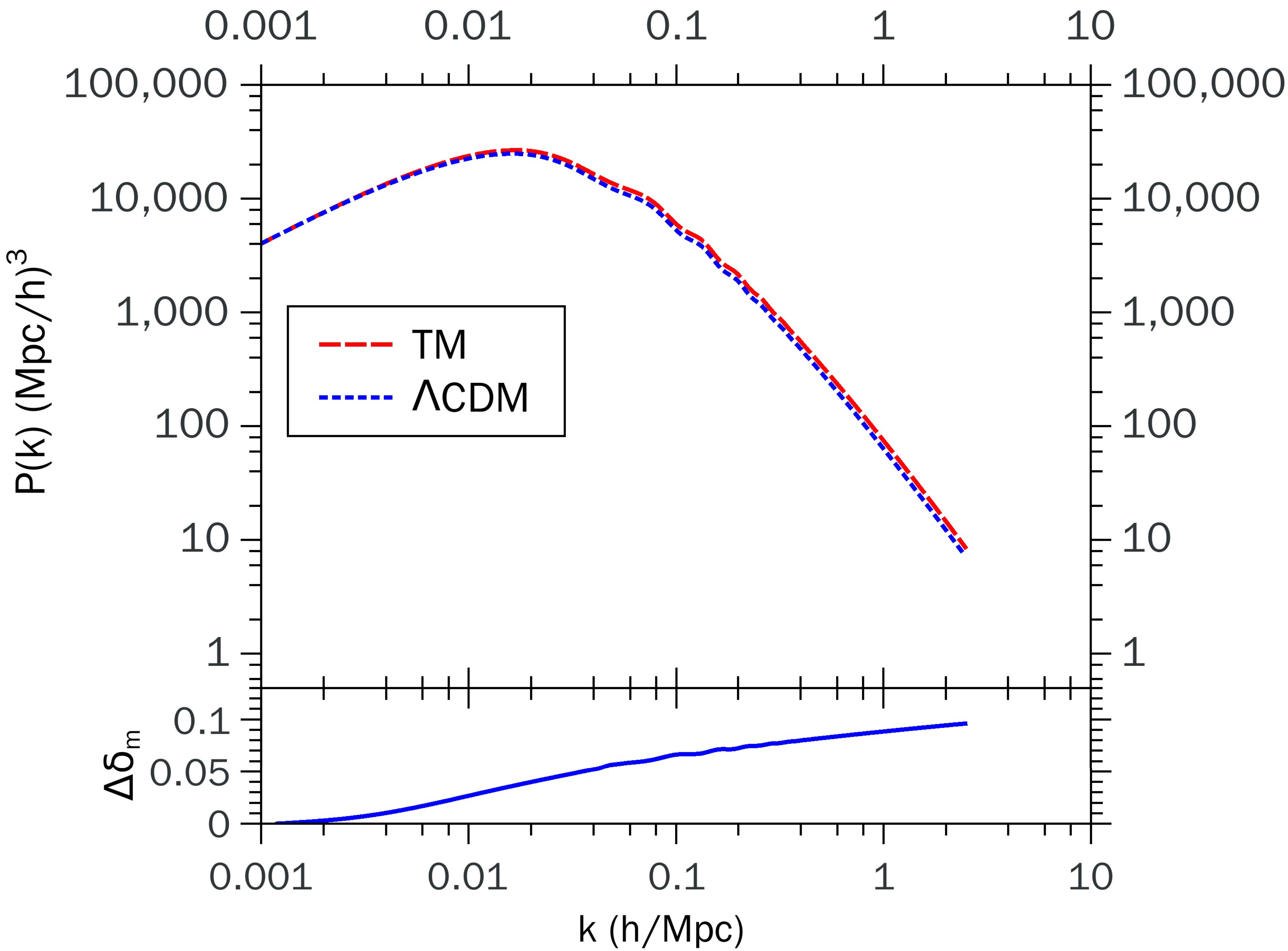


Deceleration parameter evolution:

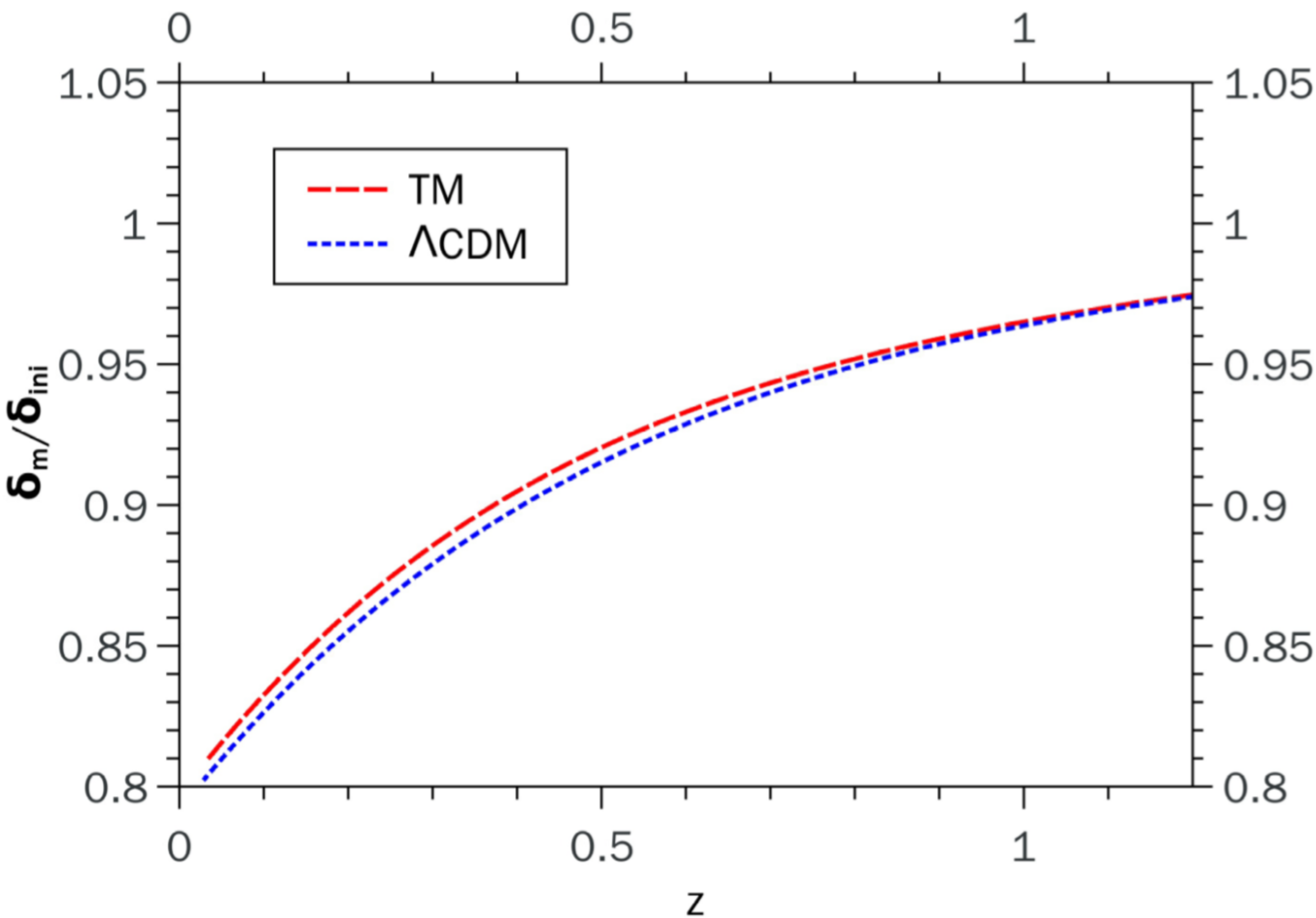
$$z(q=0) = 0.649 \quad (\Lambda\text{CDM})$$
$$z(q=0) = 0.688 \quad (\text{TM})$$



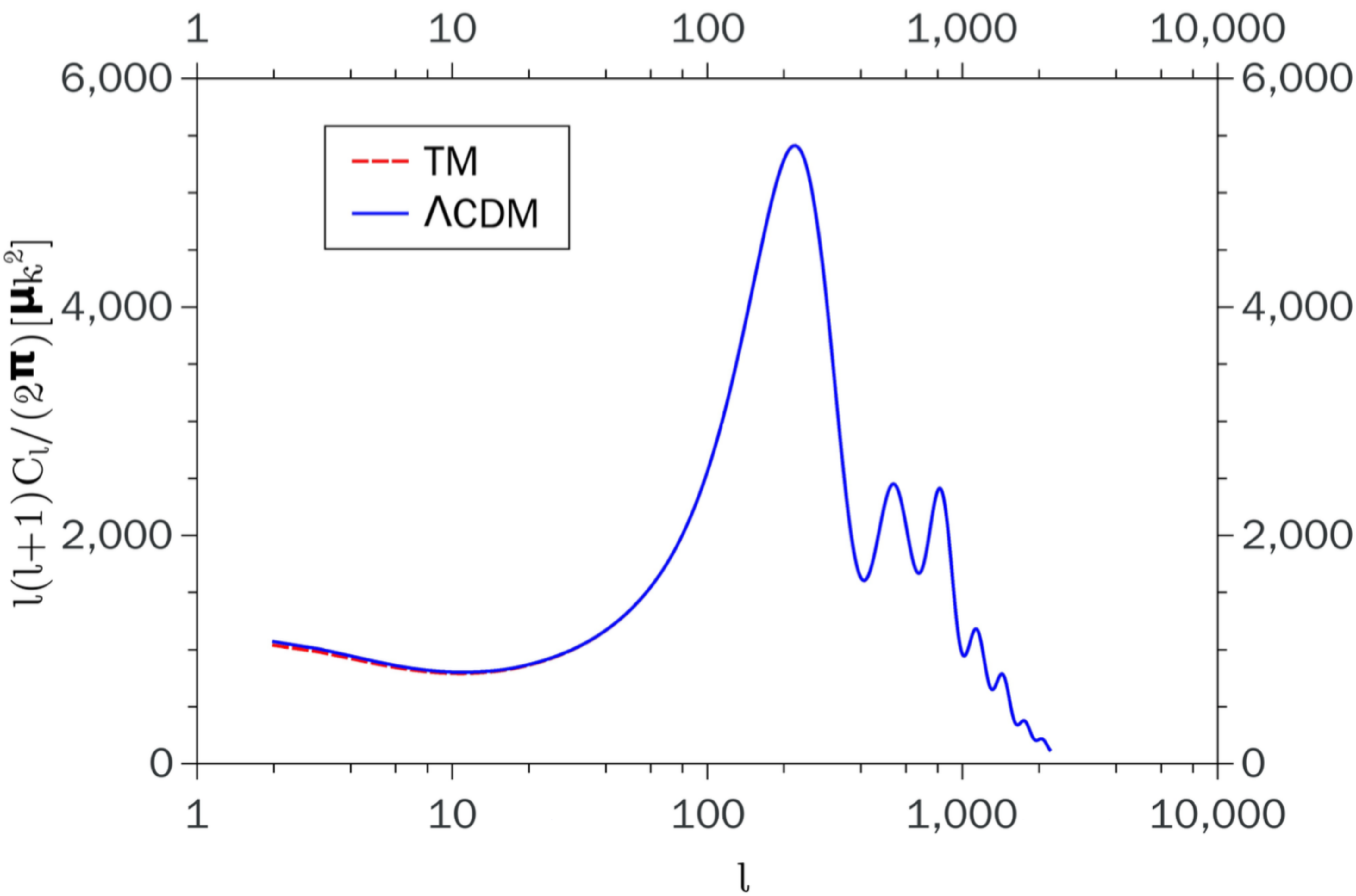
Matter Power Spectrum: $P(k) = \frac{<\delta_m^2(k)>}{(2\pi)^3}$, $\Delta\delta_m = \frac{(\delta_{m,TM} - \delta_{m,\Lambda CDM})}{\delta_{m,\Lambda CDM}}$



Density Contrast Evolution under the super-horizon limit:



Cosmic Microwave Background :



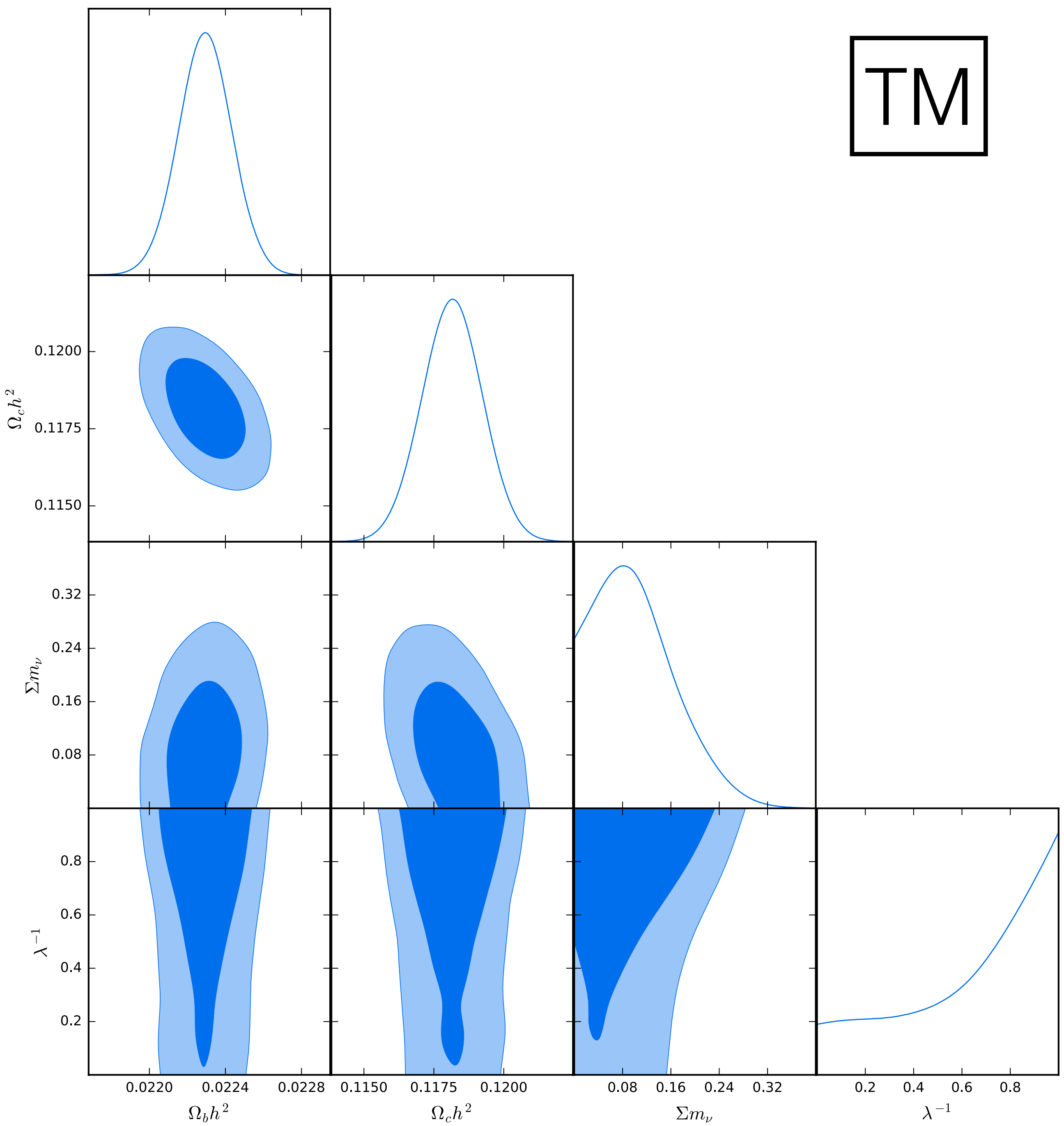
Observational Constraints

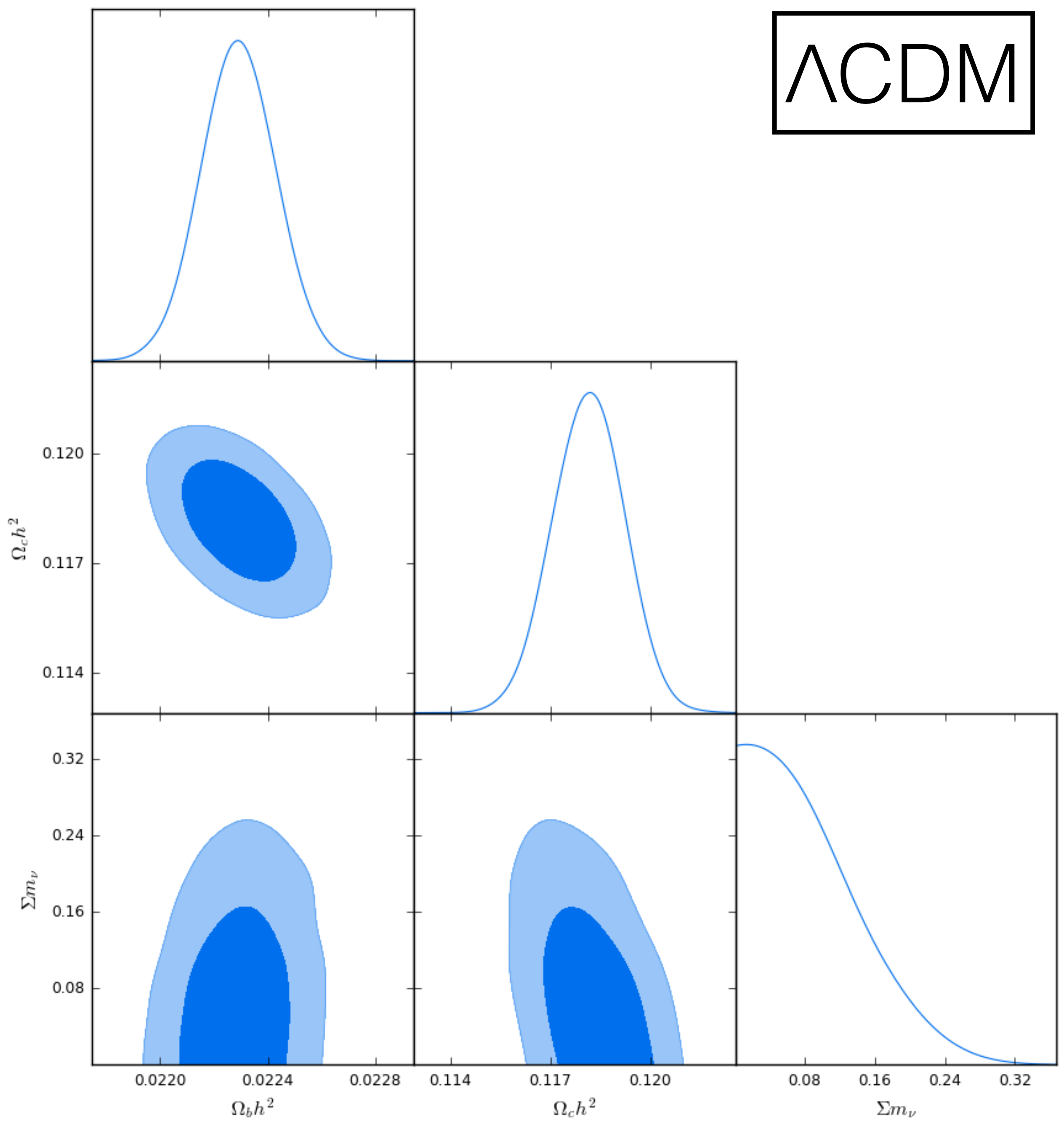
Fitting results:

Parameters	Parameter Symbols	TM	ΛCDM
Model parameter	λ^{-1}	$0.6646^{+0.33544}_{-0.54362}$	-
Density parameter of baryon	$\Omega_b h^2$	$0.02229^{+0.00028}_{-0.00028}$	$0.02229^{+0.00027}_{-0.00027}$
Density parameter of CDM	$\Omega_c h^2$	$0.11816^{+0.00212}_{-0.00212}$	$0.11816^{+0.00211}_{-0.00215}$
Total neutrino mass	Σm_ν	$0.10392^{+0.12283}_{-0.10392}$	$0.08434^{+0.11816}_{-0.08434}$
Spectral index	n_s	$0.96867^{+0.00756}_{-0.00763}$	$0.96899^{+0.00772}_{-0.00770}$
Scalar power spectrum amplitude	$\ln(10^{10} A_s)$	$3.06229^{+0.05364}_{-0.05305}$	$3.07018^{+0.05484}_{-0.05092}$
Optical depth to reionization	τ	$0.06630^{+0.02892}_{-0.02789}$	$0.07012^{+0.02944}_{-0.02690}$
Sound horizon / ang. distance	$100\ \theta_{MC}$	$1.04090^{+0.00059}_{-0.00059}$	$1.04090^{+0.00059}_{-0.00059}$
Hubble prameter	$H_0\ (km/s \cdot Mpc)$	$67.62788^{+1.13616}_{-1.21858}$	$67.71428^{+1.11201}_{-1.25371}$
Age of the universe	Age/Gyr	$13.81232^{+0.07030}_{-0.06538}$	$13.81044^{+0.07023}_{-0.06323}$
Sigma8	σ_8	$0.85866^{+0.04351}_{-0.05717}$	$0.81101^{+0.02411}_{-0.02710}$
χ^2 of the best-fit	$\chi^2_{best-fit}$	13458.82	13459.12

Contour plots:

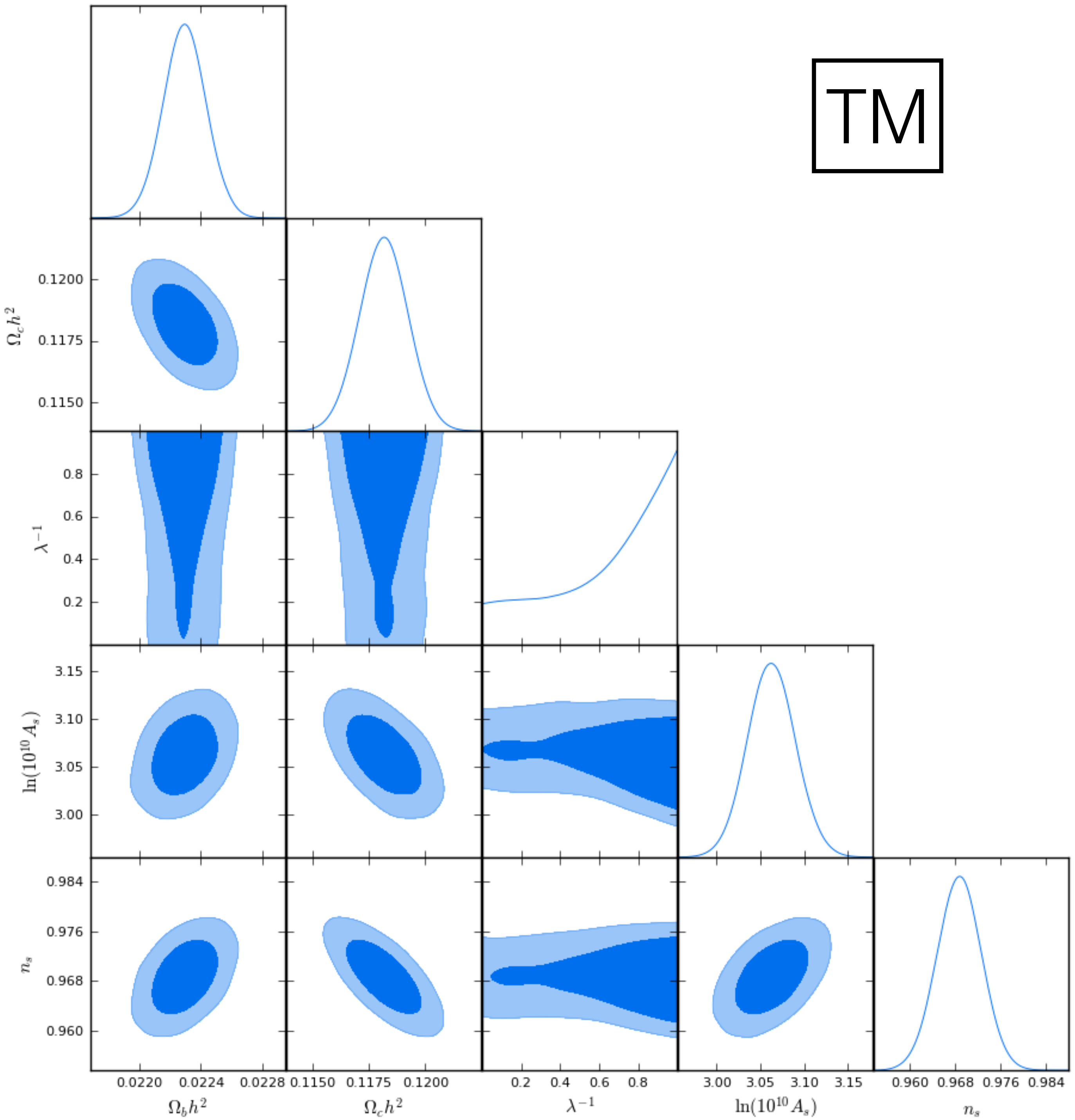
Comparing TM with ΛCDM in $\Omega_b,\, \Omega_c,\, \lambda^{-1}$ and Σm_ν

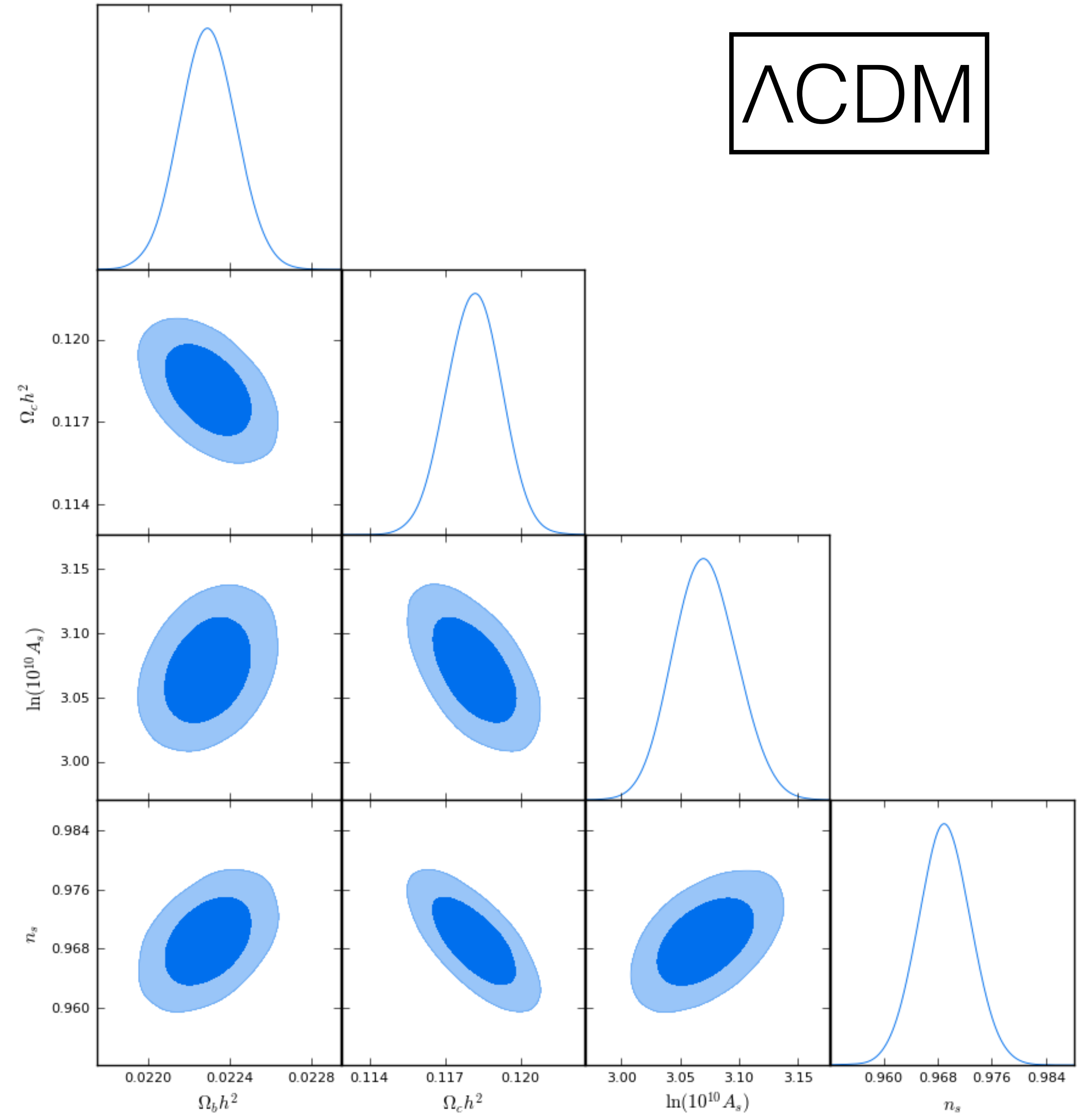




Contour plots:

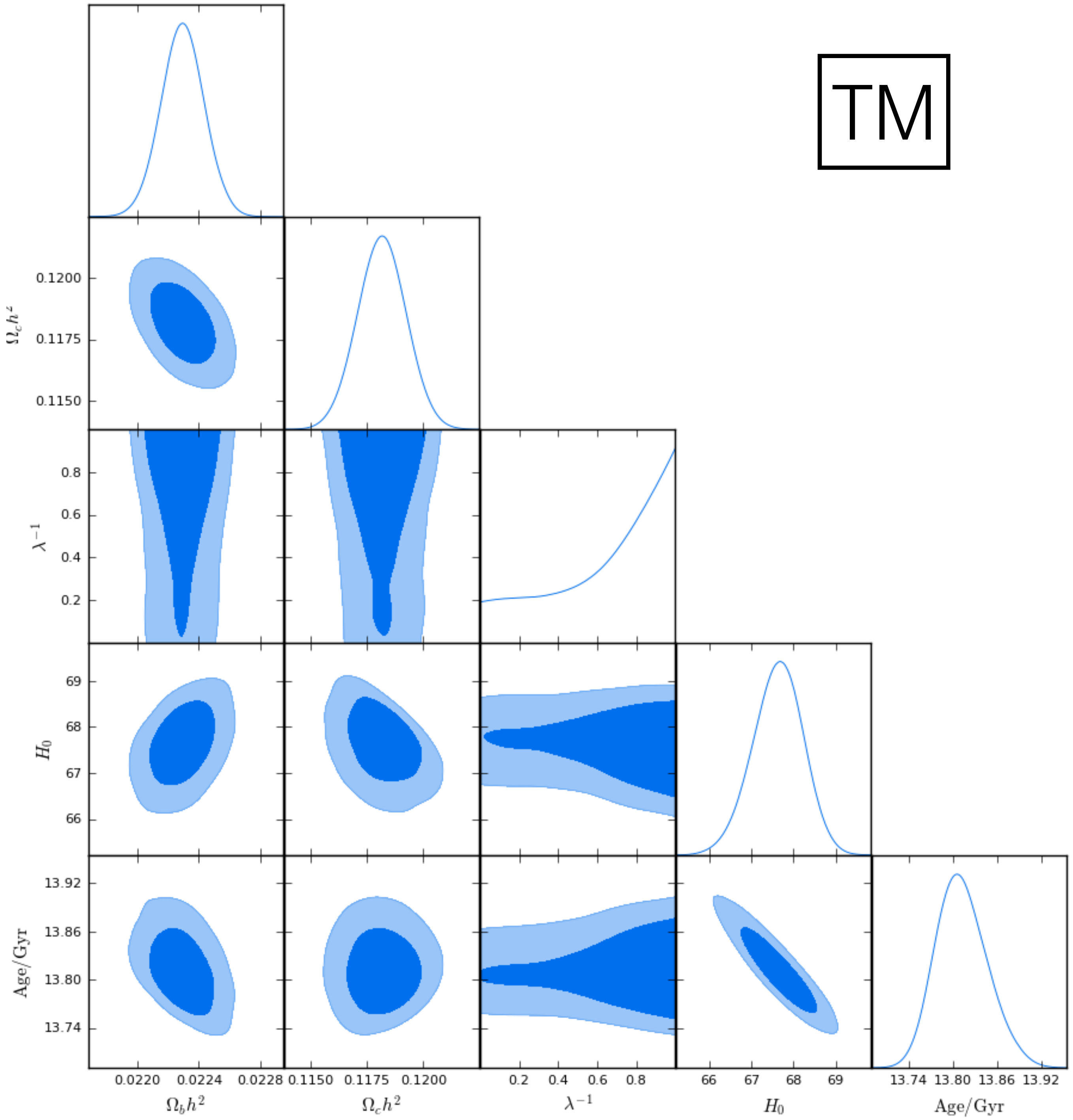
Comparing TM with ΛCDM in Ω_b, Ω_c, λ^{-1}, $ln(10^{10}A_s)$ and n_s

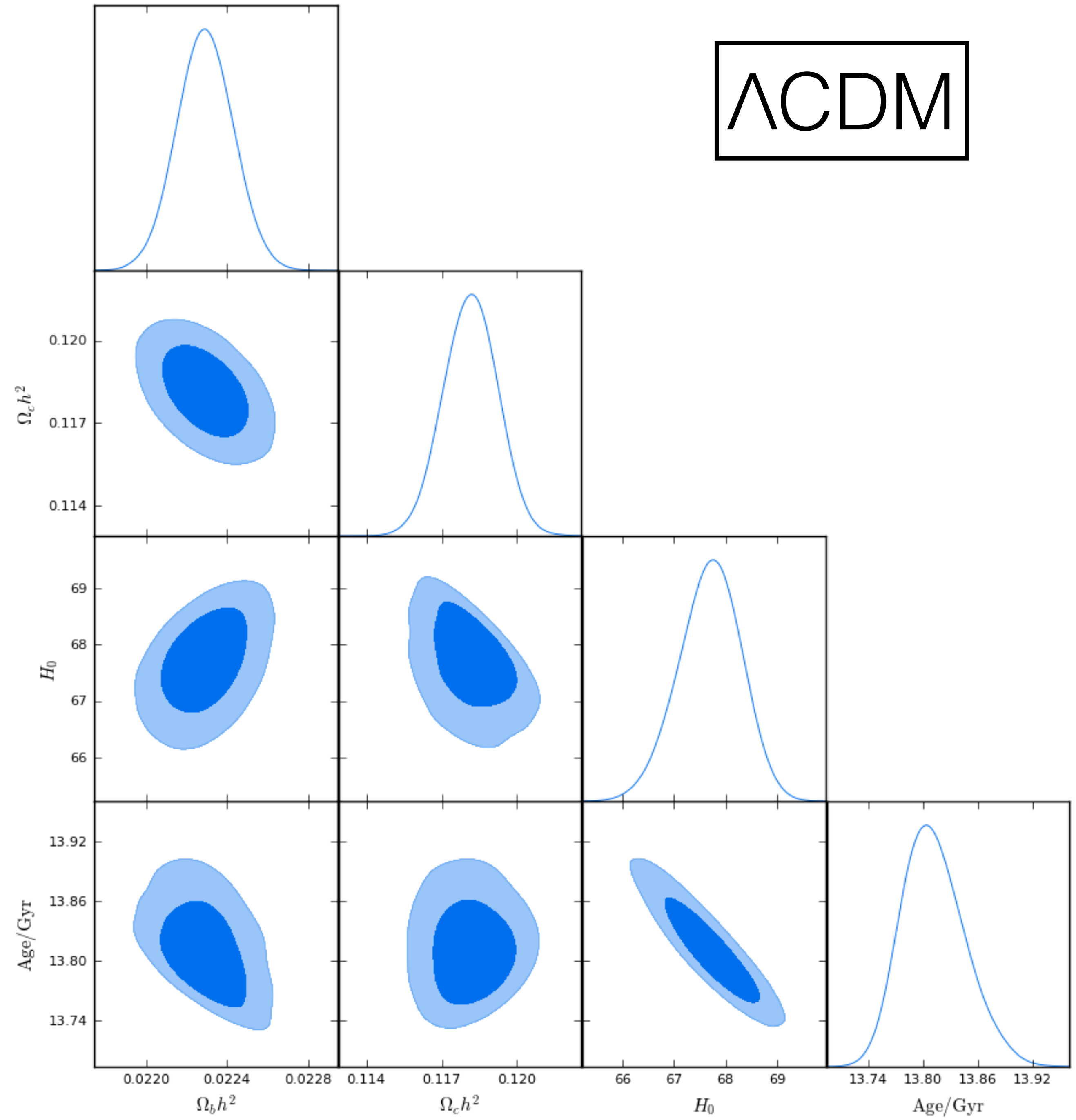




Contour plots:

Comparing TM with ΛCDM in Ω_b, Ω_c, λ^{-1}, H_0 and **age**.





Summary

Summary:

1. The TM universe performs different evolutions compared with the ΛCDM model such as the equation of state, deceleration parameter.

2. Perturbation calculations show that the matter power spectrum is strengthen in the sub-horizon region and $k \to 0$.

3. The CMB power spectrum gets smaller when ($\ell < 10$) in the Tsujikawa model by the ISW effect.

4. The MCMC result in TM indicates that the limit of the sum of the neutrino mass is relaxed and the redshift of the reionization becomes smaller.

5. The contour plot shows that the model parameter of the TM is not really sensitive to the observational data .

Backup Slides

The history of f(R) Theory:

Ruzmaĭkina, T. V. Ruzmaĭkin, A. A.	Quadratic Corrections to the Lagrangian Density of the Gravitational Field and the Singularity (1969)
H. A. Buchdahl	Non-Linear Lagrangians and Cosmological Theory (1970)
A A Starobinsky	• A New Type of Isotropic Cosmological Models Without Singularity (1980)
A A Starobinsky H -J Schmidt	On a general vacuum solution of fourth-order gravity (1987)
⋮	
Dark energy models	Tsujikawa,

σ_8 (rms density contrast in spheres of radius in 8 Mpc/h)

$$\sigma_R^2 = \frac{1}{2} \int P(k) W_R^2(k) k^2 dk$$

$$W_R = \text{ window function}$$
$$= \frac{3}{(kR)^3} (sin(kR) - kRcos(kR))$$

- If the cells have a radius of 8 Mpc/h, it turns out the σ_R is close to unity. Conventionally the normalization of the power spectrum is therefore given by quoting σ_R.

Dark Energy model: Quintessence

- quintessence (Add a scalar field Lagrangian.):

$$S = \int d^4x(\frac{\sqrt{-g}}{2\kappa^2}R + L_\phi) + S_M \, , L_\phi = \frac{-1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\phi - V(\phi)$$

- Its equation of state is always larger than −1.
- The behavior depends on the form of the potential.
- Thawing model: The field (with mass $m\phi$) has been frozen by Hubble friction (i.e. the term $H\dot{\phi}$) until recently and then it begins to evolve once H drops below $m\phi$. The equation of state of dark energy is $w\phi \simeq -1$ at early times, which is followed by the growth of $w\phi$.
- Freezing model: The field was rolling along the potential in the past, but the movement gradually slows down after the system enters the phase of cosmic acceleration.

Dark Energy model: k-essence

- K-essence:

$$S = \int d^4x \left(\frac{\sqrt{-g}}{2\kappa^2} R + P(\phi, X) \right) + S_M \,, \quad X = \frac{-1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \phi$$

- The cosmic acceleration can be realized by the kinetic energy X of the field ϕ.

$$\text{Pressure} : P_\phi = P(X, \phi)$$

$$\text{Energy Density} : \rho_\phi = 2XP_{,X} - P$$

$$w_\phi = \frac{P}{2XP_{,X} - P}$$

- Phantom: The scalar field with a negative kinetic energy.

$$P(X, \phi) = -X - V(\phi)$$

$$w_\phi = \frac{\dot{\phi}^2/2 + V(\phi)}{\dot{\phi}^2/2 - V(\phi)} < -1 \text{ for } \frac{\dot{\phi}^2}{2} < V(\phi)$$

Dark Energy model: Modified Gravity

- Scalar tensor theory (ST):

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}f(\phi, R) - \frac{1}{2}\zeta(\phi)(\nabla\phi)^2\right) + S_M$$

- The simplest ST: Brans–Dicke theory:

$$f(\phi, R) = \phi R,\ \zeta = \frac{\omega_{\mathrm{BD}}}{\phi}\ ,\ \omega_{\mathrm{BD}} : \text{the Brans–Dicke parameter}$$

- Gauss-Bonnet theory:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - f(\phi)R^2_{GB}\right) + S_M,$$

$$R^2_{GB} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

- In Cartesian coordinate system
(There is no position dependence)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2$$

- In spherical coordinate
(no theta and phi dependence):

$$\mathrm{d}s^2 = R^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right) - c^2\mathrm{d}t^2$$

Luminosity distance:

$$m - M = 5\log_{10}(D_L - 1)$$

$$\begin{aligned} D_L &= (1+z)a_0\Phi(\chi) \\ \Phi(\chi) &= \frac{1}{a_0}\int_0^{z_e}\frac{dz}{H(z)} \end{aligned}$$

Gauge transformation:

$$\Psi = \dot{\alpha} + H\alpha,$$

$$\Phi = \eta - H\alpha,$$

$$\text{where } \alpha = (\dot{h} + 6\dot{\eta})/2k^2$$

Matter perturbation (perfect fluid):

$$T_0^0 = -(\rho_M + \delta\rho_M),$$

$$T_i^0 = -(\rho_M + P_M)\partial_i v,$$

$$T_j^i = (P_M + \delta P_M)\delta_j^i$$

Perturbation in super-horizon :

$$\xi = -\Phi + Hq. \quad \text{(Curvature fluctuation)}$$

$$\xi' = \Phi' + H'q + Hq' = 0 \quad \text{(Curvature fluctuation conservation)}$$

$$\nabla_\alpha T^{i\alpha} = 0 \quad \rightarrow \quad Hq' = -\Psi \quad \text{(Momentum conservation)}$$

$$\Phi' - \Psi + H'q = 0$$

$$\Phi'' - \Psi' - \frac{H''}{H'}\Psi' - (\frac{H'}{H} - \frac{H''}{H'})\Psi = 0$$

$$\Psi = \frac{1}{1-B}(-B\Phi' - \Phi) \quad \text{(Relation between } \Phi \text{ and } \Psi\text{)}$$

$$\Phi'' + \left(1 - \frac{H''}{H'} + \frac{B'}{1-B} + B\frac{H'}{H}\right)\Phi' + \left(\frac{H'}{H} - \frac{H''}{H'} + \frac{B'}{1-B}\right)\Phi = 0\,, (k \to 0).$$

$$2\Phi_{eff} + \left(\frac{B}{2}\frac{E'}{E}\frac{E'}{4E' + E''}\right)S = \left(\frac{-1}{f_R}\frac{\kappa^2 a^2 \rho_M}{k^2}\right)\Delta_M, \text{ (Modified Poisson equation)}$$

WEP

The motion freely-falling particles are the same in a gravitational field and a uniformly accelerated frame in small enough regions of spacetime.

EEP

In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.

SEP

In small enough regions of spacetime, the laws of physics (include the gravity) reduce to those of special relativity.

Statistical Method

$$\chi^2(\boldsymbol{\theta}) = -2\ln\mathcal{L}(\boldsymbol{\theta}) + \text{const} = \sum_{i=1}^{N}\frac{(y_i - F(x_i;\boldsymbol{\theta}))^2}{\sigma_i^2} + \text{const}\ .$$