Second

Middle Term Examination

General Physics for Physics Majors

1. Show that

(i) $\nabla \cdot (\nabla \times \vec{V}) = 0$

(ii) $\nabla \times (\nabla \phi) = 0$

(iii) $\nabla \times (\nabla \times \vec{V}) = \nabla (\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$

15%

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

(i) $\nabla \cdot (\nabla \times \vec{V}) = 0$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_x\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial}{\partial x}V_z\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right] + \frac{\partial}{\partial y}\left[\frac{\partial V_x}{\partial z} - \frac{\partial}{\partial x}V_z\right] + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

$$= \frac{\partial^2}{\partial x \partial y}V_z - \frac{\partial}{\partial y}\frac{\partial}{\partial x}V_z - \frac{\partial^2}{\partial x \partial z}V_y + \frac{\partial^2}{\partial z \partial x}V_y + \frac{\partial^2}{\partial y \partial z}V_x - \frac{\partial^2}{\partial z \partial y}V_x = 0$$

Example $\vec{B} = \nabla \times \vec{A} \Rightarrow \nabla \cdot \vec{B} = 0$

(ii) $\nabla \times (\nabla \phi) = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right] + \hat{j}\left[\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2}{\partial x \partial z}\right] + \hat{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right.$$

$= 0$

Example $\nabla \times \vec{E} = 0$

$\vec{E} = -\nabla \phi$

(iii) $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] + \hat{j}\left[\frac{\partial V_x}{\partial z} - \frac{\partial}{\partial x} V_y\right] + \hat{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

$\nabla \times (\nabla \times \vec{V})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} & \frac{\partial V_x}{\partial z} - \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{vmatrix}$$

Look at the $\hat{i}$'s

$$\frac{\partial^2}{\partial y \partial x} V_y - \frac{\partial^2}{\partial y^2} V_x - \frac{\partial^2}{\partial z^2} V_x + \frac{\partial^2}{\partial z \partial x} V_z - \frac{\partial^2}{\partial x^2} V_x + \frac{\partial^2}{\partial x^2} V_x$$

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z$$

The $\hat{i}$th component of $\nabla(\nabla \cdot \vec{V})$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z\right)$$

$$= \frac{\partial^2}{\partial x^2} V_x + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2}{\partial x \partial z} V_z$$

Clearly, LHS = RHS

Use to go from Maxwell Equation

$\Rightarrow$ wave equation.

∇ is $\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$ an operator

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

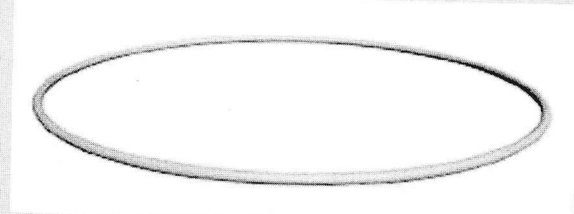
$$\nabla \times (\nabla \times \vec{V}) = (\nabla \cdot \vec{V})\nabla - (\nabla \cdot \nabla)\vec{V}$$

↳ no clear meaning

2. (a) 10%

Q: Ring of Charge

A thin rod with a uniform charge per unit length λ is bent into the shape of a circle of radius *R*



a) Choose a coordinate system for the rod. Clearly indicate your choice of origin, and axes on the diagram above.

b) Choose an infinitesimal charge element *dq* . Find an expression relating *dq* , λ, and your choice of length for *dq* .

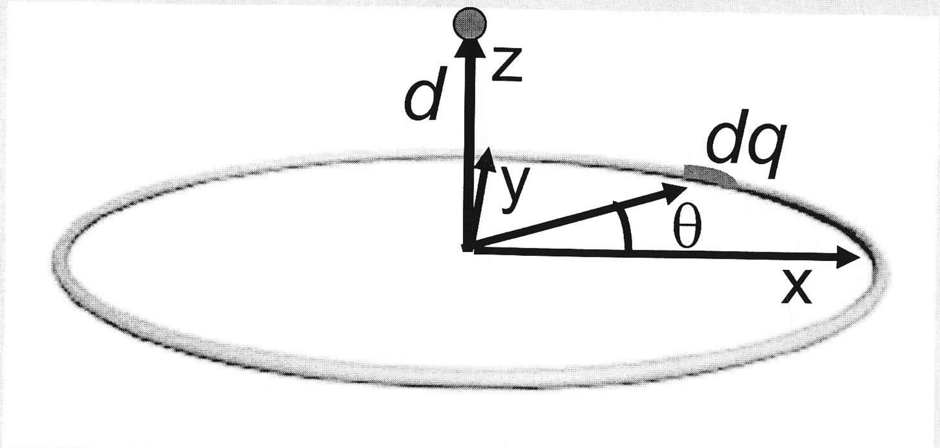
c) Find the vector components for the contribution of *dq* to the electric field along an axis perpendicular to the plane of the circle, a distance *d* above the plane of the circle. The axis passes through the center of the circle. Express the vector components in terms of your choice of unit vectors

d) What is the direction and magnitude of the electric field along the axis that passes through the center of the circle, perpendicular to the plane of the circle, and a distance *d* above the plane of the circle.

e) What is the potential at that point, assuming V(∞)=0?

2. (a) 10%

A: Ring of Charge



a) Origin & axes as pictured

b) $dq = \lambda d\ell = \lambda R d\theta$

c) $d\vec{\mathbf{E}} = \dfrac{k\,dq}{r^3}\vec{\mathbf{r}}$

$$\vec{\mathbf{r}} = -R\cos(\theta)\hat{\mathbf{i}} - R\sin(\theta)\hat{\mathbf{j}} + d\hat{\mathbf{k}}; \quad r = \sqrt{R^2 + d^2}$$

d) Horizontal components cancel, only find E_z

$$E_z = \int dE_z = \int \frac{k\,dq}{r^3} d = \frac{kd}{r^3}\int_{\theta=0}^{2\pi} \lambda R\, d\theta = \frac{kd\lambda R}{r^3} 2\pi$$

e) Find the potential by same method:

$$V(d) = \int dV = \int \frac{k\,dq}{r} = \frac{k}{r}\int_{\theta=0}^{2\pi} \lambda R\, d\theta = \frac{k\lambda R}{r} 2\pi$$

Comments

(a) Try to go into cylindeical coordinate

(b) $dq = \lambda\, dl = \lambda R\, d\theta$

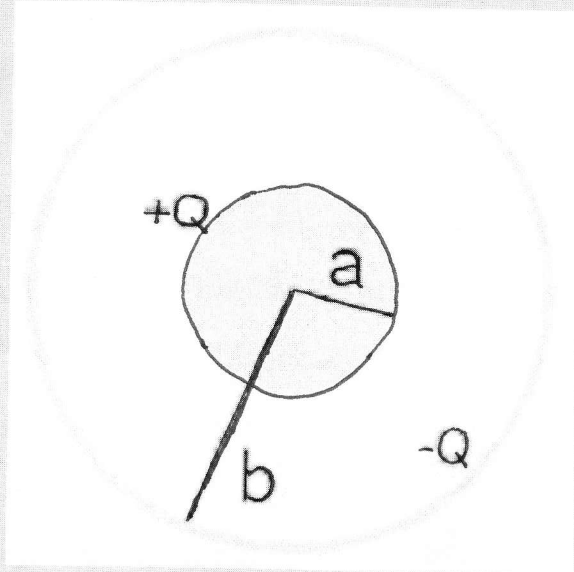
(c) R = constant $\qquad d \leftrightarrow z$

(d) Symmetry E_z only is needed

(e) Essentially Coulomb's law

2.(b) 10%

Q: Spherical Capacitor



A conducting solid sphere of radius a, carrying a charge $+Q$ is surrounded by a thin conducting spherical shell (inner radius b) with charge $-Q$.

a) What is the direction and magnitude of the electric field **E** in the three regions below. Show how you obtain your expressions.

1. $r < a$ 2. $a < r < b$ 3. $r > b$

b) What is the electric potential V(r) in these same three regions. Take the electric potential to be zero at ∞.

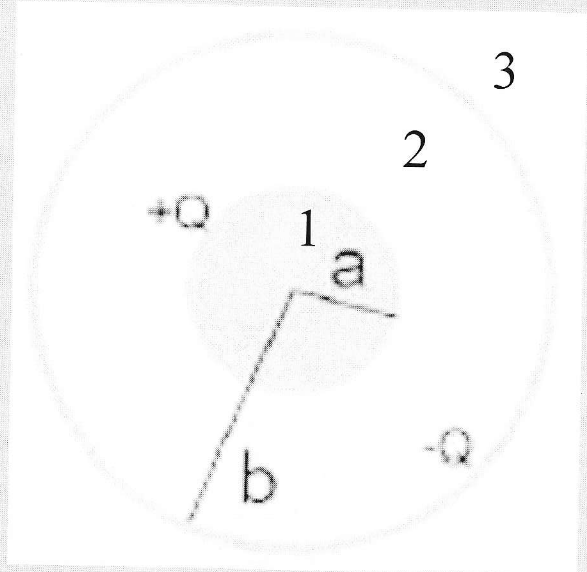
c) What is the electric potential difference between the outer shell and the inner cylinder, $\Delta V = V(b) - V(a)$?

d) What is the capacitance of this spherical capacitor?

e) If a positive charge $+2Q$ is placed anywhere on the inner sphere of radius a, what charge appears *on the outside surface* of the thin spherical shell of inner radius b?

2. (b) 10

A: Spherical Capacitor



a) By symmetry **E** is purely radial.
Choose spherical Gaussian surface

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0} = EA = E \cdot 4\pi r^2$$

1&3) $q_{in} = 0 \rightarrow \vec{\mathbf{E}} = 0$ 2) $\vec{\mathbf{E}} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

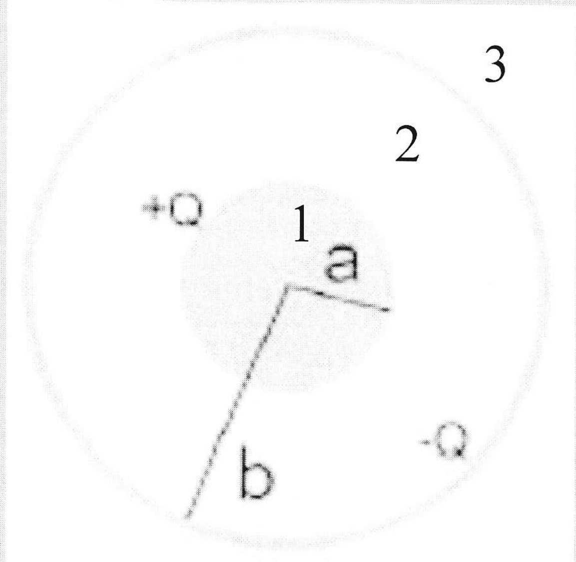
b) For V, always start from where you know it (here, ∞)

3) **E**=0 → *V* constant = 0

2) $V(r) = -\int_b^r \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$

1) **E**=0 → *V* constant = *V*(*a*) → $V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$

A: Spherical Capacitor



c) $$\Delta V = V(b) - V(a) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

d) $$C = \frac{Q}{|\Delta V|} = \frac{4\pi\varepsilon_0}{\left(a^{-1} - b^{-1}\right)}$$

e) If you place an additional +2Q charge on the inner sphere then you will induce an additional -2Q on the inner surface of the outer shell, and hence a +2Q charge on the outer surface of that shell

Answer: +2Q

Gauss law

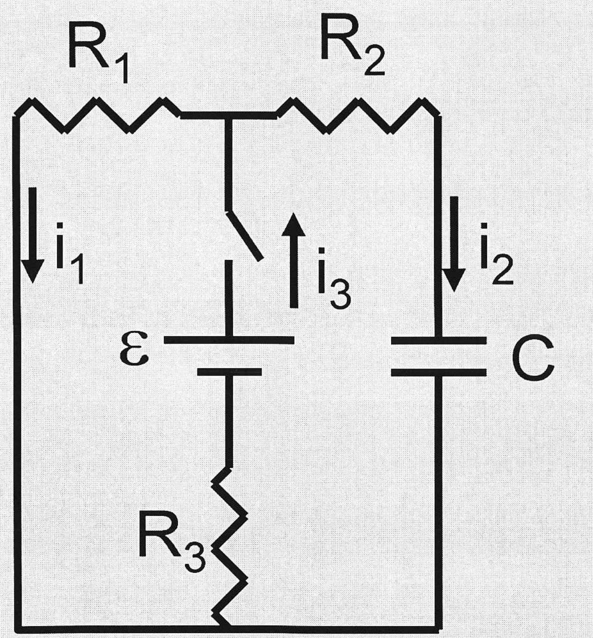
↓

note symmetry play an important role

$V(r)$ calculate along $\vec{r}$ direction

3. 15%

Problem 2: RC Circuit



Initially C is uncharged.

1. When the switch is first closed, what is the current i_3?
2. After a very long time, how much charge is stored on the capacitor?
3. Obtain a differential equation for the charge on the capacitor (Here only, let $R_1=R_2=R_3=R$)

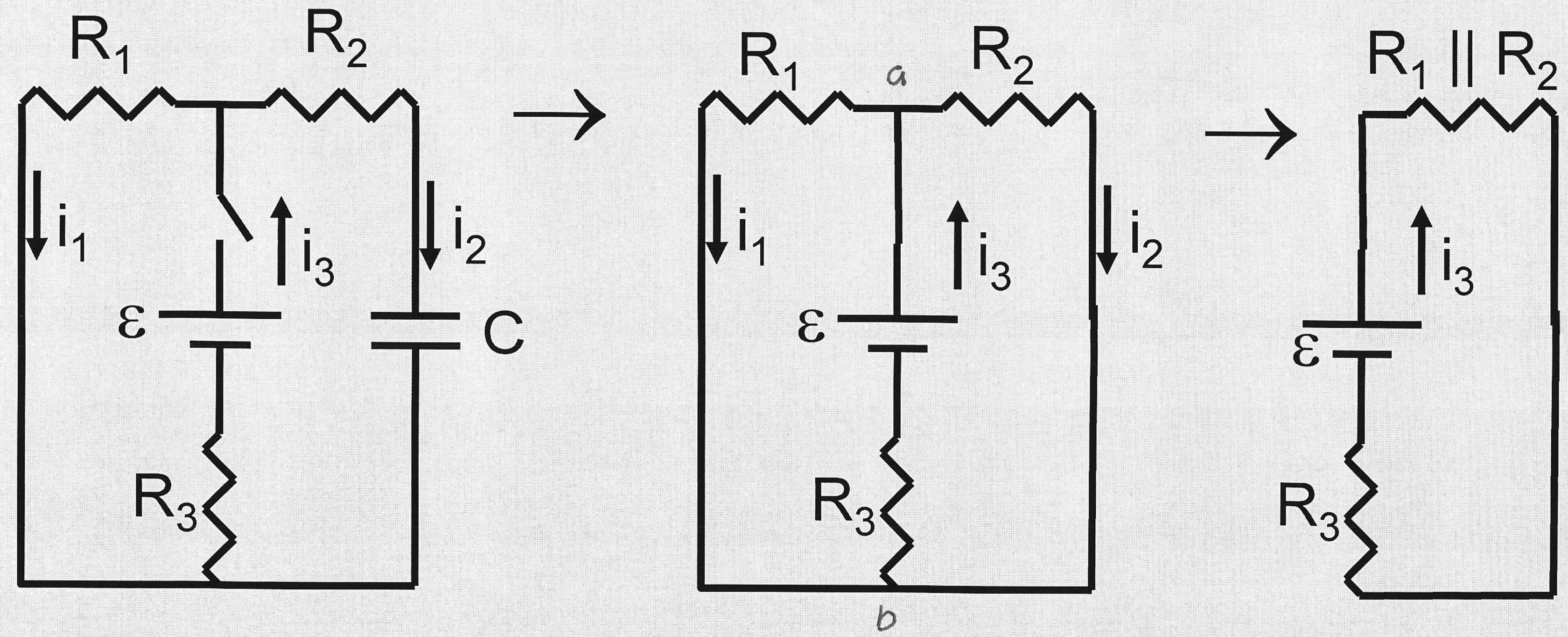
Now the switch is opened

4. Immediately after opening the switch, what is i_1? i_2? i_3?
5. How long before i_2 falls to 1/e of this initial value?

3. 15%

Solution 2: RC Circuit

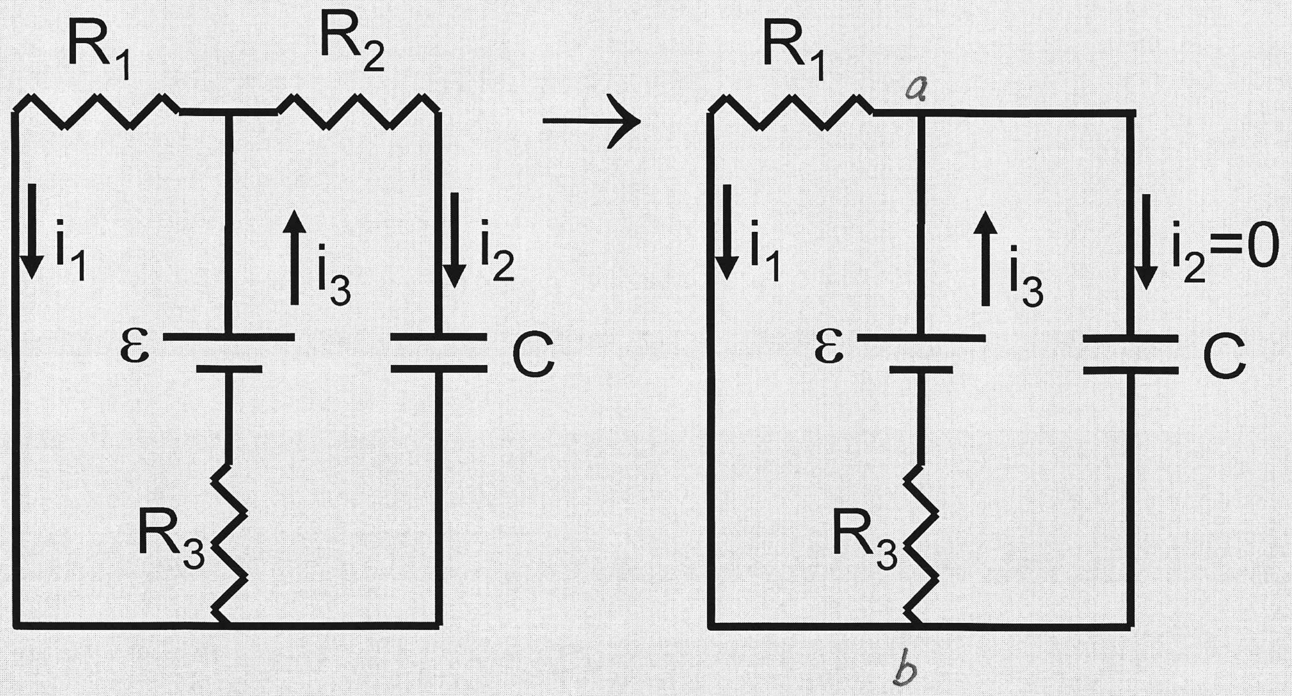
Initially C is uncharged → Looks like short



$$R_{eq} = R_3 + \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \Rightarrow \boxed{i_3 = \frac{\varepsilon}{R_{eq}}}$$

Solution 2: RC Circuit

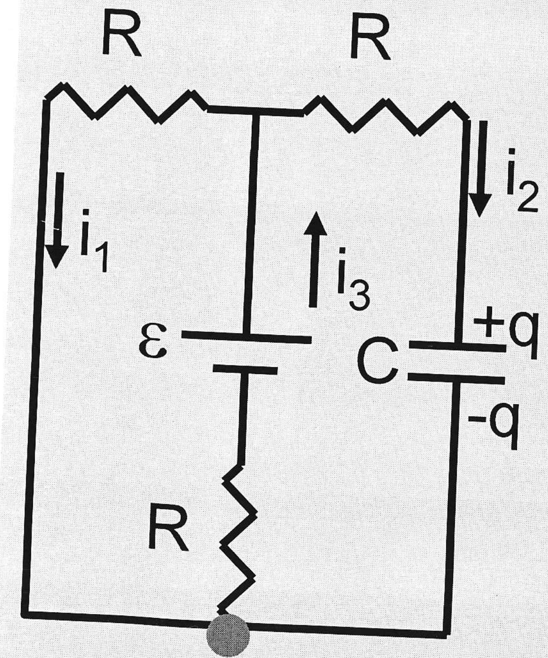
After a long time, C is full → $i_2 = 0$



$$i_1 = i_3 = \frac{\varepsilon}{R_1 + R_3}$$

$$Q = CV_C = C\left(i_1 R_1\right) = \boxed{C\varepsilon \frac{R_1}{R_1 + R_3}}$$

Solution 2: RC Circuit



Kirchhoff's Loop Rules

Left: $-i_3R+\varepsilon-i_1R=0$

Right: $-i_3R+\varepsilon-i_2R-q/C=0$

Current: $i_3=i_1+i_2$

Want to have i_2 and q only $(L-2R)$: Left Right

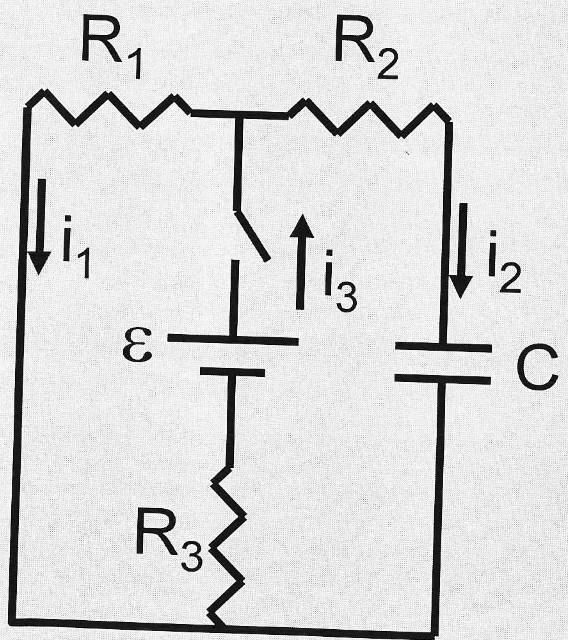
$$0=-(i_1+i_2)R+\varepsilon-i_1R+2(i_1+i_2)R-2\varepsilon+2i_2R+2q/C$$

$$=3i_2R-\varepsilon+2q/C$$

$$i_2=+\frac{dq}{dt} \rightarrow \boxed{\frac{dq}{dt}=\frac{\varepsilon}{3R}-\frac{2q}{3RC}}$$

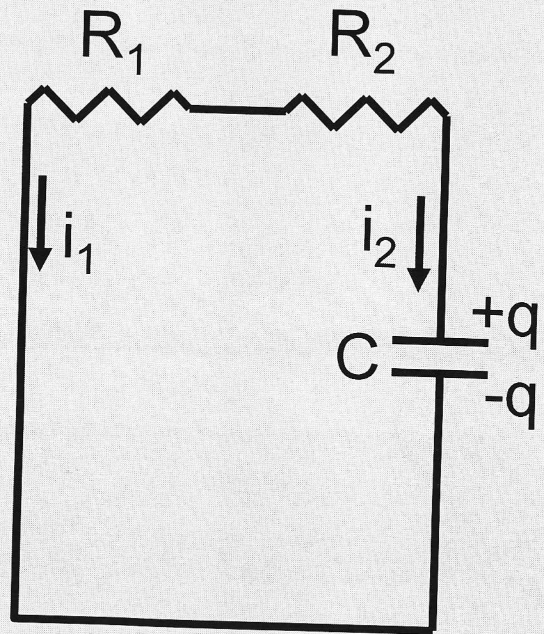
Solution 2: RC Circuit

Now open the switch.



$$i_3 = 0$$

→



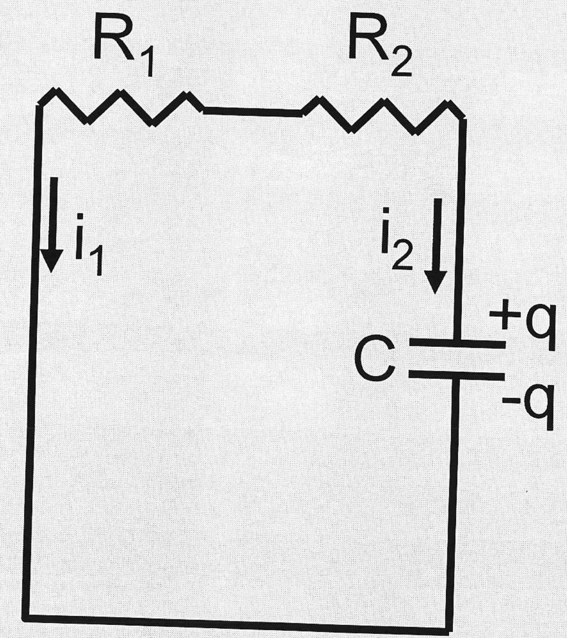
Capacitor now like a battery, with:

$$V_C = \frac{Q}{C} = \varepsilon \frac{R_1}{R_1 + R_3}$$

$$i_1 = -i_2 = \frac{V_C}{R_1 + R_2} = \varepsilon \frac{R_1}{R_1 + R_3} \frac{1}{R_1 + R_2}$$

Solution 2: RC Circuit

How long to fall to 1/e of initial current? The time constant!

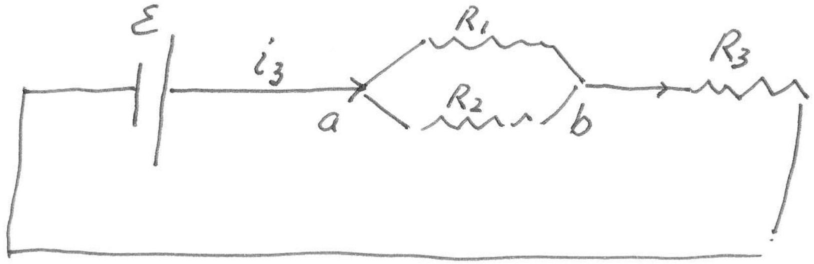


This is an easy circuit since it just looks like a resistor and capacitor in series, so:

$$\tau = (R_1 + R_2)C$$

Notice that this is different than the charging time constant, because there was another resistor in the circuit during the charging

(1) C has no charge on it ; charge is empty



$$\frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2} \qquad R_{eq} = R_3 + R_{12}$$

$$R_{12} \downarrow \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

(2) C is full → $i_2 = 0$

↓ simple

(3) Note: Kirchhoff's Loop Rules

Differential equation, first order ⟹ decay constant

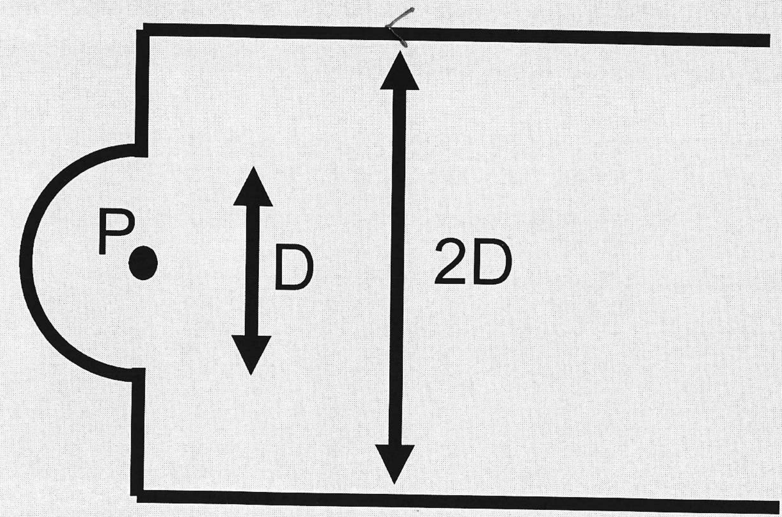
↓ RC circuit ↓ exponential decay

Only in this part

$R_1 = R_2 = R_3 = R$

4. (a) 7%

Problem 1: Wire Loop

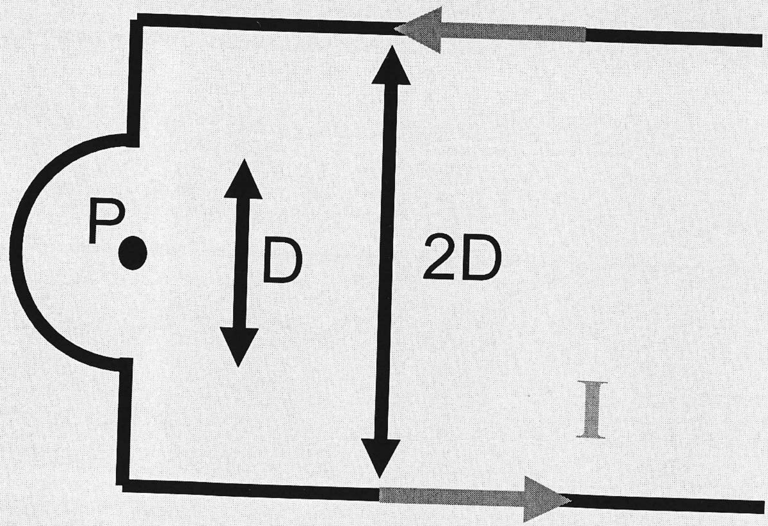


A current flowing in the circuit pictured produces a magnetic field at point P pointing out of the page with magnitude B.

a) What direction is the current flowing in the circuit?
b) What is the magnitude of the current flow?

4. (a) 7%

Solution 1: Wire Loop



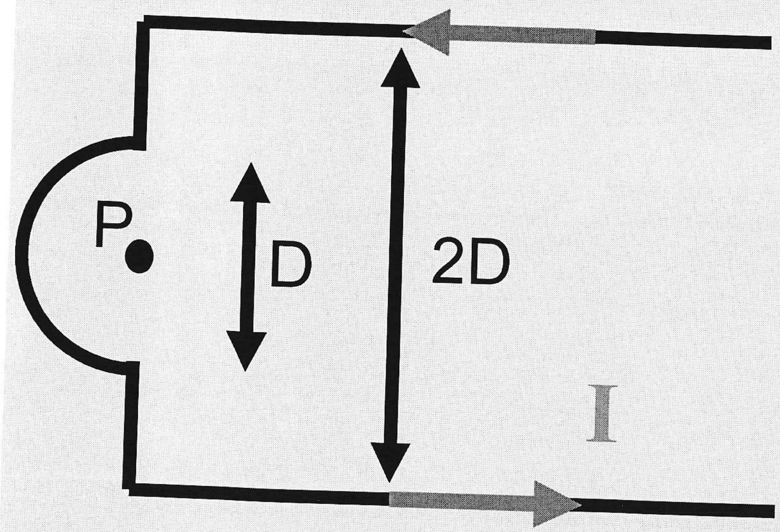
a) The current is flowing counter-clockwise, as shown above

b) There are three segments of the wire: the semi-circle, the two horizontal leads, and the two vertical leads.

The two vertical leads do not contribute to the B field (ds || r)

The two horizontal leads make an infinite wire a distance D from the field point.

Solution 1: Wire Loop



For infinite wire use Ampere's Law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} \Rightarrow B \cdot 2\pi D = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi D}$$

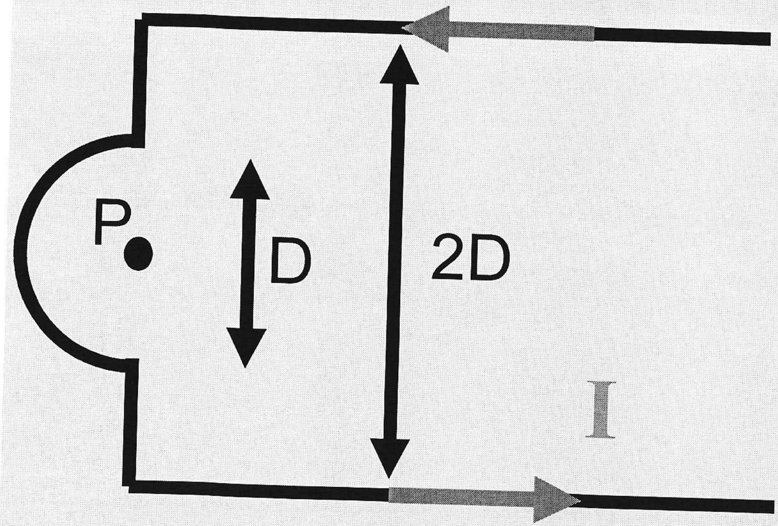
For the semi-circle use Biot-Savart:

$$\mathbf{d}\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \qquad r = \frac{D}{2} \text{ and } d\vec{\mathbf{s}} \perp \hat{\mathbf{r}}$$

$$B = \int dB = \int \frac{\mu_0}{4\pi} \frac{I\, d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

$$= \frac{\mu_0}{4\pi} \frac{I}{r^2} \left(\pi r\right) = \frac{\mu_0 I}{4r} = \frac{\mu_0 I}{2D}$$

Solution 1: Wire Loop



Adding together the two parts:

$$B = \frac{\mu_0 I}{2\pi D} + \frac{\mu_0 I}{2D} = \frac{\mu_0 I}{2D}\left(1 + \frac{1}{\pi}\right)$$

They gave us B and want I to make that B:

$$I = \frac{2DB}{\mu_0\left(1 + \frac{1}{\pi}\right)}$$

(a) $\cdot P$ out of the page

$$d\vec{B} = \frac{\mu_0 I \, d\vec{s} \times \hat{r}}{r^2}$$

$$\vec{r} = -(a\hat{i} * b\hat{j})$$

a, b are positive

$d\vec{s}$ $\quad - \hat{i}$

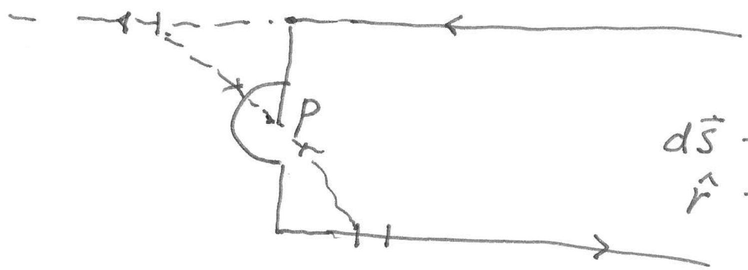
$d\vec{s} \times \vec{r} = b \quad \hat{i} \times \hat{j}$

↓

$\hat{k}$

↓

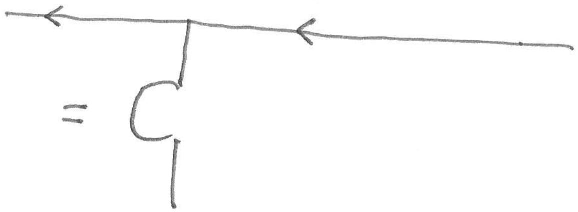
point out



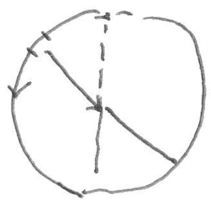
$d\vec{s} \to -d\vec{s}$

$\hat{r} \to -\hat{r}$

Biot - Savart superposition



$$\hat{\theta} \times (-\hat{r}) \longrightarrow +\hat{z}$$



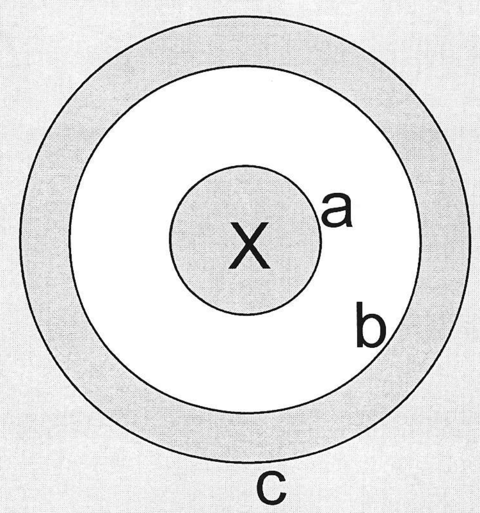
$$\hat{r} \times \hat{\theta} \rightarrow \hat{z}$$

$$-\hat{\theta} \times \hat{r} = \hat{r} \times \hat{\theta}$$

semi-circle $= \frac{1}{2}$ circle.

4. (b)

Problem 5: Coaxial Cable

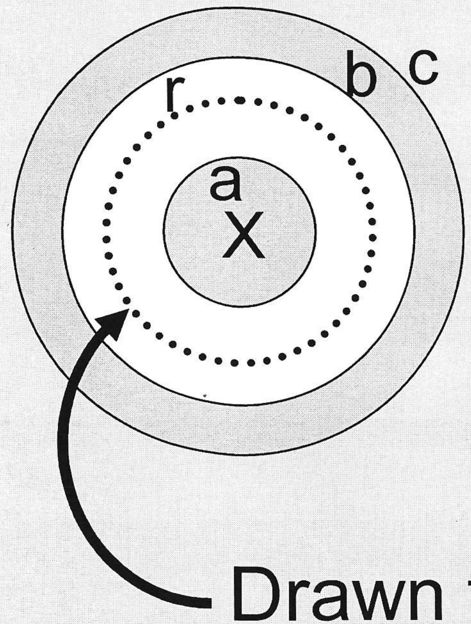


Consider a coaxial cable of with inner conductor of radius *a* and outer conductor of inner radius *b* and outer radius *c*. A current *I* flows into the page on the inner conductor and out of the page on the outer conductor.

What is the magnetic field everywhere (magnitude and direction) as a function of distance *r* from the center of the wire?

4. (b) 8%

Solution 5: Coaxial Cable



Drawn for $a < r < b$

Everywhere the magnetic field is clockwise. To figure out the magnitude use Ampere's Law:

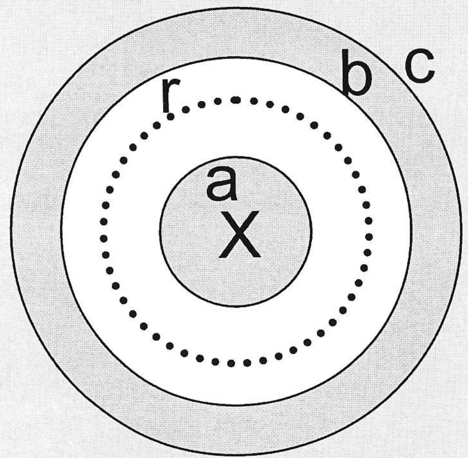
$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} \Rightarrow B \cdot 2\pi r = \mu_0 I_{enc}$$

$$\Rightarrow B = \frac{\mu_0 I_{enc}}{2\pi r}$$

The amount of current penetrating our Amperian loop depends on the radius *r*:

$$r \le a\text{:}\ I_{enc} = I\frac{r^2}{a^2} \Rightarrow \boxed{B = \frac{\mu_0 I r}{2\pi a^2}\text{ clockwise}}$$

Solution 5: Coaxial Cable



Remember: Everywhere

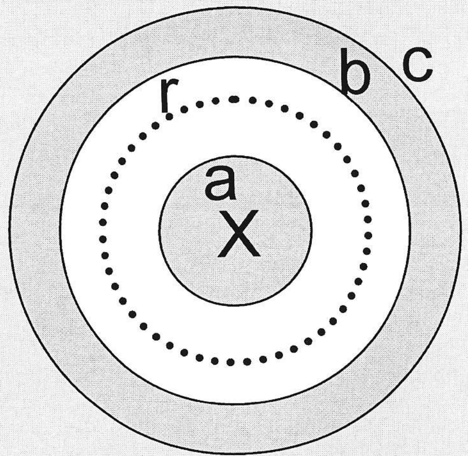
$$B = \frac{\mu_0 I_{enc}}{2\pi r} \text{ clockwise}$$

$$a \le r \le b: \; I_{Encl} = I \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 I}{2\pi r} \text{clockwise}}$$

$$b \le r \le c: \; I_{Encl} = I\left(1 - \frac{r^2 - b^2}{c^2 - b^2}\right)$$

$$\Rightarrow \quad \boxed{B = \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2 - b^2}{c^2 - b^2}\right) \text{clockwise}}$$

Solution 5: Coaxial Cable



Remember: Everywhere

$$B = \frac{\mu_0 I_{enc}}{2\pi r} \text{ clockwise}$$

$$r \geq c: \; I_{Encl} = 0 \quad \Rightarrow \quad \boxed{B = 0}$$

Straightforward application of Ampere's law

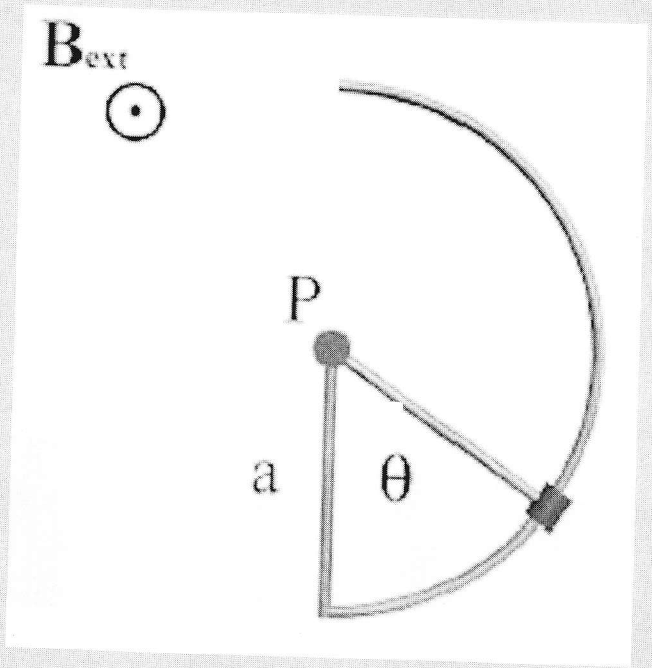
↓

symmetry

I_{enc}.

5. 15%

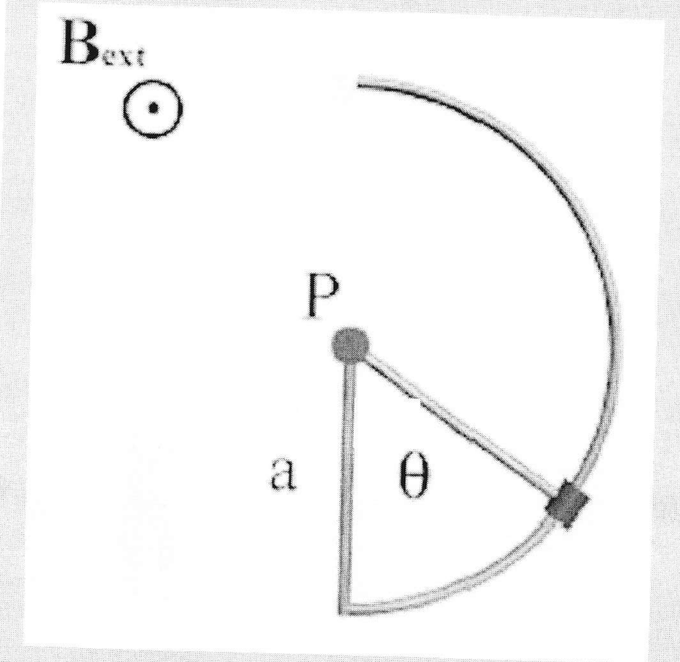
Problem 3: Pie Wedge



Consider the following pie shaped circuit. The arm is free to pivot about the center, P, and has mass *m* and resistance *R*.

1. If the angle θ decreases in time (the bar is falling), what is the direction of current?
2. If $\theta = \theta(t)$, what is the rate of change of magnetic flux through the pie-shaped circuit?

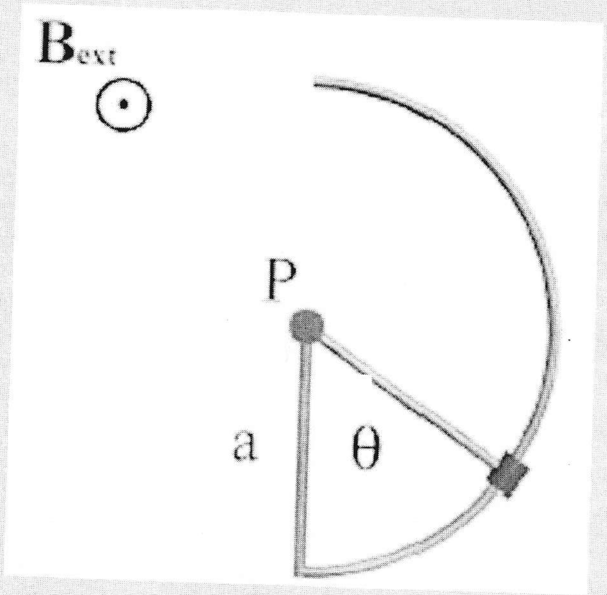
Problem 3, Part 2: Pie Wedge



3. What is the magnetic force on the bar (magnitude and direction – indicated on figure)
4. What torque does this create about P? (HINT: Assume force acts at bar center)

5.. 15%

Solution 3: Pie Wedge



1) Direction of I?

Lenz's Law says: try to oppose decreasing flux

I Counter-Clockwise (B out)

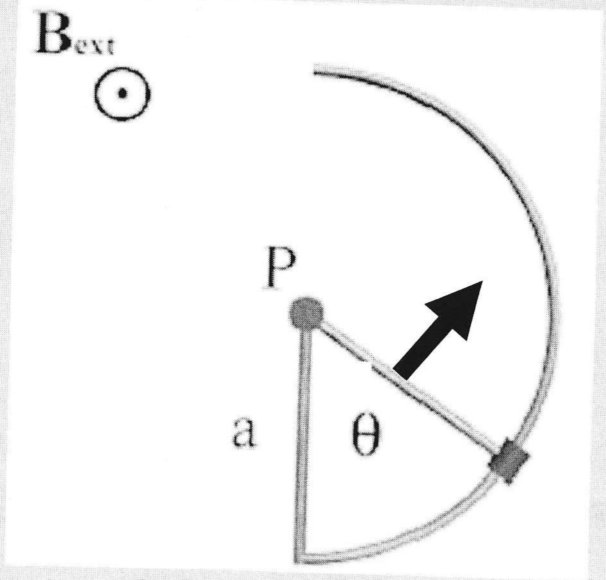
2) θ = θ(t), rate of change of magnetic flux?

$$A = \pi a^2 \left(\frac{\theta}{2\pi} \right) = \frac{\theta a^2}{2}$$

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}(BA) = B \frac{d}{dt} \frac{\theta a^2}{2}$$

$$= \frac{Ba^2}{2} \frac{d\theta}{dt}$$

Solution 3, Part 2: Pie Wedge



3) Magnetic Force?

$$d\vec{\mathbf{F}} = I d\vec{\mathbf{s}} \times \vec{\mathbf{B}} \qquad F = IaB$$

$$I = \frac{\varepsilon}{R} = \frac{1}{R}\frac{d\Phi_B}{dt} = \frac{1}{R}\frac{Ba^2}{2}\frac{d\theta}{dt}$$

$$\boxed{F = \frac{B^2 a^3}{2R}\frac{d\theta}{dt}}$$

(Dir. as pictured)

4) Torque?

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \Rightarrow \tau = \frac{a}{2} F \boxed{= \frac{B^2 a^4}{4R}\frac{d\theta}{dt}}$$

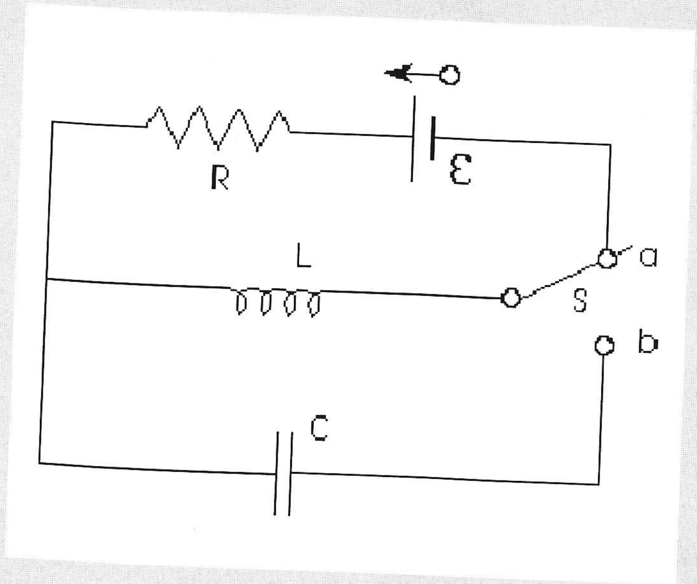
(out of page)

Application of Faraday Law

$$A = \pi a^2 \left(\frac{\theta}{2\pi} \right) = \frac{\theta a^2}{2}$$

6 20%

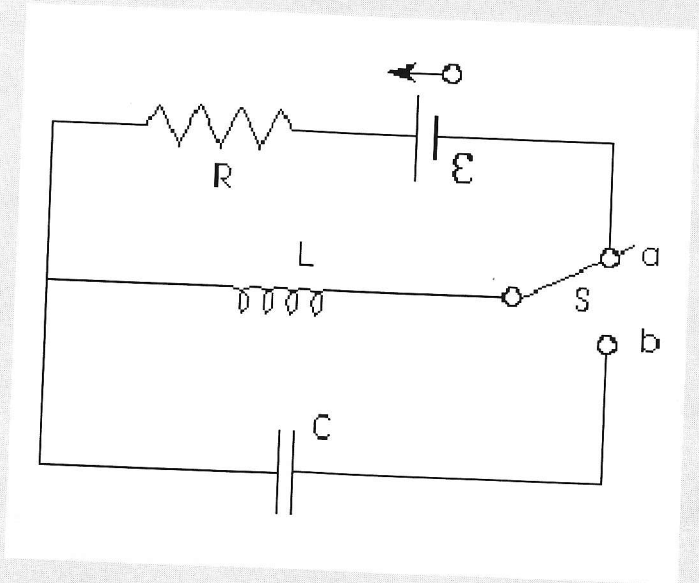
Problem 4: RLC Circuit



The switch has been in position ***a*** for a long time. The capacitor is uncharged.

1. What energy is currently stored in the magnetic field of the inductor?
2. At time $t = 0$, the switch S is thrown to position *b*. By applying Faraday's Law to the bottom loop of the above circuit, obtain a differential equation for the behavior of charge Q on the capacitor with time.

Problem 4, Part 2: RLC Circuit



3. Write down an explicit solution for *Q(t)* that satisfies your differential equation above and the initial conditions of this problem.

4. How long after $t = 0$ does it take for the electrical energy stored in the capacitor to reach its first maximum, in terms of the quantities given? At that time, what is the energy stored in the inductor? In the capacitor?

energy consideration

Switch on b

↓

decoupled from the upper loop

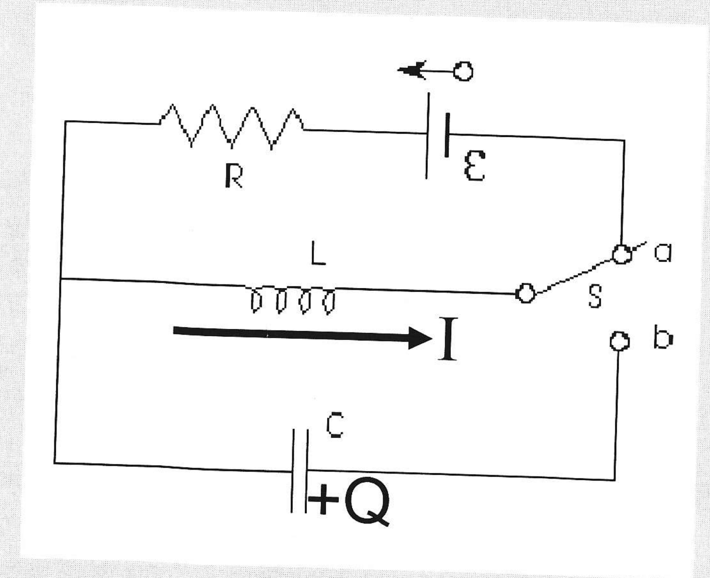
Similar to simple harmonic motion.

Initial condition $Q(t=0)$, $I(t=0)$

Energy consideration

6. 20%

Solution 4: RLC Circuit



1. Energy Stored in Inductor

$$U = \frac{1}{2} L I^2 = \frac{1}{2} L \left(\frac{\varepsilon}{R}\right)^2$$

2. Write Differential Equation

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0 \qquad I = \frac{dQ}{dt} \Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

Solution 4: RLC Circuit

3. Solution for Q(*t*): $Q(t) = Q_{max} \sin(\omega t)$

$$\boxed{\omega = \frac{1}{\sqrt{LC}}} \qquad \omega Q_{max} = I_0 = \frac{\varepsilon}{R} \Rightarrow \boxed{Q_{max} = \frac{\varepsilon\sqrt{LC}}{R}}$$

4. Time to charge capacitor

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \Rightarrow \boxed{T_{\text{Charge}} = \frac{T}{4} = \frac{\pi\sqrt{LC}}{2}}$$

Energy in inductor = 0

Energy in capacitor = Initial Energy: $U = \frac{1}{2}L\left(\frac{\varepsilon}{R}\right)^2$