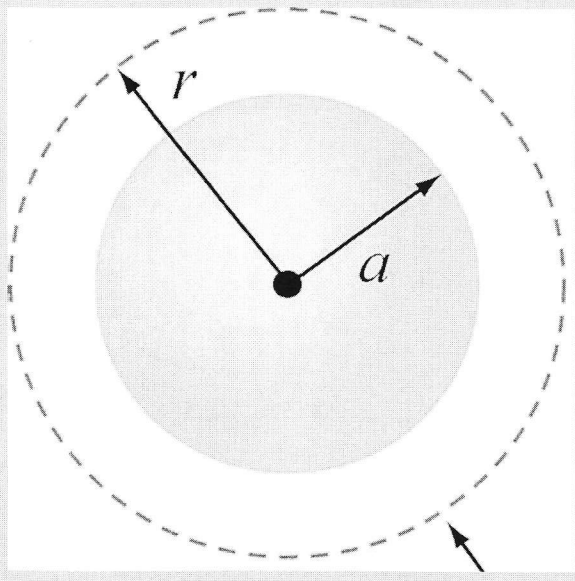
Part 3 Topics

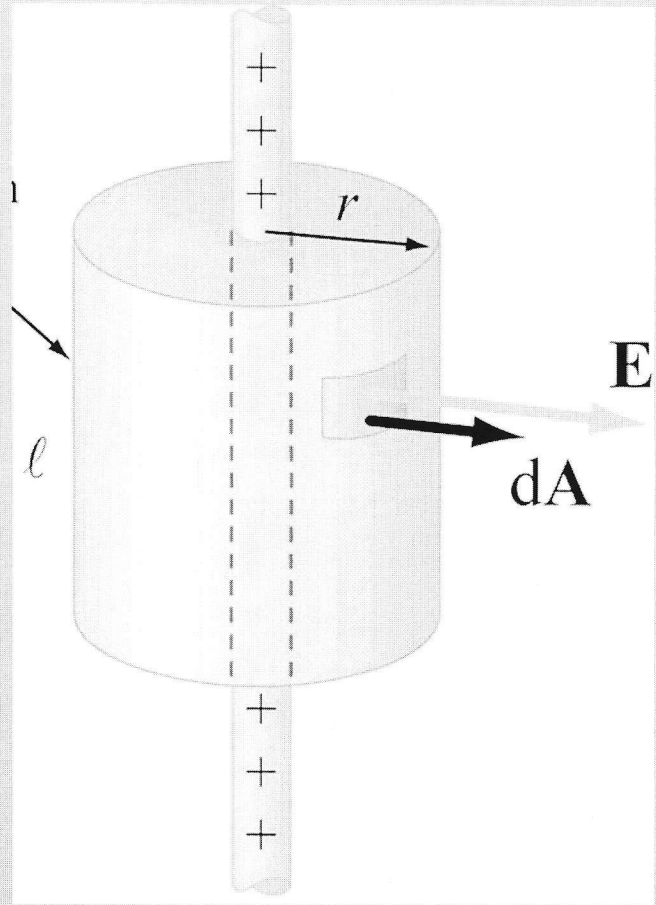
- Faraday's Law
- Self Inductance
- Mutual Inductance
  - Transformers
- Circuits
  - LR Circuits
  - Undriven (R)LC Circuits
  - Driven RLC Circuits
- Displacement Current
- Maxwell Equations

Gauss's Law:

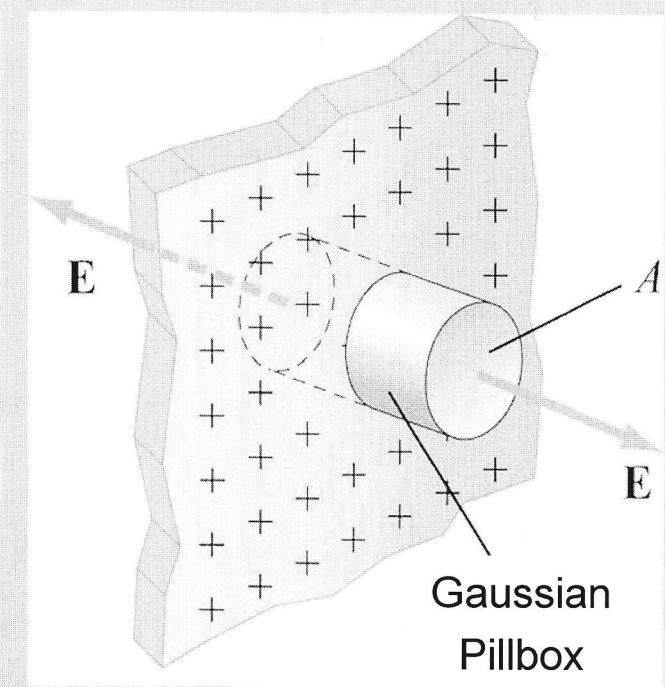
$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$



Spherical Symmetry



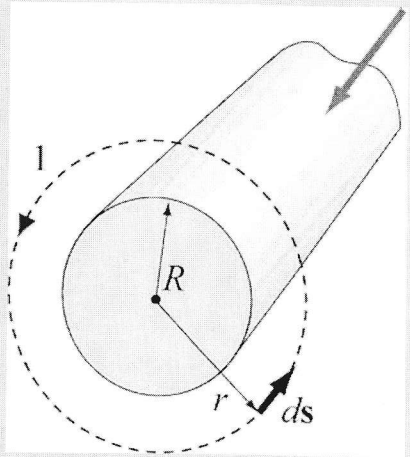
Cylindrical Symmetry

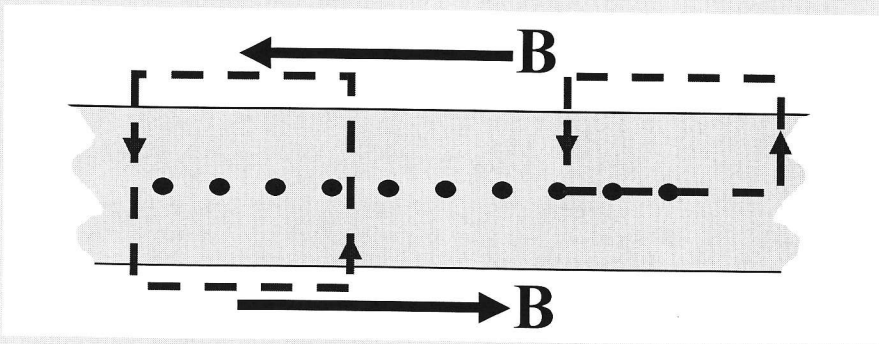


Planar Symmetry

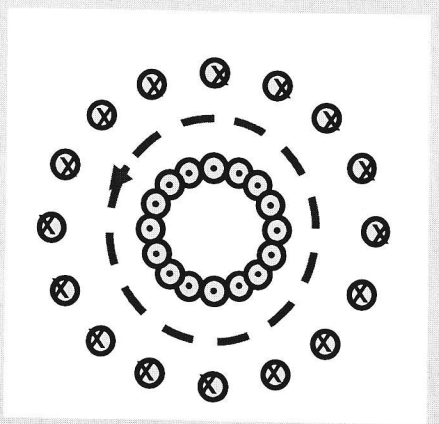
Ampere's Law: $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$

Long Circular Symmetry





(Infinite) Current Sheet

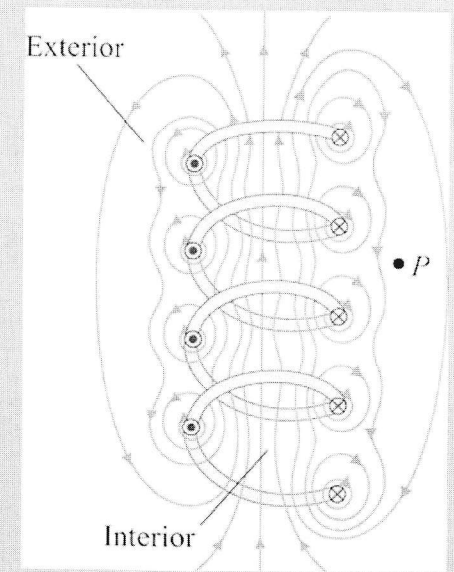


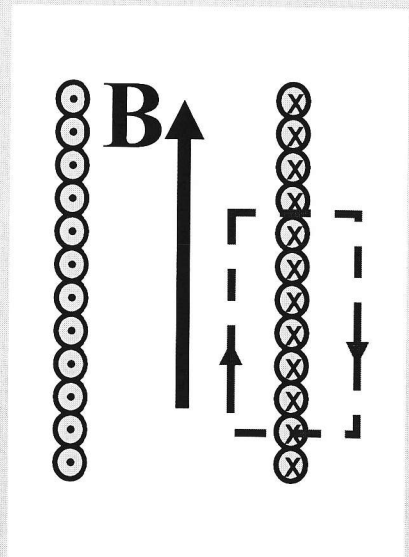
Torus/Coax

Solenoid

=

2 Current Sheets





Faraday's Law of Induction

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -N \frac{d\Phi_B}{dt}$$

Moving bar, entering field

$$= -N \frac{d}{dt}\left(BA\cos\theta\right)$$

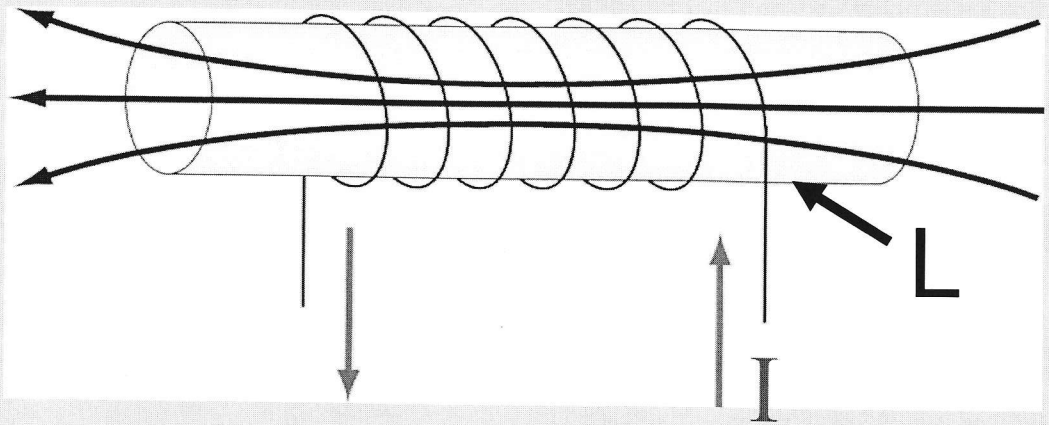
Ramp B

Rotate area in field

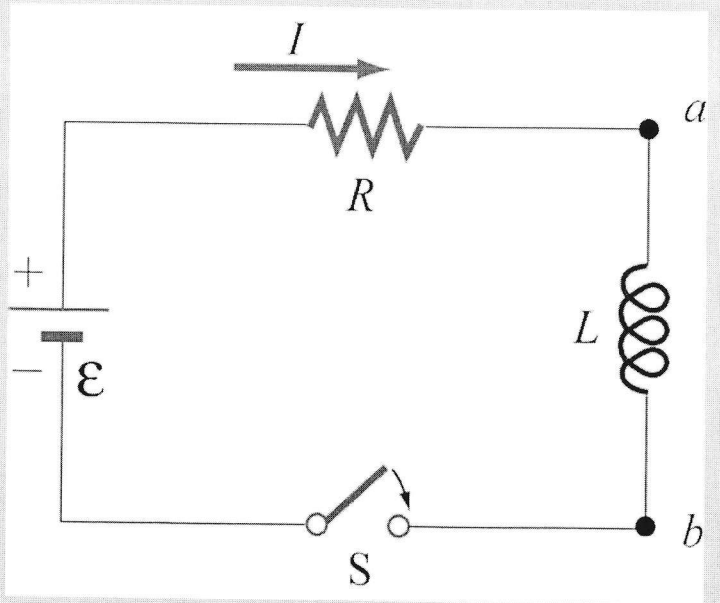
Lenz's Law:

Induced EMF is in direction that ***opposes*** the change in flux that caused it

Self Inductance & Inductors



$$L = \frac{N\Phi}{I}$$



When traveling in direction of current:

$$\mathcal{E} = -L\frac{dI}{dt}$$

Notice: This is called "Back EMF"

It is just Faraday's Law!

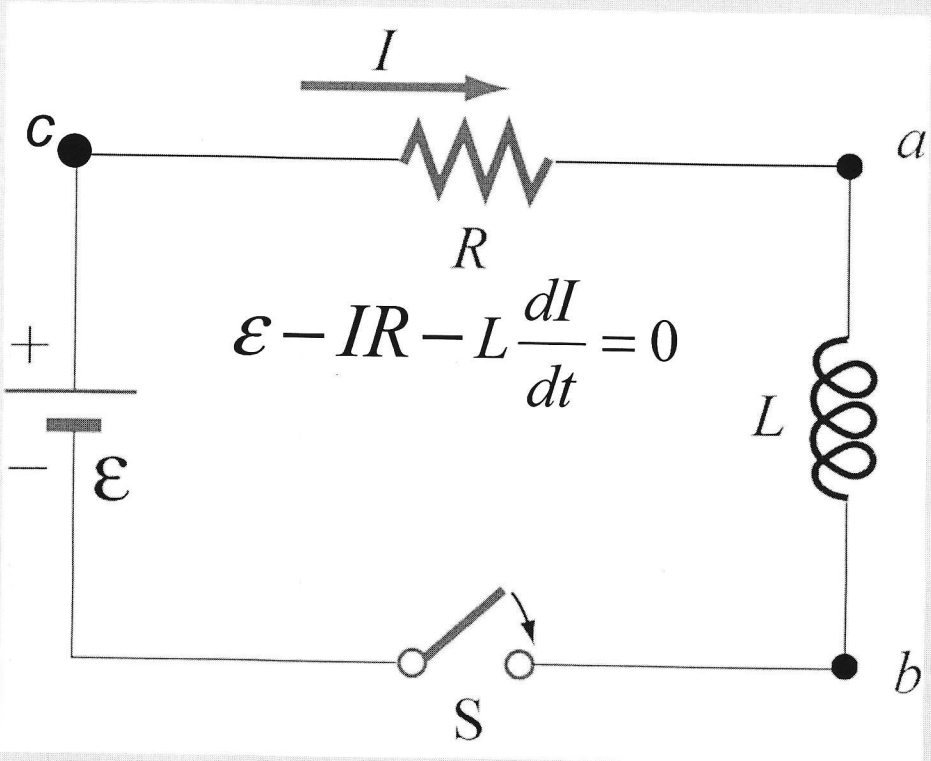
Energy Stored in Inductor

$$U_L = \frac{1}{2} L I^2$$

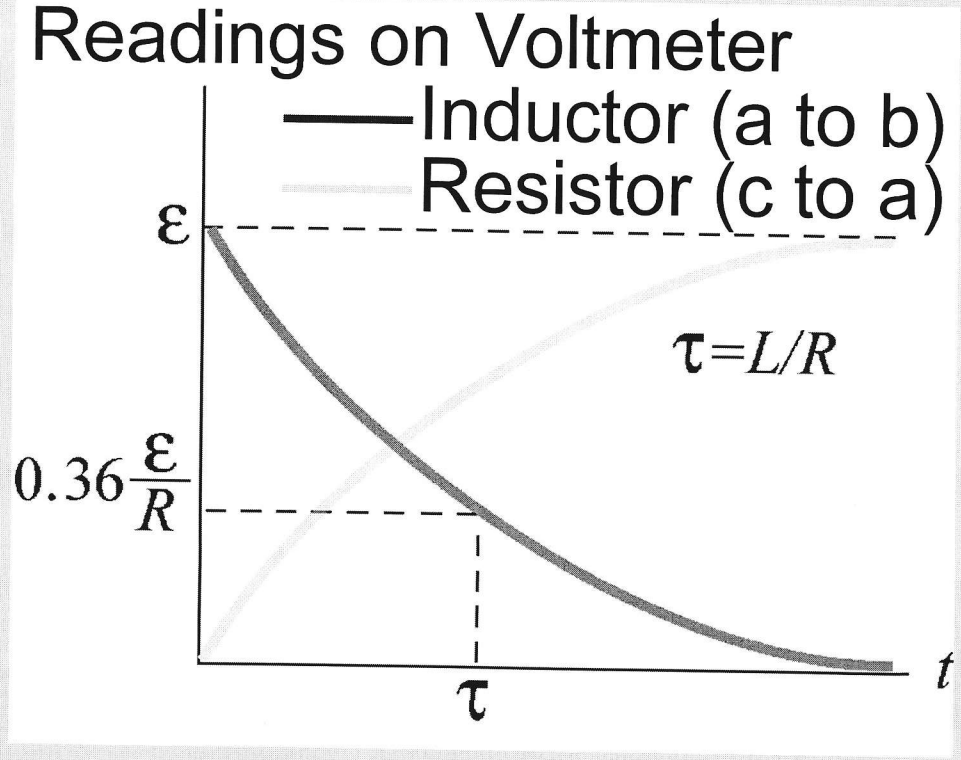
Energy is stored in the magnetic field:

$$u_B = \frac{B^2}{2\mu_o}$$: Magnetic Energy Density

LR Circuit



$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

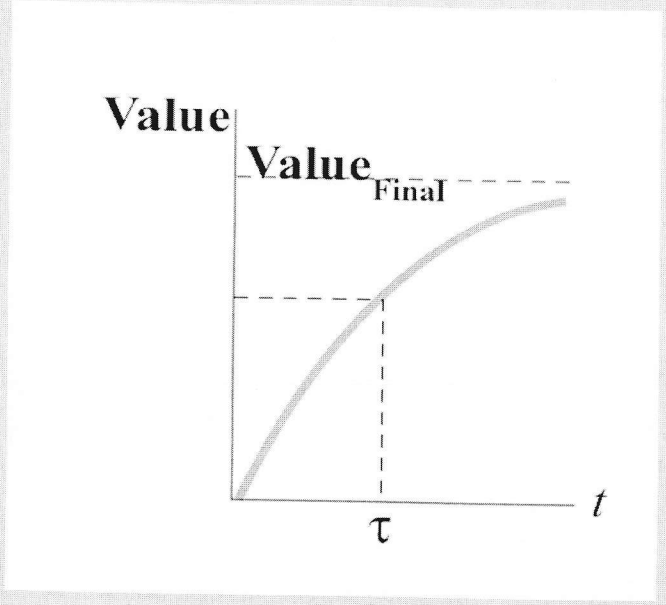


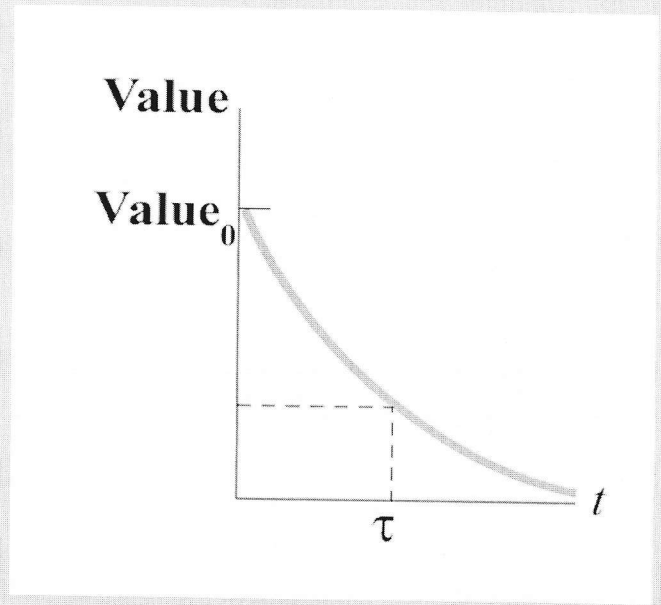
t=0⁺: Current is trying to change. Inductor works as hard as it needs to in order to stop it

t=∞: Current is steady. Inductor does nothing.

General Comment: LR/RC

All Quantities Either:



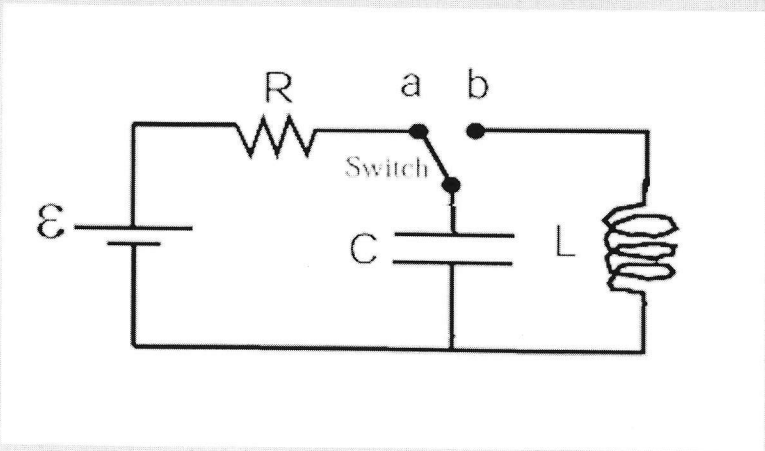


$$\text{Value}(t) = \text{Value}_{\text{Final}}\left(1 - e^{-t/\tau}\right)$$

$$\text{Value}(t) = \text{Value}_0 e^{-t/\tau}$$

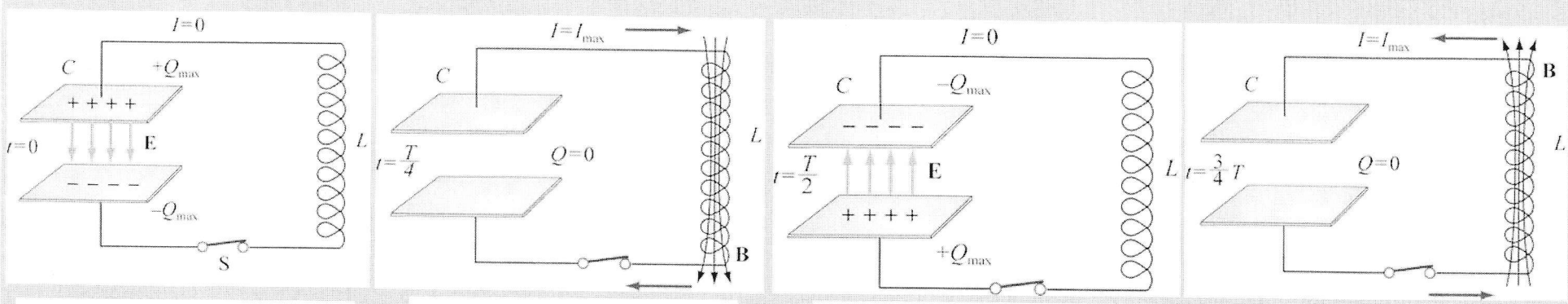
τ can be obtained from differential equation (prefactor on d/dt) e.g. τ = L/R or τ = RC

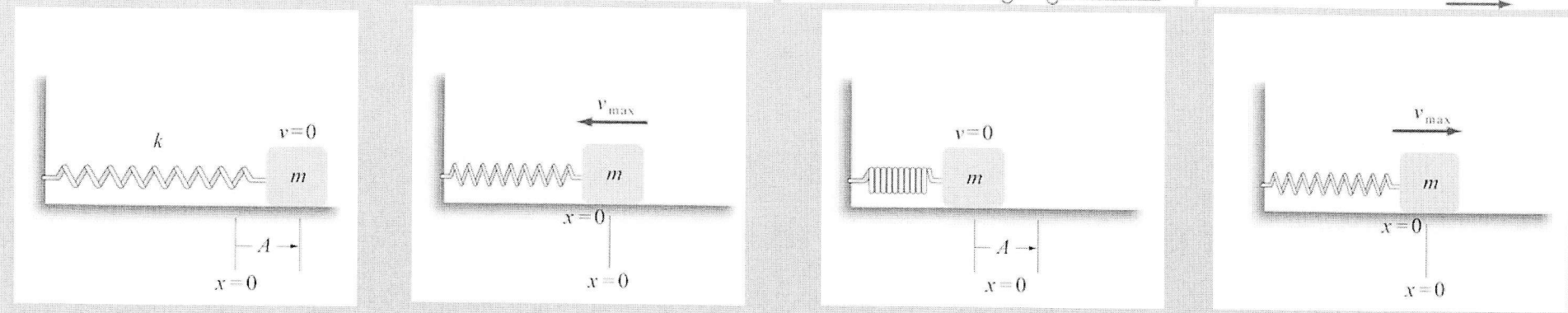
Undriven LC Circuit



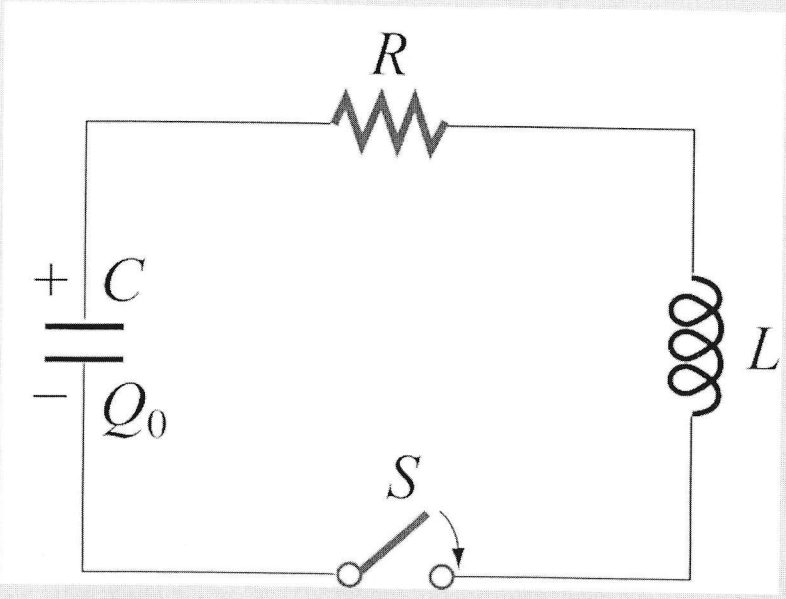
Oscillations: From charge on capacitor (Spring) to current in inductor (Mass)

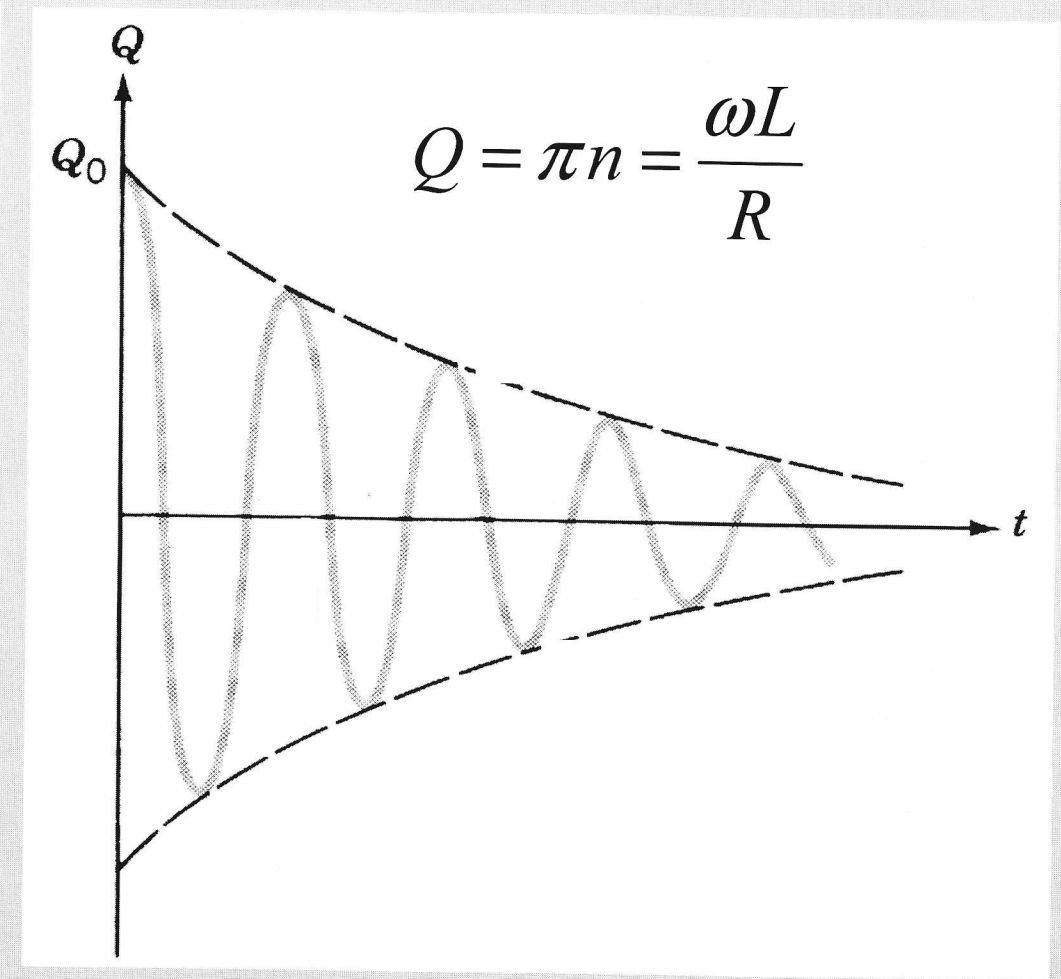
$$\omega_0 = \frac{1}{\sqrt{LC}}$$





Damped LC Oscillations



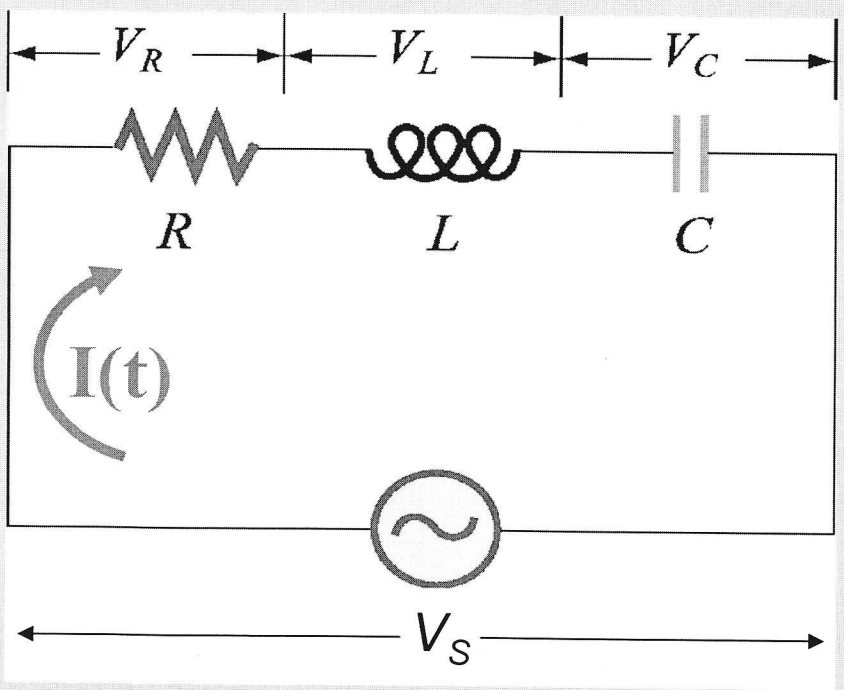


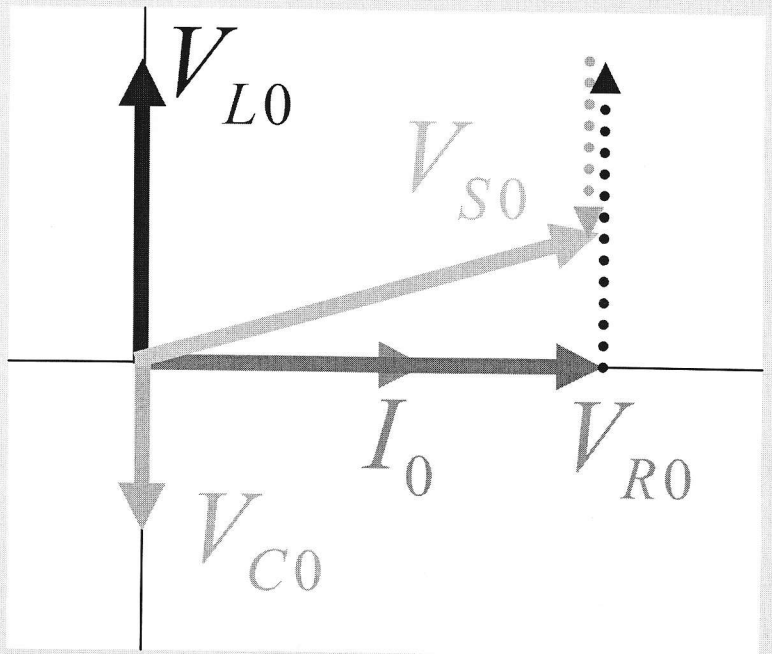
Resistor dissipates energy and system rings down over time

AC Circuits: Summary

Element	V vs I_0	Current vs. Voltage	Resistance-Reactance (Impedance)
Resistor	$V_{0R} = I_0 R$	In Phase	$R = R$
Capacitor	$V_{0C} = \frac{I_0}{\omega C}$	Leads (90°)	$X_C = \frac{1}{\omega C}$
Inductor	$V_{0L} = I_0 \omega L$	Lags (90°)	$X_L = \omega L$

Driven RLC Series Circuit

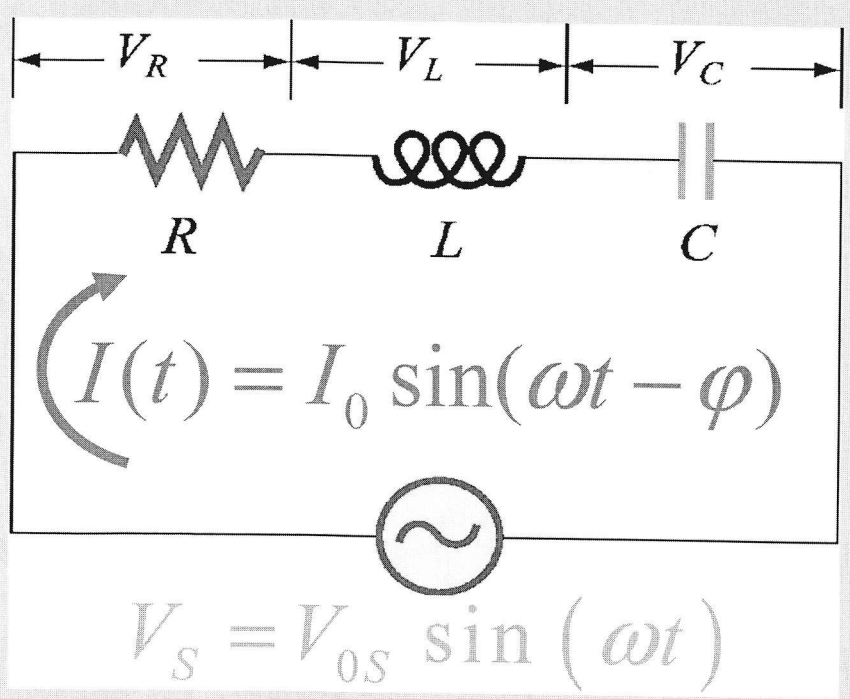


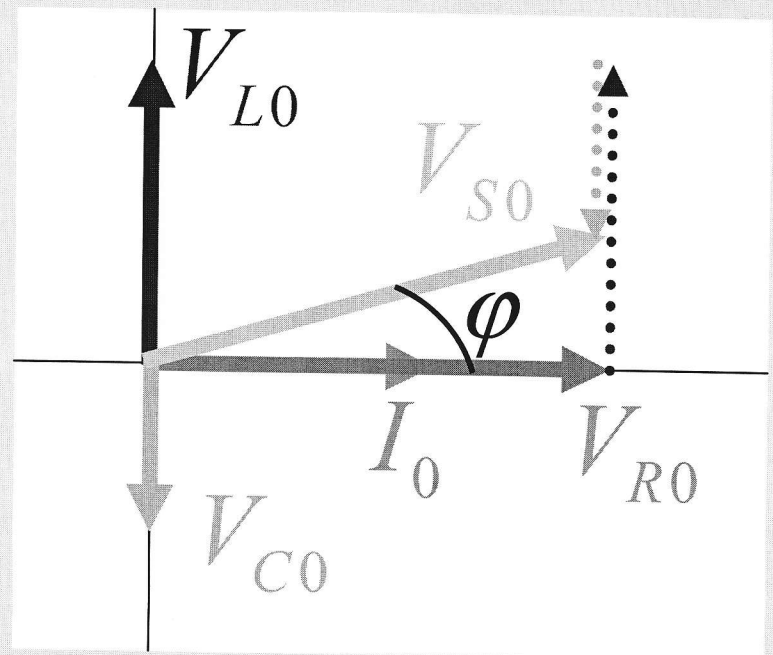


Now Solve: $V_S = V_R + V_L + V_C$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{S0} = \sqrt{V_{R0}^{\ 2} + (V_{L0} - V_{C0})^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

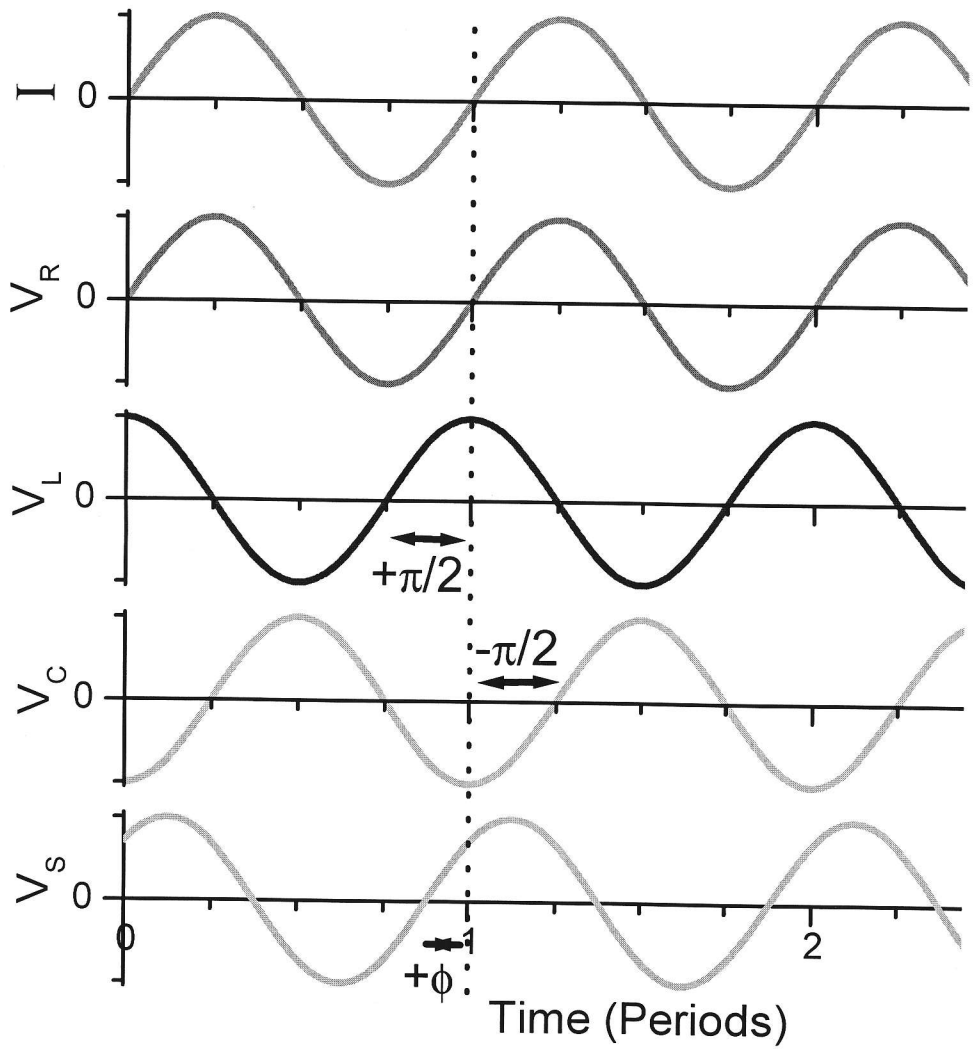
$$\boxed{I_0 = \frac{V_{S0}}{Z}}$$

$$\boxed{Z = \sqrt{R^2 + (X_L - X_C)^2}}$$

Impedance

$$\boxed{\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)}$$

Plot I, V's vs. Time



$$I(t) = I_0 \sin(\omega t)$$

$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

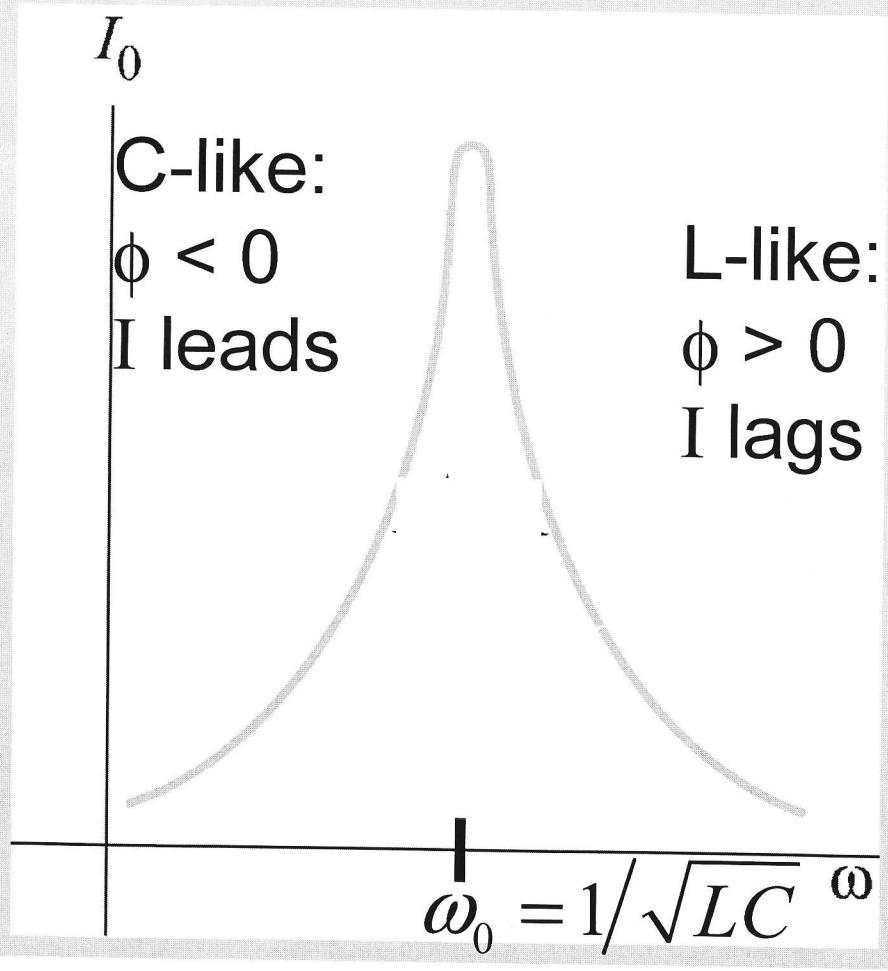
$$V_C(t) = I_0 X_C \sin\left(\omega t - \tfrac{\pi}{2}\right)$$

$$V_S(t) = V_{S0} \sin(\omega t + \varphi)$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



On resonance:

I_0 is max; $X_L = X_C$; $Z = R$;

$\phi = 0$; Power to R is max

Average Power: Resistor

$$
\begin{aligned}
< P > &= < I^2(t)R > \\
&= < I_0^2 \sin^2(\omega t - \varphi)R > \\
&= I_0^2 R < \sin^2(\omega t - \varphi) > \\
&= I_0{}^2 R\left(\tfrac{1}{2}\right)
\end{aligned}
$$

Electronic Analog: RLC Circuits

A C Circuit

The discussion we give is based on solving the differential equation.

The final results (steady state part) are written in compact form.

Usually these results are given (obtained) by phaser diagram.

Let us go back to the fundamental Equation for the LRC circuit.

$$V_0(t) - L\frac{d^2Q}{dt^2} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

(1) $Q(t)$ is a function of t

(2) $V_s(t) = V_{s0}\sin(\omega t + \alpha)$

$\downarrow$ driving voltage

The solution of $Q(t)$ is a linear combination of $\sin\omega t$ and $\cos\omega t$

$$Q(t) = Q_0 \sin(\omega t - \frac{\pi}{2} + \alpha)$$

$$= Q_0 \sin(\omega t + \alpha - \frac{\pi}{2})$$

$$= Q_0[\sin(\omega t + \alpha)\cos\frac{\pi}{2} - \cos(\omega t + \alpha)\sin\frac{\pi}{2})]$$

$$= -Q_0\cos(\omega t + \alpha) = -\frac{I_0}{\omega}\cos(\omega t + \alpha)$$

$$I(t) = \frac{dQ}{dt} = +\underbrace{Q_0\omega}_{I_0}\sin(\omega t + \alpha) = I_0\sin(\omega t + \alpha)$$

$$\frac{d^2Q}{dt^2} = Q_0\omega^2\cos(\omega t + \alpha)$$

$$= Q_0\omega^2\sin(\omega t + \alpha + \frac{\pi}{2})$$

$\Rightarrow$ $V_{0R} = I_0 R$ in phase

$$V_{0L} = L\frac{d^2Q}{dt^2} = (L\,Q_0\omega^2)\sin(\omega t + \alpha + \frac{\pi}{2})$$

$$LQ_0\omega\omega = I_0\omega L$$

$$= \underbrace{I_0\omega L}_{X_L}\sin(\omega t + \alpha + \frac{\pi}{2})$$

$$V_{oc} = \frac{Q(t)}{C} = \frac{Q_0}{C} \sin\left(\omega t + \alpha + \frac{\pi}{2}\right)$$

$$= \frac{Q_0 \omega}{\omega C} \sin\left(\omega t + \alpha + \frac{\pi}{2}\right)$$

$$= \frac{I_0}{\omega C} \sin\left(\omega t + \alpha + \frac{\pi}{2}\right)$$

$$V_s = V_R + V_L + V_C$$

$\downarrow$

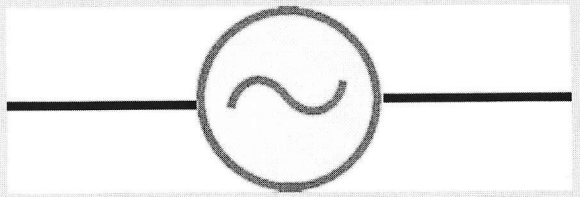
results can be read off from the phaser diagram.

See the graph.

Alternating-Current Circuit

- direct current (dc) – current flows one way (battery)
- alternating current (ac) – current oscillates

- sinusoidal voltage source

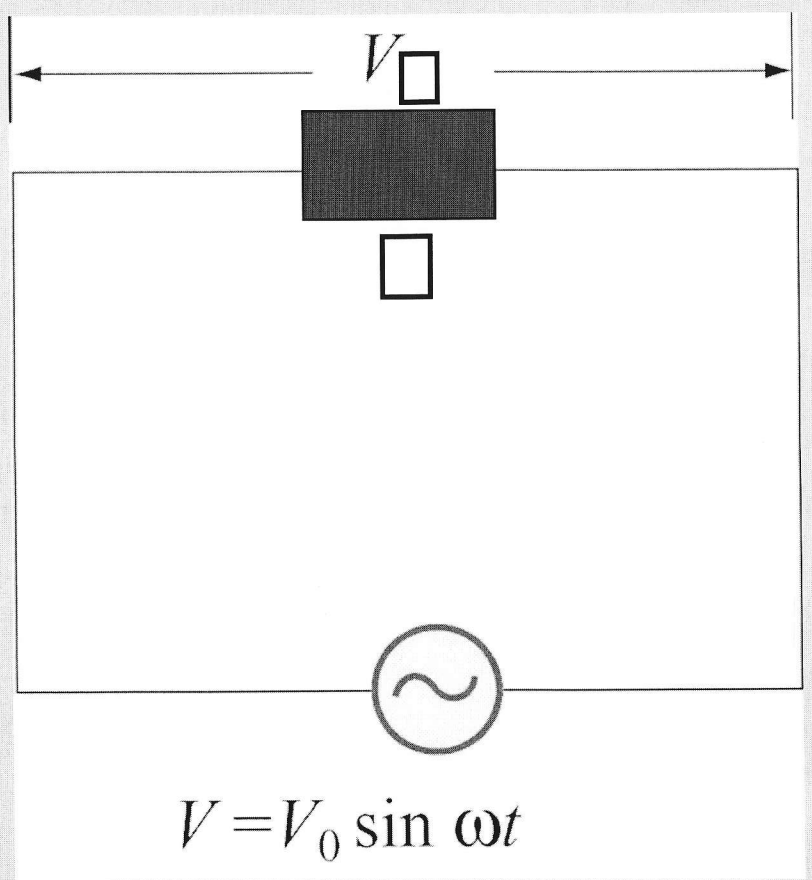


$$V(t) = V_0 \sin \omega t$$

$$\omega = 2\pi f : \text{ angular frequency}$$

$$V_0 : \text{ voltage amplitude}$$

AC Circuit: Single Element



$$V_{\square} = V$$

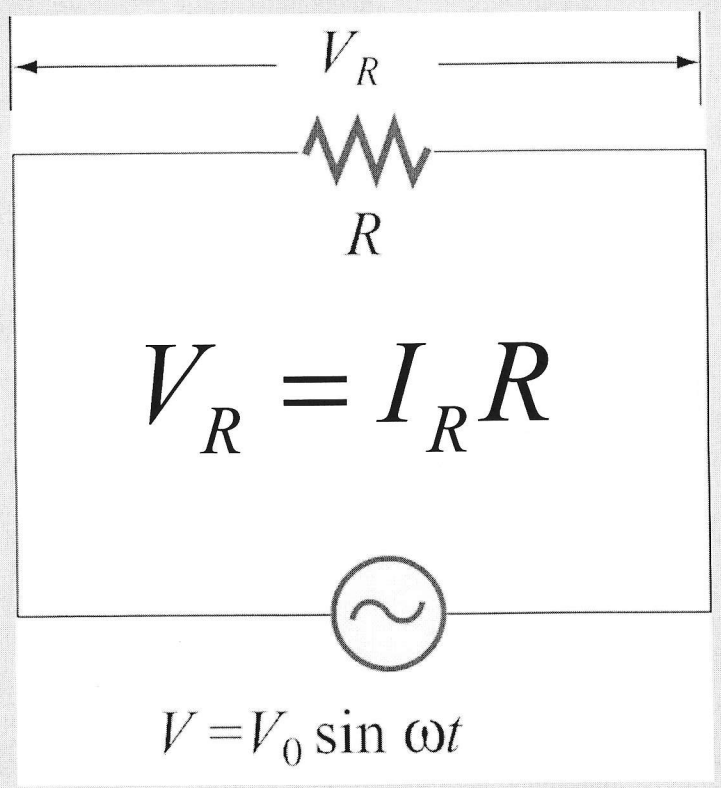
$$= V_0 \sin \omega t$$

$$I(t) = I_0 \sin(\omega t - \phi)$$

Questions:

1. What is I_0?
2. What is ϕ?

AC Circuit: Resistors



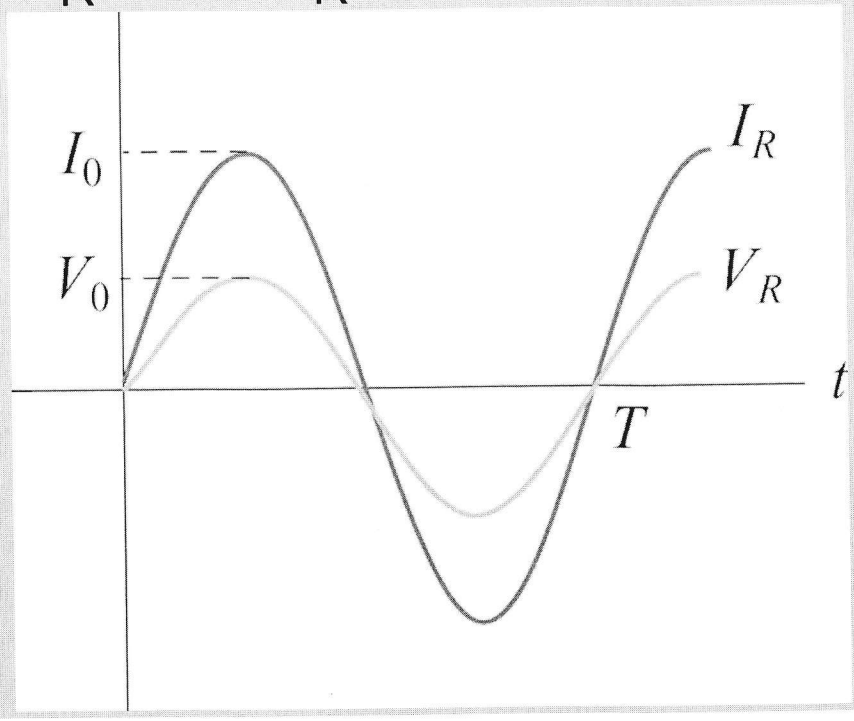
$$V_R = I_R R$$

$$I_R = \frac{V_R}{R} = \frac{V_0}{R} \sin \omega t$$

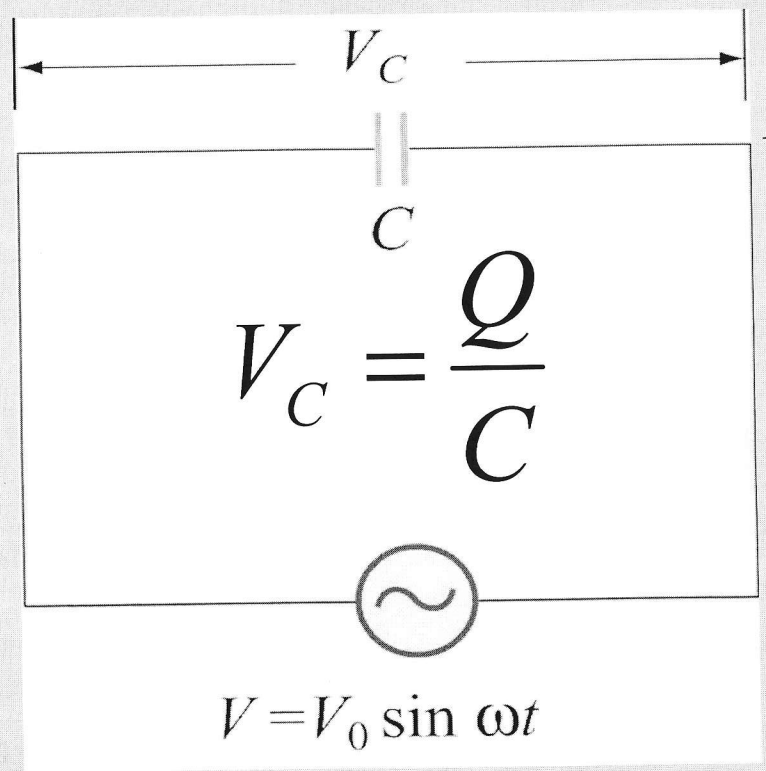
$$= I_0 \sin(\omega t - 0)$$

$$\boxed{I_0 = \frac{V_0}{R} \qquad \varphi = 0}$$

I_R and V_R are **in phase**



AC Circuit: Capacitors



$$V_C = \frac{Q}{C}$$

$$I_C(t) = \frac{dQ}{dt}$$

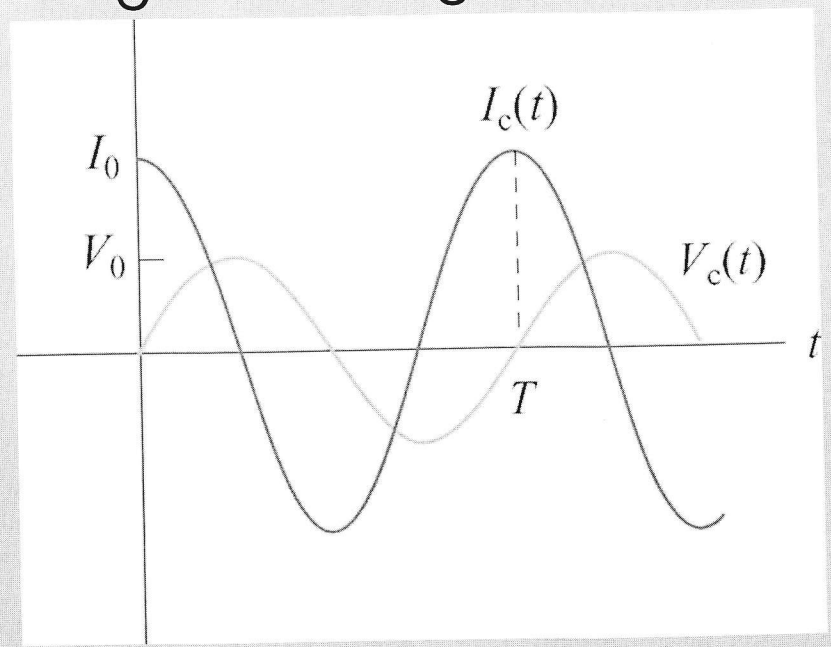
$$= \omega C V_0 \cos \omega t$$

$$= I_0 \sin(\omega t - {}^{-\pi}/_{2})$$

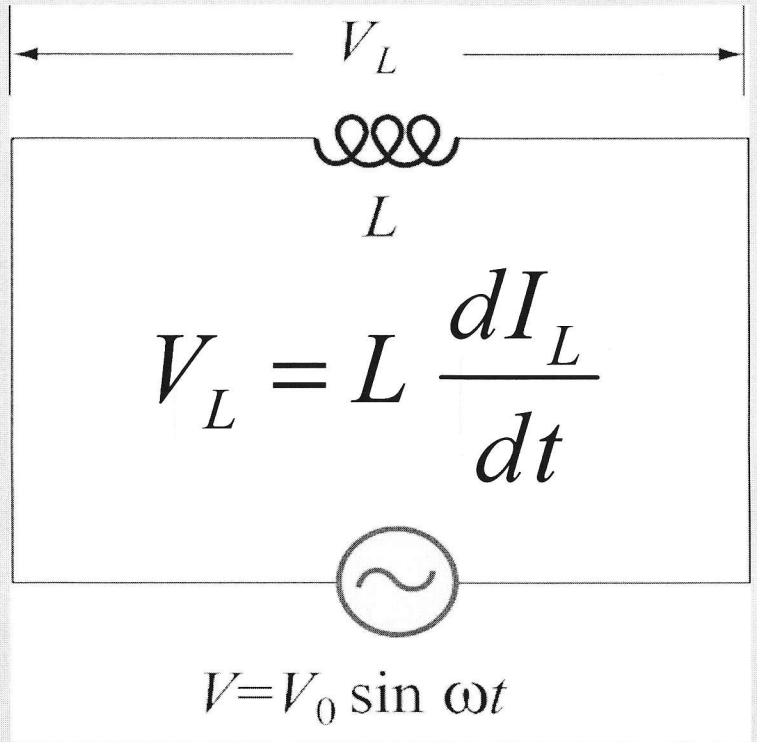
$$\boxed{\begin{aligned} I_0 &= \omega C V_0 \\ \varphi &= -\pi/2 \end{aligned}}$$

$$Q(t) = CV_C = CV_0 \sin \omega t$$

I_C leads V_C by $\pi/2$



AC Circuit: Inductors



$$V_L = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_0}{L}\sin \omega t$$

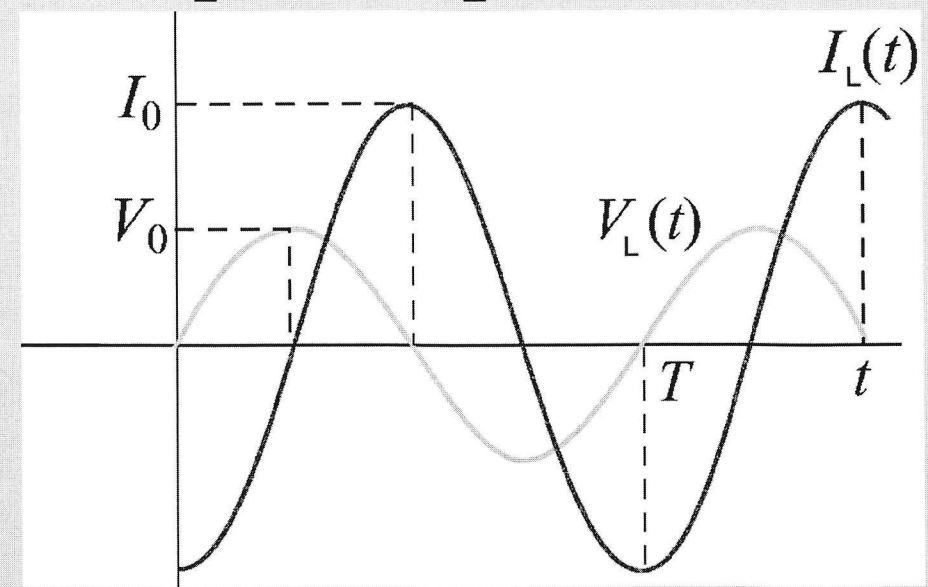
$$I_L(t) = \frac{V_0}{L}\int \sin \omega t \, dt$$

$$= -\frac{V_0}{\omega L}\cos \omega t$$

$$= I_0 \sin\left(\omega t - \pi/2\right)$$

$$\boxed{I_0 = \frac{V_0}{\omega L} \qquad \varphi = \frac{\pi}{2}}$$

I_L lags V_L by $\pi/2$



AC Circuits: Summary

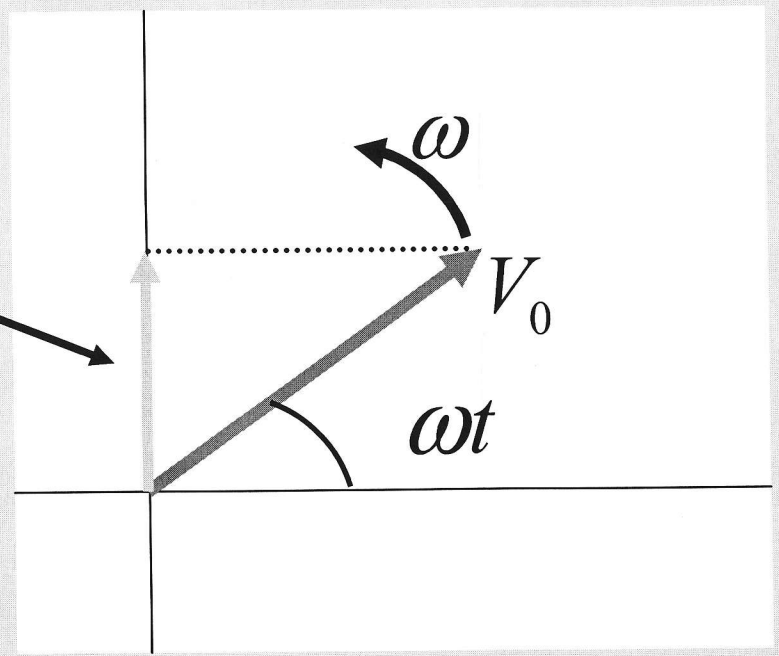
Element	I_0	Current vs. Voltage	Resistance Reactance Impedance
Resistor	$\frac{V_{0R}}{R}$	In Phase	$R = R$
Capacitor	$\omega C V_{0C}$	Leads	$X_C = \frac{1}{\omega C}$
Inductor	$\frac{V_{0L}}{\omega L}$	Lags	$X_L = \omega L$

Although derived from single element circuits, these relationships hold generally!

Phasor Diagram

Nice way of tracking magnitude & phase:

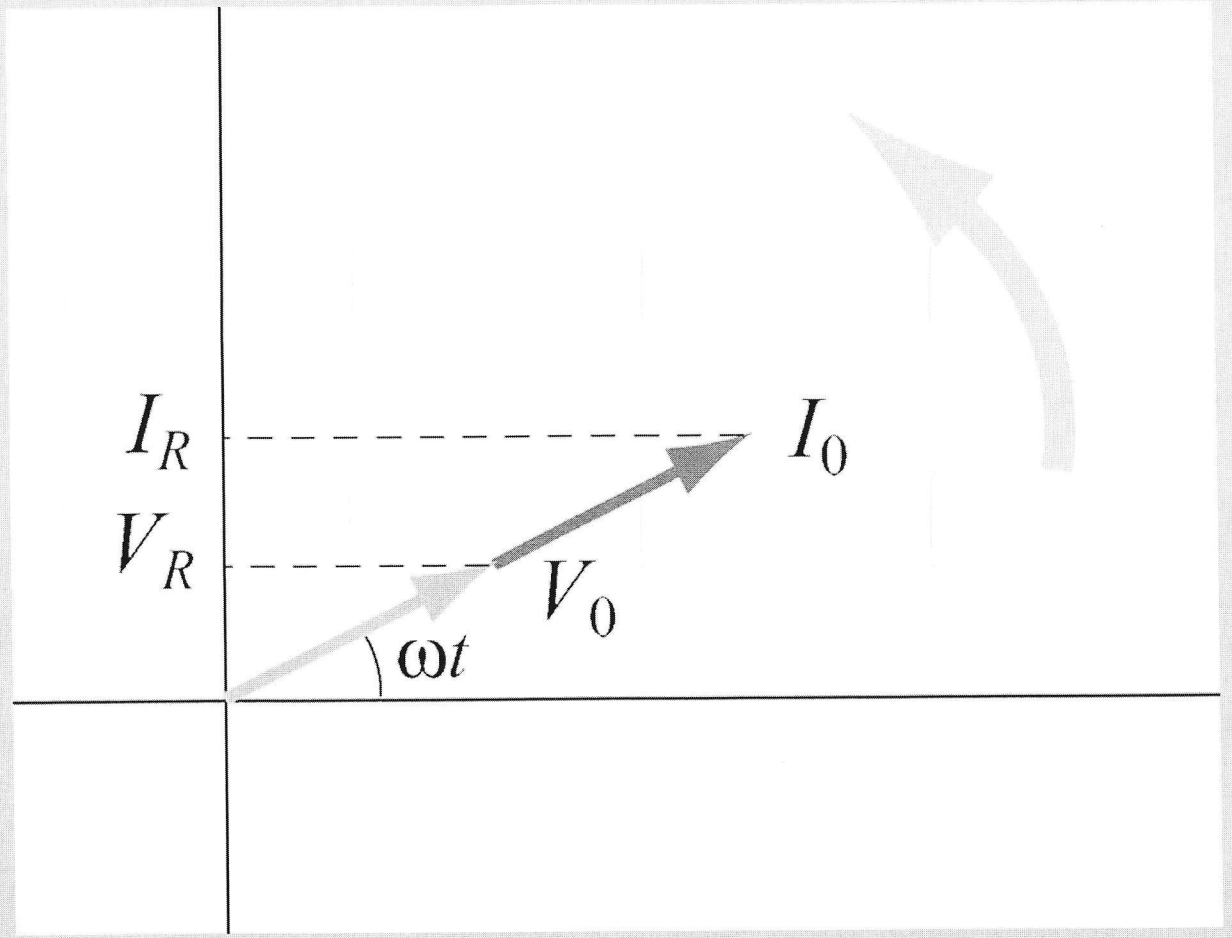
$$V(t) = V_0 \sin(\omega t)$$



Notes: (1) As the phasor (red vector) rotates, the projection (pink vector) oscillates

(2) Do both for the current and the voltage

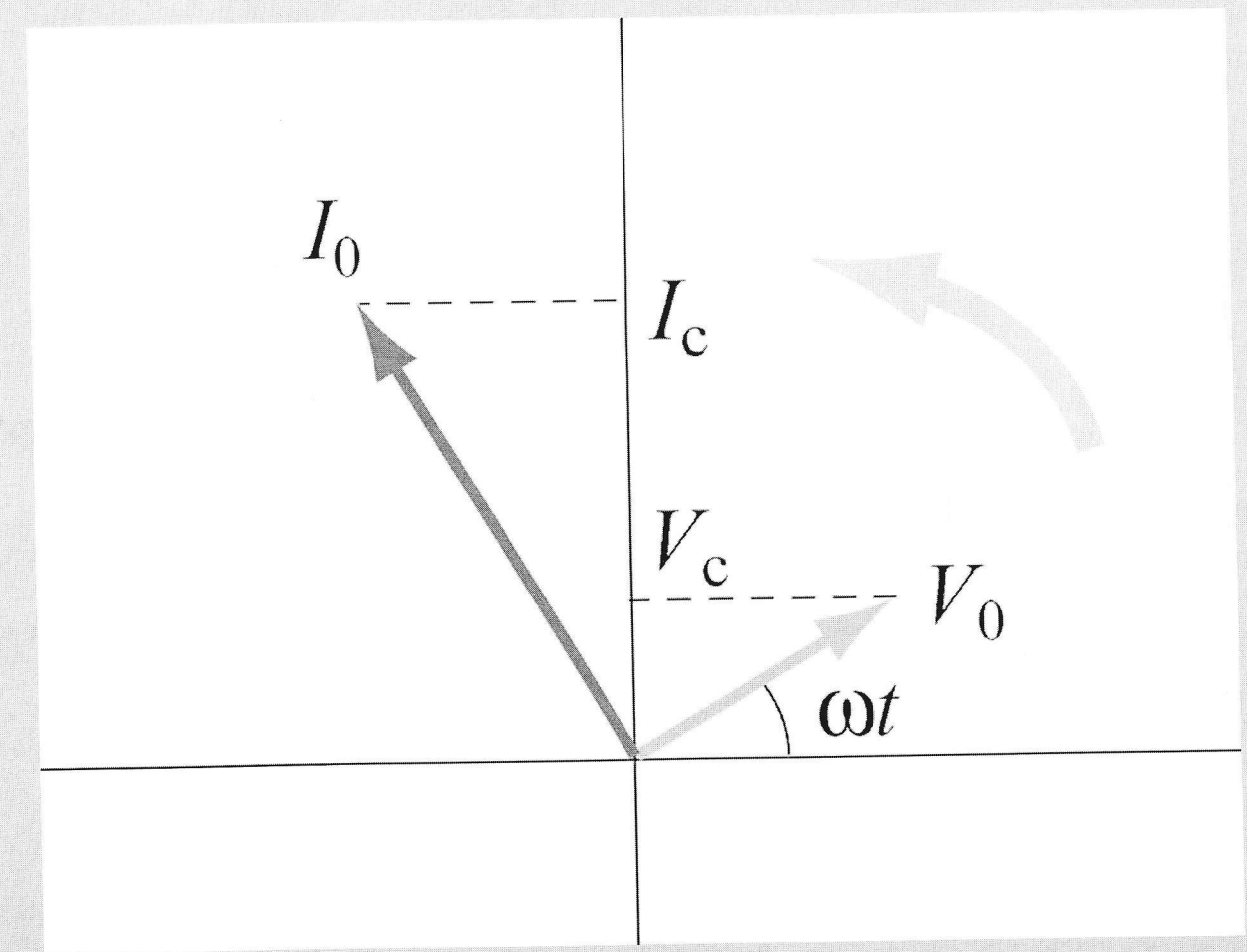
Phasor Diagram: Resistor



$$V_0 = I_0 R$$
$$\varphi = 0$$

I_R and V_R are **in phase**

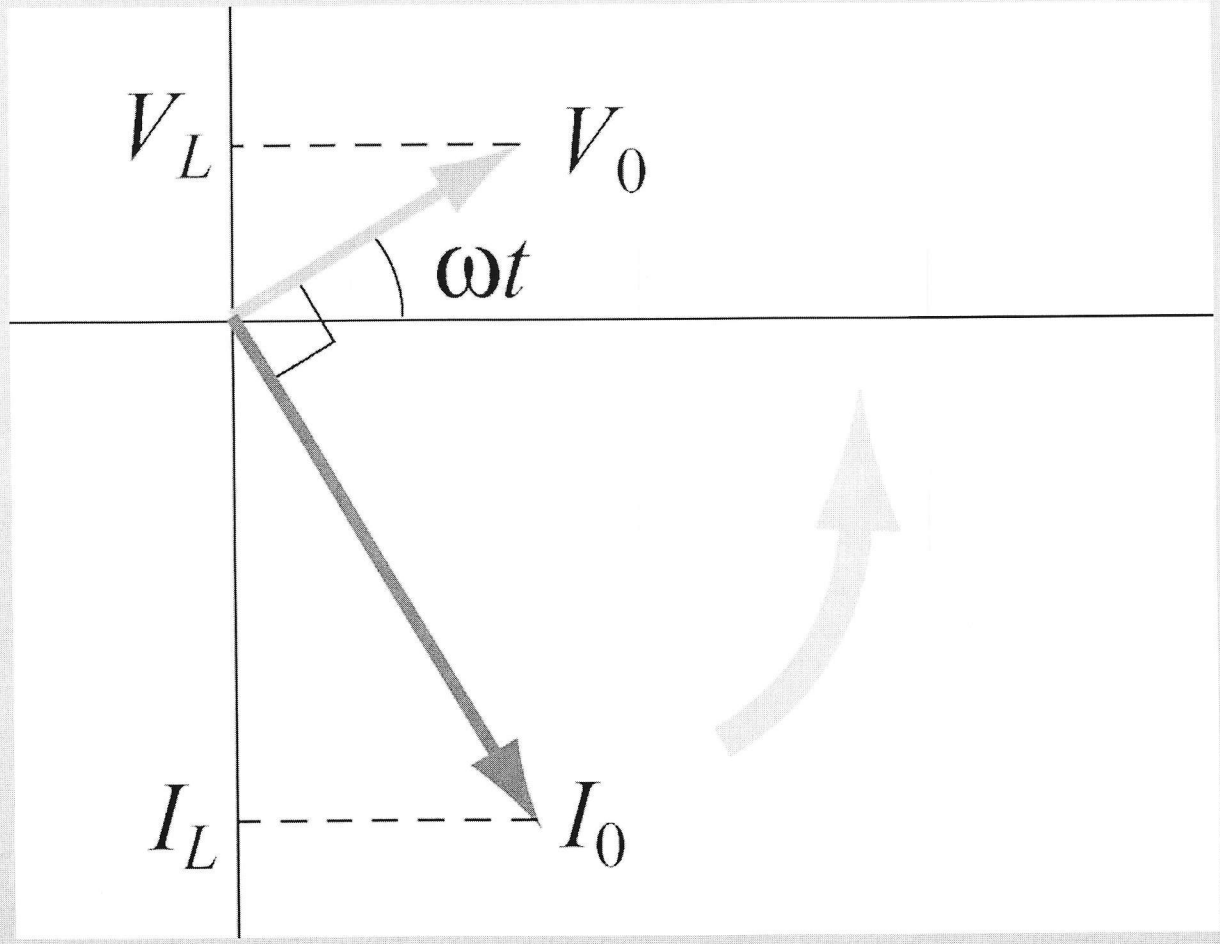
Phasor Diagram: Capacitor



$$V_0 = I_0 X_C$$
$$= I_0 \frac{1}{\omega C}$$
$$\varphi = -\pi/2$$

I_C **leads** V_C by $\pi/2$

Phasor Diagram: Inductor



$$V_0 = I_0 X_L$$
$$= I_0 \omega L$$
$$\varphi = \frac{\pi}{2}$$

I_L **lags** V_L by $\pi/2$

Question of Phase

We had fixed phase of voltage:

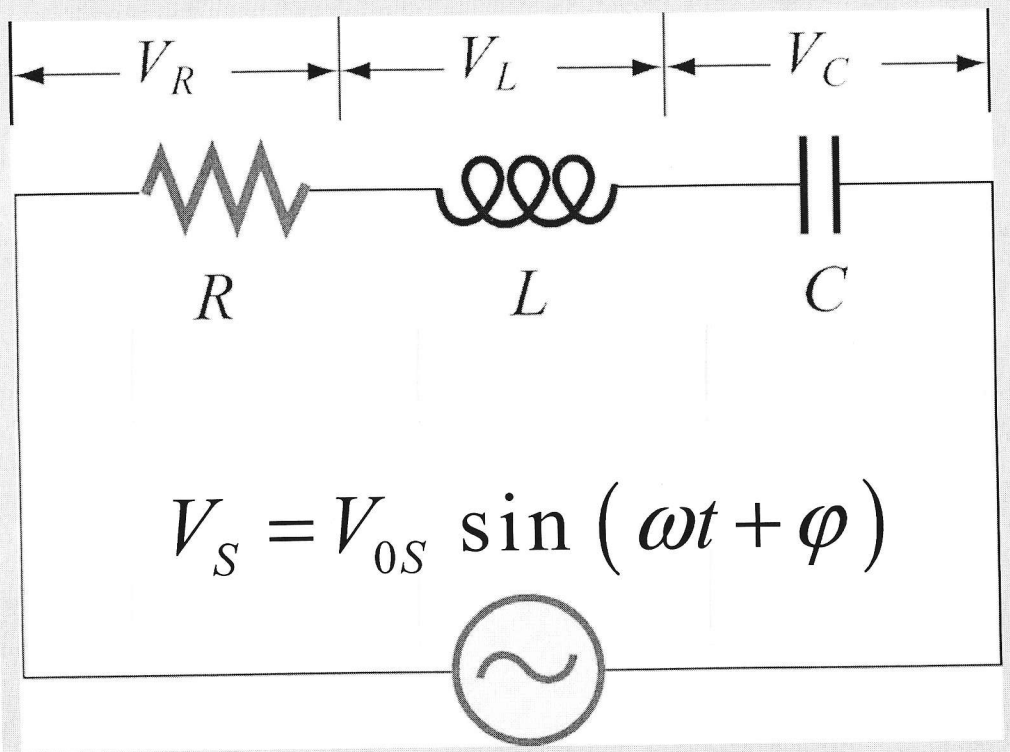
$$V = V_0 \sin \omega t \qquad I(t) = I_0 \sin(\omega t - \phi)$$

It's the same to write:

$$V = V_0 \sin(\omega t + \phi) \quad I(t) = I_0 \sin \omega t$$

(Just shifting zero of time)

Driven RLC Series Circuit



$$I(t) = I_0 \sin(\omega t)$$

$$V_R = V_{R0} \sin(\omega t)$$

$$V_L = V_{L0} \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

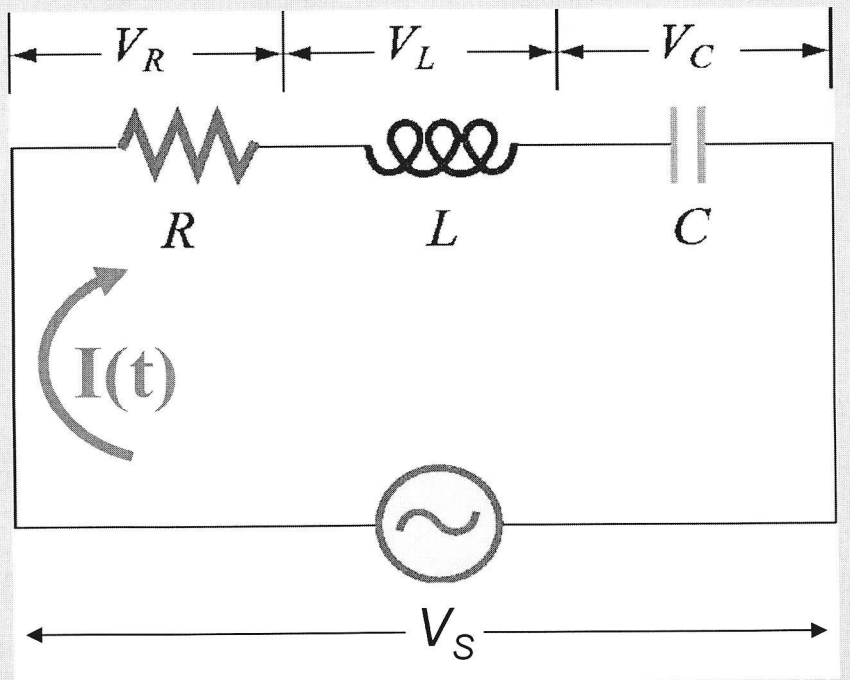
$$V_C = V_{C0} \sin\left(\omega t + \tfrac{-\pi}{2}\right)$$

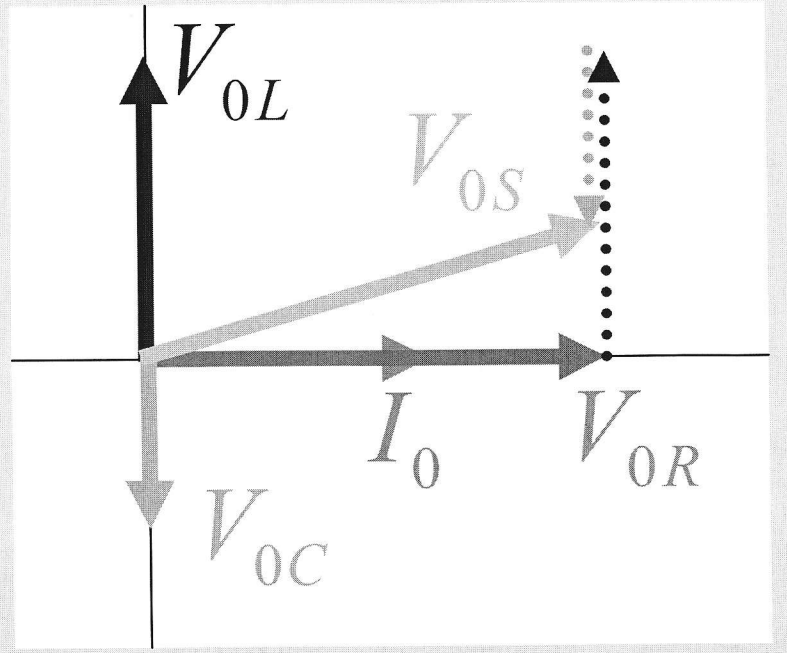
What is I_0 (and $V_{R0} = I_0 R$, $V_{L0} = I_0 X_L$, $V_{C0} = I_0 X_C$)?

What is φ? Does the current lead or lag V_s?

Must Solve: $V_S = V_R + V_L + V_C$

Driven RLC Series Circuit

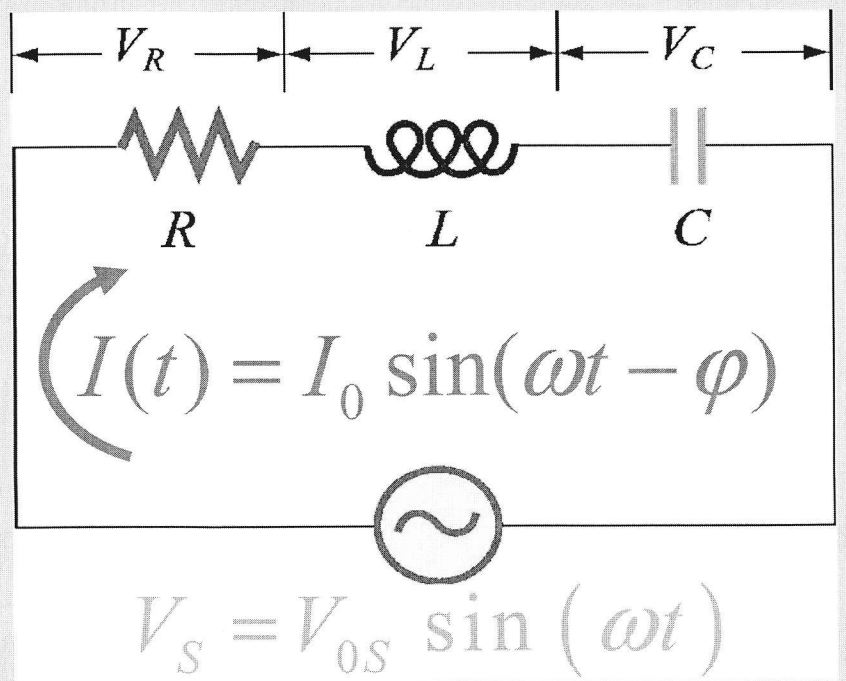


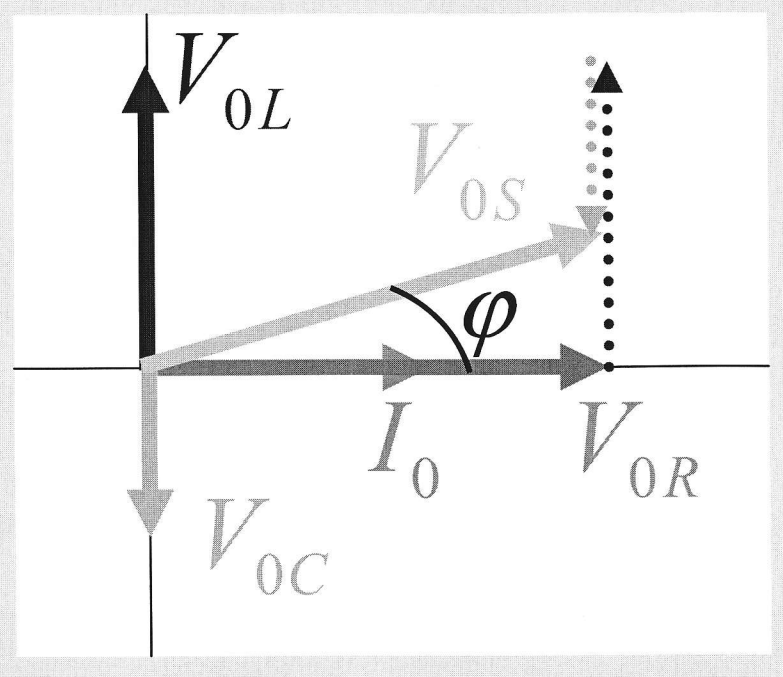


Now Solve: $V_S = V_R + V_L + V_C$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{0S} = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

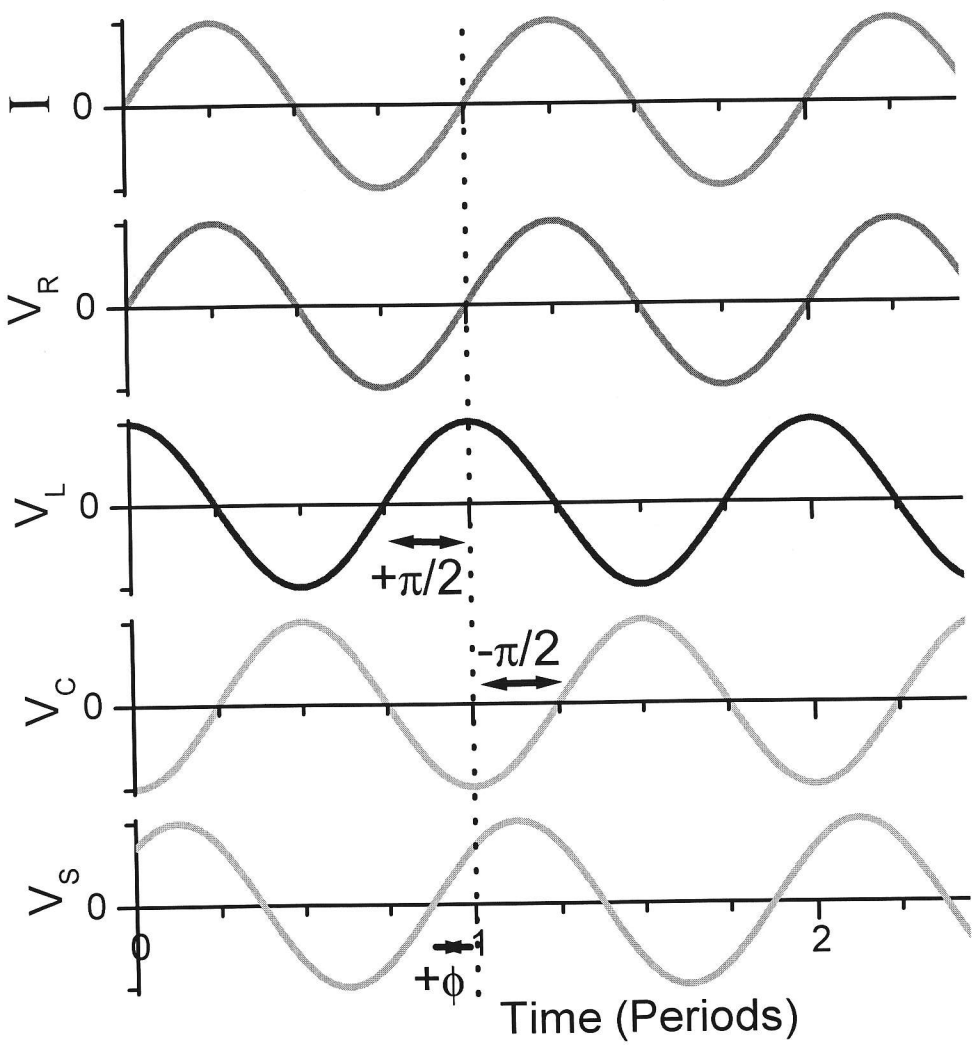
$$I_0 = \frac{V_{0S}}{Z}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Plot I, V's vs. Time



$$I(t) = I_0 \sin(\omega t)$$

$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

$$V_L(t) = I_0 X_C \sin\left(\omega t - \tfrac{\pi}{2}\right)$$

$$V_S(t) = V_{S0} \sin(\omega t + \varphi)$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

At very low frequencies, C dominates ($X_C >> X_L$):

it fills up and keeps the current low

At very high frequencies, L dominates ($X_L >> X_C$):

the current tries to change but it won't let it

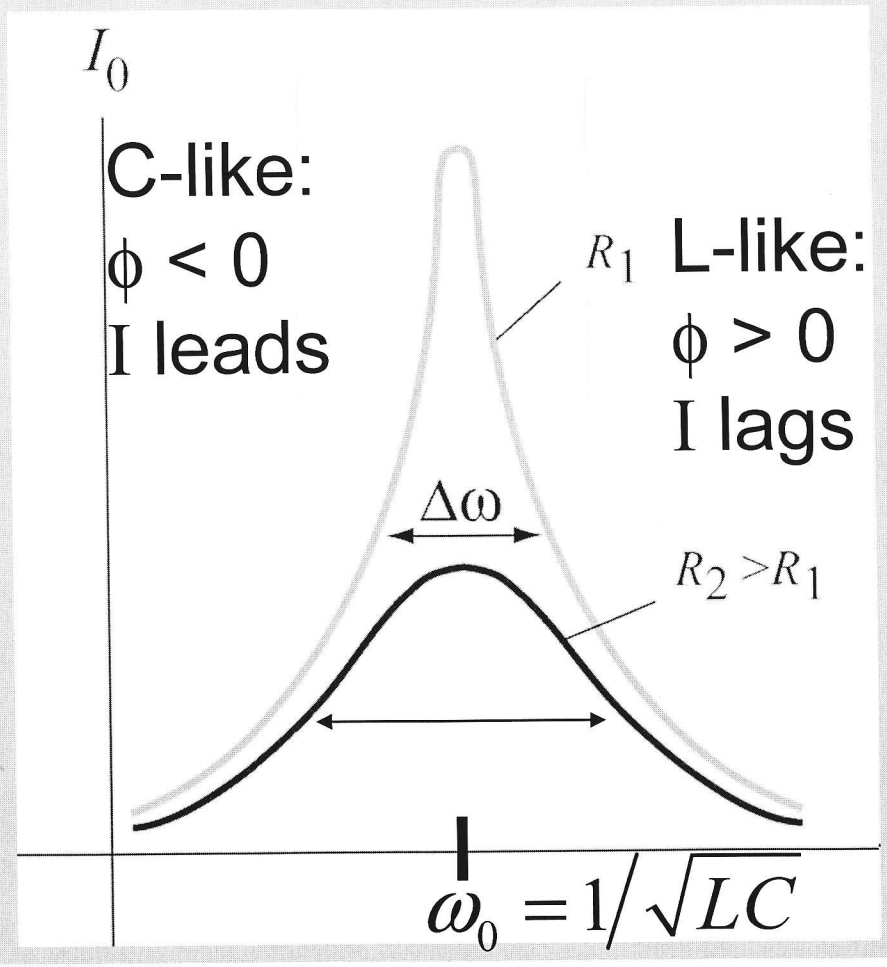
At intermediate frequencies we have **resonance**

I_0 reaches maximum when $X_L = X_C$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$