第一章

緒論

一. 物理概談及科學方法

我的 Notes 及講稿

一些 Comments on Physics and its Methods

二.

基本物理量及其量度 (單位轉換：呎→米，加侖→公升，靜止能量及質量)

準確度及有效數字

↓

請在普物實驗課討論

因次分析 (及 Scaling Analysis)

↓ 見 Notes

↓ 標度分析 見 Fowler 之 Note

數量級的粗估

見 P.9 – P.17 of Giancoli

純量與向量

↓

見第0章

*Scaling (Binding ∼ surface L^2

Weight ∼ volume L^3

Other related material

①

物理概談'

科學的要義

科學係指經由觀察、實驗、分析、歸納、理論、演繹等過程建立的有系統、組織的知識。[2] 其主要的任務是描述現象，不斷的求這描述的系統化、簡單化，統一化，一般化。而不是通常所了解的所謂"解釋"[1]

描述

- 現象的觀察和度量 ⇒ 觀念
- 觀察或度量結果的歸納和引伸
  - ⇒ 觀念間的（函數）關係
  - ⇒ "定律

創造

- 新觀念的引入
- 實驗的構想及實現
- 實驗技術的發明
- 數學方法之創新，應用及發展

理論的建立

理想氣体物態方程式[3]

P, V, T

P → 壓力

V → 体積

T → 温度

T 温度 ← 熱平衡 及其度量

$$T = f(P, V) \iff F(P, V, T) = 0$$

↓ 物態方程式

(只有均匀系統才有物態方程)

理想氣体之物態方程式

(i) 温度不變

PV = 常值　　波以耳定律 (1662) (Boyle)

(ii) 壓力不變

$\frac{V}{T}$ = 常值　　查理定律 (1787) (Charles)

給呂薩克 (1802) (Gay-Lussac)

$\frac{PV}{T}$ = 常數

氣体動力論

分子模型的假設[4]

(一) 任何有限体積之氣体是由極大數目之分子所組成。

(二) 假設各分子均相同而且可看成硬球体。分子不斷的沿各方向運動。

(三) 分子除了碰撞外,互不施力。
分子除了因與其他分子及容器之壁碰撞外沿直線運動

(四) 分子與分子及壁之碰撞均為完全彈性。
而且壁為完全平滑。

(五) 在不受外力之情況下,分子在容器中均勻分佈.

(六) 分子速度之分佈與方向無關

· 分子與容器壁之碰撞構成氣体對壁之壓力

·· 氣体的溫度與分子的平均動能成正比

⟹ 理想氣体的物態方程式

定律可應用的範圍
↓
是了解該定律
很重要的一部分

由經驗得知,一般氣体並不滿足理想氣体物態方程式。

如有些氣体,在一定範圍內,滿足凡德瓦(van der Waals)方程式。(1873)

理論上我們也必須對上述的簡單假設作修正
(如考慮分子有体積,分子間有作用力等等)
⟹ 較好的氣体物態方程式)。

科學方法 与 常識 之比較

有系統，有組織

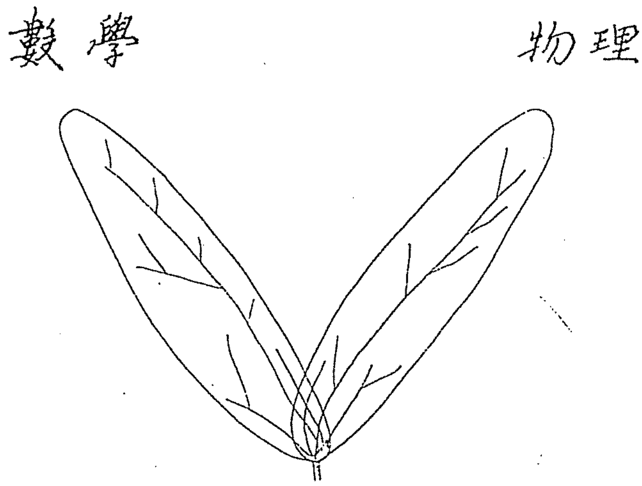
精確度較高

科學使用邏輯的演繹及數學方法

↓

精練
經濟 的 思想方法及程序
準確 的 語言

數學与物理之關係[5,6]



科學的門類

純粹科學 與 應用科學

物質
自然科學 (Physical Science)

生命科學 (Biological Science) 生命
Life
行為科學 (Behavior Science) 思想

物理學

物理是 自然科學 及生命科學的基石

科學方法是由物理首先發展出來的

伽里略 實驗科學

牛 頓 理論科學

近來高能物理利用高速電腦來處理極繁雜的資料
對未來整個科學之研究方法將有深遠的影響

物理學：研究物質在時空中各種情況下之交互作用

現今物理學研討之範圍

時間

最長 宇宙的年齡 $\sim 10^{18}$ sec

最短 強子的半衰期 $\sim 10^{-24}-10^{-25}$ sec

空

極大[8]　　宇宙的半徑　$10^{26} m = 10^{28} cm$

極小[9]　　$\sim 10^{-19} m = 10^{-17} cm$

温度

極大　　$\sim 10^{12}\ ^\circ K$[10]

極小　　$10^{-6}\ ^\circ K$

$\text{Energy}_{\text{rest}} = \text{mass}_{\text{rest}} \cdot c^2$　　$\text{rest mass} = \frac{E}{c^2}$

E　　MeV

$\text{Energy} = k_B\, T$　　$T = \frac{E}{k_B}$

k_B ↓ Boltzmann 波兹曼 常數

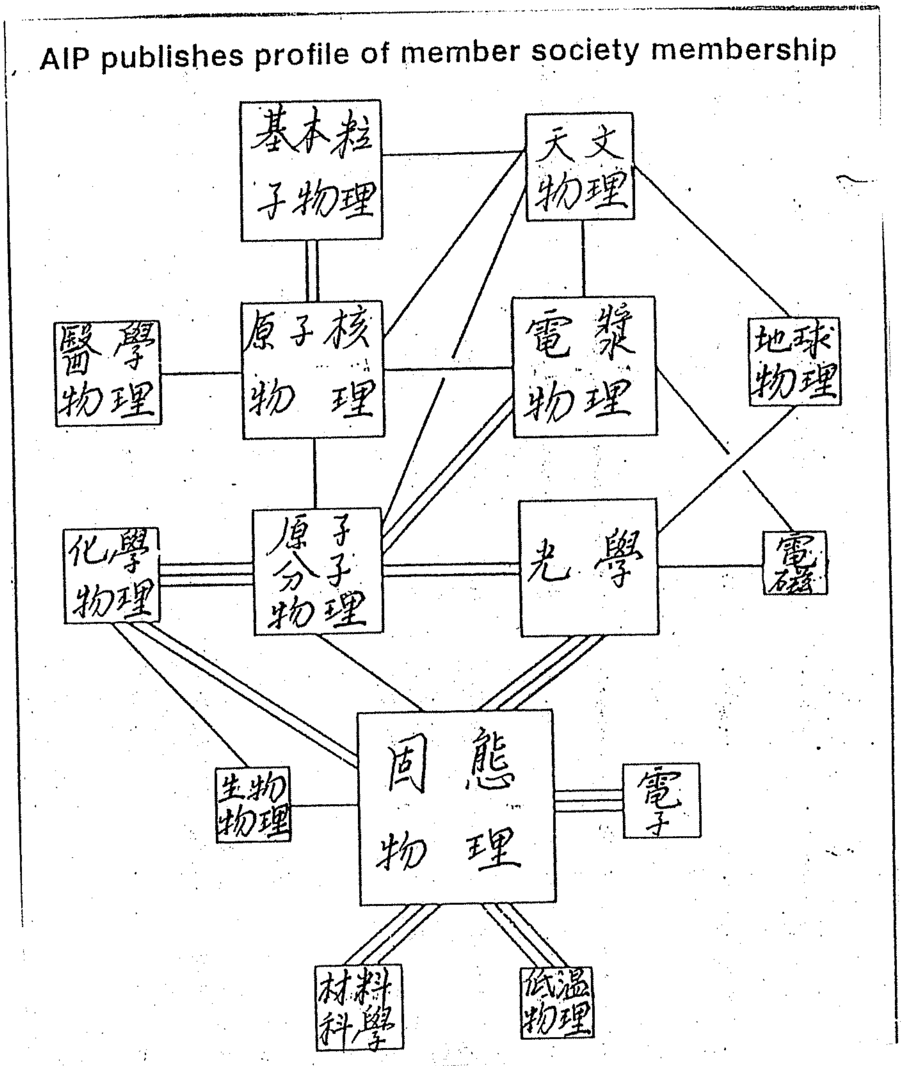
$1\ \text{MeV}/k_B \leftrightarrow 1.16 \cdot 10^{10}\ ^\circ K$

大霹靂及温度[11]

時間	温度 T ($^\circ K$)	温度 MeV/k_B
10^{-6} sec	10^{15}	10^{5}
10^{-4} sec	10^{12}	100
1- 2 sec	10^{10}	1
1. 5 min	10^{9}	0.1

物理學之分科[7]

物理學之研究領域[12]



美國物理學會各分會人數
分佈圖

方塊之大小代表各分會人數

方塊間之連線顯示各分會會員同時也從事另一分會研究之人數

*AIP (American Institute of Physics) 1983之統計

Physics Today, June 1987, 57

分類：	
編號：	8
總號：	

1993年之物理新聞[13]

Acoustics 聲學

Astrophysics 天文物理

Atomic and Molecular Physics 原子與分子物理

Biological Physics 生物物理

Chemical Physics 化學物理

Condensed Matter Physics 凝聚態物理

Physics Education 物理教育

Fluid Dynamics 流体力學

Physics and Government 物理及政府

Physics History 物理史

Medical Physics 醫學物理

Nuclear Physics 原子核物理

Optics 光學

Particle Physics 粒子物理

Plasma Physics 電漿物理
(離子体)

Polymer Physics 聚合体物理
(高分子)

清華物理系/所開課情形[14]

(四)

原子核物理　　原子及分子物理　　天文物理
基本粒子物理　　固態物理　　地球物理
　　　　　　　　　　　　　　　　電漿物理
　　　　　　　　　　　　　　　　生物物理

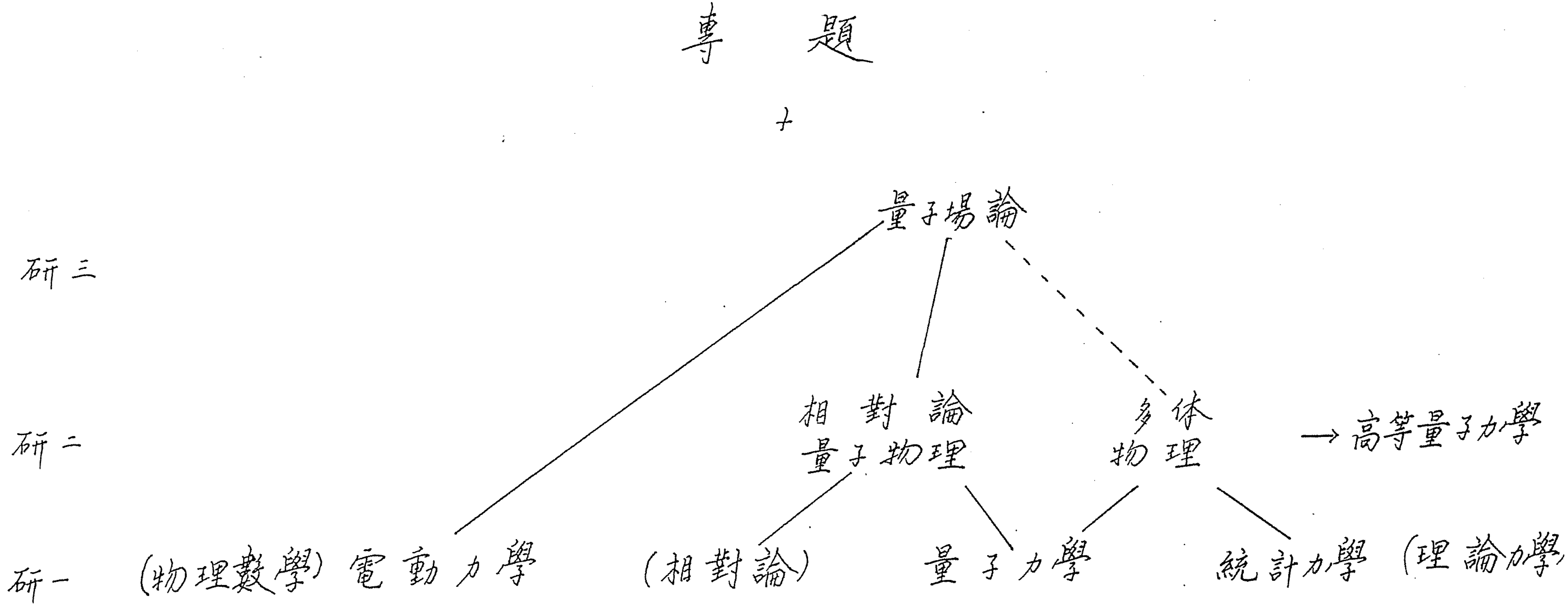
（導論系列）

(三)　數學物理　　量子物理　　相對論　　光學　　熱力學
　　　　　　　　　　　　　　　　　　　　　　氣体動力學
　　　　　　　　　　　　　　　　　　　　　　統計力學

(二)　理論力學　　電磁學

(一)　普通物理

大學部課程


專題
量子場論
研三
相對論
量子物理
多体
物理
→高等量子力學
研二
研一
(物理數學) 電動力學
(相對論)
量子力學
統計力學
(理論力學)

研究所課程

參考資料:

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2 吳大猷 "科學" "科學教育月刊" 第五十三期 第一頁
(民國七十一年十月)

3. 王竹溪 "熱力學簡明教程" 第十二頁至第二十一頁
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Second Edition Taiwan copyright 1973

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Korea Physical Society and Korea Science and Engineering Foundation (1980)

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8. Frank H. Shu "The Physical Universe" P. 11 (1982)
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10 Frank Wilczek "10^{12} Degrees in the Shade" The Sciences
Jan/Feb 1994, P. 22-30

11 L. P. Csernai "Introduction to Relativistic Heavy Ion Collisions"
P. 303 (1994)

12 Phys. Today, June 1987, P. 57

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14. 清華大學物理系/所開課情形

③ 美國物理學會之資料
各學門分佈情形

OFFICIAL 2011 UNIT MEMBERSHIP STATISTICS

UNIT	2007	2008	2009	2010	2011	2007	2008	2009	2010	2011
DIVISIONS										
Atomic, Molecular & Optical	2,780	2,832	2,885	3,007	3,049	6.01%	6.12%	6.11%	6.27%	6.32%
Astrophysics	1,979	2,114	2,164	2,356	2,404	4.27%	4.57%	4.59%	4.91%	4.98%
Biological Physics	1,850	1,881	2,048	2,134	2,064	4.00%	4.07%	4.34%	4.45%	4.28%
Computational Physics	2,049	2,129	2,201	2,299	2,410	4.43%	4.60%	4.66%	4.79%	4.99%
Condensed Matter Physics	5,387	5,592	5,717	5,964	6,040	11.64%	12.09%	12.12%	12.44%	12.51%
Chemical Physics	1,788	1,782	1,740	1,732	1,668	3.86%	3.85%	3.69%	3.61%	3.46%
Fluid Dynamics	2,655	2,735	2,745	2,773	3,169	5.74%	5.91%	5.82%	5.78%	6.57%
Polymer Physics	1,342	1,254	1,358	1,343	1,350	2.90%	2.71%	2.88%	2.80%	2.80%
Laser Science	1,331	1,363	1,384	1,357	1,417	2.88%	2.95%	2.93%	2.83%	2.94%
Materials Physics	2,419	2,453	2,680	2,853	2,851	5.23%	5.30%	5.68%	5.95%	5.91%
Nuclear Physics	2,519	2,624	2,578	2,680	2,658	5.44%	5.67%	5.46%	5.59%	5.51%
Physics of Beams	1,180	1,210	1,196	1,141	1,099	2.55%	2.62%	2.53%	2.38%	2.28%
Particles & Fields	3,371	3,470	3,461	3,565	3,465	7.28%	7.50%	7.33%	7.44%	7.18%
Plasma Physics	2,520	2,498	2,513	2,544	2,608	5.44%	5.40%	5.33%	5.31%	5.40%
TOPICAL GROUPS										
Energy Research & Applications				297	463				0.62%	0.96%
Few Body Systems	327	320	308	320	309	0.71%	0.69%	0.65%	0.67%	0.64%
Fundamental Constants	433	419	416	423	445	0.94%	0.91%	0.88%	0.88%	0.92%
Gravitation	921	1,018	1,020	1,103	1,099	1.99%	2.20%	2.16%	2.30%	2.28%
Hadronic	355	366	397	434	441	0.77%	0.79%	0.84%	0.91%	0.91%
Instrument & Measure Science	601	606	566	564	544	1.30%	1.31%	1.20%	1.18%	1.13%
Magnetism	778	836	835	898	940	1.68%	1.81%	1.77%	1.87%	1.95%
Plasma Astrophysics	365	370	368	371	391	0.79%	0.80%	0.78%	0.77%	0.81%
Quantum Information	755	886	929	1,028	1,084	1.63%	1.91%	1.97%	2.14%	2.25%
Shock Compression of Cond Matte	367	407	354	385	331	0.79%	0.88%	0.75%	0.80%	0.69%
Statistical & Non-Linear	895	944	945	982	981	1.93%	2.04%	2.00%	2.05%	2.03%
FORUMS										
Education	4,598	4,646	4,595	4,738	4,714	9.93%	10.04%	9.74%	9.88%	9.77%
Graduate Student Affairs	2,865	3,343	3,719	4,085	4,366	6.19%	7.23%	7.88%	8.52%	9.05%
History of Physics	3,854	3,928	3,776	3,797	3,744	8.33%	8.49%	8.00%	7.92%	7.76%
Industurial & Applied Physics	6,644	6,740	6,772	6,858	6,931	14.35%	14.57%	14.35%	14.30%	14.36%
International	3,437	3,608	3,655	3,761	3,759	7.42%	7.80%	7.75%	7.84%	7.79%
Physics & Society	5,548	5,805	5,874	6,061	6,110	11.98%	12.55%	12.45%	12.64%	12.66%
SECTIONS										
California - Nevada	2,072	2,305	2,302	2,490	2,624	4.48%	4.98%	4.88%	5.19%	5.44%
Four Corners	1,113	1,260	1,493	1,623	1,684	2.40%	2.72%	3.16%	3.38%	3.49%
New England	2,327	2,413	2,371	2,364	2,368	5.03%	5.22%	5.02%	4.93%	4.91%
New York State	2,290	2,436	2,503	2,502	2,511	4.95%	5.26%	5.30%	5.22%	5.20%
Northwest	1,106	1,160	1,181	1,245	1,254	2.39%	2.51%	2.50%	2.60%	2.60%
Ohio Region	1,516	1,498	1,501	1,534	1,465	3.27%	3.24%	3.18%	3.20%	3.04%
Prairie			492	673	770			1.04%	1.40%	1.60%
Southeastern	2,544	2,728	2,731	2,717	2,680	5.50%	5.90%	5.79%	5.67%	5.55%
Texas	1,502	1,534	1,535	1,575	1,755	3.24%	3.32%	3.25%	3.28%	3.64%

Official 2011 APS Membership - 48,263 2010 - 47,947 2009 - 47,189 2008 - 46,269 2007 - 46,293

Alonso - Finn Physics
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化學：利用物理定律去了解分子的構成及轉換。

分類：
編號：
總號：

②

第一章
物理概論

一、緒論：物理在科學中的地位

科學

目的（動機）

基本科學：創發性，純求知性的探索。
研究問題的方向或純由於研究者的現像力或受當世研究興趣的影響
但研究者動機和研究的問題，不以實用為目標。

應用科學：以已知之科學原理和知識對有具体目標的問題從事探討

對象：

	基本	應用
自然科學（物理）	物理，化學，天文 地球科學等	土木工程，機械工程，化學工程 電機工程，採礦，冶金，太空科學等
生物科學（生物化學、生物物理）	生物學，植物學，動物學 解剖學，微生物學，遺傳學等	農業，醫藥，護理，畜牧等 生命
社會科學（生理狀態影响心理）	人類學，社會學，心理學 經濟學等	法律，教育，政治等 思想

1. 物理學是物理科學的基石也是工程科學之基礎
例如：流体力學是水利及航空工程之基礎
熱力學是機械工程之基礎
核子物理是核子工程是核子工程之基礎
2. 物理學的研究方法是其他科學研究方法之楷模
研究方法
觀察
實驗 可控制
高 率
理論 歸納實驗結果
↓
建立模型
↓
導演及預測新實驗結果
↓
与新實驗相比
↙ ↘
与新實驗相符 / 与新實驗不符
證實模型之正確 / 修正舊模型、建立新模型。

例如 伽里略拋球實驗是實驗科學之初創
牛頓利用萬有引力定律來了解天體運動是理論科學之鼻祖
近來高能物理利用電子計算機來處理極繁雜之資料 對將來各種科學的研究應有極深遠之影響。

二. 物理學至今研討之領域及特徵

物理學是研究物質在時空之中，在各種情況下之行爲及變化 {物理是研究物質之組成及其相互作用 解釋自然現象}

時
極大 宇宙的年齡 $\sim 3\times10^{17}$ sec 10^{18} sec
極小 是基本粒子（嚴格來說是能經強交互作用衰變的強子）之半衰期 $\sim 10^{-23}$ sec.

空
極大 宇宙的半徑 $\sim 10^{28}$ cm 10^{26} m
極小 是強子的半徑 $\sim 10^{-13}$ cm. 10^{-15} m $\sim 10^{-17}$ m.

各種情況 以溫度之變化爲例
極大 10^{10} °K
極小 10^{-6} °K

⇒物理學確實是極廣泛而複雜的學問，因此物理可以分成許多支。
（我們就會兒將分別討論這些分支及其中研討之問題）
但我們在此必須強調物理之統一性。
(1) 處理和研討問題之方法是完全一致的
(2) 所有物理的研討均是以古典力學，量子力學及相對論爲基礎。
(3) 有些問題需要好幾個分支來共同解決。

三 物理學結構的研討

古典物理（十九世紀末）

$$\vec{F} = \frac{d}{dt}m\vec{v} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

（第二個等號：若質量爲常數）

作用力
(1) 重力 $\vec{F} = G\frac{m_1m_2}{r^2}\hat{r}$ 萬有引力定律
(2) 電磁力 一電荷在電磁場中所受之電磁力爲 $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ ← 羅倫斯力.

$\vec{E}$, $\vec{B}$ 是由電荷及電流分佈來決定
這組公式是馬克斯威爾公式.

由以上公式可導出在某些情況下有機械波及電磁波之存在.

熱統計學
利用微觀之分子運動來描述巨觀系統之情況及行爲
⇒ 統計力學
分子（或原子）是討論中之基本單位。

二十世紀之重大突破
(1) 分子是可分割的. 分子 → 原子 {核子 {質子 中子}, 電子} 基本粒子

Z number of protons
A−Z 〃 〃 neutrons

分類：
編號：
總號：

(2) 在討論原子、分子及更小的物質時必需要用量子力學(古典力學是量子力學的極限)
(3) 當速度趨近於光速時需用狹義相對論
(4) 當重力場極強時，需用廣義相對論
(5) 必須加入兩種新的作用力：強交互作用(核力)
弱交互作用 (β 蛻變 $n \to p + e^- + \bar{\nu}_e$)

三 物理之分類

(1) 基本粒子物理：(6.3%)
(a) 加速器物理。
(b) 高能實驗。
(c) 質點檢測器物理，
(d) 高能現象學
(e) 基本粒子理論

(2) 核子物理：(9.5%)
(a) 原子核結構
(b) 原子核性質
(c) 核子力
(d) 核子反應
(e) 原子核衰變
(f) 核輻射

(3) 原子、分子及電子物理 (5.4%)
(a) 原子物理：結構，性質，反應。
(b) 分子物理：結構，性質，反應。
(c) 電子物理：反應，光電交互作用。

(4) 凝聚体物理 (21.5%)
(a) 固態物理
(b) 液態物理　　沒有固定的型態
(c) 液晶体物理
(d) 不定体物理　　沒有固定的排列
(e) 表面物理
(f) 固態雷射

(5) 光學 (9%)
(a) 幾何光學
(b) 物理光學
(c) 雷射光學
(d) 量子光學
(e) 光學儀器

(6) 声學 (3%)
(a) 音響學
(b) 超音波學
(c) 熱振動學
(d) 噪音學

(7) 離子体(電漿)与流体物理 (5.0%)

(a) 離子体物理　　正負離子相混之氣体態 (氣体在高温時)

(b) 電漿控制學

(c) 流体力學

(8) 天文物理及相對論 (1.1%)

(a) 天文物理

(b) 相對論

(c) 宇宙論

(9) 地球与行星物理 (4%)

(a) 地殼物理

(b) 海洋物理

(c) 大氣物理

(d) 離子層物理

(e) 地磁物理

(f) 行星物理

(10) 化學物理 (1.3%)

(a) 電化學

(b) ~~流体化學~~ X光晶体學

(c) 流体化學

(11) 生物物理 (1.3%)

(a) 生物電學

(b) 細胞膜物理

(c) 保健物理

(12) 其他及雜項 (30.7%)

四種基本交互作用:
重力，電磁力，強交互作用，弱交互作用

1970 APS report.

美國	2億人口	物理學會會員	4万人	1/5000人	
中華民國	1仟6百万 ⑨ 2仟3百万	〃 〃 〃 〃 〃 〃	400人 ~2000人	→	3200人 3800人

量度及單位　　Mass, length, time, charge

Accessible and invariable.

1 m = 1,650,763.73 wavelength in vacuum of light emitted in $2P_{10} - 5d_5$ ^{86}Kr transition

現改為以光速，時間來定長度

趙凱華，羅蔚茵"力學"

1. 什么是物理学?

古希腊人把所有对自然界的观察和思考，笼统地包含在一门学问里，那就是自然哲学。科学分化为天文学、力学、物理学、化学、生物学、地质学等，只是最近几百年的事。在牛顿的时代里，科学和哲学还没有完全分家。牛顿划时代的著作名为《自然哲学的数学原理》，就是一个明证。物理学最直接地关心自然界最基本规律，所以牛顿把当时的物理学叫做自然哲学。

17 世纪牛顿在伽利略、开普勒工作的基础上，建立了完整的经典力学理论，这是现代意义下的物理学的开端。从 18 世纪到 19 世纪，在大量实验的基础上，卡诺、焦耳、开尔文、克劳修斯等建立了宏观的热力学理论，克劳修斯、麦克斯韦、玻耳兹曼等建立了说明热现象的气体分子动理论，库仑、奥斯特、安培、法拉第、麦克斯韦等建立了电磁学理论。至此，经典物理学理论体系的大厦巍然耸立。然而，正当大功甫成之际，一系列与经典物理的预言极不相容的实验事实相继出现，人们发现大厦的基础动摇了。在这些新实验事实的基础上，20 世纪初，爱因斯坦独自创立了相对论，先后在普朗克、爱因斯坦、玻尔、德布洛意、海森伯、薛定谔、玻恩等多人的努力下，创立了量子论和量子力学，奠定了近代物理学的理论基础。20 世纪随着科学的发展，从物理学中不断地分化出诸如粒子物理、原子核物理、原子分子物理、凝聚态物理、激光物理、电子物理、等离子体物理等名目繁多的新分支，以及从物理学和其它学科的杂交中生长出来的，诸如天体物理、地球物理、化学物理、生物物理等众多交叉学科。

什么是物理学？试用一句话来概括，可以说：*物理学是探讨物质的结构和运动基本规律的学科*。尽管这个相当广泛的定义仍难以刻画出当代物理学极其丰富的内涵，不过有一点是肯定的，即与其它科学相比，物理学更着重于物质世界普遍而基本的规律的追求。

物理学和天文学由来已久的血缘关系，是有目共睹的。当今物理学的研究领域里有两个尖端，一个是高能或粒子物理，另一个是天体物理。前者在最小的尺度上探索物质更深层次的结构，后者在最大的尺度上追寻宇宙的演化和起源。可是近几十年的进展表明，这两个极端竟奇妙地衔接在一起，成为一对密不可分的姊妹学科。物理学和化学从来就是并肩前进的。如果说物理化学还是它们在较为唯象的层次上的结合，则量子化学已深入到化学现象的微观机理。物理学和生物学的关系怎么样？对于如何解释生命现象的问题，历史上有过两种极端相反的看法：一是"生机论

(vitalism)”，认为生命现象是由某种“活力”主宰着，永远不能在物理和化学的基础上得到解释；另一是“还原论(reductionism)”，认为一切生命现象都可归结(或者说，还原)为物理和化学过程。1824年沃勒(F. Wöhler)成功地在实验室内用无机物合成了尿素之后，生机论动摇了。但是，能否完全用物理学和化学的原理和定律解释生命呢？回答这个问题为时尚早。不过，生命科学有自己独特的思维方式和研究手段，积累了大量知识，确立了许多定律，说把生物学“还原”为物理学和化学，是没有意义的。可是物理学研究的是物质世界普遍而基本的规律，这些规律对有机界和无机界同样适用。物理学构成所有自然科学的理论基础，其中包括生物学在内。物理学、化学和生物学相互渗透，前途是不可估量的。近四五十年在这三学科的交叉点上产生的一系列重大成就，如DNA双螺旋结构的确定、耗散结构理论的建立等，充分证明了这一点。现在人们常说，21世纪是生命科学的世纪，这话有一定道理。不过，生命科学的长足发展，必定是在与物理科学(物理学和化学)更加密切的结合中达到的。

2. 物理学与技术

社会上习惯于把科学和技术联在一起，统称“科技”，实际上二者既有密切联系，又有重要区别。科学解决理论问题，技术解决实际问题。科学要解决的问题，是发现自然界中确凿的事实和现象之间的关系，并建立理论把这些事实和关系联系起来；技术的任务则是把科学的成果应用到实际问题中去。科学主要是和未知的领域打交道，其进展，尤其是重大的突破，是难以预料的；技术是在相对成熟的领域内工作，可以作比较准确的规划。

历史上，物理学和技术的关系有两种模式。回顾以解决动力机械为主导的第一次工业革命，热机的发明和使用提供了第一种模式。17世纪末叶发明了巴本锅和蒸汽泵；18世纪末技术工人瓦特给蒸汽机增添了冷凝器，发明了活塞阀、飞轮、离心节速器等，完善了蒸汽机，使之真正成为动力。其后，蒸汽机被应用于纺织、轮船、火车；那时的热机效率只有5%~8%．1824年工程师卡诺提出他的著名定理，为提高热机效率提供了理论依据。到20世纪蒸汽机效率达到15%，内燃机效率达到40%，燃气涡轮机效率达到50%．19世纪中叶科学家迈耶、亥姆霍兹、焦耳确立了能量守恒定律，物理学家开尔文、克劳修斯建立了热力学第一、二定律。这种模式是技术向物理提出了问题，促使物理发展了理论，反过来提高了技术，即技术 → 物理 → 技术。电气化的进程提供了第二种模式。从1785年建立库仑定律，中间经过伏打、奥斯特、安培等人的努力，直到1831年法拉第发现电磁感应定律，基本上是物理上的探索，没有应用的研究。此后半个多世纪，各种交、直流发电机、电动机和电报机的研究应运而生，蓬勃地发展起来。有了1862年麦克斯韦电磁理论的建立和1888年赫兹的电磁波实验，才导致了马可尼和波波夫无线电的发明。当然，电气化反过来大大促进了物理学的发展。这种模式是物理 → 技术 → 物理。

20世纪以来，在物理和技术的关系中，上述两种模式并存，相互交叉。但几乎所有重大的新技术领域(如电子学、原子能、激光和信息技术)的创立，事前都在物理学中经过了长期的酝酿，在理论和实验上积累了大量知识，才突然迸发出来的。没有1909年卢瑟福的α粒子散射实验，就不可能有40年代以后核能的利用；只有1917年爱因斯坦提出受激发射的理论，才可能有1960年第一台激光器的诞生。当今对科学、技术，乃至社会生活各个方面都产生了巨大冲击的高技术，莫过于电子计算机，由之而引发的信息革命被誉为第二次工业革命。整个信息技术的发生、发展，其硬件部分都是以物理学的成果为基础的。大家都知道，1947年贝尔实验室的巴丁、布拉顿和肖克莱发明了晶体管，标志着信息时代的开始，1962年发明了集成电路，70年代后期出现了大规模集成电路。殊不知，在此之前至少还有20年的“史前期”，在物理学中为孕育它的诞生作了大量的理论和实验上的准备：1925－1926年建立了量子力学；1926年建立了费米-狄拉克统计法，得知固体中电子服从泡利不相容原理；1927年建立了布洛赫波的理论，得知在理想晶格中电子不发生散射；1928年索末菲提出能带的猜想；1929年派尔斯提出禁带、空穴的概念，解释了正霍耳系数的存在；同年贝特提出了费米面的概念，直至1957年才由皮帕得测量了第一个费米面，尔后剑桥学派编制了费米面一览表。总之，当前的第二次工业革命主要是按物理 → 技术 → 物理的模式进行的。

3. 物理学的方法和科学态度

现代的物理学是一门理论和实验高度结合的精确科学。物理学中有一套获得知识、组织知识和运用知识的有效步骤和方法，其要点可概括为：

(1) 提出命题

命题一般是从新的观测事实或实验事实中提炼出来的，也可能是从实际目的或已有原理中推演出来的。

(2) 推测答案

答案可以有不同的层次：建立唯象的物理模型；用已知原理和推测对现象作定性的解释；根据现有理论进行逻辑推理和数学演算，以便对现象作出定量的解释；当新事实与旧理论不符时，提出新的假说和原理去说明它，等等。

思路之后，自己背着书本把它们演算出来。这样你才能对它们成立的条件、关键的步骤、推演的技巧等有深刻的理解。

悟物穷理，就要多向自己提问：哪些是事实？哪些是推论？推论是怎样得来的？我为什么相信它？…… 问题可以正面提，也可以反向提。譬如，已知物体所受的力，可以求它的运动；知道了它的运动，反过来问它受了什么样的力。

勤于思考，悟物穷理，就要对问题建立自己的物理图像。学习物理，不做习题是不行的，但做习题不在于多，而在于精。习题做完了，不要对一下答案或交给老师去批改就了事。自己从物理上应该想一想，答案的数量级是否对头？所反映的物理过程是否合理？能否从别的角度判断自己的答案是否正确？我们应该力争能够作到，习题要么做不出来，做出来就有充分的理由相信它是对的，即使它和书上给的答案不一样。老师说你错了，你在未被说服之前要敢于和老师争辩。好的老师最欣赏的是能指出自己错误的学生。如果最后证明是你自己错了，也错个明白。

正是：书山有路勤为径，学海无涯悟作舟。

第一章 质点运动学

§1. 引 言

1.1 力学的研究对象

在各种形态的物质运动中，最简单的一种是物体位置随时间的变动。宏观物体之间（或物体内各部分之间）的相对位置变动，例如，各种交通工具的行驶，大气和河水的流动、天体的运行等，称为机械运动（mechanical motion）❶。

力学（mechanics）的研究对象是机械运动。经典力学研究的是在弱引力场中宏观物体的低速运动。通常把力学分为运动学（kinematics）、动力学（dynamics）和静力学（statics）。运动学只描述物体的运动，不涉及引起运动和改变运动的原因；动力学则研究物体的运动与物体间相互作用的内在联系；静力学研究物体在相互作用下的平衡问题。

1.2 质 点

在物理学中，为了突出研究对象的主要性质，暂不考虑一些次要的因素，经常引入一些理想化的模型来代替实际的物体。“质点”就是一个理想化的模型。

在研究机械运动时，物体的形状和大小是千差万别的。对有些场合（如落体受到空气的阻力问题），物体的形状和大小是重要的；但在很多问题中，这些差别对物体运动的影响不大，若不涉及物体的转动和形变，我们可暂不考虑它们的形状和大小，把它们当作一个具有质量的点（即质点）来处理。例如，人们常把单摆的摆球、在电场中运动的带电粒子等当作质点。又如，同样是地球，在研究它绕日公转时，可以将它看作质点；在研究它的自转问题时，就不能把它当作质点处理了。此外，当我们研究一些比较复杂的物体（如刚体、流体）运动时，虽然不能把整个物体看成质点，但在处理方法上可把复杂物体看成由许多质点组成，在解决质点运动问题的基础上来研究这些复杂物体的运动。

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❶ 英文 mechanical 一词有“机械的”和“力学的”双重涵义，故有人主张把 mechanical motion 叫做“力学运动”。但因“机械运动”一词沿用已久，并且也说得通，未改。

分類：
編號：
總號：

雜　記（普物）

The Universe is Lawful
Lord is subtle. - Einstein.

Scientific Method

Observation
Experimentation, data analysis
Phenomenology
Hypothesis → Law
Model → Theory
Computational physics
Foundation
Formalism

Classification

Kinematics
Dynamics

四大力學
古典力學 → 流体力學

電動力學（光學）

統計力學 → 熱力學

量子力學

分類：	
編號：	
總號：	

量度及單位

單位的要求：易得 Accessible
不變 Invariable.

指數之符號

P	10^{15}
T	10^{12}
G	10^{9}
M	10^{6}
K	10^{3}
m	10^{-3}
μ	10^{-6}
n	10^{-9}
p	10^{-12}
f	10^{-15}

例：

(a) $1\ \mu m = 10^{-6}\ m$
$1\ fm = 10^{-15}\ m$

(b) 電子的靜止能量 $= m_0 c^2 = 0.511$ MeV ($\downarrow 10^6$)
J/ψ 粒子的靜止能量 = 3.1 GeV

(c) 加速器的能量
清華 3 MeV 質子加速器
↓
加速質子使其動能達到 $3 \cdot 10^6$ eV
↓
電子伏特

同步加速器（SRRC） 1.3 GeV 電子加速器
↓
加速電子使其動能達到 $1.3 \cdot 10^9$ eV

Fermi lab. 1 TeV 質子加速器
↓
10^{12} eV

大距離 光年, parsec (parallel ↓ 秒差距) order of magnitude → 數量級

間接方法 → 注意應用範圍 數據處理, 數據顯示, histogram (→ 分佈表)
實驗誤差, 有效數字

分類：	
編號：	
總號：	

近代物理之交互作用
↓
核交互作用

因次分析

單擺

		因次
週期	t	T
擺長	ℓ	L
擺的質量	m	M
重力加速度	g	LT^{-2}
角振幅	θ	無因次

$$T = t = \ell^{\alpha} m^{\beta} g^{\gamma} = (L)^{\alpha}(M)^{\beta}(LT^{-2})^{\gamma}$$

$$\beta = 0$$
$$\alpha + \gamma = 0$$
$$-2\gamma = 1$$
$$\Rightarrow \quad \alpha = \tfrac{1}{2}, \quad \beta = 0, \quad \gamma = -\tfrac{1}{2}$$

$$t \rightarrow \phi(\theta)\sqrt{\frac{\ell}{g}}$$

↑ 無法從因式分析中得到資料

液体珠在它自身表面張力下之振動 王贛駿
（特別需要強調的是在重力場以外）

		因次
週期	t	T
表面張力	s	MT^{-2}
液体之密度	d	ML^{-3}
液珠之半徑	r	L

$MT^{-2} \leftarrow s = \frac{F}{\ell} = \frac{M\frac{L}{T^2}}{L}$ （s ↓ surface tension）

$$T = t = s^{\alpha} d^{\beta} r^{\gamma} = (MT^{-2})^{\alpha}(ML^{-3})^{\beta}(L)^{\gamma}$$

$$\alpha + \beta = 0$$
$$-3\beta + \gamma = 0$$
$$-2\alpha = 1$$
$$\Rightarrow \quad \alpha = -\tfrac{1}{2}, \quad \beta = \tfrac{1}{2}, \quad \gamma = \tfrac{3}{2}$$

$$\Rightarrow \quad t = \begin{pmatrix}\text{dimensionless} \\ \text{constant}\end{pmatrix} \cdot \sqrt{\frac{r^3 d}{s}}$$

Rayleigh Nature 95, 66 (1915)

主要的假設：沒有其他有因次的變數
↑
物理上之考慮

What is Physics

Physics is the use of scientific method to find out the basic principles governing light and matter, and to discover the implication of those laws.

There are rules by which the universe function.

Matter and Light

The Scientific Method.

Basic principles

① Science is a cycle of theory and experiment

Experiment ⟵ Theory

Experiment created under certain condition (idealization or abstraction) → a useful theory → new experiments under new conditions

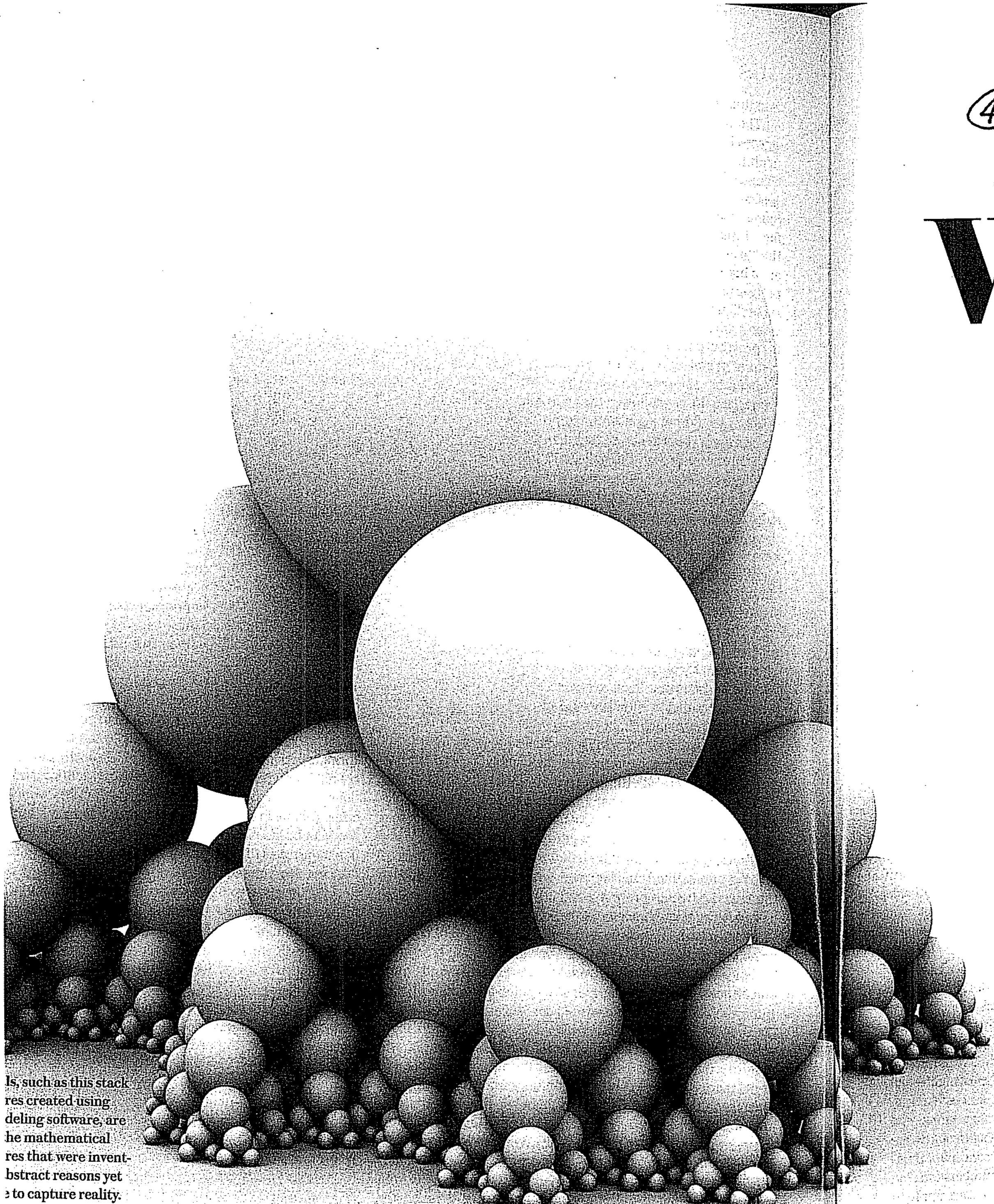
under new conditions theory is not a good approximation or not valid at all. → new theory

② Theories should both predict and explain

Theory should be testable

Explainatory value means that many phenomena should be accounted for with few basic principles

③ Experiments should be reproducible.



ls, such as this stack
res created using
deling software, are
he mathematical
res that were invent-
bstract reasons yet
e to capture reality.

(4)

Mario Livio is a theoretical astrophysicist at the Space Telescope Science Institute in Baltimore. He has studied a wide range of cosmic phenomena, ranging from dark energy and supernova explosions to extrasolar planets and accretion onto white dwarfs, neutron stars and black holes.

PHILOSOPHY OF SCIENCE

Why Math Works

Is math invented or discovered? A leading astrophysicist suggests that the answer to the millennia-old question is both

By Mario Livio

MOST OF US TAKE IT FOR GRANTED that math works—that scientists can devise formulas to describe subatomic events or that engineers can calculate paths for spacecraft. We accept the view, initially espoused by Galileo, that mathematics is the language of science and expect that its grammar explains experimental results and even predicts novel phenomena. The power of mathematics, though, is nothing short of astonishing. Consider, for example, Scottish physicist James Clerk Maxwell's famed equations: not only do these four expressions summarize all that was known of electromagnetism in the 1860s, they also anticipated the existence of radio waves two decades before German physicist Heinrich Hertz detected them. Very few languages are as effective, able to articulate volumes' worth of material so succinctly and with such precision. Albert Einstein pondered, "How is it possible that mathematics, a product of human thought that is independent of experience, fits so excellently the objects of physical reality?"

As a working theoretical astrophysicist, I encounter the seemingly "unreasonable effectiveness of mathematics," as Nobel laureate physicist Eugene Wigner called it in 1960, in every step of my job. Whether I am struggling to understand which progenitor systems produce the stellar explosions known as type Ia supernovae or calculating the fate of Earth when our sun ultimately becomes a red giant, the tools I use and the models I develop are mathematical. The uncanny way

IN BRIEF

The deepest mysteries are often the things we take for granted. Most people never think twice about the fact that scientists use mathematics to describe and explain the world. But why should that be the case?

Math concepts developed for purely abstract reasons turn out to explain real phenomena. Their utility, as physicist Eugene Wigner once wrote, "is a wonderful gift which we neither understand nor deserve."

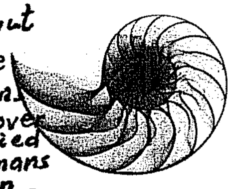
Part of the puzzle is the question of whether mathematics is an invention (a creation of the human mind) or a discovery (something that exists independently of us). The author suggests it is both.

Illustration by Tom Beddard

at math captures the natural world has fascinated me through- ıt my career, and about 10 years ago I resolved to look into the ıue more deeply.

At the core of this mystery lies an argument that mathemati- ıns, physicists, philosophers and cognitive scientists have had r centuries: Is math an invented set of tools, as Einstein be- ıved? Or does it actually exist in some abstract realm, with hu- ıns merely discovering its truths? Many great mathemati- ıns—including David Hilbert, Georg Cantor and the group own as Nicolas Bourbaki—have shared Einstein's view, associ- ed with a school of thought called Formalism. But other illustri- ıs thinkers—among them Godfrey Harold Hardy, Roger Pen- se and Kurt Gödel—have held the opposite view, Platonism.

This debate about the nature of mathematics rages on today ıd seems to elude an answer. I believe that by asking simply ıether mathematics is invented or discovered, we ignore the ıssibility of a more intricate answer: both invention and dis- ıvery play a crucial role. I posit that together they account for ıy math works so well. Although eliminating the dichotomy tween invention and discovery does not fully explain the un- ısonable effectiveness of mathematics, the problem is so pro- ınd that even a partial step toward solving it is progress.



The universe has regularities, known as symmetries, that let physicists describe it mathematically. And no one knows why.

INVENTION *AND* DISCOVERY

THEMATICS is unreasonably effective in two distinct ways, one I ink of as active and the other as passive. Sometimes scientists ıate methods specifically for quantifying real-world phenome- . For example, Isaac Newton formulated calculus for the pur- ıse of capturing motion and change, breaking them up into in- ıtesimally small frame-by-frame sequences. Of course, such ac- e inventions are effective; the tools are, after all, made to order. ıat is surprising, however, is their stupendous accuracy in some ıses. Take, for instance, quantum electrodynamics, the mathe- ıtical theory developed to describe how light and matter inter- t. When scientists use it to calculate the magnetic moment of electron, the theoretical value agrees with the most recent perimental value—measured at 1.00115965218073 in the ap- opriate units in 2008—to within a few parts per trillion!

Even more astonishing, perhaps, mathematicians sometimes velop entire fields of study with no application in mind, and yet ıades, even centuries, later physicists discover that these very ınches make sense of their observations. Examples of this kind passive effectiveness abound. French mathematician Évariste lois, for example, developed group theory in the early 1800s for sole purpose of determining the solvability of polynomial ıations. Very broadly, groups are algebraic structures made up sets of objects (say, the integers) united under some operation r instance, addition) that obey specific rules (among them the stence of an identity element such as 0, which, when added to y integer, gives back that same integer). In 20th-century phys- , this rather abstract field turned out to be the most fruitful y of categorizing elementary particles—the building blocks of ıtter. In the 1960s physicists Murray Gell-Mann and Yuval 'eman independently showed that a specific group, referred to SU(3), mirrored a behavior of subatomic particles called had- ıs—a connection that ultimately laid the foundations for the ıdern theory of how atomic nuclei are held together.

The study of knots offers another beautiful example of passive ıectiveness. Mathematical knots are similar to everyday knots, except that they have no loose ends. In the 1860s Lord Kelvin hoped to describe atoms as knotted tubes of ether. That misguided model failed to connect with reality, but mathematicians continued to analyze knots for many decades merely as an esoteric arm of pure mathematics. Amazingly, knot theory now provides important insights into string theory and loop quantum gravity—our current best attempts at articulating a theory of space-time that reconciles quantum mechanics with general relativity. Similarly, English mathematician Hardy's discoveries in number theory advanced the field of cryptography, despite Hardy's earlier proclamation that "no one has yet discovered any warlike purpose to be served by the theory of numbers." And in 1854 Bernhard Riemann described non-Euclidean geometries—curious spaces in which parallel lines converge or diverge. More than half a century later Einstein invoked those geometries to build his general theory of relativity.

A pattern emerges: humans invent mathematical concepts by way of abstracting elements from the world around them—shapes, lines, sets, groups, and so forth—either for some specific purpose or simply for fun. They then go on to discover the connections among those concepts. Because this process of inventing and discovering is man-made—unlike the kind of discovery to which the Platonists subscribe—our mathematics is ultimately based on our perceptions and the mental pictures we can conjure. For instance, we possess an innate talent, called subitizing, for instantly recognizing quantity, which undoubtedly led to the concept of number. We are very good at perceiving the edges of individual objects and at distinguishing between straight and curved lines and between different shapes, such as circles and ellipses—abilities that probably led to the development of arithmetic and geometry. So, too, the repeated human experience of cause and effect at least partially contributed to the creation of logic and, with it, the notion that certain statements imply the validity of others.

SELECTION AND EVOLUTION

MICHAEL ATIYAH, one of the greatest mathematicians of the 20th century, has presented an elegant thought experiment that reveals just how perception colors which mathematical concepts we embrace—even ones as seemingly fundamental as numbers. German mathematician Leopold Kronecker famously declared, "God created the natural numbers, all else is the work of man." But imagine if the intelligence in our world resided not with humankind but rather with a singular, isolated jellyfish, floating deep in the Pacific Ocean. Everything in its experience would be continuous, from the flow of the surrounding water to its fluctuating temperature and pressure. In such an environment, lacking individual objects or indeed anything discrete, would the concept of number arise? If there were nothing to count, would numbers exist?

Like the jellyfish, we adopt mathematical tools that apply to our world—a fact that has undoubtedly contributed to the perceived effectiveness of mathematics. Scientists do not choose analytical methods arbitrarily but rather on the basis of how well they predict the results of their experiments. When a tennis ball machine shoots out balls, you can use the natural numbers 1, 2, 3, and so on, to describe the flux of balls. When firefighters use a hose, however, they must invoke other concepts, such as volume or weight, to render a meaningful description of the stream. So, too, when distinct subatomic particles collide in a particle accelerator, physicists turn to measures such as energy and momentum and not to the end number of particles, which would reveal only partial information about how the original particles collided because additional particles can be created in the process.

Over time only the best models survive. Failed models—such as French philosopher René Descartes's attempt to describe the motion of the planets by vortices of cosmic matter—die in their infancy. In contrast, successful models evolve as new information becomes available. For instance, very accurate measurements of the precession of the planet Mercury necessitated an overhaul of Newton's theory of gravity in the form of Einstein's general relativity. All successful mathematical concepts have a long shelf life: the formula for the surface area of a sphere remains as correct today as it was when Archimedes proved it around 250 B.C. As a result, scientists of any era can search through a vast arsenal of formalisms to find the most appropriate methods.

Not only do scientists cherry-pick solutions, they also tend to select problems that are amenable to mathematical treatment. There exists, however, a whole host of phenomena for which no accurate mathematical predictions are possible, sometimes not even in principle. In economics, for example, many variables—the detailed psychology of the masses, to name one—do not easily lend themselves to quantitative analysis. The predictive value of any theory relies on the constancy of the underlying relations among variables. Our analyses also fail to fully capture systems that develop chaos, in which the tiniest change in the initial conditions may produce entirely different end results, prohibiting any long-term predictions. Mathematicians have developed statistics and probability to deal with such shortcomings, but mathematics itself is limited, as Austrian logician Gödel famously proved.

SYMMETRY OF NATURE

THIS CAREFUL SELECTION of problems and solutions only partially accounts for mathematics's success in describing the laws of nature. Such laws must exist in the first place! Luckily for mathematicians and physicists alike, universal laws appear to govern our cosmos: an atom 12 billion light-years away behaves just like an atom on Earth; light in the distant past and light today share the same traits; and the same gravitational forces that shaped the universe's initial structures hold sway over present-day galaxies. Mathematicians and physicists have invented the concept of symmetry to describe this kind of immunity to change.

The laws of physics seem to display symmetry with respect to space and time: They do not depend on where, from which angle, or when we examine them. They are also identical to all observers, irrespective of whether these observers are at rest, moving at constant speeds or accelerating. Consequently, the same laws explain our results, whether the experiments occur in China, Alabama or the Andromeda galaxy—and whether we conduct our experiment today or someone else does a billion years from now. If the universe did not possess these symmetries, any attempt to decipher nature's grand design—any mathematical model built on our observations—would be doomed because we would have to continuously repeat experiments at every point in space and time.

Even more subtle symmetries, called gauge symmetries, prevail within the laws that describe the subatomic world. For instance, because of the fuzziness of the quantum realm, a given particle can be a negatively charged electron or an electrically neutral neutrino, or a mixture of both—until we measure the electric charge that distinguishes between the two. As it turns out, the laws of nature take the same form when we interchange electrons for neutrinos or any mix of the two. The same holds true for interchanges of other fundamental particles. Without such gauge symmetries, it would have been very difficult to provide a theory of the fundamental workings of the cosmos. We would be similarly stuck without locality—the fact that objects in our universe are influenced directly only by their immediate surroundings rather than by distant phenomena. Thanks to locality, we can attempt to assemble a mathematical model of the universe much as we might put together a jigsaw puzzle, starting with a description of the most basic forces among elementary particles and then building on additional pieces of knowledge.

Our current best mathematical attempt at unifying all interactions calls for yet another symmetry, known as supersymmetry. In a universe based on supersymmetry, every known particle must have an as yet undiscovered partner. If such partners are discovered (for instance, once the Large Hadron Collider at CERN near Geneva reaches its full energy), it will be yet another triumph for the effectiveness of mathematics.

I started with two basic, interrelated questions: Is mathematics invented or discovered? And what gives mathematics its explanatory and predictive powers? I believe that we know the answer to the first question. Mathematics is an intricate fusion of inventions and discoveries. Concepts are generally invented, and even though all the correct relations among them existed before their discovery, humans still chose which ones to study. The second question turns out to be even more complex. There is no doubt that the selection of topics we address mathematically has played an important role in math's perceived effectiveness. But mathematics would not work at all were there no universal features to be discovered. You may now ask: Why are there universal laws of nature at all? Or equivalently: Why is our universe governed by certain symmetries and by locality? I truly do not know the answers, except to note that perhaps in a universe without these properties, complexity and life would have never emerged, and we would not be here to ask the question. SA

MORE TO EXPLORE

The Unreasonable Effectiveness of Mathematics in the Natural Sciences. Eugene Wigner in *Communications in Pure and Applied Mathematics*, Vol. 13, No. 1, pages 1–14; February 1960.

Pi in the Sky: Counting, Thinking, and Being. John D. Barrow. Back Bay Books, 1992.

Creation v. Discovery. Michael Atiyah in *Times Higher Education Supplement*; September 29, 1995.

Is God a Mathematician? Mario Livio. Simon & Schuster, 2010.

SCIENTIFIC AMERICAN ONLINE

Is mathematics invented, discovered, both or neither? See examples of remarkable mathematical structures that invite this question at ScientificAmerican.com/aug11/livio

EDWARD CHARLES LE GRICE Getty Images

Isolated Systems and Reductionism

To avoid having to study everything at once scientists isolate things they are trying to study

Subdividing the universe into smaller and smaller part

The method of splitting things into smaller and smaller parts

↓

studying how those parts influence each others

reductionism

The hope is that seemingly complex rules governing the ~~smaller~~ larger units can be better understood in terms of simple rules governing the smaller part

↓

past experience show this approach has been successful.

定律 (Law)

定理 (Theorem)

定義 (Definition)

歸納 (Induction)
Key: Limit of Applicability

演繹 (Deduction)
Key: validity

Key: operational 操作性
precision 準確
useful.

Examples

1–5 Converting Units

Any quantity we measure, such as a length, a speed, or an electric current, consists of a number *and* a unit. Often we are given a quantity in one set of units, but we want it expressed in another set of units. For example, suppose we measure that a table is 21.5 inches wide, and we want to express this in centimeters. We must use a **conversion factor**, which in this case is (by definition) exactly

$$1\text{ in.} = 2.54\text{ cm}$$

or, written another way,

$$1 = 2.54\text{ cm/in.}$$

Since multiplying by one does not change anything, the width of our table, in cm, is

$$21.5\text{ inches} = (21.5\,\cancel{\text{in.}}) \times \left(2.54\,\frac{\text{cm}}{\cancel{\text{in.}}}\right) = 54.6\text{ cm}.$$

Note how the units (inches in this case) cancelled out. A Table containing many unit conversions is found inside the front cover of this book. Let's consider some Examples.

PHYSICS APPLIED
The world's tallest peaks

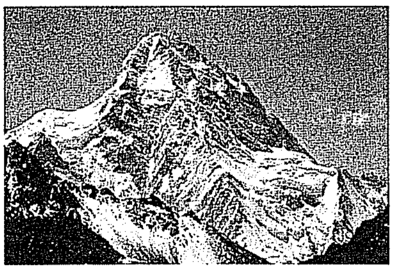


FIGURE 1–6 The world's second highest peak, K2, whose summit is considered the most difficult of the "8000-ers." K2 is seen here from the north (China).

TABLE 1–6
The 8000-m Peaks

Peak	Height (m)
Mt. Everest	8850
K2	8611
Kangchenjunga	8586
Lhotse	8516
Makalu	8462
Cho Oyu	8201
Dhaulagiri	8167
Manaslu	8156
Nanga Parbat	8125
Annapurna	8091
Gasherbrum I	8068
Broad Peak	8047
Gasherbrum II	8035
Shisha Pangma	8013

EXAMPLE 1–2 **The 8000-m peaks.** The fourteen tallest peaks in the world (Fig. 1–6 and Table 1–6) are referred to as "eight-thousanders," meaning their summits are over 8000 m above sea level. What is the elevation, in feet, of an elevation of 8000 m?

APPROACH We need simply to convert meters to feet, and we can start with the conversion factor 1 in. = 2.54 cm, which is exact. That is, 1 in. = 2.5400 cm to any number of significant figures, because it is *defined* to be.

SOLUTION One foot is 12 in., so we can write

$$1\text{ ft} = (12\,\cancel{\text{in.}})\left(2.54\,\frac{\text{cm}}{\cancel{\text{in.}}}\right) = 30.48\text{ cm} = 0.3048\text{ m},$$

which is exact. Note how the units cancel (colored slashes). We can rewrite this equation to find the number of feet in 1 meter:

$$1\text{ m} = \frac{1\text{ ft}}{0.3048} = 3.28084\text{ ft}.$$

We multiply this equation by 8000.0 (to have five significant figures):

$$8000.0\text{ m} = (8000.0\,\cancel{\text{m}})\left(3.28084\,\frac{\text{ft}}{\cancel{\text{m}}}\right) = 26{,}247\text{ ft}.$$

An elevation of 8000 m is 26,247 ft above sea level.

NOTE We could have done the conversion all in one line:

$$8000.0\text{ m} = (8000.0\,\cancel{\text{m}})\left(\frac{100\,\cancel{\text{cm}}}{1\,\cancel{\text{m}}}\right)\left(\frac{1\,\cancel{\text{in.}}}{2.54\,\cancel{\text{cm}}}\right)\left(\frac{1\text{ ft}}{12\,\cancel{\text{in.}}}\right) = 26{,}247\text{ ft}.$$

The key is to multiply conversion factors, each equal to one (= 1.0000), and to make sure the units cancel.

EXERCISE E There are only 14 eight-thousand-meter peaks in the world (see Example 1–2), and their names and elevations are given in Table 1–6. They are all in the Himalaya mountain range in India, Pakistan, Tibet, and China. Determine the elevation of the world's three highest peaks in feet.

EXAMPLE 1–3 **Apartment area.** You have seen a nice apartment whose floor area is 880 square feet (ft^2). What is its area in square meters?

APPROACH We use the same conversion factor, 1 in. = 2.54 cm, but this time we have to use it twice.

SOLUTION Because 1 in. = 2.54 cm = 0.0254 m, then $1\,ft^2 = (12\,in.)^2(0.0254\,m/in.)^2 = 0.0929\,m^2$. So $880\,ft^2 = (880\,ft^2)(0.0929\,m^2/ft^2) \approx 82\,m^2$.

NOTE As a rule of thumb, an area given in ft^2 is roughly 10 times the number of square meters (more precisely, about 10.8×).

EXAMPLE 1–4 **Speeds.** Where the posted speed limit is 55 miles per hour (mi/h or mph), what is this speed (*a*) in meters per second (m/s) and (*b*) in kilometers per hour (km/h)?

APPROACH We again use the conversion factor 1 in. = 2.54 cm, and we recall that there are 5280 ft in a mile and 12 inches in a foot; also, one hour contains (60 min/h) × (60 s/min) = 3600 s/h.

SOLUTION (*a*) We can write 1 mile as

$$1\,\text{mi} = (5280\,\cancel{\text{ft}})\left(12\,\frac{\cancel{\text{in.}}}{\cancel{\text{ft}}}\right)\left(2.54\,\frac{\cancel{\text{cm}}}{\cancel{\text{in.}}}\right)\left(\frac{1\,\text{m}}{100\,\cancel{\text{cm}}}\right) = 1609\,\text{m}.$$

We also know that 1 hour contains 3600 s, so

$$55\,\frac{\text{mi}}{\text{h}} = \left(55\,\frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\right)\left(1609\,\frac{\text{m}}{\cancel{\text{mi}}}\right)\left(\frac{1\,\cancel{\text{h}}}{3600\,\text{s}}\right) = 25\,\frac{\text{m}}{\text{s}},$$

where we rounded off to two significant figures.

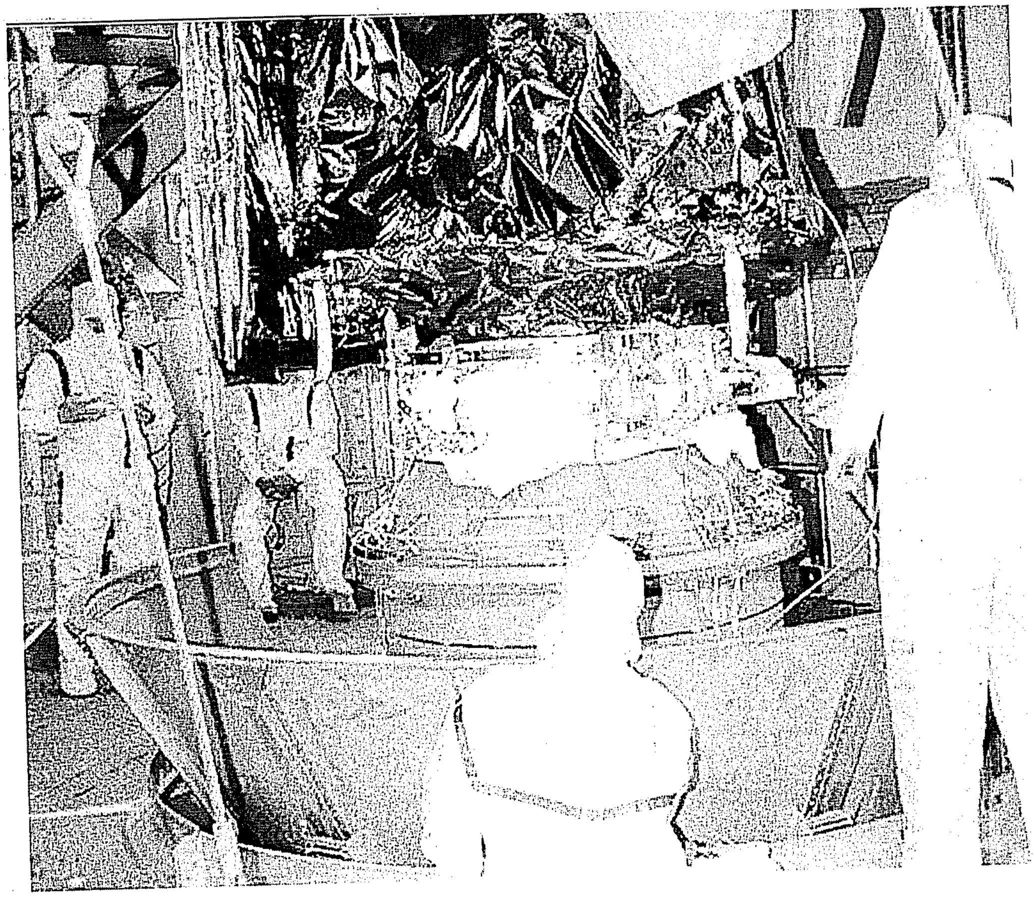
(*b*) Now we use 1 mi = 1609 m = 1.609 km; then

$$55\,\frac{\text{mi}}{\text{h}} = \left(55\,\frac{\cancel{\text{mi}}}{\text{h}}\right)\left(1.609\,\frac{\text{km}}{\cancel{\text{mi}}}\right) = 88\,\frac{\text{km}}{\text{h}}.$$

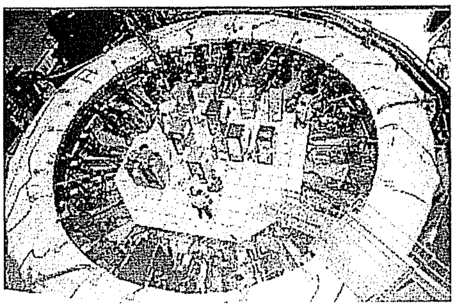
NOTE Each conversion factor is equal to one. You can look up most conversion factors in the Table inside the front cover.

EXERCISE F Would a driver traveling at 15 m/s in a 35 mi/h zone be exceeding the speed limit?

When changing units, you can avoid making an error in the use of conversion factors by checking that units cancel out properly. For example, in our conversion of 1 mi to 1609 m in Example 1–4(*a*), if we had incorrectly used the factor $\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)$ instead of $\left(\frac{1\,\text{m}}{100\,\text{cm}}\right)$, the centimeter units would not have cancelled out; we would not have ended up with meters.



The Mars Climate Orbiter is prepared for its mission. The laws of physics are the same everywhere, even on Mars, so the probe could be designed based on the laws of physics as discovered on earth. There is unfortunately another reason why this spacecraft is relevant to the topics of this chapter: it was destroyed attempting to enter Mars' atmosphere because engineers at Lockheed Martin forgot to convert data on engine thrusts from pounds into the metric unit of force (newtons) before giving the information to NASA. Conversions are important!



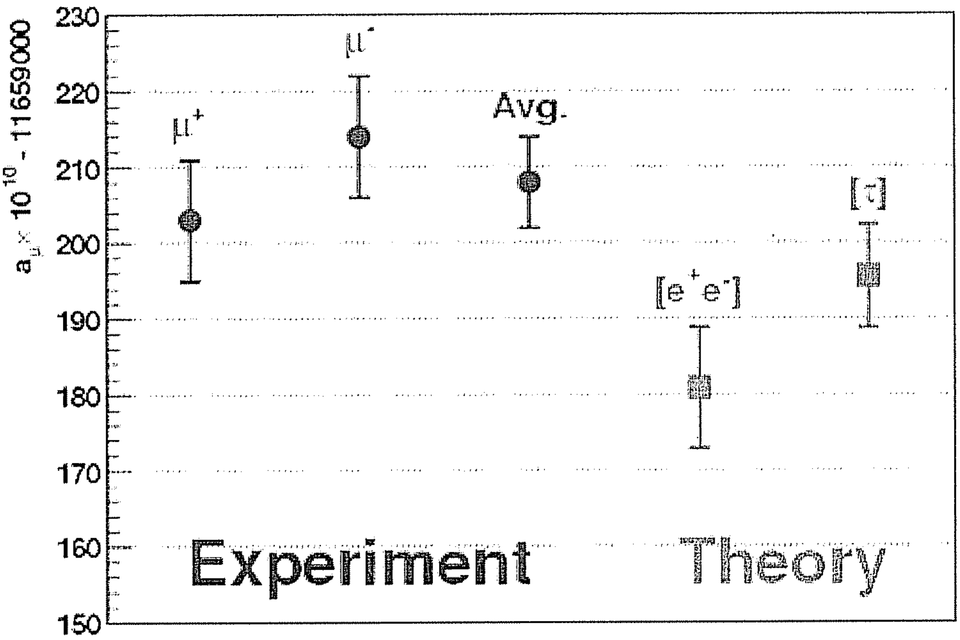
Significant Figure

The E821 Muon (g-2) Home Page

- News & Information
- The Physics of g-2
- Publications
- Collaboration List
- g-2 Pictures
- Expt. Documentation
- Run Schedule and Meetings
- Job Opportunities
- Visitor Information
- Future Experiments
- Other g-2 Web Pages
- Internal

The (g-2) value of the negative muon was announced January 8, 2004!

a_{μ^-} (BNL'01) = 11 659 214 (8)(3) × 10^{-10} (0.7 ppm)



a_μ (exp) = 11 659 208 (6) × 10^{-10} (0.5 ppm)

☆ Our paper (.ps,.pdf) based on the 2001 data set has been published in Physical Review Letters 92; 1618102 (2004).
☆ Read the BNL Press Release of Thursday, January 8, 2004 here.
☆ The webcast of the announcement may be found on this page.

The Muon (g-2) Experiment at Brookhaven National Laboratory is stringently testing the Standard Model by measuring the Anomalous Magnetic Moment of the Muon to unprecedented precision.

The experiment is run by an international collaboration of more than 60 physicists from 11 institutes in the United States, Germany, The Netherlands, Russia and Japan.

To learn about the physics of the g-2 measurement, click here.

You are visitor number [an error occurred while processing this directive]
Web contact: Ernst Sichtermann

Privacy and Security Notice

$$\mu = \mu_B \, g \, s \quad (s \downarrow \tfrac{1}{2})$$

$$a_{\mu^\pm} = \frac{g-2}{2}$$

$$\mu_B = \frac{e\hbar}{mc}$$

分類：
編號：
總號：

Order of Magnitude Estimate

1–6 Order of Magnitude: Rapid Estimating

We are sometimes interested only in an approximate value for a quantity. This might be because an accurate calculation would take more time than it is worth or would require additional data that are not available. In other cases, we may want to make a rough estimate in order to check an accurate calculation made on a calculator, to make sure that no blunders were made when the numbers were entered.

A rough estimate is made by rounding off all numbers to one significant figure and its power of 10, and after the calculation is made, again only one significant figure is kept. Such an estimate is called an **order-of-magnitude estimate** and can be accurate within a factor of 10, and often better. In fact, the phrase "order of magnitude" is sometimes used to refer simply to the power of 10.

SOLVED PROBLEM 1.2 View from the Willis Tower

PROBLEM

It goes without saying that one can see farther from a tower than from ground level; the higher the tower, the farther one can see. The Willis Tower in Chicago has an observation deck, which is 412 m above ground. How far can one see out over Lake Michigan from this observation deck under perfect weather conditions? (Assume eye level is at 413 m above the level of the lake.)

SOLUTION

THINK

As we have stressed before, this is the most important step in the problem-solving process. A little preparation at this stage can save a lot of work at a later stage. Perfect weather conditions are specified, so fog or haze is not a limiting factor. What else could determine how far one can see? If the air is clear, one can see mountains that are quite far away. Why mountains? Because they are very tall. But the landscape around Chicago is flat. What then could limit the viewing range? Nothing, really; one can see all the way to the horizon. And what is the deciding factor for where the horizon is? It is the curvature of the Earth. Let's make a sketch to make this a little clearer.

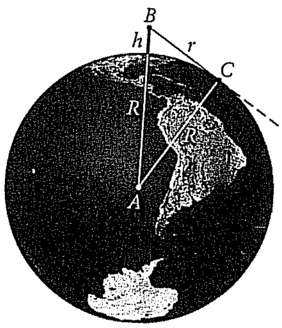


FIGURE 1.10 Distance from the top of the Willis Tower (B) to the horizon (C).

SKETCH

Our sketch does not have to be elaborate, but it needs to show a simple version of the Willis Tower on the surface of the Earth. It is not important that the sketch be to scale, and we elect to greatly exaggerate the height of the tower relative to the size of the Earth. See Figure 1.10.

It seems obvious from this sketch that the farthest point (point C) that one can see from the top of the Willis Tower (point B) is where the line of sight just touches the surface of the Earth tangentially. Any point on Earth's surface farther away from the Willis Tower is hidden from view (below the dashed line segment). The viewing range is then given by the distance r between that surface point C and the observation deck (point B) on top of the tower, at height h. Included in the sketch is also a line from the center of Earth (point A) to the foot of the Willis Tower. It has length R, which is the radius of Earth. Another line of the same length, R, is drawn to the point where the line of sight touches the Earth's surface tangentially.

RESEARCH

As you can see from the sketch, a line drawn from the center of the Earth to the point where the line of sight touches the surface (A to C) will form a right angle with that line of sight (B to C); that is, the three points A, B, and C form the corners of a right triangle. This is the key insight, which enables us to use trigonometry and the Pythagorean theorem to attack the solution of this problem. Examining the sketch in Figure 1.10, we find

$$r^2 + R^2 = (R+h)^2.$$

SIMPLIFY

Remember, we want to find the distance to the horizon, for which we used the symbol r in the previous equation. Isolating that variable on one side of our equation gives

$$r^2 = (R+h)^2 - R^2.$$

Now we can simplify the square and obtain

$$r^2 = R^2 + 2hR + h^2 - R^2 = 2hR + h^2.$$

Finally, we take the square root and obtain our final algebraic answer:

$$r = \sqrt{2hR + h^2}.$$

CALCULATE

Now we are ready to insert numbers. The accepted value for the radius of Earth is $R = 6370$ km $= 6.37 \cdot 10^6$ m, and $h = 413$ m $= 4.13 \cdot 10^2$ m was given in the problem. This leads to

$$r = \sqrt{2(4.13 \cdot 10^2 \text{ m})(6.37 \cdot 10^6 \text{ m}) + (4.13 \cdot 10^2 \text{ m})^2} = 7.25382 \cdot 10^4 \text{ m}.$$

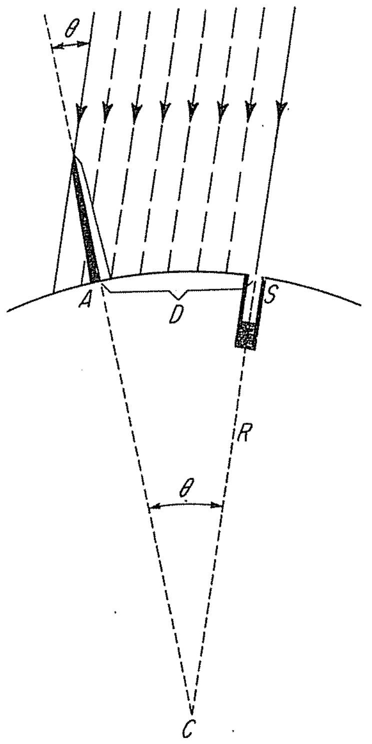


Fig. 2–3. *Measuring the Radius of the Earth.* In this diagram we assume that Alexandria (A) and Syene (S) are on the same meridian line and that Syene is exactly on the Tropic of Cancer. In these circumstances at noon of the summer solstice the sun's rays are vertical at Syene, whereas they make a measurable angle θ with the vertical at Alexandria. The angle θ is evidently equal to the angle subtended at the center of the earth (C) by the arc AS.

1.3. The change in the direction of the zenith as the traveler journeys to north or south is well adapted to a measurement of the actual radius of the earth. If the earth's surface is spherical, verticals passing through any two points A and S on that surface should meet at the center of the earth, C (Fig. 2–3). Let θ denote the angle in degrees between the verticals and let D denote the distance from A to S, i.e., the length of the circular arc subtended by θ at the earth's surface. The ratio of D to the circumference of the earth must be equal to the ratio of θ to 360°. Denoting the radius of the earth by R, we compute its value by means of the simple proportion

$$\frac{2\pi R}{D} = \frac{360}{\theta}. \tag{2.1}$$

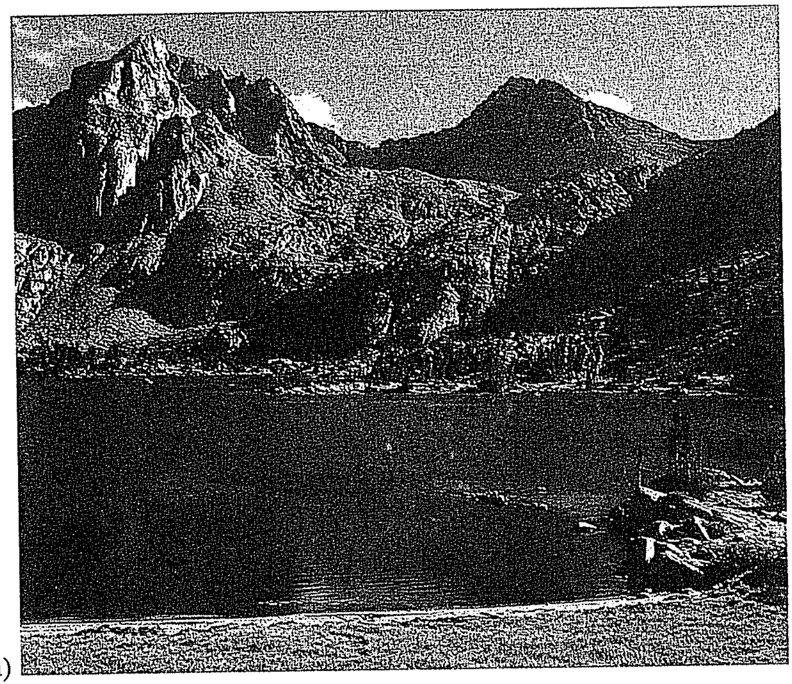
If A is due north of S, the two verticals are in a common meridian plane. To measure the angle between them we have only to compare the angles which they make with rays of light from any heavenly body as it crosses the meridian. (In making the first measurements of the radius R it was necessary to assume that *the distances of the sun and stars from the earth are so great in comparison with its size that rays from any of these heavenly sources incident at different points on the earth's surface can be treated as parallel.* This crucially important assumption is fully justified by modern measurements of the distances to the sun and stars.)

One of the first estimates of R was made by Eratosthenes, an Alexandrian scientist contemporary to Archimedes, about 230 B.C. He estimated the distance from Alexandria (A) to Syene (S), supposedly on the Tropic of Cancer due south of Alexandria, to be 5000 "stadia." At the summer solstice the sun was reflected vertically from the water in a very deep well in Syene and thus was almost directly overhead, but at Alexandria, according to Eratosthenes, it was one fiftieth part of 360° south of the zenith. In other words, his value of θ, measured presumably by a noon shadow, was 360/50, or 7.2°. On this basis the circumference of the earth is 250,000 stadia. There has been uncertainty regarding the length to be assigned to the stadium as used by Eratosthenes, since the Greek and Egyptian values differed. However, if we follow Dreyer[4] and call it 516.73 feet, the circumference of the earth as measured works out to be 24,460 miles. The corresponding radius is 3890 miles. Modern measurements give 24,860 and 3957 miles, respectively, for these distances. The agreement is startling, but largely accidental. Later, and perhaps more careful, measurements did not agree very well.[5]

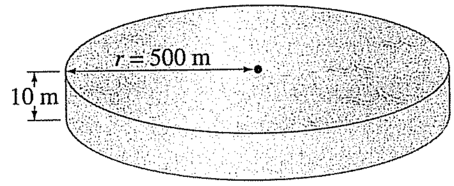
[3] J. L. E. Dreyer, *op. cit.*, pp. 117–118.

[4] J. L. E. Dreyer, *op. cit.*, p. 175.

[5] According to Samuel Eliot Morison, the mistake of Columbus in underestimating the size of the earth was due to a misinterpretation of the unit used in an estimate made by Arabian geographers in the ninth century and published by Alfragan. See S. E. Morison, *Admiral of the Ocean Sea* (Little, Brown and Co., Boston, 1942), Vol. 1, p. 87.



(a)



(b)

FIGURE 1–7 Example 1–5. (a) How much water is in this lake? (Photo is of one of the Rae Lakes in the Sierra Nevada of California.) (b) Model of the lake as a cylinder. [We could go one step further and estimate the mass or weight of this lake. We will see later that water has a density of $1000\ \mathrm{kg/m^3}$, so this lake has a mass of about $(10^3\ \mathrm{kg/m^3})(10^7\ \mathrm{m^3}) \approx 10^{10}\ \mathrm{kg}$, which is about 10 billion kg or 10 million metric tons. (A metric ton is 1000 kg, about 2200 lbs, slightly larger than a British ton, 2000 lbs.)]

PHYSICS APPLIED
Estimating the volume (or mass) of a lake; see also Fig. 1–7

EXAMPLE 1–5 **ESTIMATE** **Volume of a lake.** Estimate how much water there is in a particular lake, Fig. 1–7a, which is roughly circular, about 1 km across, and you guess it has an average depth of about 10 m.

APPROACH No lake is a perfect circle, nor can lakes be expected to have a perfectly flat bottom. We are only estimating here. To estimate the volume, we can use a simple model of the lake as a cylinder: we multiply the average depth of the lake times its roughly circular surface area, as if the lake were a cylinder (Fig. 1–7b).

SOLUTION The volume V of a cylinder is the product of its height h times the area of its base: $V = h\pi r^2$, where r is the radius of the circular base.† The radius r is $\frac{1}{2}\ \mathrm{km} = 500\ \mathrm{m}$, so the volume is approximately

$$V = h\pi r^2 \approx (10\ \mathrm{m}) \times (3) \times (5 \times 10^2\ \mathrm{m})^2 \approx 8 \times 10^6\ \mathrm{m^3} \approx 10^7\ \mathrm{m^3},$$

where π was rounded off to 3. So the volume is on the order of $10^7\ \mathrm{m^3}$, ten million cubic meters. Because of all the estimates that went into this calculation, the order-of-magnitude estimate $(10^7\ \mathrm{m^3})$ is probably better to quote than the $8 \times 10^6\ \mathrm{m^3}$ figure.

NOTE To express our result in U.S. gallons, we see in the Table on the inside front cover that $1\ \text{liter} = 10^{-3}\ \mathrm{m^3} \approx \frac{1}{4}\ \text{gallon}$. Hence, the lake contains $(8 \times 10^6\ \mathrm{m^3})(1\ \text{gallon}/4 \times 10^{-3}\ \mathrm{m^3}) \approx 2 \times 10^9$ gallons of water.

PROBLEM SOLVING
Use symmetry when possible

EXAMPLE 1–6 **ESTIMATE** **Thickness of a page.** Estimate the thickness of a page of this book.

APPROACH At first you might think that a special measuring device, a micrometer (Fig. 1–8), is needed to measure the thickness of one page since an ordinary ruler clearly won't do. But we can use a trick or, to put it in physics terms, make use of a *symmetry*: we can make the reasonable assumption that all the pages of this book are equal in thickness.

SOLUTION We can use a ruler to measure hundreds of pages at once. If you measure the thickness of the first 500 pages of this book (page 1 to page 500), you might get something like 1.5 cm. Note that 500 numbered pages,

†Formulas like this for volume, area, etc., are found inside the back cover of this book.

counted front and back, is 250 separate sheets of paper. So one page must have a thickness of about

$$\frac{1.5\text{ cm}}{250\text{ pages}} \approx 6 \times 10^{-3}\text{ cm} = 6 \times 10^{-2}\text{ mm},$$

or less than a tenth of a millimeter (0.1 mm).

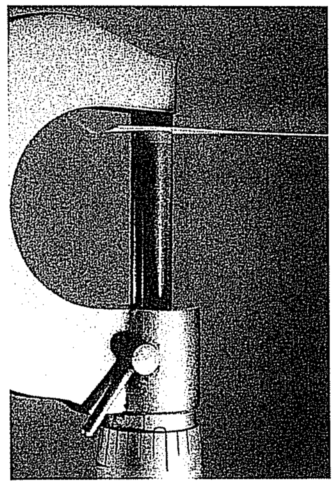


FIGURE 1–8 Example 1–6. Micrometer used for measuring small thicknesses.

EXAMPLE 1–7 **ESTIMATE** **Height by triangulation.** Estimate the height of the building shown in Fig. 1–9, by "triangulation," with the help of a bus-stop pole and a friend.

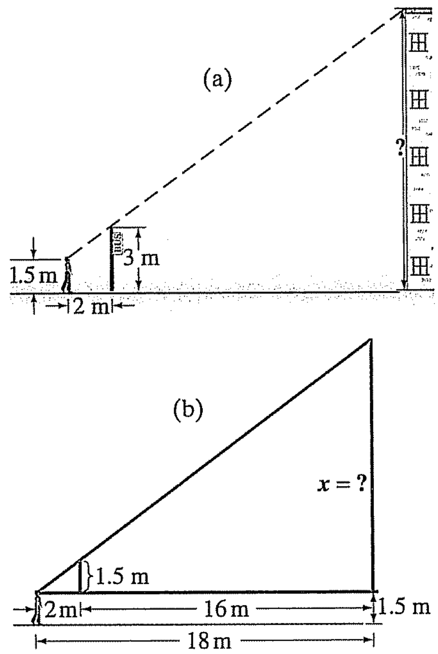
APPROACH By standing your friend next to the pole, you estimate the height of the pole to be 3 m. You next step away from the pole until the top of the pole is in line with the top of the building, Fig. 1–9a. You are 5 ft 6 in. tall, so your eyes are about 1.5 m above the ground. Your friend is taller, and when she stretches out her arms, one hand touches you, and the other touches the pole, so you estimate that distance as 2 m (Fig. 1–9a). You then pace off the distance from the pole to the base of the building with big, 1-m-long steps, and you get a total of 16 steps or 16 m.

SOLUTION Now you draw, to scale, the diagram shown in Fig. 1–9b using these measurements. You can measure, right on the diagram, the last side of the triangle to be about $x = 13$ m. Alternatively, you can use similar triangles to obtain the height x:

$$\frac{1.5\text{ m}}{2\text{ m}} = \frac{x}{18\text{ m}}, \quad \text{so} \quad x \approx 13\tfrac{1}{2}\text{ m}.$$

Finally you add in your eye height of 1.5 m above the ground to get your final result: the building is about 15 m tall.

FIGURE 1–9 Example 1–7. Diagrams are really useful!



EXAMPLE 1–8 **ESTIMATE** **Estimating the radius of Earth.** Believe it or not, you can estimate the radius of the Earth without having to go into space (see the photograph on page 1). If you have ever been on the shore of a large lake, you may have noticed that you cannot see the beaches, piers, or rocks at water level across the lake on the opposite shore. The lake seems to bulge out between you and the opposite shore—a good clue that the Earth is round. Suppose you climb a stepladder and discover that when your eyes are 10 ft (3.0 m) above the water, you can just see the rocks at water level on the opposite shore. From a map, you estimate the distance to the opposite shore as $d \approx 6.1$ km. Use Fig. 1–10 with $h = 3.0$ m to estimate the radius R of the Earth.

APPROACH We use simple geometry, including the theorem of Pythagoras, $c^2 = a^2 + b^2$, where c is the length of the hypotenuse of any right triangle, and a and b are the lengths of the other two sides.

SOLUTION For the right triangle of Fig. 1–10, the two sides are the radius of the Earth R and the distance $d = 6.1\text{ km} = 6100\text{ m}$. The hypotenuse is approximately the length $R + h$, where $h = 3.0$ m. By the Pythagorean theorem,

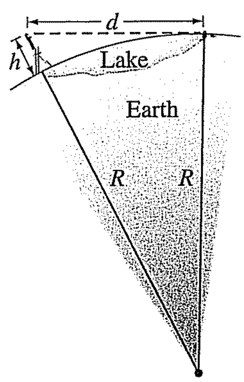
$$\begin{aligned} R^2 + d^2 &\approx (R + h)^2 \\ &\approx R^2 + 2hR + h^2. \end{aligned}$$

We solve algebraically for R, after cancelling R^2 on both sides:

$$R \approx \frac{d^2 - h^2}{2h} = \frac{(6100\text{ m})^2 - (3.0\text{ m})^2}{6.0\text{ m}} = 6.2 \times 10^6\text{ m} = 6200\text{ km}.$$

NOTE Precise measurements give 6380 km. But look at your achievement! With a few simple rough measurements and simple geometry, you made a good estimate of the Earth's radius. You did not need to go out in space, nor did you need a very long measuring tape. Now you know the answer to the Chapter-Opening Question on p. 1.

FIGURE 1–10 Example 1–8, but not to scale. You can see small rocks at water level on the opposite shore of a lake 6.1 km wide if you stand on a stepladder.



* 1–7 Dimensions and Dimensional Analysis

When we speak of the **dimensions** of a quantity, we are referring to the type of base units or base quantities that make it up. The dimensions of area, for example, are always length squared, abbreviated $[L^2]$, using square brackets; the units can be square meters, square feet, cm^2, and so on. Velocity, on the other hand, can be measured in units of km/h, m/s, or mi/h, but the dimensions are always a length $[L]$ divided by a time $[T]$: that is, $[L/T]$.

The formula for a quantity may be different in different cases, but the dimensions remain the same. For example, the area of a triangle of base b and height h is $A = \frac{1}{2}bh$, whereas the area of a circle of radius r is $A = \pi r^2$. The formulas are different in the two cases, but the dimensions of area are always $[L^2]$.

Dimensions can be used as a help in working out relationships, a procedure referred to as **dimensional analysis**. One useful technique is the use of dimensions to check if a relationship is *incorrect*. Note that we add or subtract quantities only if they have the same dimensions (we don't add centimeters and hours); and the quantities on each side of an equals sign must have the same dimensions. (In numerical calculations, the units must also be the same on both sides of an equation.)

For example, suppose you derived the equation $v = v_0 + \frac{1}{2}at^2$, where v is the speed of an object after a time t, v_0 is the object's initial speed, and the object undergoes an acceleration a. Let's do a dimensional check to see if this equation

†A check of the San Francisco Yellow Pages (done after this calculation) reveals about 50 listings. Each of these listings may employ more than one tuner, but on the other hand, each may also do repairs as well as tuning. In any case, our estimate is reasonable.

*Some Sections of this book, such as this one, may be considered *optional* at the discretion of the instructor, and they are marked with an asterisk (*). See the Preface for more details.

could be correct or is surely incorrect. Note that numerical factors, like the $\frac{1}{2}$ here, do not affect dimensional checks. We write a dimensional equation as follows, remembering that the dimensions of speed are $[L/T]$ and (as we shall see in Chapter 2) the dimensions of acceleration are $[L/T^2]$:

$$\left[\frac{L}{T}\right] \stackrel{?}{=} \left[\frac{L}{T}\right] + \left[\frac{L}{T^2}\right][T^2] = \left[\frac{L}{T}\right] + [L].$$

The dimensions are incorrect: on the right side, we have the sum of quantities whose dimensions are not the same. Thus we conclude that an error was made in the derivation of the original equation.

A dimensional check can only tell you when a relationship is wrong. It can't tell you if it is completely right. For example, a dimensionless numerical factor (such as $\frac{1}{2}$ or 2π) could be missing.

Dimensional analysis can also be used as a quick check on an equation you are not sure about. For example, suppose that you can't remember whether the equation for the period of a simple pendulum T (the time to make one back-and-forth swing) of length ℓ is $T = 2\pi\sqrt{\ell/g}$ or $T = 2\pi\sqrt{g/\ell}$, where g is the acceleration due to gravity and, like all accelerations, has dimensions $[L/T^2]$. (Do not worry about these formulas—the correct one will be derived in Chapter 14; what we are concerned about here is a person's recalling whether it contains ℓ/g or g/ℓ.) A dimensional check shows that the former (ℓ/g) is correct:

$$[T] = \sqrt{\frac{[L]}{[L/T^2]}} = \sqrt{[T^2]} = [T],$$

whereas the latter (g/ℓ) is not:

$$[T] \neq \sqrt{\frac{[L/T^2]}{[L]}} = \sqrt{\frac{1}{[T^2]}} = \frac{1}{[T]}.$$

Note that the constant 2π has no dimensions and so can't be checked using dimensions.

Further uses of dimensional analysis are found in Appendix C.

EXAMPLE 1–10 Planck length. The smallest meaningful measure of length is called the "Planck length," and is defined in terms of three fundamental constants in nature, the speed of light $c = 3.00 \times 10^8\,\mathrm{m/s}$, the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3/kg\cdot s^2}$, and Planck's constant $h = 6.63 \times 10^{-34}\,\mathrm{kg\cdot m^2/s}$. The Planck length λ_P (λ is the Greek letter "lambda") is given by the following combination of these three constants:

$$\lambda_P = \sqrt{\frac{Gh}{c^3}}.$$

Show that the dimensions of λ_P are length $[L]$, and find the order of magnitude of λ_P.

APPROACH We rewrite the above equation in terms of dimensions. The dimensions of c are $[L/T]$, of G are $[L^3/MT^2]$, and of h are $[ML^2/T]$.

SOLUTION The dimensions of λ_P are

$$\sqrt{\frac{[L^3/MT^2][ML^2/T]}{[L^3/T^3]}} = \sqrt{[L^2]} = [L]$$

which is a length. The value of the Planck length is

$$\lambda_P = \sqrt{\frac{Gh}{c^3}} = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/kg\cdot s^2})(6.63 \times 10^{-34}\,\mathrm{kg\cdot m^2/s})}{(3.0 \times 10^8\,\mathrm{m/s})^3}} \approx 4 \times 10^{-35}\,\mathrm{m},$$

which is on the order of 10^{-34} or 10^{-35} m.

NOTE Some recent theories (Chapters 43 and 44) suggest that the smallest particles (quarks, leptons) have sizes on the order of the Planck length, 10^{-35} m. These theories also suggest that the "Big Bang," with which the Universe is believed to have begun, started from an initial size on the order of the Planck length.

分類：
編號：
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Scaling

↓

extension of the textbook

*1–7 THE USES OF SCALING

Scaling Laws

In Section 1–3, dimensional analysis was used as a check in problem solving. There are two other important applications of dimensional analysis. In the first of these, we can discover scaling laws. *Scaling laws* reveal how a change in one dimensional parameter of a problem leads to changes in another parameter with the same dimensions. Let us consider a problem that Galileo studied in his book *Dialogue Concerning Two New Sciences*, namely, the scaling properties of animals: What must giants look like? Consider a bone (Fig. 1–20) whose length is ℓ, characterizing the height, or linear size, of an animal. The width of the bone is b. It is a

fact that the strength of such a bone is proportional to its cross-sectional area; that is,

$$\text{strength} = c_1 b^2.$$

The constant factor c_1 is characteristic of the material, and it does not change if the size of the animal changes. This strength must be such as to support the weight or, equivalently, the *mass m* of the animal. But the mass, as long as the animal's composition does not change, is proportional to the *volume*, and the volume is in turn proportional to ℓ^3, just as the volume of a cube is equal to the length of the side cubed. Thus

$$m = c_2 \ell^3.$$

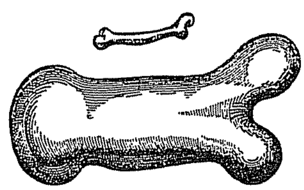


FIGURE 1–20 The scaling of bones. (From Galileo, *Dialogue Concerning Two New Sciences*, 1638.)

The precise value of c_2 depends on the precise shape of the animal and need not concern us. If now the strength of the bone must be proportional to the mass, we have

$$\text{strength} = c_3 m.$$

Combining these three equations, we have

$$c_1 b^2 = c_3 m = c_3 c_2 \ell^3;$$

$$c = \frac{\ell^3}{b^2}, \quad \text{where} \quad c = \frac{c_1}{c_2 c_3}. \qquad (1\text{–}16)$$

The constant c is characteristic of our animal's internal structure, but not of its size.

Now what happens if we try to change the size parameter ℓ by a factor f? The constant c cannot change, so b must change to compensate for ℓ changing. Equation (1–16) can be satisfied when we change ℓ to $f \times \ell$ only if b changes by a factor f'. We find f' by insisting that the ratio ℓ^3/b^2 be constant:

$$\frac{\ell^3}{b^2} = \frac{(f\ell)^3}{(f'b)^2} = \frac{f^3}{(f')^2}\frac{\ell^3}{b^2};$$

$$\frac{f^3}{f'^2} = 1 \quad \text{or} \quad f' = f^{3/2}. \qquad (1\text{–}17)$$

If the length of the bone changes by a factor f, then its width must change by a 16
factor $f^{3/2}$. For example, if the length is changed by a factor 3, as in Fig. 1–20, which comes from Galileo's book, then according to Eq. (1–17), the width of the bone must change by a factor $(3)^{3/2} \simeq 5$. The scaling argument shows that the giant animal cannot have the same proportions as its smaller model (Fig. 1–21) and helps us to understand why hippopotamuses will never be mistaken for overgrown chihuahuas.

Understanding the scaling behavior of physical systems is of primary importance for engineering. Such understanding allows filmmakers to film ripples in a bathtub in slow motion and convince us that they are mighty waves on the ocean, or tells us how the stiffness of a beam varies with its length and width. Scaling arguments tell us how to strengthen structures or how to test small models of large aircraft in wind tunnels.

Relations Based on Dimensional Analysis

The second major application of dimensional analysis is in the derivation of new relations between physical quantities. Consider as an example the simple pen-

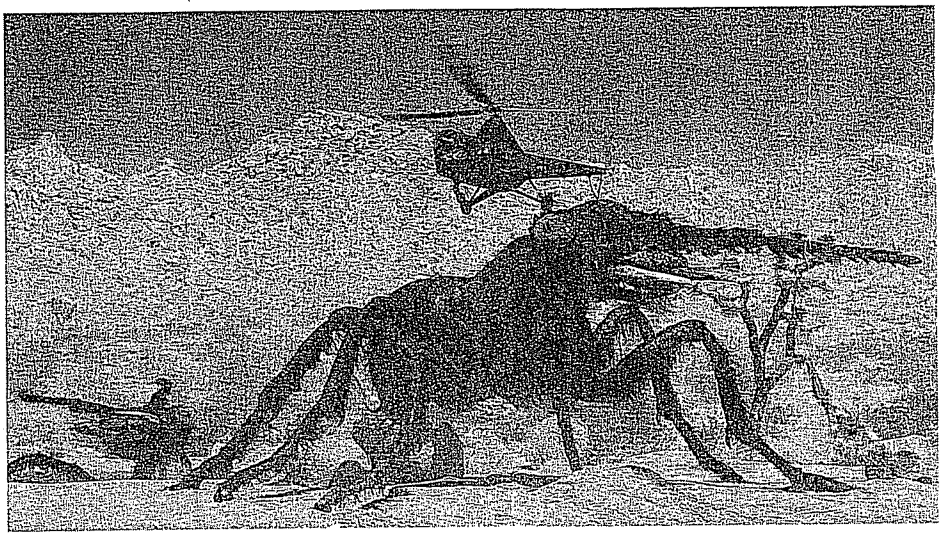


FIGURE 1–21 The giant ants of the 1950s movie *Them* have the same proportions as their normal counterparts do. Galileo's scaling analysis shows that such creatures could not exist, because their structure could not support their weight.

dulum, which consists, in idealized form, of a small bob of mass m on the end of a light string of length ℓ. A pendulum swings because, when it is displaced away from the vertical direction, gravity pulls it back down. It overshoots the minimum, going to the other side, and repeats its motion. One full cycle of this motion takes a time τ or T, which is called the *period*. The problem we pose is this: How does the period depend on the physical parameters of the pendulum?

To answer this question we must gather a list of physical parameters that might be relevant to the period. This requires some knowledge. Could the period of the pendulum depend on the *internal* structure of the bob or of the string? We must know enough to answer no, because changing the string or changing the bob from lead to plastic (of the same mass) does not change the period. Nor do we expect air resistance to play a large role. The list we gather includes the mass m, with dimension $[M]$; the length ℓ of the string, with dimension $[L]$; and the acceleration of gravity, g, discussed in Section 1–4. The latter quantity has dimension $[g] = [LT^{-2}]$. There are no other dimensional quantities on which the period of the pendulum should depend.

The dimension of the period is time $[T]$. We now look for an algebraic combination of m, ℓ, and g that has the dimension of τ. We want to find q, r, and s so that

$$[\tau] = [m^q][\ell^r][g^s],$$

or, in terms of the dimensions,

$$[T] = [M^q][L^r][L^s T^{-2s}].$$

Now there are no powers of $[M]$ on the left-hand side, so $q = 0$, and the mass does not enter at all. In order to match the powers of $[T]$, $s = -\frac{1}{2}$, and then, in order that the powers of $[L]$ on the right-hand side cancel, $r = \frac{1}{2}$. In other words, the algebraic combination $\sqrt{\ell/g}$ is the only combination of the parameters that has the same dimension as that of the period,

$$\tau \propto \sqrt{\frac{\ell}{g}}.$$

(The symbol $\propto$ indicates proportionality.) This is an illuminating result. It does not tell us the dimensionless numerical coefficient by which the square root is

multiplied to give an equality for τ rather than a proportionality. But it does give the dependence of τ on ℓ and g, and it does say that the mass does not enter into the result. One measurement of the period of a pendulum with known length would determine the unknown numerical coefficient. We might even hope that the coefficient is not too far from unity. This is in fact very often the case!

Is some dependence on the angle of the pendulum's swing possible? This angle is dimensionless and so could enter in any way. For large swings of a pendulum, there is indeed further dependence on the angle of the swing, so dimensional analysis is not the whole story. Application of dimensional analysis in this way is not always simple. Enough experience and knowledge of the physical situation is required to know just what quantities are relevant to the problem at hand. Nevertheless, the technique has proven itself throughout the history of science; and many times, when the problem is so complicated that it can be solved only numerically, it is an invaluable tool.

SUMMARY

The range of quantities relevant to our understanding of the physical world is so large that it is useful to employ scientific notation. In this notation, any number can be represented by a decimal number from 1 to 10, multiplied by some power of 10.

The quantities that appear in physics and engineering have units as well as sizes. The International System of units, or SI, provides reproducible and precise definitions of mass, length, and time. The SI units are, respectively, kilograms (kg), meters (m), and seconds (s). Some quantities are used so often that their units are given a special name (for example, force is measured in newtons, N, in SI), but these units are derived units: They can always be expressed in terms of the three primary units (for example, $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$). Units that appear in equations can be manipulated algebraically; after conversion to a single unit system, the units of both sides of any correct equation will match.

Mass, length, and time are the quantities with the three primary dimensions, abbreviated M, L, T. Dimensions should not be confused with units, which refer to a particular choice of unit system. Any physical quantity has dimensions that are rational combinations of the primary dimensions. Dimensions can be manipulated algebraically, and both sides of any correct equation will have the same dimensions. When we analyze the dimensions of an equation, we are performing a dimensional analysis. Dimensional analysis is useful for checking the answers to problems, for learning scaling laws, and for discovering relations between physical quantities.

Numbers that represent physical quantities can be measured only to a certain accuracy. An explicit way to indicate this accuracy is to write a physical quantity x as a central value $\pm$ an uncertainty. Calculations involving physical quantities are meaningful only to within the known accuracy of those quantities. When several numbers of different accuracies are involved in a calculation, the least accurate quantity is the primary determinant of the accuracy of the result. A second way to indicate the known accuracy of a physical quantity is with the number of significant figures with which it is expressed. This is the number of digits between 1 and 10 by which the power of 10 in scientific notation is multiplied.

The ability to estimate is one that should be cultivated. An educated first

guess is a valuable start to the solution of any problem. An order-of-magnitude calculation is such a guess. Similarly, when you arrive at an answer to a physical problem, it is always wise to ask yourself if it makes sense.

Some, but not all, physical quantities include directional information. Temperature and time do not. Such quantities are scalars: They have magnitude (including the appropriate dimensions and units) only. Displacement and velocity do include directional information; such quantities are represented by vectors. The vectors **A** and **B** are mathematical objects with both magnitude and direction. They obey the rule

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{1-2}$$

Vectors can be expressed in graphical form; we draw them as arrows of length equal to their magnitude within a particular coordinate system. The simplest such system is the Cartesian system, with mutually perpendicular x-, y-, and z-axes for three-dimensional space. We can express any vector **V** in terms of the unit vectors, vectors of unit length, for a given coordinate system. Thus the unit vectors **i**, **j**, and **k** point along the x-, y-, and z-axes, respectively. Then

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}. \tag{1-14}$$

The quantities V_x, V_y, and V_z are the components of **V**. The magnitude of **V** is

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}. \tag{1-15}$$

Vector equations are relations that equate different vectors. In such an equation, the components of the vectors are equal on both sides, so that in three dimensions a vector equation stands for three separate equations.

Scaling: Why Giants Don't Exist

Michael Fowler, UVa 10/12/06

Galileo begins "*Two New Sciences*" with the striking observation that if two ships, one large and one small, have identical proportions and are constructed of the same materials, so that one is purely a scaled up version of the other in every respect, nevertheless the larger one will require proportionately more scaffolding and support on launching to prevent its breaking apart under its own weight. He goes on to point out that similar considerations apply to animals, the larger ones being more vulnerable to stress from their own weight (page 4):

> Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? ... and just as smaller animals are proportionately stronger and more robust than the larger, so also smaller plants are able to stand up better than the larger. I am certain you both know that an oak two hundred cubits high would not be able to sustain its own branches if they were distributed as in a tree of ordinary size; and that nature cannot produce a horse as large as twenty ordinary horses or a giant ten times taller than an ordinary man unless by miracle or by greatly altering the proportions of his limbs and especially his bones, which would have to be considerably enlarged over the ordinary.

For more of the text, click here.

To see what Galileo is driving at here, consider a chandelier lighting fixture, with bulbs and shades on a wooden frame suspended from the middle of the ceiling by a thin rope, just sufficient to take its weight (taking the electrical supply wires to have negligible strength for this purpose). Suppose you like the design of this particular fixture, and would like to make an exactly similar one for a room twice as large in every dimension. The obvious approach is simply to double the dimensions of all components. Assuming essentially all the weight is in the wooden frame, its height, length and breadth will all be doubled, so its volume—and hence its weight—will increase eightfold. Now think about the rope between the chandelier and the ceiling. The new rope will be eight times bigger than the old rope just as the wooden frame was. But the weight-bearing capacity of a uniform rope does *not* depend on its length (unless it is so long that its own weight becomes important, which we take not to be the case here). How much weight a rope of given material will bear depends on the *cross-sectional area* of the rope, which is just a count of the number of rope fibers available to carry the weight. The crucial point is that if the rope has all its dimensions doubled, this cross-sectional area, and hence its weight-carrying capacity, is only increased *fourfold*. Therefore, the doubled rope will not be able to hold up the doubled chandelier, the weight of which increased eightfold. For the chandelier to stay up, it will be necessary to use a new rope which is considerably fatter than that given by just doubling the dimensions of the original rope.

This same problem arises when a weight is supported by a pillar of some kind. If enough weight is piled on to a stone pillar, it begins to crack and crumble. For a uniform material, the weight it can carry is proportional to the cross-sectional area. Thinking about doubling all the dimensions of a stone building supported on stone pillars, we see that the weights are all increased eightfold, but the supporting capacities only go up fourfold. Obviously, there is a definite limit to how many times the dimensions can be doubled and we still have a stable building.

As Galileo points out, this all applies to animals and humans too (page 130): "(large) increase in height can be accomplished only by employing a material which is harder and stronger than usual, or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animals suggests a monstrosity."

He even draws a picture:

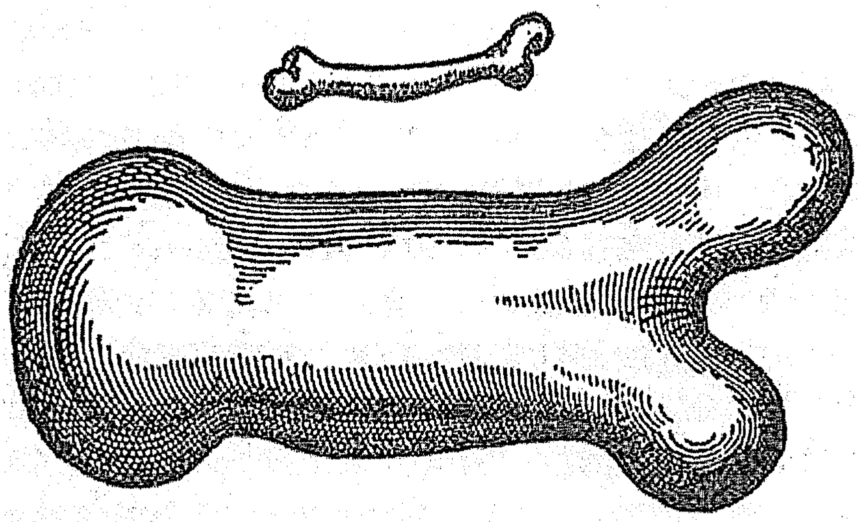


Fig. 27

Galileo understood that you cannot have a creature looking a lot like an ordinary gorilla except that it's sixty feet high. What about Harry Potter's friend Hagrid? Apparently he's twice normal height (according to the book) and three times normal width (although he doesn't look it on this link). But even that's not enough extra width (if the bone width is in proportion).

There is a famous essay on this point by the biologist J. B. S. Haldane, in which he talks of the more venerable giants in Pilgrim's Progress, who were ten times bigger than humans in every dimension, so their weight would have been a thousand times larger, say eighty tons or so. As Haldane says, their thighbones would only have a hundred times the cross section of a human thighbone, which is known to break if stressed by ten times the weight it normally carries. So these giants would break their thighbones on their first step. Or course, big creatures could get around this if they could evolve a stronger skeletal material, but so far this hasn't happened.

Another example of the importance of size used by Galileo comes from considering a round stone falling through water at its terminal speed. What happens if we consider a stone of the same material and shape, but one-tenth the radius? It falls much more slowly. Its weight is down by a factor of one-thousand, but the surface area, which gives rise to the frictional retardation, is only down by a factor of one hundred. Thus a fine powder in water---mud, in other words---may take days to settle, even though a stone of the same material will fall the same distance in a second or two. The point here is that as we look on smaller scales, gravity becomes less and less important compared with viscosity, or air resistance—this is why an insect is not harmed by falling from a tree.

This ratio of surface area to volume has also played a crucial role in evolution, as pointed out by Haldane. Almost all life is made up of cells which have quite similar oxygen requirements. A microscopic creature, such as the tiny worm *rotifer*, absorbs oxygen over its entire surface, and the oxygen rapidly diffuses to all the cells. As larger creatures evolved, if the shape stayed the same more or less, the surface area went down relative to the volume, so it became more difficult to absorb enough oxygen. Insects, for example, have many tiny blind tubes over the surface of their bodies which air enters and diffuses into finer tubes to reach all parts of the body. The limitations on how well air will diffuse are determined by the properties of air, and diffusion beyond a quarter-inch or so takes a long time, so this limits the size of insects. Giant ants like those in the old movie "*Them*" wouldn't be able to breathe!

The evolutionary breakthrough to larger size animals came with the development of blood circulation as a means of distributing oxygen (and other nutrients). Even so, for animals of our size, there has to be a tremendous surface area available for oxygen absorption. This was achieved by the development of lungs—the lungs of an adult human have a surface area of a hundred square meters approximately. Going back to the microscopic worm rotifer, it has a simple straight tube gut to absorb nutrients from food. Again, if larger creatures have about the same requirements per cell, and the gut surface absorbs nutrients at the same rate, problems arise because the surface area of the gut increases more slowly than the number of cells needing to be fed as the size of the creature is increased. this problem is handled by replacing the straight tube gut by one with many convolutions, in which also the smooth surface is replaced by one with many tiny folds to increase surface area. Thus many of the complications of internal human anatomy can be understood as strategies that have evolved for increasing available surface area per cell for oxygen and nutrient absorption towards what it is for simpler but much smaller creatures.

On the other hand, there is some good news about being big—it makes it feasible to maintain a constant body temperature. This has several advantages. For example, it is easier to evolve efficient muscles if they are only required to function in a narrow range of temperatures than if they must perform well over a wide range of temperatures. However, this temperature control comes at a price. Warm blooded creatures (unlike insects) must devote a substantial part of their food energy simply to keeping warm. For an adult human, this is a pound or two of food per day. For a mouse, which has about one-twentieth the dimensions of a human, and hence twenty times the surface area per unit volume, the required food for maintaining the same body temperature is twenty times as much as a fraction of body weight, and a mouse must consume a quarter of its own body weight daily just to stay warm. This is why, in the arctic land of Spitzbergen, the smallest mammal is the fox.

How high can a giant flea jump? Suppose we know that a regular flea can jump to a height of three feet, and a giant flea is one hundred times larger in all dimensions, so its weight is up by a factor of a million. Its amount of muscle is also up by a factor of a million, and when it jumps it rapidly transforms chemical energy stored in the muscle into kinetic energy, which then goes to gravitational potential energy on the upward flight. But the amount of energy stored in the muscle and the weight to be lifted are up by the same factor, so we conclude that the giant flea can also jump three feet! We can also use this argument in reverse—a shrunken human (as in I shrunk the kids) could jump the same height as a normal human, again about three feet, say. So the tiny housewife trapped in her kitchen sink in the movie could have just jumped out, which she'd better do fast, because she's probably very hungry!

Question: from *The Economist*, Sept 16, 1995 page 74: "the average 16-year-old Japanese girl has grown 4% heavier since 1975, although she is only 1% taller." Just how much plumper does she look? What percent increase would keep her shape exactly the same?

Teaching note: I began the lecture with five questions in a powerpoint presentation, to be answered using clickers. The idea was to get the class thinking about how areas and volumes increase when an object increases in size, keeping the same proportions. To understand how doubling the diameter of a circle increases its area fourfold, imagine the circle just fitting inside a square. It's obvious what happens for squares—and also that the circle takes up the same percentage of the square's area no matter what size they are, provided it just fits. Then a cube, and a ball in a cubical box. Think first about a 2x2x2 cube made of a child's cubical building blocks. Visualize both *volume* and *area* increase from 1x1x1.

The last two questions were asked later, at the appropriate point in the class.

Galileo's *Two New Sciences*, pages 1 - 4

FIRST DAY

INTERLOCUTORS: SALVIATI, SAGREDO AND SIMPLICIO

SALV. The constant activity which you Venetians display in your famous arsenal suggests to the studious mind a large field for investigation, especially that part of the work which involves mechanics; for in this department all types of instruments and machines are constantly being constructed by many artisans, among whom there must be some who, partly by inherited experience and partly by their own observations, have become highly expert and clever in explanation.

SAGR. You are quite right. Indeed, I myself, being curious by nature, frequently visit this place for the mere pleasure of observing the work of those who, on account of their superiority over other artisans, we call "first rank men." Conference with them has often helped me in the investigation of certain effects including not only those which are striking, but also those which are recondite and almost incredible. At times also I have been put to confusion and driven to despair of ever explaining something for which I could not account, but which my senses told me to be true. And notwithstanding the fact that what the old man told us a little while ago is proverbial and commonly accepted, yet it seemed to me altogether false, like many another saying which is current among the ignorant; for I think they introduce these expressions in order to give the appearance of knowing something about matters which they do not understand.

SALV. You refer, perhaps, to that last remark of his when we asked the reason why they employed stocks, scaffolding and bracing of larger dimensions for launching a big vessel than they do for a small one; and he answered that they did this in order to avoid the danger of the ship parting under its own heavy weight, a danger to which small boats are not subject?

SAGR. Yes, that is what I mean; and I refer especially to his last assertion which I have always regarded as a false, though current, opinion; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale. Now, since mechanics has its foundation in geometry, where mere size cuts no figure, I do not see that the properties of circles, triangles, cylinders, cones and other solid figures will change with their size. If, therefore, a large machine be constructed in such a way that its parts bear to one another the same ratio as in a smaller one, and if the smaller is sufficiently strong for the purpose for which it was designed, I do not see why the larger also should not be able to withstand any severe and destructive tests to which it may be subjected.

SALV. The common opinion is here absolutely wrong. Indeed, it is so far wrong that precisely the opposite is true, namely, that many machines can be constructed even more perfectly on a large scale than on a small; thus, for instance, a clock which indicates and strikes the hour can be made more accurate on a large scale than on a small. There are some intelligent people who maintain this same opinion, but on more reasonable grounds, when they cut loose from geometry and argue that the better performance of the large machine is owing to the imperfections and variations of the material. Here I trust you will not charge me with arrogance if I say that imperfections in the material, even those which are great enough to invalidate the clearest mathematical proof, are not sufficient to explain the deviations observed between machines in the concrete and in the abstract. Yet I shall say it and will affirm that, even if the imperfections did not exist and matter were absolutely perfect, unalterable and free from all accidental variations, still the mere fact that it is matter makes the larger machine, built of the same material and in the same proportion as the smaller, correspond with exactness to the smaller in every respect except that it will not be so strong or so resistant against violent treatment; the larger the machine, the greater its weakness. Since I assume matter to be unchangeable and always the same, it is clear that we are no less able to treat this constant and invariable property in a rigid manner than if it belonged to simple and pure mathematics. Therefore, Sagredo, you would do well to change the opinion which you, and perhaps also many other students of mechanics, have entertained concerning the ability of machines and structures to resist external disturbances, thinking that when they are built of the same material and maintain the same ratio between parts, they are able equally, or rather proportionally, to resist or yield to such external disturbances and blows. For we can demonstrate by geometry that the large machine is not proportionately stronger than the small. Finally, we may say that, for every machine and structure, whether artificial or natural, there is set a necessary limit beyond which neither art nor nature can pass; it is here understood, of course, that the material is the same and the proportion preserved.

SAGR. My brain already reels. My mind, like a cloud momentarily illuminated by a lightning-flash, is for an instant filled with an unusual light, which now beckons to me and which now suddenly mingles and obscures strange, crude ideas. From what you have said it appears to me impossible to build two similar structures of the same material, but of different sizes and have them proportionately strong; and if this were so, it would not be possible to find two single poles made of the same wood which shall be alike in strength and resistance but unlike in size.

SALV. So it is, Sagredo. And to make sure that we understand each other, I say that if we take a wooden rod of a certain length and size, fitted, say, into a wall at right angles, i. e., parallel to the horizon, it may be reduced to such a length that it will just support itself; so that if a hair's breadth be added to its length it will break under its own weight and will be the only rod of the kind in the world.* Thus if, for instance, its length be a hundred times its breadth, you will not be able to find another rod whose length is also a hundred times its breadth and which, like the former, is just able to sustain its own weight and no more: all the larger ones will break while all the shorter ones will be strong enough to support something more than their own weight. And this which I have said about the ability to support itself must be understood to apply also to other tests; so that if a piece of scantling will carry the weight of ten similar to itself, a beam having the same proportions will not be able to support ten similar beams.

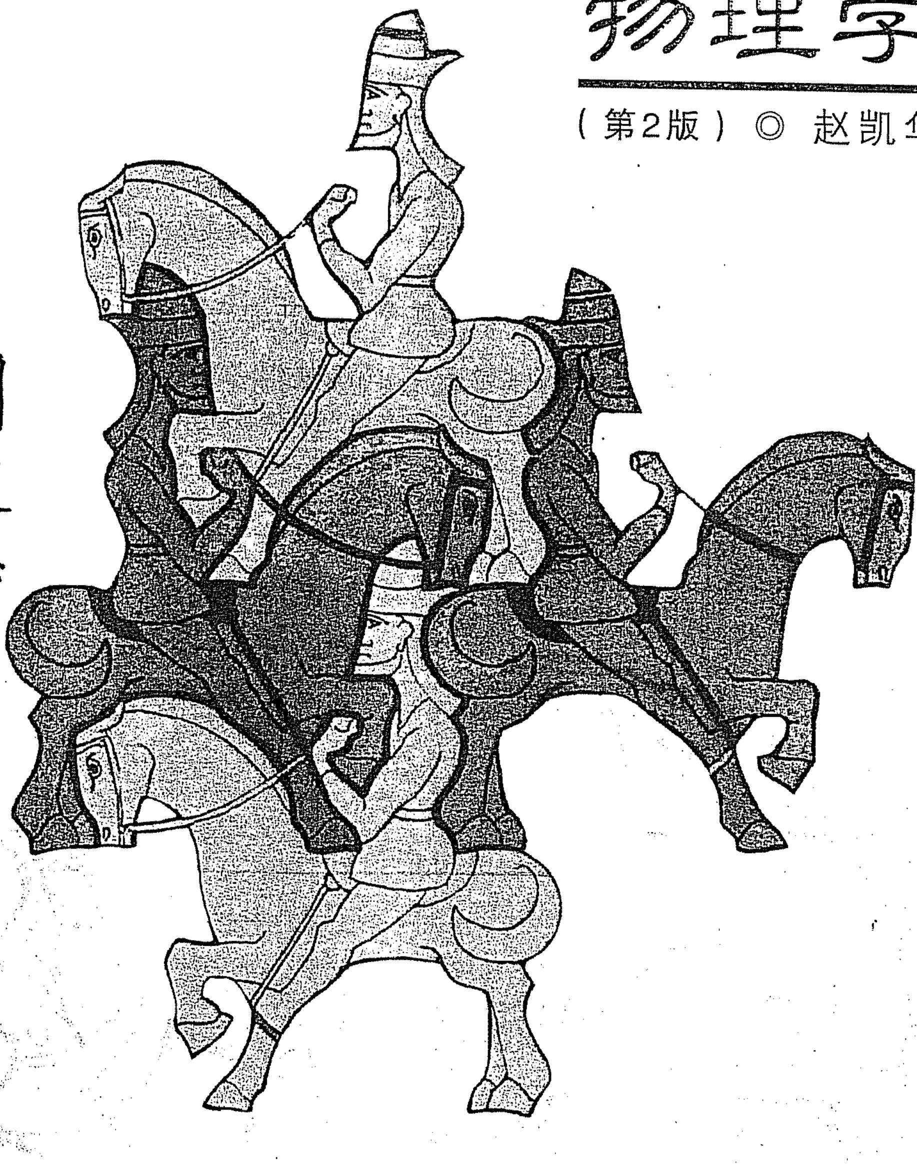
Please observe, gentlemen, how facts which at first seem improbable will, even on scant explanation, drop the cloak which has hidden them and stand forth in naked and simple beauty. Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon. Do not children fall with impunity from heights which would cost their elders a broken leg or perhaps a fractured skull? And just as smaller animals are proportionately stronger and more robust than the larger, so also smaller plants are able to stand up better than larger. I am certain you both know that an oak two hundred cubits high ,would not be able to sustain its own branches if they were distributed as in a tree of ordinary size; and that nature cannot produce a horse as large as twenty ordinary horses or a giant ten times taller than an ordinary man unless by miracle (*note this phrase - Galileo is trying to cover himself*) or by greatly altering the proportions of his limbs and especially of his bones, which would have to be considerably enlarged over the ordinary. Likewise the current belief that, in the case of artificial machines the very large and the very small are equally feasible and lasting is a manifest error. Thus, for example, a small obelisk or column or other solid figure can certainly be laid down or set up without danger of breaking, while the large ones will go to pieces under the slightest provocation, and that purely on account of their own weight.

*The author here apparently means that the solution is unique.

定性与半定量物理学

（第2版）◎ 赵凯华

閻愛德



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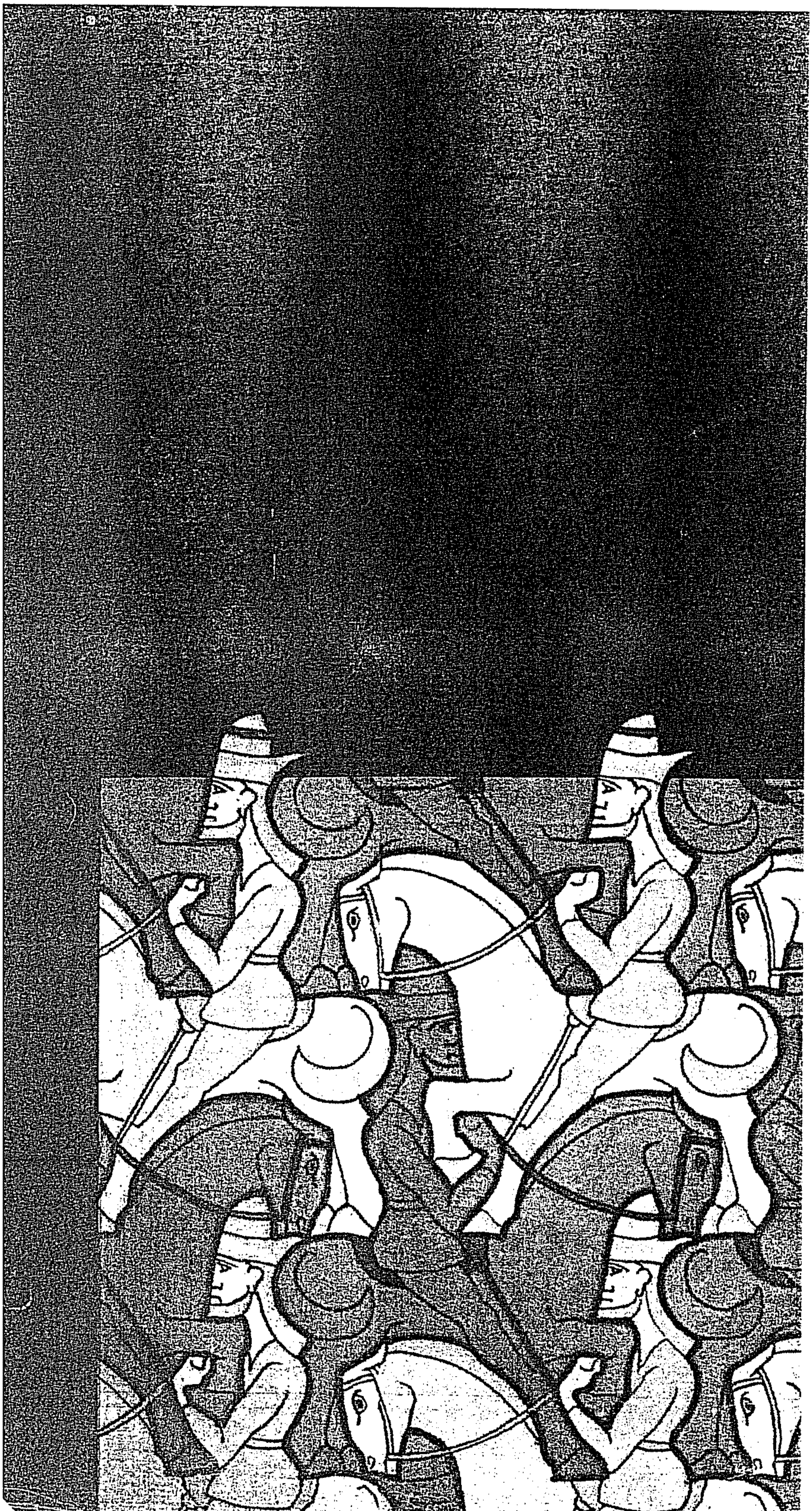

第二章 量纲分析和标度律

§1. 量纲分析基本原理及例题

在通常的物理教科书中，量纲理论往往只是顺便提及和一带而过。关于量纲的应用，在教学中也只介绍单位的换算、检查公式的对错等少数方面。在本章里我们将介绍量纲分析方法的理论基础和更多方面的应用。

由于各物理量之间存在着规律性的联系，我们不必对每个物理量的单位都独立地予以规定。我们可以选定一些物理量作为“基本量”，并为每个基本量规定一个“基本量度单位”，其它物理量的量度单位则可按照它们与基本量之间的关系式（定义或定律）导出，这些物理量称为“导出量”，它们的单位称为“导出单位”。按照此种方法构成的一套单位，构成一定的“单位制”。在不同的单位制中，不仅基本量的选取可以不同，基本量的数目也可以不同。例如 CGS 单位制中有三个基本量，而 MKSA 单位制中有四个基本量。

在选定了单位制之后，导出量的量度单位就可由基本量度单位表达出来，这种表达式称为该导出量的“量纲式”，设 $X_1, X_2, \cdots, X_m$ 是所选单位制中的 m 个基本单位，用 $[P]$ 代表导出量 P 的量纲式，则

$$[P] = X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m}, \tag{2.1}$$

指数 $(a_1, a_2, \cdots, a_m)$ 称为物理量 P 的“量纲”。显然，一个物理量的量纲与单位制的选择有关。未给定单位制，就谈不上量纲。

量纲可以看成是某个“矢量空间”中的“矢量”。对(2.1)式两端取对数，则有

$$\ln[P] = a_1 \ln X_1 + a_2 \ln X_2 + \cdots + a_m \ln X_m, \tag{2.2}$$

这里，若我们把 $\ln X_1, \ln X_2, \cdots, \ln X_m$ 看作 m 维空间的“正交基矢”，则 $(a_1, a_2, \cdots, a_m)$ 就是“矢量” $\ln[P]$ 在基矢上的投影，或者说，是它的“分量”。今后为了简便，我们把量纲式写成

$$\ln[P] \sim (a_1, a_2, \cdots, a_m). \tag{2.3}$$

所谓几个物理量的量纲彼此独立，是指无法用它们幂次的乘积组成无量纲量。用矢量的语言表达，这很清楚，就是代表它们量纲的矢量彼此线性无关。从几何的观点看，两个矢量线性无关，就是它们不共线；三个矢量线性无关，就是它们不共面 …… 在 m 维空间内最多有 m 个彼此线性无关的矢量。m 个矢量 $(a_{1i}, a_{2i}, \cdots, a_{mi})$ $(i = 1, 2, \cdots, m)$ 线性无关的条件是由它们组成的行列式不等于零：

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{21} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} \neq 0. \tag{2.4}$$

1.1 Π 定理

量纲分析法的理论基础是“Π 定理”,这定理是 Buckingham 在 1914 年提出的[1]:设某物理问题内涉及 n 个物理量(包括物理常量)$P_1, P_2, \cdots, P_n$,而我们所选的单位制中有 m 个基本量($n>m$),则由此可组成 $n-m$ 个无量纲的量 $\Pi_1, \Pi_2, \cdots, \Pi_{n-m}$. 在物理量 $P_1, P_2, \cdots, P_n$ 之间存在的函数关系式

$$f(P_1, P_2, \cdots, P_n) = 0 \tag{2.5}$$

可表达成相应的无量纲形式

$$F(\Pi_1, \Pi_2, \cdots, \Pi_{n-m}) = 0. \tag{2.6}$$

(在 $n=m$ 的情况下有两种可能:若 $P_1, P_2, \cdots, P_m$ 的量纲彼此独立,则不能由它们组成无量纲的量;若不独立,则还可能组成无量纲的量。)

定理的证明如下:

设 n 个物理量的量纲为

$$\ln[P_i] \sim (a_{1i}, a_{2i}, \cdots, a_{mi}) \quad (i=1, 2, \cdots, n), \tag{2.7}$$

其中最多只能有 m 个是线性无关的。我们假定它们是其中的前 m 个,则其余 $n-m$ 个物理量中的任何一个都可表示成它们的线性组合,即

$$\ln[P_{m+j}] = x_1\ln[P_1] + x_2\ln[P_2] + \cdots + x_m\ln[P_m], \quad (j=1, 2, \cdots, n-m) \tag{2.8}$$

写成分量形式,用矩阵表示,则有

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{1,m+j} \\ a_{2,m+j} \\ \vdots \\ a_{m,m+j} \end{pmatrix} \quad (j=1, 2, \cdots, n-m), \tag{2.9}$$

---

[1] Buckingham, E., *Phys. Rev.*, **4**(4), 345(1914); *J. Wash. Acad., Sci.*, **3**, 347(1914). Π 为希腊字母,读作 pi.

由于等式左端方阵的行列式不等于零，故对于每个j有一组解$(x_{1j}, x_{2j}, \cdots, x_{mj})$，共$n-m$组。这就是说，我们有

$$[P_{m+j}] = [P_1]^{x_{1j}}[P_2]^{x_{2j}}\cdots[P_m]^{x_{mj}} \quad (j=1, 2, \cdots, n-m) \tag{2.10}$$

或者说

$$\Pi_j = P_1^{-x_{1j}}P_2^{-x_{2j}}\cdots P_m^{-x_{mj}}P_{m+j} \quad (j=1, 2, \cdots, n-m) \tag{2.11}$$

是一些无量纲的量，这样的无量纲量共有$n-m$个。

现在我们设想把P_1', P_2', $\cdots$, P_m'的量度单位分别改变为原来的$1/\alpha_1$, $1/\alpha_2$, $\cdots$, $1/\alpha_m$，则在此新单位制中这些量的数值P_1, P_2, $\cdots$, P_m与原来的数值P_1, P_2, $\cdots$, P_m有如下关系：

$$P_i' = \alpha_i P_i \quad (i=1, 2, \cdots, m), \tag{2.12}$$

量纲关系式(2.10)表明，物理量P_{m+1}, P_{m+2}, $\cdots$, P_n在新旧单位制之间的数值关系为

$$P_{m+j}' = \alpha_1^{x_{1j}}\alpha_2^{x_{2j}}\cdots\alpha_m^{x_{mj}}P_{m+j} \quad (j=1, 2, \cdots, n-m), \tag{2.13}$$

取$\alpha_i = P_i^{-1}$，由(2.12)及(2.13)式有

$$\begin{cases} P_i' = 1 & (i=1, 2, \cdots, m); \\ P_{m+j}' = \Pi_j & (j=1, 2, \cdots, n-m). \end{cases}$$

函数关系(2.5)不应受量度单位变换的影响，亦即我们有

$$f(P_1', P_2', \cdots, P_n') = 0,$$

对于上述特殊选择，有

$$f(\overbrace{1,1,\cdots,1}^{m\text{个}},\Pi_1,\Pi_2,\cdots,\Pi_{n-m}) \equiv F(\Pi_1,\Pi_2,\cdots,\Pi_{n-m}),$$

这便是前面的(2.6)式，于是定理证讫。

Π 定理可以表示成另一等价形式，这一形式在很多场合更便于使用。在一定问题中物体系的发展和演化往往由若干个变量决定，不妨叫它们做“主定参量”。在上面的推演中，$\ln[P_1]$, $\ln[P_2]$, $\cdots$, $\ln[P_m]$实际上起着一组新基矢的作用，我们尽量选为代表主定参量的量纲矢量。如果在其它的物理量中此时此刻我们感兴趣的是其中的某一个，譬如P_{m+1}，则我们可以从(2.6)式中把Π_1解出来：

$$\Pi_1 = \Phi(\Pi_2, \cdots, \Pi_{n-m}),$$

因$\Pi_1 = P_1^{-x_{11}}P_2^{-x_{21}}\cdots P_m^{-x_{m1}}P_{m+1}$，并将$P_{m+1}$解出，于是有

$$P_{m+1} = P_1^{x_{11}}P_2^{x_{21}}\cdots P_m^{x_{m1}}\Phi(\Pi_2, \cdots, \Pi_{n-m}). \tag{2.14}$$

这便是 Π 定理另一种形式的表述。

1.2 例 题

下面我们将介绍一系列运用 Π 定理作量纲分析的例题。

例题 1 用量纲法证明勾股弦定理。

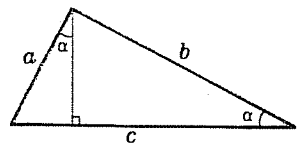

图 2-1 例题 1—— 勾股弦定理

解： 一个直角三角形的面积且可由它的一边（譬如斜边 c）和一个锐角（譬如 α）所决定。α 是无量纲的，按(2.14)式，我们有

$$A = c^2\Phi(\alpha).$$

如图 2-1 所示，作 c 边的垂线将三角形分成两个与原来相似的小直角三角形，它们各有一个同样的锐角 α，故它们的面积应分别为

$$A_1 = a^2\Phi(\alpha),\quad A_2 = b^2\Phi(\alpha).$$

由 $A = A_1 + A_2$ 得

$$c^2\Phi(\alpha) = a^2\Phi(\alpha) + b^2\Phi(\alpha),$$

消去 $\Phi(\alpha)$，即得

$$c^2 = a^2 + b^2. \quad ▌$$

这便是著名的勾股弦定理，西方称之为毕达哥拉斯(Pythagoras)定理。这是现代初等数学课程中必不可少的内容，读者是否想到，它竟然可以用 Π 定理如此简洁地证明！

上面的例题是个几何问题，在物理学中，仅仅靠量纲分析，也可得到某些重要结论。虽然不一定像上面的例题那样，在每一问题中都能得到完全的定量结果，但往往与它只差一个无量纲的未知函数或未知系数。有时，借助于量纲以外其它来源的知识和推理（如已知的特例或实验规律），还可不太难地进一步获知那未知函数的某些特征，甚至将它完全确定下来。

在下面的例题中，除个别的（主要是电磁学的）问题外，我们都选取质量(M)、长度(L)和时间(T)作为基本量，故 $m = 3$.

例题 2 细棒的转动惯量。

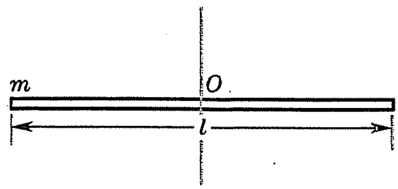

图 2-2 例题 2—— 细棒的转动惯量

解： 设均匀细棒长度为 l，质量为 m，求绕通过其中点 O 的转轴的转动惯量 I（图 2-2）。

转动惯量的量纲式为

$$[I] = \mathrm{ML}^2,$$

故任意形状的物体的转动惯量可写

$$I = ml^2\Phi(\{\Pi\}),$$

其中 m 是该物体的总质量，l 是它的某个特征长度. $\{\Pi\}$ 代表一组能确定其几何形状的无量纲参量，如长方形两边长之比，三角形底与高之比，或某些角度及它们的三角函数等。对于几何形状相似的物体，函数 $\Phi(\{\Pi\})$ 是等同的。对于那些只用一个特征长度即可完全确定的几何形体，如正方形或立方体、圆或球、细棒等，Φ 退化为一个未知常数，下面用 k 表示。故细棒的

转动惯量可写成

$$I = kml^2. \tag{2.15}$$

进一步确定 k，尚需其它知识。这里我们利用转动惯量的平行轴定理：

$$I = I_C + md^2, \tag{2.16}$$

这里 I_C 是绕通过质心某个特定轴的转动惯量。若将此轴平行移动距离 d，则绕新轴的转动惯量 I 由上式决定。

设(2.15)式中 I 代表细棒的 I_C，即绕中点 O 的转动惯量。将轴移至端点，则 $d = l/2$，按(2.16)式转动惯量将为

$$I' = \left(k + \frac{1}{4}\right)ml^2. \tag{2.17}$$

现在我们设想，将细棒平分为两段，每段质量为 $m/2$，长度为 $l/2$，按(2.17)式它们绕通过其端点 O 的转轴的转动惯量皆为

$$I'' = \frac{1}{8}\left(k + \frac{1}{4}\right)ml^2,$$

由于两段绕同一轴的转动惯量之和等于总的转动惯量 $I = kml^2$，即

$$2I'' = I \quad 或 \quad \frac{2}{8}\left(k + \frac{1}{4}\right) = k,$$

由此得 $k = 1/12$，代入(2.15)式和(2.17)式，得细棒的转动惯量为

$$\begin{cases} 通过中心 \quad I = \frac{1}{12}ml^2, & (2.18) \\ 通过一端 \quad I = \frac{1}{3}ml^2. & (2.19) \end{cases}$$

这结果与通常用积分方法计算出来的一致。▌

用上题的方法还可求出更复杂情况，如长方形、三角形、多边形、圆等的转动惯量，而无需作繁复的积分。[1]

例题3 单摆的周期。

解： 单摆是大家都很熟悉的例子，其周期 T 的表达式平常需要通过解微分方程求得，这里我们用量纲分析来讨论它。这种方法的关键是选好与问题有关的变量。在本题中我们选单摆的质量 m、重力加速度 g 和摆长 l 为独立变量，此外再考虑周期 T 和总能量 E 两个量。本题的量纲关系比前两题稍复杂一些，我们采用比较正规的方法来分析。把上述各物理量的量纲列成矩阵如下，每纵列代表一个变量的量纲矢量：

---

[1] R. Rabinoff, *Am. J. Phys.*, **53**(1985), 501；译文见《大学物理》1987 年第 7 期，第 31 页。

	m	g	l	T	E
M	1	0	0	0	1
L	0	1	1	0	2
T	0	−2	0	1	−2

解代数方程组

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -2 & 0\end{pmatrix}\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix} \quad 和 \quad \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -2 & 0\end{pmatrix}\begin{pmatrix}x_{12}\\x_{22}\\x_{32}\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix},$$

得

$$\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}=\begin{pmatrix}0\\-1/2\\1/2\end{pmatrix} \quad 和 \quad \begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

由此获得 $n-m=2$ 个无量纲组合

$$\Pi_1 = T\sqrt{\frac{g}{l}}, \qquad \Pi_1 = \frac{E}{mgl}.$$

按照(2.14)式我们可以写出

$$T = \sqrt{\frac{l}{g}}\,\Phi\left(\frac{E}{mgl}\right), \tag{2.20}$$

这里的函数 Φ 不能用量纲法进一步定出了。mgl 代表摆角为 90° 时相对于最低点的势能，Π_2 代表对应此情况的约化能量。在摆幅很小时 $\Pi_2 \ll 1$，可将函数按宗量的幂次展开：

$$\Phi = C_0 + C_1\left(\frac{E}{mgl}\right) + C_2\left(\frac{E}{mgl}\right)^2 + \cdots,$$

于是

$$T = C_0\sqrt{\frac{l}{g}}\left[1 + C_1'\left(\frac{E}{mgl}\right) + C_2'\left(\frac{E}{mgl}\right)^2 + \cdots\right], \tag{2.21}$$

式中 C_0，$C_1' = C_1/C_0$，$C_2' = C_2/C_0$ 等都是纯数。在小摆幅的极限下，上式可只保留第一项，这时周期 T 既与质量 m 无关，又与摆幅无关。这两点都不是显然的。

顺便指出，理论推导表明，(2.20)式的解析表达式为全椭圆积分

$$\Phi(\Pi) = 4K(\sqrt{\Pi/2}), \tag{2.22}$$

$\Pi \to 0$ 时 $K \to \pi/2$，从而 $\Phi \to C_0 = 2\pi$，这是大家熟知的结果。大摆幅时周期随约化能量 Π_2 变化的曲线示于图2-3. 可以

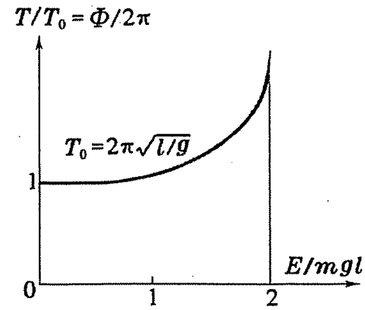

图 2-3 例题 3—— 单摆的周期

看出,在相当大的摆幅范围内,周期 T 偏离小摆幅的极限 $T_0=2\pi\sqrt{l/g}$ 不大。在 $E/mgl=1$ 时(摆角 90°), T 增加 18%; $E/mgl=2$ 时(摆角 180°,即从圆周的最高点摆下), $T\to\infty$. ▮

例题 4　一道 N 个质点碰撞的考题。

质量分别为 $m_1, \cdots, m_N$ 的 N 个质点静止地放在一无限大无摩擦的水平平面上,它们排在一条直线上,位置分别为 $x_1, \cdots, x_N$. 另有一质量为 m 的质点 P 位于此直线上某处,具有沿此直线的初速 v_i. 待所有质点不再碰撞以后, P 经历了 n 次碰撞,并获得末速 v_f. 如果其初速为 $3v_i$, 质点 P 将经历多少次碰撞? 其末速将为何? 解释你的答案的理由。

解:　设质点的初始位置为 x. 本题中有 $N+1$ 个位置变量 $x, x_1, \cdots, x_N$ 和 $N+1$ 个质量 $m, m_1, \cdots, m_N$, 及一系列无量纲恢复系数 $\{e\}$, 加上质点 P 的初速 v_i, 即构成本题全部主定参量,它们可完全决定质点 P 的碰撞次数 n 和末速 v_f. 由上述 $2N+3$ 个有量纲的量容易构造出 $2N$ 个无量纲量: $x_1/x, \cdots, x_N/x$; $m_1/m, \cdots, m_N/m$. 剩下有独立量纲的量为 x, m, v_i, 它们的量纲表如下:

	x	m	v_i
M	0	1	0
L	1	0	1
T	0	0	−1

其行列式不为零,因而不可能再有无量纲的组合。无量纲量 n 和 v_f/v_i 只能是上述无量纲量的函数:

$$n=\Phi\left(\frac{x_1}{x}, \cdots, \frac{x_N}{x}, \frac{m_1}{m}, \cdots, \frac{m_N}{m}, \{e\}\right), \tag{2.23}$$

$$\frac{v_f}{v_i}=\Psi\left(\frac{x_1}{x}, \cdots, \frac{x_N}{x}, \frac{m_1}{m}, \cdots, \frac{m_N}{m}, \{e\}\right). \tag{2.24}$$

它们的数值与 v_i 无关。故当 $v_i\to 3v_i$ 时, 碰撞次数 n 不变,末速 $v_f\to 3v_f$. ▮

例题 5　试用量纲法由开普勒第三定律推论万有引力的性质。

解:　开普勒第二定律表明行星的角动量守恒。亦即,它所受的是指向太阳的有心力。试设此力 f 与距离 r 的关系具有倒幂次形式,并与行星的质量 m 成正比:

$$f=\frac{mC}{r^n}, \tag{2.25}$$

并设椭圆轨道的半长轴 a 和周期 T 除与比例常量 C 有关外,还依赖于行星的能量 E 和轨道角动量 L,现利用 Π 定理来分析开普勒第三定律中出现的

a^3/T^2 比值。为此先把一些有关参量的量纲列成下表：

	C	E	L	a^3/T^2
M	0	1	1	0
L	$n+1$	2	2	3
T	−2	−2	−1	−2

从这个量纲表可算出，由这些量可以组成一个无量纲量：

$$C^{\frac{3}{n+1}}\left(\frac{E}{L}\right)^{\frac{2(n-2)}{n+1}}\left(\frac{a^3}{T^2}\right)^{-1}.$$

根据 Π 定理，我们有

$$\frac{a^3}{T^2} = C^{\frac{3}{n+1}}\left(\frac{E}{L}\right)^{\frac{2(n-2)}{n+1}}\Phi(\Pi), \tag{2.26}$$

式中 Π 是与椭圆轨道形状有关的其它无量纲参量（如偏心率），Φ 是 Π 的某个无量纲函数。然而开普勒第三定律宣称 a^3/T^2= 太阳系常量，与行星的性质（如 E、L）无关，故上式中 $\Phi=1$，$n=2$，C 为太阳系常量（与行星的质量 m 无关）。亦即(2.25)式中的力 f 与 r 的平方成反比，与行星的质量 m 成正比。▌

例题 6 一道量子真空涨落的考题。

两块无限大平行平面壁相距 z，皆由理想导体构成。从经典理论看，两壁间是没有相互作用的，但若计及（相对论性）电磁场的量子真空涨落效应，两壁间单位面积上的作用力 p 与距离 z 呈怎样的函数关系？

解： 除了距离 z 外，这里涉及电磁场，有关的参量为真空中的光速 c；这里还涉及量子效应，有关的参量还有普朗克常量 h. 从量纲表

	z	c	h	p
M	0	0	1	1
L	0	−1	2	−1
T	0	−1	−1	−2

可以解出 $[p]=[z]^{-4}[c][h]$，即

$$p \propto \frac{ch}{z^4}. \tag{2.27}$$

即 p 反比于 z 的四次方。▌

例题 7 试用量纲法导出黏度 η、扩散系数 D、热导率 κ 和比热 c、流体密度 ρ、平均热运动速率 $\overline{v}$、平均自由程 $\overline{\lambda}$ 之间的关系。

答： η 和 $\kappa/c \propto \rho\overline{v}\,\overline{\lambda}$， $D \propto \overline{v}\,\overline{\lambda}$. ▌

用最粗略的平均自由程概念，得到上列三式的比例系数都是 1/3. ❶ 严

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❶ 见任何一本普通物理热学书。

格的理论只有从确定分布函数入手，计算是相当繁复的。所得系数，对 η 来说是0.499；对 κ/c 来说，还要再乘以2.5 ~2.522；对 D 来说，情况就更复杂了。[1] 但不管怎样，上题内给出的定性关系，在数量级上是正确的。

在普朗克提出量子论之前，维恩(W. Wien)曾根据热力学原理证明，黑体的辐射频谱必有如下的函数形式：

$$r(\nu,\ T)=c\nu^3\Phi\left(\frac{\nu}{T}\right), \tag{2.28}$$

式中 c 为真空中光速，ν 为频率，T 为热力学温度。只靠热力学不能进一步把函数 Φ 的形式确定下来。但是由此我们可以得到斯特藩-玻耳兹曼定律和维恩位移定律。为此我们先求温度 T 时的辐射本领：

$$R(T)=\int_0^\infty r(\nu,\ T)\mathrm{d}\nu=c\int_0^\infty \nu^3\Phi\left(\frac{\nu}{T}\right)\mathrm{d}\nu=cT^4\int_0^\infty\left(\frac{\nu}{T}\right)^3\Phi\left(\frac{\nu}{T}\right)\mathrm{d}\left(\frac{\nu}{T}\right)$$

$$=cT^4\int_0^\infty x^3\Phi(x)\mathrm{d}x,$$

上式右端的定积分是个与 T 无关的常数，于是我们得到了 $R=\sigma T^4$ 的结论(斯特藩-玻耳兹曼定律)，但比例系数 σ 不知。

为了导出维恩位移定律，先把黑体的辐射频谱(2.27)式改用 $\lambda=c/\nu$ 表示。因 $\mathrm{d}\nu=-\frac{c}{\lambda^2}\mathrm{d}\lambda$，由 $r(\lambda,\ T)\mathrm{d}\lambda=|r(\nu,\ T)\mathrm{d}\nu|$ 的条件得

$$r(\lambda,\ T)=\frac{c^5}{\lambda^5}\Phi\left(\frac{c}{\lambda T}\right); \tag{2.28'}$$

在温度 T 时 $r(\lambda,\ T)$ 的极大值位于

$$\frac{\partial r(\lambda,\ T)}{\partial\lambda}=-\frac{5c^5}{\lambda^6}\Phi\left(\frac{c}{\lambda T}\right)-\frac{c^6}{\lambda^7 T}\Phi'\left(\frac{c}{\lambda T}\right)=0$$

的地方，Φ' 代表 $\mathrm{d}\Phi(x)/\mathrm{d}x$. 这相当于求下式的根：

$$5\Phi(x)+x\Phi'(x)=0\quad (x=c/\lambda T),$$

设此根位于 $x=c/\lambda T=x_0$ 处，于是

$$\lambda_{\max}T=c/x_0\quad（维恩位移定律）.$$

由于函数 Φ 未知，常数 c/x_0 也未知。

例题8 黑体辐射定律。

解： 今天，在普朗克之后我们已经知道黑体辐射的量子本性，上述两条定律可以很简捷地由量纲分析导出。为了便于作量纲分析，我们宁可用具有能量量纲的 kT(k 为玻耳兹曼常量)代替 T. 除此之外，参与本问题的普适常量还应该有 c 和 h(普朗克常量)。故可假定 $R(T)$ 仅由 kT、c、h、三个

[1] 王竹溪. 统计物理学导论. 北京：高等教育出版社，1965，第四章。

主定参量所决定。现将它们的量纲表列出如下：

	kT	c	h	R
M	1	0	1	1
L	2	1	2	0
T	−2	−1	−1	−3

由此得

$$[R]=[kT]^4[c]^{-2}[h]^{-3},$$

即 $R\propto T^4$，且得斯特藩-玻耳兹曼常数 $\sigma\propto k^4/c^2h^3$（利用普朗克公式所得结果为 $\sigma=2\pi^5k^4/15c^2h^3$，即上面未能定出的比例常数为 $2\pi/15\doteq 40.80$）。

在维恩位移定律中 λ_{max} 决定于 k, c, h, 由量纲法可得

$$[\lambda_{max}]=[kT]^{-1}[c][h],$$

即 λ_{max} = 常量 b，维恩常量 $b\propto ch/k$（利用普朗克公式可导出 b 的表达式，其结果为 $b=0.2014\,ch/k$）。▌

由此可见，有了普朗克常量 h 的概念之后，可以不用热力学，仅从量纲上就能得出两条黑体辐射定律。虽然仍不能将 σ、b 两个常数最后解出，但它们与普适常量 k, c, h 之间的依赖关系已弄清楚了。

有一根长度为 l、截面半径为 r 的圆棒（见图2－4a），把它置于两个支撑点上。棒在自重的作用下稍有弯曲（图2－4b）。若考察左边半段棒，我们会注意到，来自左面支点的垂直作用力 N 和这半段棒的重量 $W/2$ 大小相等、方向相反，形成一个顺时针方向的力偶。但是由于棒处于平衡状态，所以右边半段棒对它施加的力必产生一个大小相等、方向相反的力偶矩，称为此棒的“内力矩”。

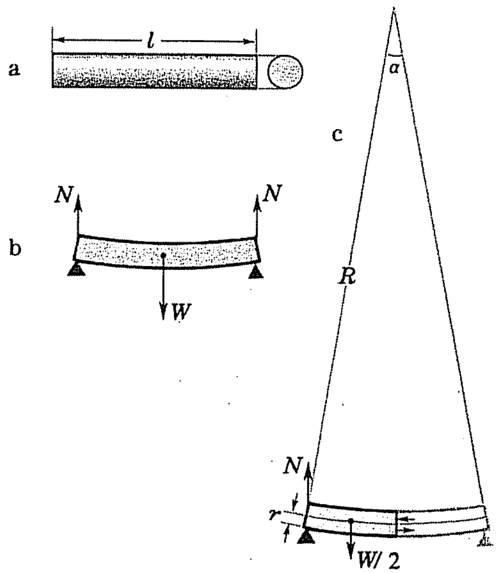

图2－4 例题9——弯曲圆棒的内力矩

首先让我们来分析一下，内力矩是怎样产生的。当圆棒弯曲时，上侧缩短，内部产生压应力；下侧伸长，内部产生张应力（图2－4c）。这一张一压，就形成了内力矩。设上、下侧长度的改变量为 Δl，则应变为 $\Delta l/l$，内力矩应正比于材料的

杨氏模量 Y 和应变 $\Delta l/l$. 设棒对曲率中心的张角为 α，则 $l=R\alpha$，$\Delta l=r\alpha$，故 $\Delta l/l=r/R$.

例题 9 圆棒弯曲时内力矩与半径的关系。

解： 用量纲法来分析，内力矩 $M_{内}$ 的量纲为

$$[M_{内}]=\mathrm{ML^2T^{-2}},$$

杨氏模量的量纲为

$$[Y]=\mathrm{ML^{-1}T^{-2}},$$

由此不难看出

$$\Pi_1=\frac{M_{内}}{Yr^3}$$

是个无量纲量。另一个无量纲的量就是 $\Pi_2=r/R$. 按照 Π 定理，我们有

$$L_{内}=Yr^3\Phi\left(\frac{r}{R}\right),$$

按胡克定律，$M_{内}\propto\Delta l/l=r/R$，故应取 $\Phi\propto r/R$，于是得

$$M_{内}\propto Yr^4/R, \tag{2.29}$$

即内力矩正比于圆棒半径的四次方。理论计算表明，上式中的比例系数是 $\pi/4$，即 $M_{内}=\pi Yr^4/4R$. ▌

例题 10 决定水面波的因素有二：一是表面张力，它促使表面的任何弯曲展平；二是重力；它促使水面的任何倾斜恢复水平。试用量纲分析法确定，当以上每个单一因素起作用时，波速 v 与约化波长 $\lambda\!\!\!^-=\lambda/2\pi$ 的依赖关系。哪个因素对长波起主要作用？哪个因素对短波起主要作用？

解： 表面张力起作用时，$v\propto\sqrt{\gamma/\rho\lambda\!\!\!^-}$（短波时起主要作用）；重力起作用时，$v\propto\sqrt{g\lambda\!\!\!^-}$（长波时起主要作用）。式中 γ 为表面张力系数，ρ 为水的密度，g 为重力加速度。▌

例题 11 上题中讨论水面的重力波时，未考虑水的深度 h 的影响，亦即它相当于 $h\gg\lambda\!\!\!^-$ 的深水极限情形。在 $h\ll\lambda\!\!\!^-$ 的相反情形里，v 将不取决于波长，而取决于深度。试用量纲法分析 v 与 h 的依赖关系。

答： $v\propto\sqrt{gh}$. ▌

海啸（tsunami）是由地震引起的一种海洋巨浪。以 2004 年 12 月印度洋的海啸为例，其波速与现代喷气式客机相当，即每小时几百千米，周期 $T\approx$ 40 min，故波长也有几百千米。地球上海洋的平均深度 $h\approx3.8$ km，远小于波长，故海啸属于浅水波，其波速应写成 $v=\sqrt{gh}$，以海洋的平均深度 h 代入，得 $v\approx193$ m/s $=695$ km/h，从而波长 $\lambda\approx463$ km. 在开阔的洋面上其振幅只有几厘米到几十厘米高，所以在那里的船只不会有什么异样感觉。但是当海浪冲上岸时，随着滩头的深度 h 的减小，波速 v 一步一步地减小，结果后浪

催前浪,把浪头一步一步地逼高,直达几十米,酿成滔天大祸。

根据 Π 定理可以预期,对于任意的 h 和 λ, 水面波速的表达式可写成

$$v=\sqrt{g\lambda}\,\Phi(\Pi),$$

这里 $\Pi=\lambda/h$, 上面两题的结果告诉我们, $\Phi(\Pi)$具有如下极限行为:

$$\begin{cases}\Pi\to 0\text{ 时}, & \Phi(\Pi)\to\text{常数},\\ \Pi\to\infty\text{ 时}, & \Phi(\Pi)\to\Pi^{-1/2}.\end{cases}$$

流体力学的理论计算表明, $\Phi=\tanh\Pi^{-1/2}$, 从而

$$v=\sqrt{g\lambda}\tanh\sqrt{h/\lambda}. \tag{2.30}$$

上面的量纲分析定性地与此式符合。

从以上各种例子可以看出,用量纲的方法常常很简便,甚至不需要知道具体的定律和机制,便可得到一些重要的概念和信息。然而,正确地运用它并不总是那么容易。难的是主定参量的选择,这要求相当的经验和对现象本质透彻的了解。特别是在那些未知的领域内,更需要有物理上的直觉和洞察力,才能处理得妥帖。下面回顾一段历史上的插曲,是很有教益的。

§2. 量纲法的进一步讨论

2.1 单位制中基本量个数的多寡有什么影响?

瑞利(Lord Rayleigh)是最早提倡使用量纲法的先驱者之一。1915 年他用量纲法讨论流动着的流体的导热问题,他的推理后来遭到许多人的批评,他自己未能对此作出令人信服的辩解。

瑞利当时讨论的问题可表述如下:在无穷大空间内作匀速平行运动的不可压缩无黏性流体中,置放一个给定形状的固定物体,它与远处流体保持恒定的温度差 θ. 待流场达到恒定后,试分析影响单位时间内由该物体传出热量 H 的因素。瑞利是这样选择主定参量的:物体的特征线度 l,远处未受扰流场中的流速 v,温度差 θ,流体单位体积的热容 c 和热导率 κ. 瑞利所选的单位制是五元的:除质量 M、长度 L、时间 T 之外,还有温度 Θ 和热量 Q(选这样的单位制当然是无可非议的)。以上各参量的量纲列于下表:

	l	v	θ	c	κ	H
M	0	0	0	0	0	0
L	1	1	0	−3	1	0
T	0	−1	0	0	1	−1
Θ	0	0	1	−1	1	0
Q	0	0	0	1	1	1

这里共 6 个变量($n=6$), 5 个基本量,本应有 $n-m=1$ 个无量纲组合量。但

这里 5×5 的行列式显然等于 0，因为第一行都是 0，所以由前 5 个参量还可多组成一个无量纲量. 这里的两个无量纲量为

$$\Pi_1 = \frac{H}{\kappa l \theta}, \qquad \Pi_2 = \frac{lvc}{\kappa}.$$

故有

$$H = \kappa l \theta \Phi\left(\frac{lvc}{\kappa}\right). \tag{2.31}$$

它表明，在给定 l, κ 的情况下，H 正比于温差 θ；此外它只取决于 v 和 c 的乘积，即对于不同的 v 和 c，只要 v 与 c 乘积一样，H 的值就相同。

略布欣斯基（Рябушинский）对瑞利这个推论作了批评. 他也取这 5 个主定参量，但认为，若取通常的 M、L、T 三个基本量，温度和热量都应具有能量的量纲，于是这里可以决出三个无量纲量所以有

	l	v	κ	c	κ	H
M	0	0	1	0	0	1
L	1	1	2	−3	1	2
T	0	−1	−2	0	−1	−3

这里可以决出三个无量纲量

$$\Pi_1 = \frac{H}{\kappa l \theta}, \quad \Pi_2 = \frac{lvc}{\kappa}, \quad \Pi_3 = cl^3.$$

所以有

$$H = \kappa l \theta \Phi\left(\frac{lvc}{\kappa}, cl^3\right). \tag{2.32}$$

这表明，在 θ, l, κ 给定后，H 不仅取决于 v 和 c 的乘积，它还取决于 c 本身。

在对略布欣斯基的答复中瑞利写道：[1]"略布欣斯基所提出的问题，与其说是属于我所涉及的相似性原理的应用这一方面的，不如说是属于逻辑方面的。问题很值得进一步探讨。我的结论是在通常的傅里叶热传导方程的基础上得出的，在方程中温度和热量被看做是 *sui generis*（不同质的）量。如果由分子理论提供的关于热的本质的进一步认识，使我们处于比以前研究个别问题时还要坏的境地，那的确是一个佯谬，佯谬的解决看来应该是，在傅里叶方程中包含着被在略布欣斯基的论证中忽视的关于热和温度本质的某些东西。"

看了这段答辩，读者感到问题澄清了吗？正如后来 Bridgman 说的，瑞利的答复难以令人满意，他对这个矛盾全然没有给出解释。瑞利和略布欣

[1] 瑞利的原作见 *Nature*, V.95, 1915, p.69；略布欣斯基的批评见 *Nature*, V.95, 1915, p.105；瑞利的答复见 *Nature*, V.95, 1915, p.646；这段争论载入下列书籍：Bridgman, P. W., *Dimensional Analysis*, New Haven, Yale University Press, 1932.

斯基二人中，究竟谁是谁非呢？一般说来，当我们选取更多的基本量时，都会出现一些新的物理常数。例如用国际 MKSA 的电磁学单位制($n=4$)，就比用高斯单位制($n=3$)多个物理常数 ε_0. 瑞利采用的五元单位制中包含两个新的物理常量：热功当量 J 和玻耳兹曼常量 k，它们在瑞利的单位制中的量纲式分别为

$$[J]=\mathrm{ML^2/T^2Q},\qquad [k]=\mathrm{ML^2/T^2\Theta}.$$

选取主定参量时，应该把它们也包括进去，这样一来，我们就可比瑞利多得到一个无量纲量

$$\Pi_3=\frac{Jcl^3}{k},$$

从而瑞利的(2.30)式应修改为

$$H=\kappa l\theta\Phi\left(\frac{lvc}{\kappa},\frac{Jcl^3}{k}\right).\tag{2.31′}$$

由于 J、k 是普适常数，这公式与略布欣斯基的(2.31)式基本上一样. 所以，看来略布欣斯基是对的。

然而，流体是不可压缩和无黏性的，在这个问题中没有热能与机械能的转化，力学过程与热过程独立进行，因此在这里热功当量 J 一定不重要，也就是说，在(2.30′)式中的 Φ 函数实际上不依赖于 $\Pi_3=\dfrac{Jcl^3}{k}$. 这样，我们又回到了瑞利的结论！

从上面的讨论可以看出，运用量纲法分析问题时，选基本量个数较多的单位制，有助于把问题分析得更加深入，但不要忘记把增加的普适常数列入主定参量中。同时在分析问题时，需要更多一些物理思考。

2.2 物理方程中常量的压缩与恢复❶

在作理论演算时，为了书写简便，人们常把公式中的物理常数压缩掉一些。通常的说法是“令公式中的某些常数为 1”，从而不再出现。例如我们处理这样一个问题：将一质量为 m 的质点悬挂在劲度系数为 k 的弹簧下端，l 为弹簧的原长，g 为重力加速度。将质点从未伸长的弹簧下端由静止状态释放，求弹簧长度 x 随时间 t 的变化。

牛顿运动方程为

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-k(x-l)+mg,\tag{2.33}$$

初始条件为

$$[x]_{t=0}=l,\tag{2.34}$$

$$\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]_{t=0}=0.\tag{2.35}$$

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❶ 可参考 E. A. Desloge, *Am. J. Phys.*, **52**(4), 312(1984).

令 $m=k=1$,则微分方程化为

$$\frac{d^2x}{dt^2}=-x+l+1, \tag{2.36}$$

初始条件形式不变。由此求得问题的解为

$$x=l+1-\cos t. \tag{2.37}$$

这种压缩物理常数的作法在量子场论中最常用。在那里总出现光速 c 和约化普朗克常量 $\hbar$($\hbar=h/2\pi$)。人们常说“令 $c=\hbar=1$”,或者说“取 $c=\hbar=1$ 的单位制”, 于是质点的能量公式化为玻尔磁子的公式化为

$$E=\sqrt{p^2+m^2} \quad (p \text{ 为动量}, m \text{ 为质量}) \tag{2.38}$$

玻尔磁子的公式为

$$\mu_B=\frac{e}{2m} \quad (e \text{ 为电子电荷}, m \text{ 为电子质量}) \tag{2.39}$$

等等。c 和 $\hbar$ 都从公式里消失了。这种单位制叫做“自然单位制”。

我们不禁要问:所谓“令 $c=\hbar=1$”” 或“取 $c=\hbar=1$ 的单位制” 是什么意思?必要的时候,如何将这些常数恢复起来?本节就来讨论这个问题。

让我们回顾一下 §1 中有关 Π 定理证明的段落。在具有 m 个基本量的单位制中,我们可以把 m 个量纲彼此独立的量 $P_1, P_2, \cdots, P_m$ 的量度单位改变为原来的 $1/\alpha_1, 1/\alpha_2, \cdots, 1/\alpha_m$, 这时在新单位制中这些量的数值 $P'_1, P'_2, \cdots, P'_m$分别改变为

$$P'_i=\alpha_i P_i \quad (i=1, 2, \cdots, m). \tag{2.40}$$

由量纲式(2.10)可知,其它量 $P_{m+1}, P_{m+2}, \cdots, P_n$ 在新旧单位制之间的数值关系为

$$P'_{m+j}=\alpha_1^{x_1 j}\alpha_2^{x_2 j}\cdots\alpha_m^{x_m j}P_{m+j} \quad (j=1, 2, \cdots, n-m). \tag{2.41}$$

取 $\alpha_i=P_i^{-1}$, 则有

$$\begin{cases} P'_i=1 & (i=1, 2, \cdots, m); \\ P'_{m+j}=\Pi_j & (j=1, 2, \cdots, n-m). \end{cases}$$

这里 Π_j 是无量纲的。经过上述单位变换,经过上述单位变换,原有的函数关系或方程

$$f(P'_1, P'_2, \cdots, P'_n)=f(1,1, \cdots, 1, \Pi_1 \cdots, \Pi_{n-m})=0$$

变为

$$f(P_1, P_2, \cdots, P_n)=0.$$

上面所述压缩方程式中常数的办法就是这个意思。例如当人们说“令 $m=k=g=1$” 或“取 $m=k=g=1$ 的单位制” 时,就相当于取 $P_1=m$, $P_2=k$, $P_3=g$ 为新基,令 $\alpha_1=1/m$, $\alpha_2=1/k$, $\alpha_3=1/g$, 从而有 $P'_1=m'=1$, $P'_2=k'=1$,

$P'_3=g'=1$. 所以更准确的说法应该是在新的单位制中 $m'=k'=g'=1$. 余下的量,如 x, t 等(相当于 P_{m+1}, P_{m+2}, …)也都换为如 x', t' 等(相当于 P'_{m+1}, P'_{m+2}, …), 它们都已是无量纲的了。在(2.33)—(2.35)各式中所有的符号都要理解为带撇的。

被压缩掉的常数如何恢复?这要靠(2.13)式。列出上面弹簧振子例题中全部物理量的量纲式:

	m	k	g	x	l	t
M	1	1	0	0	0	0
L	0	0	1	1	0	0
T	0	−2	−2	0	0	1

由此得到

$$[x]=[l]=[k]^{-1}[m][g],$$
$$[t]=[k]^{-1/2}[m]^{1/2},$$

从而

$$x'=\frac{kx}{mg},\qquad l'=\frac{kl}{mg},\qquad t'=\sqrt{\frac{k}{m}}t,$$

将它们代入加撇的(2.35)式, 即

$$x'=l'+1-\cos t'.$$

得

$$x=l+\frac{mg}{k}\left(1-\cos\sqrt{\frac{k}{m}}t\right) \tag{2.42}$$

在此表达式中常数 k, m, g 得到了恢复。

在量子场论中采用的自然单位制中也可以压缩掉三个物理常量. 除 c, $\hbar$ 外第三个取什么?有人取质量,有人取能量,等等,在各种书籍文献中没有统一的习惯。其实也可像在(2.36)、(2.37)式中那样,不再压缩第三个常数。这相当于不取 $\alpha_3=P_3^{-1}$, 而是取 $\alpha_3=1$. 设 $P_1=c$, $P_2=\hbar$, $P_3=m$, 及 $\alpha_1=1/c$, $\alpha_2=1/\hbar$, $\alpha_3=1$. 现在我们来恢复(2.36)、(2.37)式中常数 c、$\hbar$. 先写出有关量的量纲式来,对电磁量我们采用高斯单位制:

	c	$\hbar$	m	E	p	e	μ_B
M	0	1	1	1	1	1/2	1/2
L	1	2	0	2	1	3/2	5/2
T	−1	−1	0	−2	−1	−1	−1

由此得

$$[E]=[c]^2[m],\qquad [p]=[c][m],$$

$$[e]=[c]^{1/2}[\hbar]^{1/2},\qquad [\mu_B]=[c]^{-1/2}[\hbar]^{3/2}[m]^{-1}.$$

从而在新单位制中

$$E' = E/c^2, \qquad p' = p/c,$$

$$e' = e/\sqrt{c\hbar}, \qquad \mu'_B = \mu_B\sqrt{c/\hbar^2}.$$

将它们代入加撇的(2.36)、(2.37)式,即

$$E' = \sqrt{p'^2 + m'^2}, \qquad \mu'_B = e'/2m',$$

就可得到

$$E = c\sqrt{p^2 + m^2c^2}, \tag{2.43}$$

和

$$\mu_B = \frac{e\hbar}{2mc}. \tag{2.44}$$

式中的常数 c、$\hbar$ 得到了恢复。

§3. 模拟试验与物理相似性原理

在工程技术以及其它许多领域中,人们常希望利用模拟试验来代替对实际现象的研究,例如用水代替石油来研究它们在管道中的流动,把设计好的飞机缩小成模型放在风洞中试验其性能,等等。这样做不仅在经济上有很大好处,并带来很大方便,而且还使我们可能在一定程度上预言某些目前尚无法达到的条件下出现的情况。怎样才能使我们模拟试验的结果真的对实际有指导意义呢?关键是要作量纲分析。先看流体力学中的两个例子。

3.1 流体的相似性原理之一——管道流动问题

考虑水平管道内流体的流动问题。设管道的横截面为圆形,直径为 d,从而面积为 $S = \pi d^2/4$. 与我们当前所考虑的问题有关的物理量,还有流体的密度 ρ、黏度 η、平均流速 $\bar{v} = Q_V/S$,以及压强梯度 $\Delta p/\Delta l$. 选 ρ、d、$\bar{v}$ 为主定参量,列出所有这些物理量的量纲如下:

	ρ	d	$\bar{v}$	η	$\Delta p/\Delta l$
M	1	0	0	1	1
L	−3	1	1	−1	−2
T	0	0	−1	−1	−2

由此决定出两个无量纲参量:

$$\Pi_1 = \frac{\eta}{\rho d\bar{v}}, \qquad \Pi_2 = \frac{(\Delta p/\Delta l)\, d}{\rho \bar{v}^2}.$$

这里 Π_1 的倒数称为**雷诺数**(Reynolds number),记作 $\mathscr{R}$,是流体力学中的一个非常重要的无量纲量。根据 Π 定理,管道截面上的总压力梯度可写作

$$\frac{\Delta F}{\Delta l} = \frac{\Delta p}{\Delta l} S = \frac{\rho \bar{v}^2 S}{d} \Phi(\mathscr{R}) = \frac{\pi \rho \bar{v}^2 d}{4} \Phi(\mathscr{R}), \tag{2.45}$$

这里的无量纲函数 $\Phi(\mathscr{R})$称为**管道阻力系数**,它只依赖于雷诺数,与流体的

其它具体性质无关。如果我们希望要流量的表达式，则有

$$Q_V = \bar{v} S = \frac{d}{\rho \bar{v} \Phi(\mathscr{R})} \frac{\Delta F}{\Delta l} = \frac{\pi d^3}{4\rho \bar{v} \Phi(\mathscr{R})} \frac{\Delta p}{\Delta l} = \frac{\pi d^4}{4\eta \mathscr{R} \Phi(\mathscr{R})} \frac{\Delta p}{\Delta l},$$

或
$$Q_V = \Gamma(\mathscr{R}) \frac{d^4}{\eta} \frac{\Delta p}{\Delta l}, \tag{2.46}$$

式中的另一个无量纲函数 $\Gamma(\mathscr{R}) = \frac{\pi}{4\mathscr{R}\Phi(\mathscr{R})}$.

从以上(2.44)和(2.45)式可以看出，全部问题已归结为求函数关系 $\Phi(\mathscr{R})$或 $\Gamma(\mathscr{R})$. 如果我们在一定直径的管道中用实验方法测水在其中流动时阻力系数 $\Phi(\mathscr{R})$与雷诺数 $\mathscr{R}$ 的依赖关系，所得数据可在研究其它液体(如石油、水银)在不同直径的管道中流动时加以利用。甚至可在许多情况下(速度远小于声速，从而压缩性不重要时)运用到空气在管道中的流动。图2－5很好地说明了这一点，水和空气的实验数据的确差不多落在同一曲线上。

流体的运动有两种截然不同的形式——层流和湍流. 图2－5所示实验数据表明，函数 $\Phi(\mathscr{R})$有两支，一支对应于层流，另一支对应于湍流。$\mathscr{R}$ 较小的一段曲线对应于层流，它是哈根(G. Hagen)于1839年和泊肃叶(J. L. M. Poiseuille)于1840年用实验建立的。在直管道中作层流流动的流线是平行直线，流体没有任何加速度，可以预期流量 Q_V 与惯性(密度 ρ)没有关系，亦即(2.44)式中的 $\Gamma(\mathscr{R})$为常数，或者说阻力系数 $\Phi(\mathscr{R}) \propto \mathscr{R}^{-1}$. 理论上可以证明，在这种情况下的流量公式为

$$Q_V = \frac{\pi d^4}{128\eta} \frac{\Delta p}{\Delta l}, \tag{2.47}$$

上式称为**泊肃叶公式**。与泊肃叶公式对比即可看出，$\Gamma(\mathscr{R}) = \frac{\pi}{128}$，从而 $\Phi(\mathscr{R}) = \frac{32}{\mathscr{R}} \propto \frac{1}{\mathscr{R}}$. 图2－5中左边的一段曲线就是按照泊肃叶公式画的，可以看出，在小雷诺数时它与实验数据符合得很好。可是在雷诺数 $\mathscr{R}$=2000～2600范围内实验数据分散了，存在一个过渡区，由层流向湍流过渡。在更大的雷诺数区域里实验重新聚敛到一条曲线上，这是湍流的区域。在湍流的区域里 $\Phi(\mathscr{R})$服从一条与泊肃叶公式的预言完全不同的规律。由层流向湍流过渡的雷诺数称为**临界雷诺数**，我们记作 $\mathscr{R}_\star$. 临界雷诺数不是一个很确定的数值，在不同的情况下其数量级都可以相差很大。

3.2 流体的相似性原理之二——运动物体受阻问题

现在我们来看在流体中以匀速 v 运动的物体所受的阻力问题，这也是

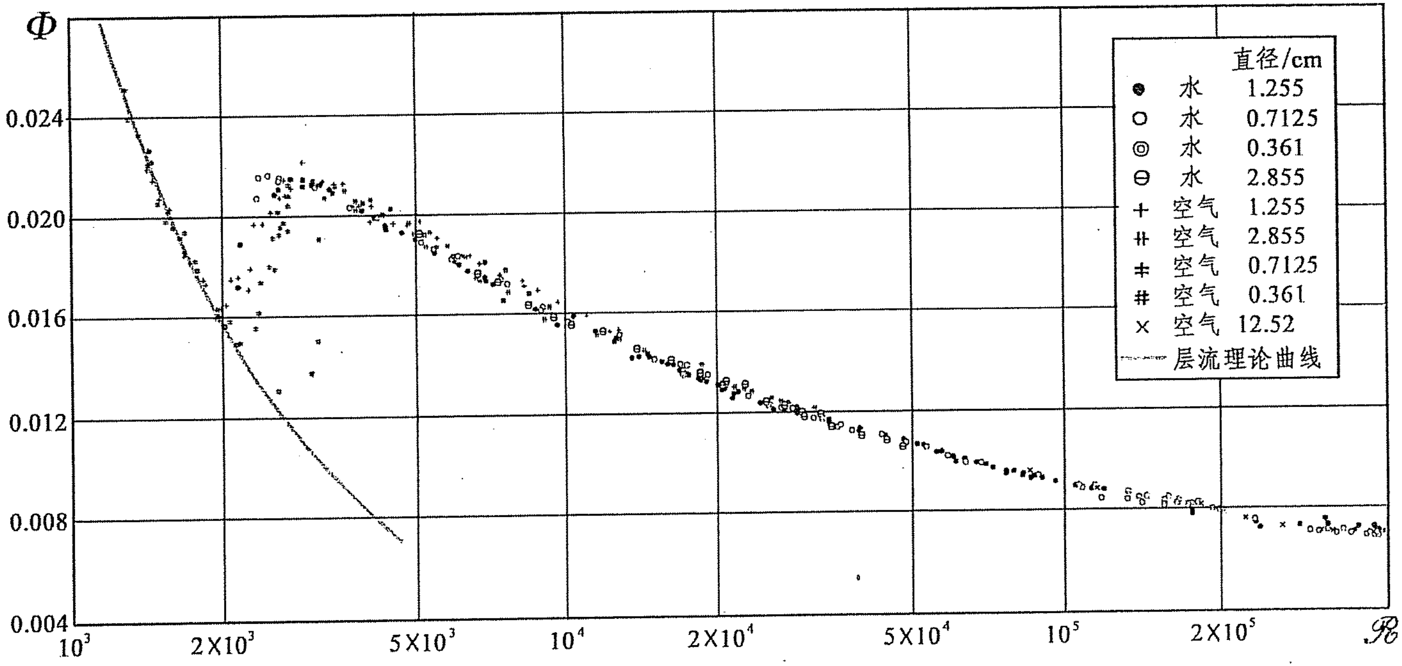

图 2-5 圆形管道的阻力系数

物体不动，流体以速度 v 流动时给物体的曳引力。姑且认为曳引力 f 正比于单位体积内流体所含的动能 $\frac{1}{2}\rho v^2$ 和物体的横截面积 S，于是有

$$f = C_d \frac{\rho v^2 S}{2}, \tag{2.48}$$

式中 C_d 叫做曳引系数(drag coefficient)，从量纲上看，它应是个无量纲的系数，只可能与无量纲的雷诺数 $\mathscr{R}$ 有关。对于球体，$S=\pi r^2=\pi d^2/4$. 图2－6所示为对球体的曳引系数 C_d 与雷诺数函数关系的实验曲线，由图可见，实验数据还是很好地聚敛在同一条曲线上的。当 $\mathscr{R}\ll 1$ 时，$C_d \propto \mathscr{R}^{-1}$；当 $\mathscr{R}\approx 10^3 \sim 10^5$ 时，C_d 几乎与 $\mathscr{R}$ 无关。故曳引系数可以写成[1]

$$C_d = \frac{C_0}{\mathscr{R}}(1 + C_1 \mathscr{R}), \tag{2.49}$$

也就是说，对于小雷诺数，阻力 $f \propto \eta$ 和 v；对于大雷诺数，$f \propto v^2$，与黏度无关。

从图2－6还可看出，当雷诺数再增大到一定程度，曳引系数 C_d 会突然急剧下降，这现象叫做曳引力崩溃(drag crisis)。

对于在黏性流体中作缓慢运动的小球，其所受阻力有个著名的公式——斯托克斯公式：

$$f = 6\pi\eta r v, \tag{2.50}$$

式中 $r=d/2$ 为小球半径。这公式是用层流的理论计算出来的，根据它可以定出(2.48)式中的 $C_0=24$(进一步的理论计算得 $C_1=3/16$)。

例题12　小孩玩的气球皮重10g，吹胀后直径达30cm，试按斯托克斯公式计算它在空气中的下落速度。你认为结果符合实际吗？

解：　当重力与空气阻力平衡时：

$$mg = 6\pi\eta r v,$$

气球以速度 v 匀速下落：

$$v = \frac{mg}{6\pi\eta r}.$$

室温下空气的黏度 $\eta = 1.8\times10^{-5}\,\mathrm{N\cdot s/m^2}$，代入上式，得

$$v = \frac{10^{-2}\,\mathrm{kg}\times 9.8\,\mathrm{m/s^2}}{6\pi\times1.8\times10^{-5}\,\mathrm{N\cdot s/m^2}\times0.15\,\mathrm{m}} = 1.9\times10^3\,\mathrm{m/s}.$$

这速率接近声速的6倍，显然是荒唐的。▌

问题出在哪里？经验告诉我们，在空气中飘浮的气球，下落是很慢的，速率最多每秒几十厘米。按 $v\approx50\,\mathrm{m/s}$ 估算，取空气密度 $\rho=1.29\,\mathrm{kg/m^3}$，得

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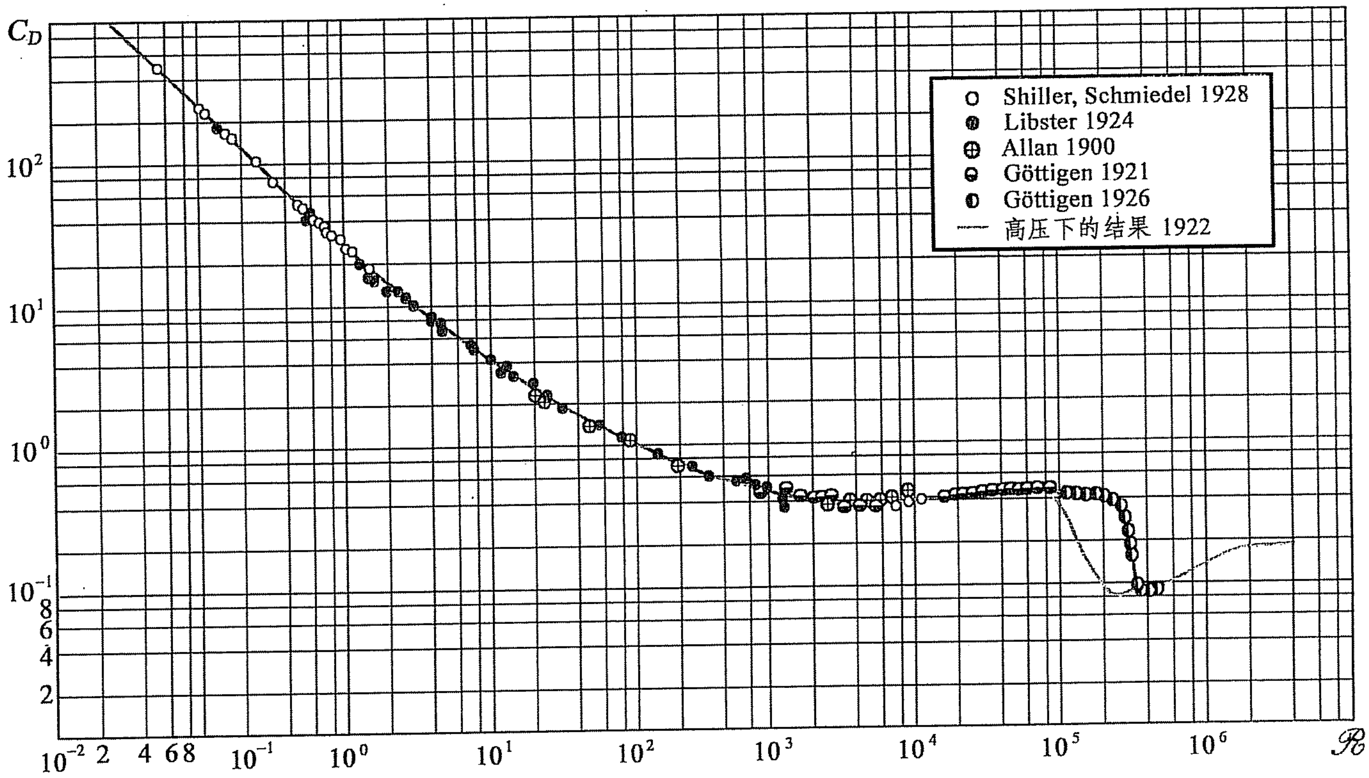
[1] 参见：Л. Г. 洛强斯基. 液体与气体力学. 下册. 林鸿荪等译. 北京：人民教育出版社，1959. 542.



C_D
10^2
10^1
10^0
10^{-1}
8
6
4
2
10^{-2} 2 4 6 8 10^{-1} 10^0 10^1 10^2 10^3 10^4 10^5 10^6 $\mathscr{R}$
Shiller, Schmiedel 1928
Libster 1924
Allan 1900
Göttigen 1921
Göttigen 1926
高压下的结果 1922

图 2–6 球体的曳引系数

气球下落时的雷诺数

$$\mathscr{R}=\frac{\rho d v}{\eta}=\frac{1.29\,\mathrm{kg/m^3}\times 0.3\,\mathrm{m}\times 0.5\,\mathrm{m/s}}{1.8\times 10^{-5}\,\mathrm{N\cdot s/m^2}}\approx 10^5\,!$$

原来速率虽“小”，雷诺数却不小。对于这样大的雷诺数，斯托克斯公式早已不成立了，这时我们需要用高雷诺数的公式。在(2.48)式中略去第一项，得曳引系数 $C_d=C_0C_1=24\times3/16=9/2$，代入(2.47)式，令其中阻力 f 与重力 mg 平衡，得气球下落的终极速度(terminal velocity)为

$$v=\frac{4}{3}\sqrt{\frac{mg}{\pi d^2}}\approx 0.8\,\mathrm{m/s},$$

这数值就比较接近真实情况了。

从上面所述我们看到，无量纲参数 $\mathscr{R}=\rho v l/\eta$ 在流体力学里的重要性。这里 l 是我们所讨论的问题里的特征长度，如管道的直径、飞行物的几何线度等。新设计的飞机是要在风洞(wind tunnel)里做模拟实验的。模型飞机的尺寸 l 变小了，要保持雷诺数不变，其它参量就得改变，或者加大 v，或者加大 ρ，或者减小 η. 气体动理学理论(kinetic theory of gases)告诉我们，在一定的温度下 η 与 ρ 无关，故可以加大空气密度和风速来维持雷诺数不变。所以，在现代航空技术中人们建造压缩空气在其中作高速循环的密封型风洞来做模拟试验。

3.3 静力学模拟的相似性原理

不要以为只有动力学模拟中才有保持物理相似性的困难，其实在静力学中也存在类似的问题。现在我们来考虑工程上弹性结构(如桥梁桁架)的模拟问题。各向同性建筑材料的弹性性能由两个参量来表征。在这里我们选杨氏模量 Y 和泊松比 σ，前者的量纲为 $\mathrm{ML^{-1}T^{-2}}$，后者无量纲。如果此机构是在重力下达到平衡的，则单位体积的重量 ρg 将是一个重要的参量。加上特征长度 L 和负载力 P，共五个参量。除原有的一个无量纲量 σ 外，稍加分析我们即可发现，在剩下的四个量中只有两个的量纲彼此独立，我们还可以找到另外两个无量纲量

$$\Pi_1=\frac{P}{L^3\rho g},\qquad \Pi_2=\frac{Y}{L\rho g},$$

若模型采用与 实物相同的材料来制做，则 Y、σ、ρ 不变，重力加速度 g 通常也是不变的。令 P 按正比于 L^3 的比例缩小，可保证 Π_1 不变，但怎样才能在 L 缩小时保证 Π_2 不变？从上式看来好像没什么办法了。实际上出路尚有一条，加大 g！把模型装在离心机上甩，用惯性离心力来模拟重力，以增大有效的 g. 实际中正是这样做的(参见图 2－7)。

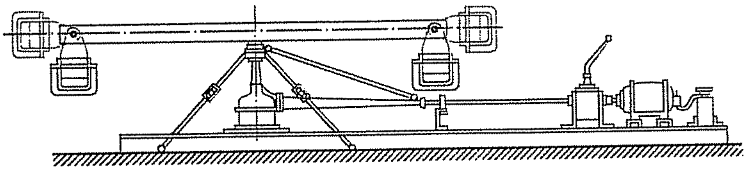


图 2-7 模型试验用的离心机

3.4 原子弹爆炸火球

1945年7月16日美国在新墨西哥州试爆了第一颗原子弹，1947年美国军方将一系列反映原子弹爆炸后的火球随时间扩展的照片解密，在报刊杂志上公布了(见图2－8和图2－9)，但该原子弹释放的总能量仍旧保密。

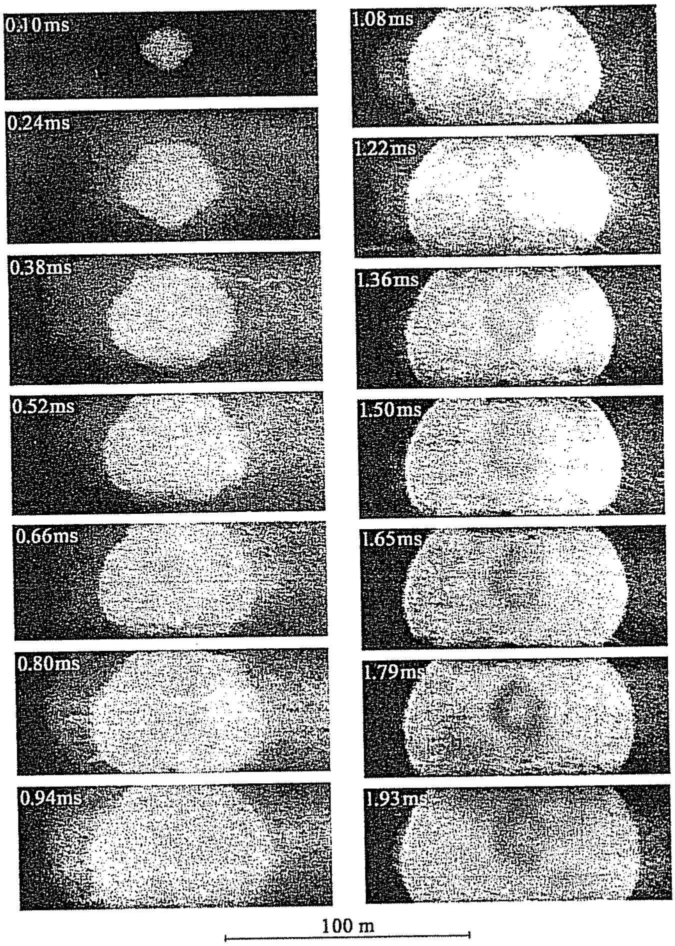


图 2-8 原子弹火球照片序列之一

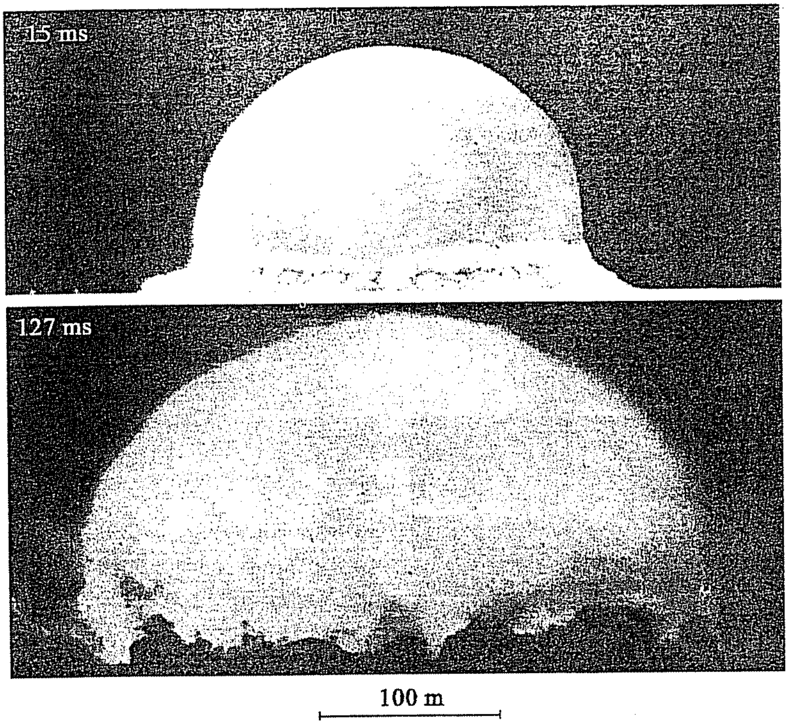


图 2-9 原子弹火球照片序列之二

英国力学家泰勒(G. I. Taylor)1950 年发表了一篇文章[1]，根据这些照片序列居然将原子弹的能量算出来了。这惹得美国军方很不高兴，客气地抱怨他不该发表这一结果。

泰勒是怎样计算的？他利用的是量纲分析。

火球半径 R 随时间 t 变化的数据记录如下表：

t/ms	R/m	t/ms	R/m	t/ms	R/m	t/ms	R/m	t/ms	R/m
0. 10	11. 1	0. 80	34. 2	1. 50	44. 4	3. 53	61. 1	15. 0	106. 5
0. 24	19. 9	0. 94	36. 3	1. 65	46. 0	3. 80	62. 9	25. 0	130. 0
0. 38	25. 4	1. 08	38. 9	1. 79	46. 9	4. 07	64. 3	34. 0	145. 0
0. 52	28. 8	1. 22	41. 0	1. 93	48. 7	4. 34	65. 6	53. 0	175. 0
0. 66	31. 9	1. 36	42. 8	3. 26	59. 0	4. 61	67. 3	62. 0	185. 0

用对数坐标作二者关系的曲线如图 2 - 10，我们发现它是一条斜率为 0. 4 的

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[1] G. I. Taylor, *Proc. Roy. Soc.* , **A201** (1065), 175, 1950.

G. Batchelor, *The Life and Legacy of G. I. Taylor*, Cambridge University Press, 1996.

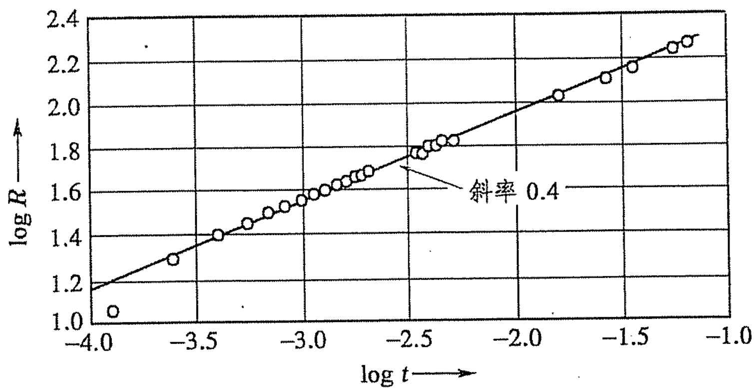

图 2-10 原子弹爆炸火球 R-t 关系

直线。

泰勒研究流体力学方程时，得到一种自相似运动(self-similar motion)的解，气体中从一点发生强爆炸时产生的冲击波属于这种自相似运动，它的解只依赖能量 E、气体密度 ρ 和定压定体热容比 γ 等三个参量。若用量纲分析，再加上 R 和 t，五个变量中 γ 是无量纲的，另外四个可组成一个无量纲量：

$$\Pi=\frac{Et^2}{\rho R^5},$$

按 Π 定理 R 和 t 的关系可写成

$$R=\left(\frac{E}{\rho}\right)^{1/5}t^{2/5}\Phi(\gamma), \tag{2.51}$$

即 $R\propto t^{2/5}$. 令泰勒感到此结果没有把握的是在原子弹火球中温度极高，从而 γ 会变。在常温下空气分子是双原子分子，不计振动自由度，$\gamma=7/5=1.40$. 如果在高温下分子振动自由度激发起来，则 $\gamma=9/7\approx1.29$，然而若分子解离成单原子，则 $\gamma=5/3\approx1.67$. 如果火球各处 γ 不均匀，自相似运动的条件遭到破坏，上述量纲分析的结果就不可信了。当他用从序列照片得到的数据作 $\log R$-$\log t$ 图得到 $-2/5=-0.4$ 的斜率时，信心大增，可能是某种平均效果使 γ 可以看做是均匀的。从上式把 E 解出来，得

$$E=\Psi(\gamma)\frac{\rho R^5}{t^2}, \tag{2.52}$$

式中 $\Psi(\gamma)=\Phi^{-5}(\gamma)$ 与 $\Phi(\gamma)$ 同为 γ 的无量纲函数。若假定 $\Psi(\gamma)$ 的数量级为 1，取 $\rho=1.25\,\mathrm{kg/m^3}$，平均说来 $R^5/t^2=6.67\times10^{13}\,\mathrm{m^5/s^2}$，则由上式可估计出原子弹爆炸时释放出能量的数量级：

$$E\sim\frac{\rho R^5}{t^2}\approx8.34\times10^{13}\,\mathrm{J}\sim10^{14}\,\mathrm{J}.$$

泰勒曾为无量纲函数 $\Psi(\gamma)$ 推导过一个理论公式，按此公式计算，不同 γ 值下 $\Psi(\gamma)$ 取值如下，按(2.52)式算出相应的能量值的也列在下面。

γ	1.20	1.30	1.40	1.667
$\Psi(\gamma)$	1.727	1.167	0.856	0.487
$E/10^{13}$ J	14.4	9.74	7.14	4.06
TNT 当量 * / 万吨	3.4	2.29	1.68	0.95

* 1 吨 TNT 当量 $= 10^6$ kcal $= 4.18\times10^9$ J.

如果取等效的 γ 值为 1.40，则 $E=7.14\times10^{13}$ J $=1.68$ 吨 TNT 当量，这数值是比较符合实际的。

§4. 生物界的标度律

4.1 大人国、小人国的比喻

未受过物理训练的人，常以为任何事物的尺度都可以随意地放大和缩小。古今中外，有关大人国、小人国的构想屡见不鲜。先从我国古籍中摘出几条看看。晋张华撰《博物志·异人》云："龙伯国人，长三十丈，生万八千岁而死"；又《博物志·外国》云："大人国，其人孕三十六年"；汉东方朔撰《神异经·西北荒经》云："西北荒中有小人，长一分，其君朱衣玄冠，乘辂(音 hé，挽车横木)车马，引为威仪"。这类描述大多记载了上古时代的神话，其中着重的是身躯的大小，间或涉及生命的节奏。从量纲的角度看，长度 L 和时间 T 二者之间该有什么比例关系？谁也说不清。较近的著作描写得细致得多。18 世纪英国作家斯威夫特(J. Swift)写了一本有名的讽刺小说《格列佛游记(Gulliver's travels)》，书中述说了主人公格列佛在小人国和大人国的奇遇。固然这同样是无稽之谈，由于叙述较具体，我们不妨故作认真，给它来一番物理分析，倒也是一次很好的相似性原理的练习。

为了方便，我们假定这些大人和小人的几何线度 l 都与正常人差一个数量级，但同为在地球上生活的血肉之躯，体内的成分和物理、化学过程都是相似的。

首先探讨一下大人国的公民。设身体的平均密度与我们相同，从而他们的体重 $\propto l^3$，而骨骼的截面积 $\propto l^2$，因此单位截面上承担的静态负荷 $\propto l$，即等于我们的 10 倍。运动使骨骼增加另一部分负荷，除重力外，运动时受到的惯性力正比于加速度。按《游记》书中描写，在大人国里的生活节奏与我们差不多，故一切速度和加速度 $\propto l$，从而动态负荷又大了 10 倍，即在单位骨骼截面上承担的负荷可能等于我们的 100 倍。所以大人国里的公民决不敢作稍微剧烈一点的动作，否则他们骨折的可能性比我们大 100 倍。

从高度跌下来时，重力 $\propto mg \propto l^3$，而空气阻力 $\propto$ 面积 $\propto l^2$，故最后达到的极限速度 $\propto l$，对大人国的公民来说，也大了10倍，即重力对他们的威胁比我们大得多。与此相反，小人国里的公民在这一点上是享有优越性的，他们在剧烈的运动中骨折的危险比我们小100倍，重力给他们带来的威胁也要小得多。《游记》中描写，小人国中有一人仓惶地从躺着的格列佛的腰部往下跳，竟跌伤了。其实，哪怕他从站着的格列佛的肩膀上跳下来，也不会受伤，虽然按比例，这高度相当他们的几层楼。正像我们的一只老鼠从天花板上跌落一样，不是什么危险也没有吗？

然而，小人国里的公民也有自己的苦恼。我们的体温是恒定的(37°C)，与周围的环境保持一定的温差 ΔT. 按照牛顿冷却定律，单位时间从体表散失的热量正比于皮肤的面积($\propto l^2$)和 ΔT，这要靠体内发生的热量来补偿，这热量正比于体重或体积($\propto l^3$)。可见，与自身的体重相比，小人国里的公民必须摄入比我们大10倍发热量的食物。此外，按衍射公式计算，眼睛最小分辨角的理论极限 $\theta \propto \lambda/D$，这里 λ 是可见光的波长，D 为瞳孔的直径。可见，小人国公民的最小分辨角比我们大10倍。假定他们的明视距离比我们的小10倍，则在他们书上印的字应和我们的一样大。但是他们的书页线度比我们的小10倍，如果我们的书上每行印30个字的话，他们的书上每行只印3个字，每页里的行数也要相应地少10倍。

沿此线索，读者恐怕还会想到更多的方面，说明几何相似的东西在物理上常常是不相似的。地球上人类身躯的尺度只能是现在的数量级，过多地放大和缩小，在物理上都是不现实的。

4.2 单一特征长度的标度律

上述对大人国、小人国的分析完全可用于我们现实世界里各种大小的动物。

在日常的直觉中，人们习惯于用几何相似地放大(或缩小)的倍数去推论其后果，譬如，曾经流传过一种说法：跳蚤可以跳一米高，若它长得像人那样大，就能跳一千多米高。这里错误地按几何线度放大的倍数去推算其它某种后果。尽管早在三百多年前伽利略就指出这种观点是错误的，但谬种仍在不断流传。合乎实际的假设应该是，对所有的动物，每单位质量的肌肉所提供的能量大体是同量级的。这样，跳到高度 h 需要的能量 $mgh \propto m \propto l^3$，而肌肉能够提供的能量也正比于 $m \propto l^3$，从而 h 的数量级与 l 无关。即各种动物，无论大小，能跳的高度在数量级上大体一样。像人那样大的跳蚤最多也只能跳几米高！

伽利略是物理学的开山祖师，在标度变换下物理规律不是不变的，这

一思想首先也是伽利略提出的。在他看来，这一思想和他发现的运动定律同样重要。他把两者都写进了他最重要的著作之中。这本著作是《关于两种新科学的对话(Dialog on Two New Sciences)》。这里的图2－11就是从该书中复制而来的。图中是两块狗骨头，小的一块是正常狗的骨头，大的一块是一种假想巨犬的骨头。这只巨犬的长、宽、高都比正常狗大一倍。伽利略指出，为了能够支撑自身的体重，这只巨犬的骨头必须具有如此一副长粗不成比例的模样。在《关于两种新科学的对话》中还记载了威尼斯造船厂一位有经验工匠的话："在最大的船只下水时必须格外注意，以避免大船在它们自身的巨大重量下发生开裂的危险。"这段话引发了书中对话者的争论，问题的实质是，将小船的设计按比例几何相似地按比例放大，造出的船是否变得不结实了？用量纲来分析，这是显然的。船的重量（包括自重与载荷）$\propto$ 体积$\propto l^3$，而船的骨架的横截面积 $S \propto l^2$，从而这里的标度律是单位面积上的负荷 $\propto l$，船愈大，单位面积上的负荷愈大，开裂的危险也就愈大。这个道理可以用到其它许多地方，譬如马从两倍于它身高的地方跌下来会摔断骨头，而猫可以从五六倍于自己身高的地方跳下来安全无恙，老鼠从天花板上跌落下来（这相当于它身高的几十倍），什么危险也没有。大象粗壮的四条腿，与其它小型动物的腿是不成比例的。鲸鱼这样大的哺乳动物只能生活在海里，在岸上搁浅，失去了水的浮力，它们就会被自身的重量压死。

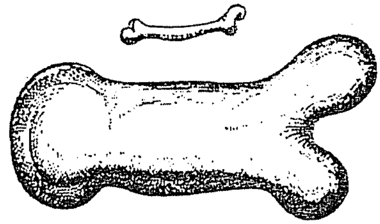

图2－11　伽利略《关于两种新科学的对话》里的插图

例题13　如果炖一只4.5kg的火鸡需要3小时，炖一只9kg的火鸡需要多长时间？

答：　3小时$\times 2^{2/3} \approx 5$小时。这种估计与一些著名的烹饪指南上所说的相当符合。▌

鸵鸟是当今世界上最大的鸟，有人说它不会飞是因为翅膀退化了。如果鸵鸟长了一副与它身体的大小成比例的翅膀，是否它就能飞起来了呢？

飞翔的必要条件是空气的上举力f至少与体重$W=mg$平衡。按高雷诺数的情况估算，量纲法给出如下的比例关系：

$$f = CS\rho v^2, \tag{2.53}$$

式中S是翅膀的面积，ρ为空气密度，v是起飞或飞行的速度，C是个无量纲的比例系数。能起飞的条件是$f > W$，即

$$v > \sqrt{\frac{mg}{CS\rho}}.$$

我们作一个简单的几何相似性假设，设鸟的几何线度为 l，质量 $m \propto$ 体积 $\propto l^3$，$S \propto l^2$，于是起飞的临界速度 $v \propto \sqrt{l}$。燕子的最小飞行速率大约是 20 km/h，而鸵鸟的体长 l 大约是燕子的 25 倍，从而起飞的临界速度大 $\sqrt{25}=5$ 倍，即需要 100 km/h，这已接近飞机起飞速率的量级，再擅长奔跑的鸵鸟也是做不到的。实际上鸵鸟的奔跑速度大约只有 40 km/h。

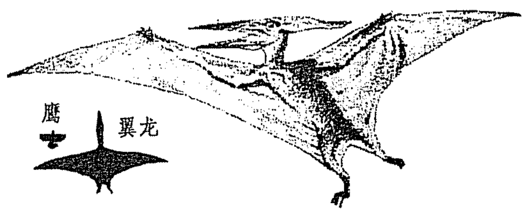

图 2-12 经艺术家复原的翼龙形象及其与鹰的比例

上面的比例关系还告诉我们，像麻雀这样的小鸟和天鹅这样的大飞禽，起飞和着陆的姿势是不会相同的。小鸟只需从枝头跳到空中用翅膀拍打一两下，就能达到飞行的临界速率；大鸟则首先要沿着地面或水面奔跑或滑行一段，或者从一个高的栖木向下俯冲，才能达到飞行的临界速度。当今最大的飞行动物是信天翁，它的翅膀全长 3 m 多，通常是借助于上升的气流在高空作长时间的翱翔的。在恐龙时代有一种会飞的爬虫类动物，叫做翼龙（图 2-12），其翅膀全长竟达 16 m，看来这样大的动物是不能起飞的，除非爬上峭壁或栖息在高大的林梢，像滑翔机那样从那里滑翔。

本小节所用的标度律都是以单一特征长度 l 为基础的，这有时过于简单化了。下面作些改进。

4.3 圆柱的标度律

对于一定半径 r 的实心圆柱体，竖起来的时候其高度 l 有个临界值 l_c，超过它，在自重力的作用下直立的姿态不再是稳定的，它开始弯折。为了定性地估算这个 l_c，我们考虑弹性势能和重力势能的变化。如图 2-13，我们用柱长 l 对曲率中心所张的角度 θ 来描述形变，按(2.28)式 $M_内 \propto Yr^4/R$，

$$M_内 \propto \frac{\pi Yr^4}{R} \propto \frac{Yr^4}{l}\theta, \quad \left(\theta = \frac{l}{R}\right)$$

从而弹性势能的增量为

$$\Delta E_{p弹} = \int_0^\theta M_内 \, d\theta \propto \frac{Yr^4}{l}\theta^2, \tag{2.54}$$

弯折时柱的重心下降量为（见图 2-13）

$$\Delta h = \frac{l}{2} - R\sin\frac{\theta}{2} = \frac{l}{2} - \frac{R\theta}{2}\left[1 - \frac{1}{3!}\left(\frac{\theta}{2}\right)^2\right] = \frac{l\theta^2}{48}.$$

柱体的重量为 $W=\pi r^2 l\rho g$，故重力势能的改变为

$$\Delta E_{p重}=-W\Delta h=-\frac{\pi r^2 l^2 \rho g}{48}\theta^2, \tag{2.55}$$

式中负号表示 $E_{p重}$ 减少。当 $|\Delta E_{p重}|\geqslant\Delta E_{p弹}$ 时，直立柱体失稳，故 l_c 由 $|\Delta E_{p重}|=\Delta E_{p弹}$ 决定。从(2.53)和(2.54)式

$$r^2 l_c^2 \rho g\theta^2 \propto Yr^2\theta^2/l_c,$$

即

$$l_c \propto \left(\frac{Yr^2}{\rho g}\right)^{1/3} \propto r^{2/3}. \tag{2.56}$$

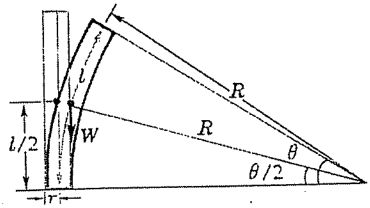

图 2－13 直立圆柱的弯折

此式表明，一圆柱体在自重力作用下能抗弯折的临界高度并没有它的半径 r 增长得快。例如，当半径加倍时，其临界高度只增大 $2^{2/3}=1.59$ 倍。

把(2.54)式运用到树木的高度与粗细关系的问题上，是饶有兴味的。当然，树不是光杆，其上还有树冠；此外，决定树高的因素也未必就是它的抗弯能力。图 2－14 给出一些北美洲树木的 l-d 关系的数据（$d=2r$ 为直径）。上面那条实线代表抗弯折临界高度，纵横坐标都是按对数标度的，此直线的斜率等于 2/3. 虚线的方程也具有 $l=Cd^{2/3}$ 的形式，为拟合那些实际的分散数据点，这里取 $C=34.9$. 看来没有数据点出现在那条代表理论极限的实线之上，且拟合曲线的斜率也接近 2/3. 以上结果加强了我们这样的信念，即抗弯折强度是决定树木高粗比的关键因素。

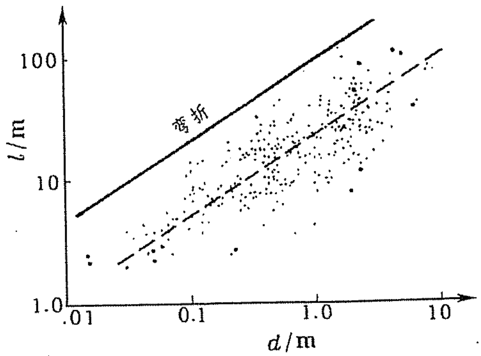

图 2－14 一些树木长粗比的数据

如果像 4.2 节里那样，只用同一个特征长度 l 来表征动物的体型，我们得到体表面积 $S\propto l^2$ 和质量 $m\propto$ 体积 $\propto l^3$ 的结论，即 S 与 m 的关系应为

$$S\propto m^{2/3}=m^{0.67}.$$

人们预期，代谢率（即体内利用食物产生热量的速率）也按同一比例，即 $\propto m^{2/3}=m^{0.67}$. 但是 1932 年 Kleiber 对哺乳动物的实验观察结果并不是这样的，而是 $S\propto m^{0.63}$（图 2－15）和产热率 $\propto m^{0.75}$（图 2－16）。差别虽不大，但已足以对上述单一特征长度的模型提出挑战。

为了改善我们的模型，应该考虑如下事实，即动物躯体的大部分更接近

圆柱状，这或许是一种经得住弯折的构造。若果真如此，身体各部分的长度 l 与半径 r 应有如下比例关系：

$$l \propto r^{2/3} \quad 或 \quad r \propto l^{3/2},$$

圆柱体的体积 $\pi r^2 l \propto l^4$，从而质量 $m \propto l^4$，或者说 $l \propto m^{1/4}$，因此体表面积（主要是柱形的侧面积）

$$S \propto rl \propto m^{5/8}.$$

5/8 = 0.625，非常接近 0.63.

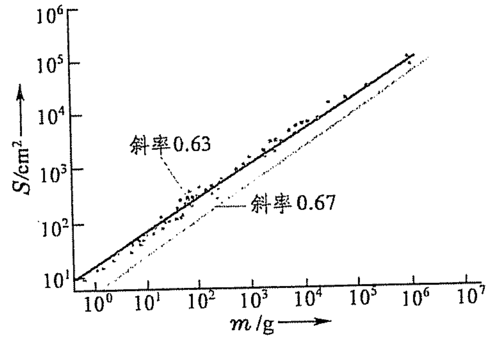

图 2-15 各种大小哺乳动物的体表面积

为求得代谢速率，我们不仅要计算面积，而且要计算用于使肌肉弯曲的功率。这功率 P 等于所施的力乘以肌肉收缩的速率 v. 实验发现：所有哺乳动物每单位面积肌肉发出的力 $\sigma = F/S$ 是相同的；对于随意肌纤维，肌肉收缩的速率也相同，故

$$P \propto \sigma S' v \propto S',$$

这里 S' 是肌肉的横截面积，它正比于 $r^2 \propto m^{3/4}$，于是

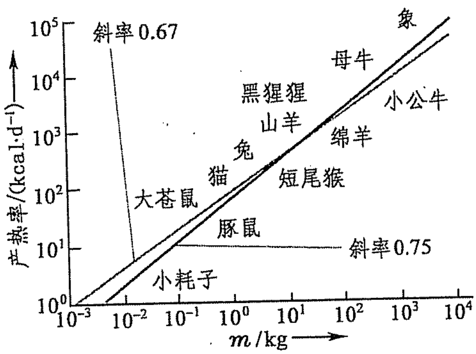

图 2-16 各种大小哺乳动物的代谢率

$$P \propto m^{3/4} = m^{0.75}.$$

如果我们假设产热率正比于功率 P，图 2 - 16 的实验结果便得到了解释。由于代谢过程要利用通过肺壁所吸收的氧气，所以肺的表面积也正比于 $m^{0.75}$. 这结果也得到实验上很好的证实，在生物学中称为 Kleiber 定律。[1]

每次心跳所抽运的血液量正比于心脏的体积，假定它 $\propto m$，则单位时间抽运的血液量 $\propto m\nu$，这里 ν 是心跳的频率。血液的抽运量应与需求相适应，即应有 $m\nu \propto m^{0.75}$，由此得

$$\nu \propto m^{-0.25}.$$

这意味着大型哺乳动物应具有较低的脉搏频率。实验上也观察到了这一点。如果我们进一步对哺乳动物的心率和它们的寿命作比较，就可发现，寿

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[1] M. Kleiber, *The Fire of Life*, Wiley, N. Y. 1961.

命与心跳周期相比，对所有哺乳动物都大体相同。这就是说，不管躯体大小如何，每一哺乳动物一生中心跳的次数大体相同。[1]

上面以相似律(2.54)为基础来分析哺乳动物身体各部分的比例关系，并不是无可非议的。有人提出，抗骨折才是决定性的因素。若那样，则(2.52)式应代之以 $l_c \propto r^3$. 于是后面一系列的结论都要作相应的改变。生物学的问题较物理学复杂得多，对这个问题的讨论我们就此搁笔了。

§5. 生长的标度律与自组织

5.1 生长的标度律

我们在第一章 1.3 标度变换一节中谈到，鹦鹉螺的曲线具有标度不变性：$\ln r \propto \theta$，或者说，$r \propto e^\theta$. 鹦鹉螺是沿螺线转着圈生长的，在一定意义上说，θ 相当于时间 t，所以上述标度不变律反映了鹦鹉螺随时间 t 按指数律生长。这种规律在生物界具有相当的普遍性，只要增长率正比于自身的大小，指数生长律便是必然的。

指数增长律不仅适用于生物个体的生长，也存在于种群或人口的增殖问题中。以 N 代表人口，$\mathrm{d}N/\mathrm{d}t$ 为人口的增长率，如果人口增长率正比于现有人口：

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N \quad (\lambda > 0), \tag{2.57}$$

此微分方程的解为

$$N = N_0 e^{\lambda t}, \tag{2.58}$$

这个人口增长律是 1798 年英国经济学家马尔萨斯(Thomas Malthus)提出来的。

马尔萨斯的理论曾经对生物学家达尔文(Charles Darwin)提出“物竞天择”的进化论产生过巨大的影响。达尔文在自传里写道：“1838 年 10 月，即在我开始作系统的考察 15 个月后，我作为消遣偶然地读到了马尔萨斯的《人口论》。当时对动物和植物习性长期而持续的观察已使我很好地意识到，生存竞争是普遍存在的。马尔萨斯的理论立即打动了我，使我认识到，在这种环境下，有利的变异将被保留下来，不利的变异将被淘汰掉。于是我终于形成了一个能用来指导我工作的理论。”马尔萨斯的理论之所以震动了达尔文，是因为它揭示出，自然界的动物和植物可以产生比能存活下来多得多的后代，人类也不例外。按马尔萨斯的理论，如果对人口的增长不加控制的话，贫困与饥荒必将导致人类的毁灭。对马尔萨斯的理论有的社会革命家不以为然，他们认为：“人人都有两只手”，人口是生产力，越多越好，在

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[1] T. McMahon, *Science*, **179** (1973), 1201.

“真正优越的”社会制度下，人类社会的一切弊端和灾祸都会根除。是这样吗？实践给了这种理论无情的打击。

1846 年比利时数学家 Verhulst 发展了马尔萨斯的理论，将马尔萨斯方程(2.57)修改为

$$\frac{\mathrm{d}N}{\mathrm{d}t}=\lambda\left(1-\frac{N}{N_{max}}\right)N \quad (\lambda>0), \tag{2.59}$$

这里 N_{max} 是为资源和环境允许的人口上限，超过此限，人们将难以生存和增殖。上式左端括弧内的第二项表达了这一点：当 $N \to N_{max}$ 时，人口趋于零增长；$N>N_{max}$ 时，人口出现负增长。微分方程(2.59)称为**逻辑斯蒂(logistic)方程**，其解为

$$N=\frac{N_{max}}{1+\left(\frac{N_{max}}{N}-1\right)e^{-\lambda t}}. \tag{2.60}$$

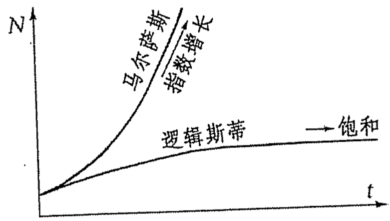


图 2-17 马尔萨斯和逻辑斯蒂人口增长律

图 2-17 所示为马尔萨斯方程和逻辑斯蒂方程解的曲线。可以看出，当 $N \ll N_{max}$ 时，N 近似地按指数增长；当 N 增长到足够大时，它趋于饱和值 N_{max}。

5.2 斐波那契数列与黄金分割

生物界的指数增长律还有一个离散的迭代理论。1202 年意大利数学家斐波那契(Fibonacci)出了一道算术题：“有人将一对兔子关在被围墙圈起来的所在。假设每月每对兔子生一对小兔，小兔一个月后即能生产，一年内将生产多少对兔子？”这个问题的答案见右表：第一个月只有一对成兔，一个月后它们生了一对小兔；到了第三个月，老兔又生一对小兔，上个月生的幼兔成熟了，此时共有成兔两对，幼兔一对；如此类推，即得表中的数字序列。表中的成兔数列

月份	成兔对数	幼兔对数
1	**1**	0
2	**1**	1
3	**2**	1
4	**3**	2
5	**5**	3
6	**8**	5
7	**13**	8
8	**21**	13
9	**34**	21
10	**55**	34
11	**89**	55
12	**144**	89

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 86, 144, …

称为**斐波那契数列**，它可用迭代法构成：1+1=2, 1+2=3, 2+3=5, 3+5=8, …，即相邻两项之和构成下一项。用公式来表达，有

$$F_{n+1}=F_n+F_{n-1} \quad (n=1, 2, 3, \cdots), \tag{2.61}$$

只要给出初始两项

$$F_0=1, \quad F_1=1, \tag{2.62}$$

整个斐波那契数列便可迭代出来。

当 n 很大时，数列的增长渐近地趋近于指数律。我们可以按图 2－18 所示那样，先作一个边长为 1 的正方形，再作一个边长为 1 的正方形与之并列，然后作边长为 2 的正方形与前两个正方形并列，再作边长为 3 的正方形与前两个正方形并列，如此等等。用光滑弧线将各正方形对角顶点连起来，就成为一条美丽的螺线，它看起来与鹦鹉螺的对数螺线差不多，愈到外圈愈像。

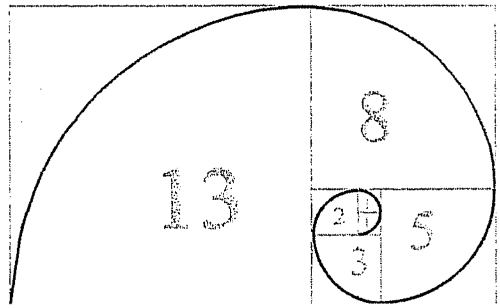


图 2－18 按斐波那契数列迭代而成的螺线

有趣的是，斐波那契数列与黄金分割有一定的联系。古希腊数学家欧几里得(Euclid)曾如下命题：如图 2－19 所示，将一线段 AB 分割为 AC 和 CB 两段，使 $\overline{AC}:\overline{CB}=\overline{AB}:\overline{AC}$，两段之比若何？设此比值为 $x:1$，则 $x/1=1/(x+1)$，即 x 满足下列二次代数方程：

图 2－19 黄金分割问题

$$x^2-x-1=0, \tag{2.63}$$

它的两个根为

$$x=\begin{cases}1+\tau=\dfrac{1+\sqrt{5}}{2}=1.618034\cdots,\\ -\tau=\dfrac{1-\sqrt{5}}{2}=-0.618034\cdots.\end{cases} \tag{2.64}$$

它们是具有无穷多位不循环小数的无理数。按此比例的分割，后世称为**黄金分割**。

由于 $1+\tau$ 和 $-\tau$ 都满足代数方程(2.61)式，不难看出

$$F_n\propto(1+\tau)^n \quad 和 \quad F_n\propto(-\tau)^n$$

都满足斐波那契数列的迭代关系式(2.61)，它们的线性组合

$$F_n=a(1+\tau)^n+b(-\tau)^n \tag{2.65}$$

也满足(2.61)式。将(2.64)式代人初始条件(2.62)式：

$$a+b=1,\quad a(1+\tau)+b(-\tau)=1,$$

即可将系数 a、b 确定下来：

$$a=\frac{1+\tau}{1+2\tau}=\frac{1+\tau}{\sqrt{5}},\qquad b=\frac{\tau}{\sqrt{5}}.$$

将 a、b 的数值代入(2.65)式,即得斐波那契数列 F_n 与黄金分割 τ 的关系:

$$F_n=\frac{1}{\sqrt{5}}\left[(1+\tau)^{n+1}-(-\tau)^{n+1}\right]=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \quad (2.66)$$

现在我们来看看 $n\to\infty$ 时斐波那契数列中相邻项比值的极限。因

$$\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n=0,$$

故由(2.66)式可得

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}=1.618034\cdots, \quad (2.67)$$

$$\lim_{n\to\infty}\frac{F_n}{F_{n+1}}=\frac{2}{1+\sqrt{5}}=\frac{1-\sqrt{5}}{2}=0.618034\cdots. \quad (2.68)$$

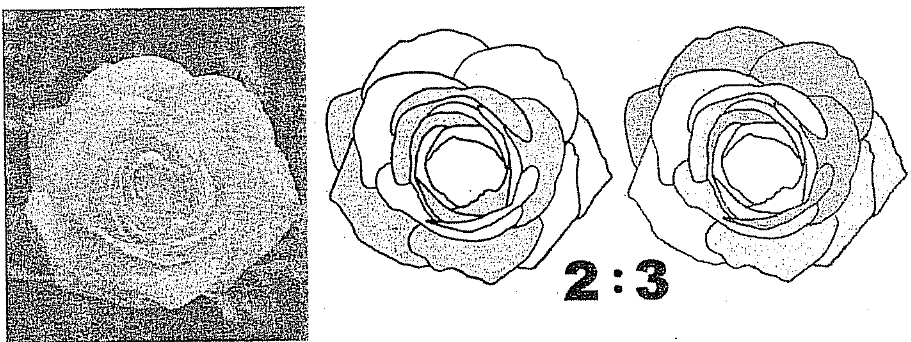
此外,由迭代关系(2.61)式还可得到另一个关系式:

$$F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}. \quad (2.69)$$

此式不难用数学归纳法予以证明。

5.3 植物的花序与斐波那契数列[1]

从孩提时代起我们就为自然界中一些与整数相联系的事物感到惊奇:三叶草具有三片叶,丁香花四瓣,梅花五瓣,雪花却是六角的。在玫瑰花(图 2－20a)的花瓣排布中你能发现什么简单的整数关系吗?细致的观察表明,这里有两套螺旋线:顺时针的二条(图 2－20b),逆时针的三条(图 2－20b),它们的比例是 2:3。



a　　　b　　　c

图 2－20 玫瑰花瓣的排布

[1] J. O. Mattila, *Proceedings of the International Workshop on teaching non-linear phenomena at schools and universities*, Lake Balaton, Hungary, April 1987, p. 143.

如果你闲来步入来松林，偶尔从地上拾起一个松果，你会现它也具有双螺旋结构（图 2－21）。数一数，顺、逆时针螺线数目之比为5:8。对于杉果（图2－22），两套双螺线数目之比是 8∶13。如果我们好奇地数一数菠萝的螺纹，两套双螺线数目之比也是 813 或 13:8。蒲公英是一种不太引人注意的小草，它的果实是一个毛茸茸的白色小球，风一吹，一颗颗带有“降落伞”的种子飘到远方，观察一下剩下的花蒂，

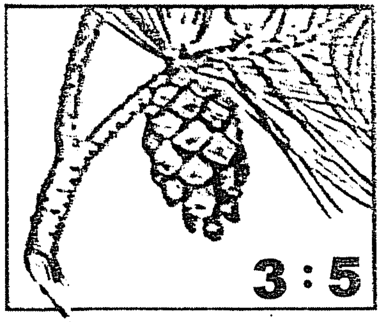


图 2－21 松果的双螺线结构

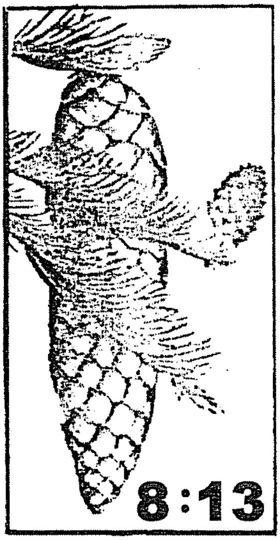


图 2－22 杉果的双螺线结构

图 2－23 蒲公英花蒂的双螺线结构

发现它也具有双螺旋结构（图 2－23）。数一数，两套双螺线数目之比是 21:34 或 34:21。也许你会联想到向日葵（图 2－24），数一数它花盘上两套双螺线数目之比，结果是 55:89 或 89:55。请

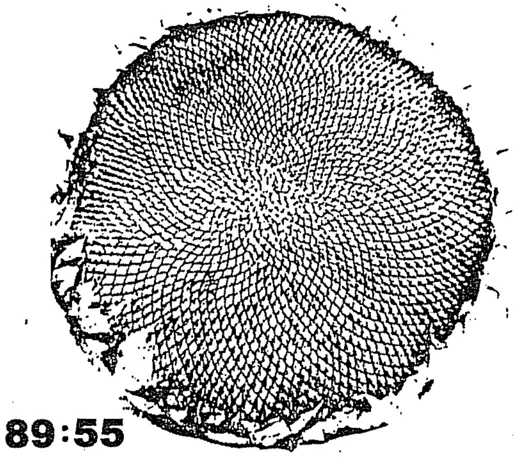


图 2－24 向日葵花盘的双螺线结构

看，2、3、5、8、13、21、34、55、89，这不正是斐波那契数列吗？我们惊叹，造物主竟是如此高明的数学家！

上述神秘现象背后生理过程的动力学机理是什么？这问题早就引起科学家的注意，提出了不少模型。19 世纪植物学家 W. Hofmeister 最先对茎的顶部做系统的观察，并对生长的动力学提出一些建议。他的意见可归纳成如下几点：

1. 芽苞生于茎的顶部边缘；
2. 由于生长，芽苞中心相对于顶点以速率 $V(r)$ 朝外运动；
3. 新芽按周期 T 产生；
4. 新芽苞在顶部边缘有最大发展空间（即离前面的芽苞都较远）的位置上产生。

20 世纪 90 年代物理学家 S. Douady 和 Y. Couder[1] 在此基础上将上述模型精确化，进一步做定量研究。他们假设，茎的顶部是轴对称的，其周围存在一个半径为 R 的区域（图 2－25），在其外芽苞相对于顶点沿纯径向运动，不再重组它们的角位置。他们发现，这里有一个无量纲的参数 $G=V_0T/R$，其中 $V_0=V(R)$，即芽苞在顶部边缘上的速度。植物学中还有另一个无量纲的参数 $P=r_{n+1}/r_n$，即相继出生的两个芽苞到顶点距离之比。如果我们假定生长速度 $V(R)$ 正比于 r：

$$\frac{\mathrm{d}r}{\mathrm{d}t}=V(r)=\frac{V_0}{R}r\propto r, \quad (2.70)$$

则 r 随时间 t 作指数增长：

$$r=R\mathrm{e}^{V_0t/R}, \quad (2.71)$$

这是符合植物生长实际的。由(2.71)式知

$$r_{n-1}=R\exp\left[\frac{(n+1)V_0T}{R}\right],$$

$$r_n=R\exp\left(\frac{nV_0T}{R}\right),$$

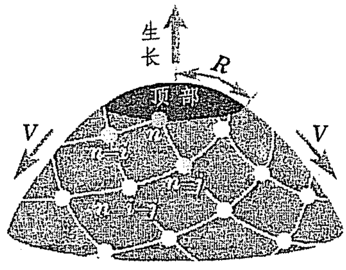


图 2–25 茎的顶部生长参量

从而 $$P=\frac{r_{n-1}}{r_n}=\exp\left(-\frac{V_0T}{R}\right)=\mathrm{e}^G, \quad 即 \quad G=\ln P.$$

即 G 与 P 是同一参量，但在实际中 P 比 G 容易测量。植物学家证实，植物生长序的发展主要是 P（或者说 G）连续递减的结果。

[1] S. Douady & Y. Couder, *Phys. Rev. Lett.*, **68**(1992), 2098; *J. Theor. Biol.*, **178**(1996), 255.

Douady 和 Couder 的研究方法有三:(1) 物理实验模拟,(2) 硬球模型的理论计算,(3) 计算机模拟。

(1) 物理实验模拟

实验装置如图 2-26 所示,在塑料圆池内盛有黏滞液体(硅酮),置于亥姆霍兹线圈 C_1、C_2 产生的磁场 $\boldsymbol{H}$ 中。在池的中央上方有一滴管,定期滴下一滴磁性液体(悬浮有铁磁微粒的液体)。液滴在磁场中磁化,成为磁偶极子。由于磁场沿径向增加,偶极子受到磁场的梯度力而沿径向朝外漂移。池的边缘有深沟以收容漂移过来的液滴。此外,各偶极子因取向相同而彼此排斥。在池的中央底部有一圆形隆起,使液滴刚滴下时处于不稳定状态,它将在先前滴下的偶极子的排斥作用下滑向一边。在此实验里,以磁偶极子模拟一个个芽苞,以它们之间的排斥力模拟芽苞争取生长空间时彼此远离的趋向。液滴的瞬时位置用可相机拍摄下来,如图 2-27 所示,图中编码代表液滴下滴的时间顺序。无量纲参数 G 的大小可通过液滴下滴的周期来调节。在实验中我们关注两相继液滴径矢之间的夹角 φ,称为发散角。

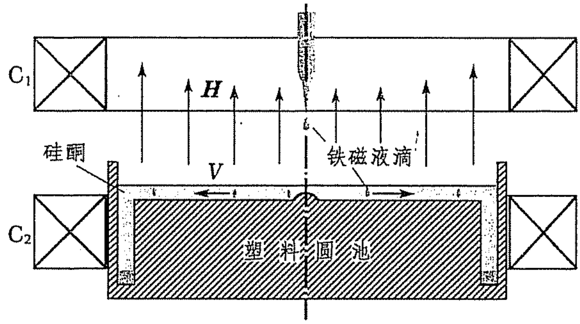
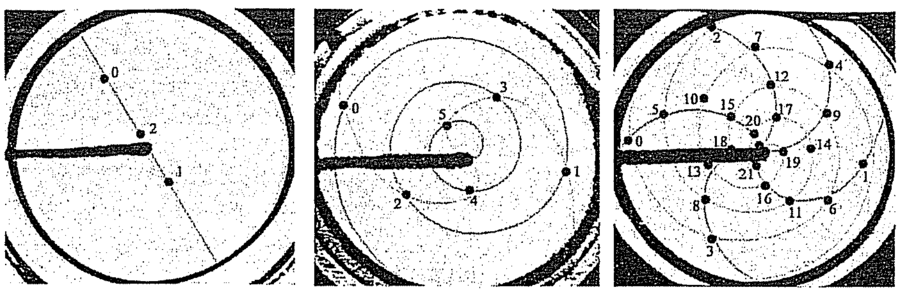


图 2-26 模拟植物生长序的物理实验装置



a b c

图 2-27 在模拟实验中形成的图样

实验开始时令 $G=1$(图 2-27a),这时相继滴下的液滴距离都比较远,只有前一个液滴的排斥力起作用,顶部圆周上距前一液滴最远的位置是与之 相对的位置,即 $\varphi=180°$,液滴都在一条直线上。在图 2-27b 里 G 减小到 0.7,相继液滴的距离因之减小。虽然液滴 1 仍出现在液滴 0 的相对位置

上，但液滴2出现时，就得同时考虑液滴1和液滴0的存在。这时液滴1较近而液滴0较远，前者的排斥作用比后者大，使液滴2出现在180°>φ>90°之间的位置。观测表明，这时φ=150°. 可以预料，当G进一步减小时，φ亦减小。在图2–27c中，G减小到0.15，φ=139°. 然而随着G的减小，φ不会无限制地减小。下文将说明，数值模拟表明(见图2–28)，G较小时，φ一开始从180°急剧下降到某个数值，然后有些小的振荡，最后趋于一个稳定值，此值大约在137.5°左右。理论上可以证明，当$G\to0$时，$\varphi\to360°\times(1-0.618034)\approx137.508°$，即对圆周角进行黄金分割。

从图2－27b和c中已可看出，各液滴的位置可用两组环绕方向相反的螺线串起来，其数目在图2－27b中$i=1$，$j=2$；在图2－27c中$i=3$，$j=5$。它们都是相接的斐波那契数。

(2) 硬盘模型的理论计算

1902年G. van Iterson曾用硬盘代表芽苞铺设柱面、锥面、平面的方法来产生植物生长序图样。在这里我们按下法产生几何图案：假定新出生

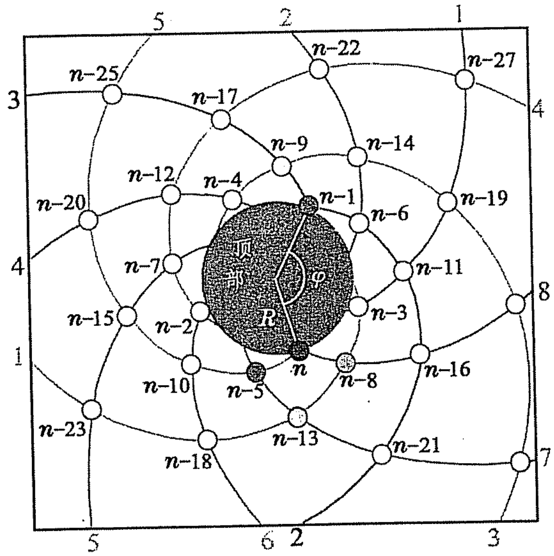


图2–28 硬盘模型铺设的图样($i=5$, $j=8$)

的单元n与前面出生的最近单元$n=i$和$n=j$等距(参见图2－28中的n和$n=5$和$n=8$)。若将T、R标准化为1，则无量纲的参数G的意义就是边缘上的速度V_0. 这样，新出生的单元n在顶部边缘上，它距中心O的距离为1，第$n=i$和$n=j$个单元距中心O的距离分别为e^{iG}和e^{jG}. 进一步假定所有发散角都是相同的，若所有角度都从单元n算起，则第$n=i$和$n=j$个单元的角度分别为$i\varphi$和$j\varphi$. 于是用余弦定理等距离假设可表达为

$$\sqrt{e^{2iG}+1-2e^{iG}\cos(i\varphi)}$$
$$=\sqrt{e^{2jG}+1-2e^{jG}\cos(j\varphi)},$$

取等式两端的平方，消去 1，得

$$e^{2iG}-2e^{iG}\cos(i\varphi)$$
$$=e^{2jG}-2e^{jG}\cos(j\varphi). \quad (2.72)$$

这便是几何模型的基本公式。由此公式，给出 G 值和螺线数目 i 和 j，即求出相应的发散角 φ. 不过对于同一 G 值，有时有多个不同的 (i, j) 分支解。下面的计算机模拟表明，如果从空白的初始条件（没有预先布置好的单元存在）出发，且 G 从 1 单调下降，系统总是沿 (i, j) 符合斐波那契数列发展的。

一般说来由 n, $n-i$, $n-j$, $n-i-j$ 为顶点构成的四边形（见图 2–28 中 n, $n-5$, $n-8$, $n-13$）中，n 到 $n-i-j$ 的距离大于 n 到 $n-i$ 或 n 到 $n-j$ 的距离。然而当 G 减小到一定关节点，三个距离会变得相等。请看图 2 – 29。G 减小意味着每个单元到中心 O 距离缩短。这就是说，每个单元向中心靠拢，愈远的缩进得愈多（图 a）。到了图 b 和图 c，n 到 $n-i$, $n-j$, $n-i-j$ 三个距离已相等，由 n, $n-i$, $n-j$, $n-i-j$ 为顶点的四边形过渡到由 n, $n-i-j$, $n-j$, $n-i-2j$ 为顶点构成的四边形。若 G 再减小，如图 d 所示，已成为由 n, $n-i$, $n-j$, $n-i-j$ 为顶点的四边形过渡到由 n, $n-i-j$, $n-j$, $n-i-2j$ 为顶点构成的四边形，即双螺线数已从 (i, j) 过渡到 $(i+j, j)$。这便是双螺线数的发展遵循斐波那契数列规则的原因。由数值计算得到的关节点 G 和 φ 值列于下表中。

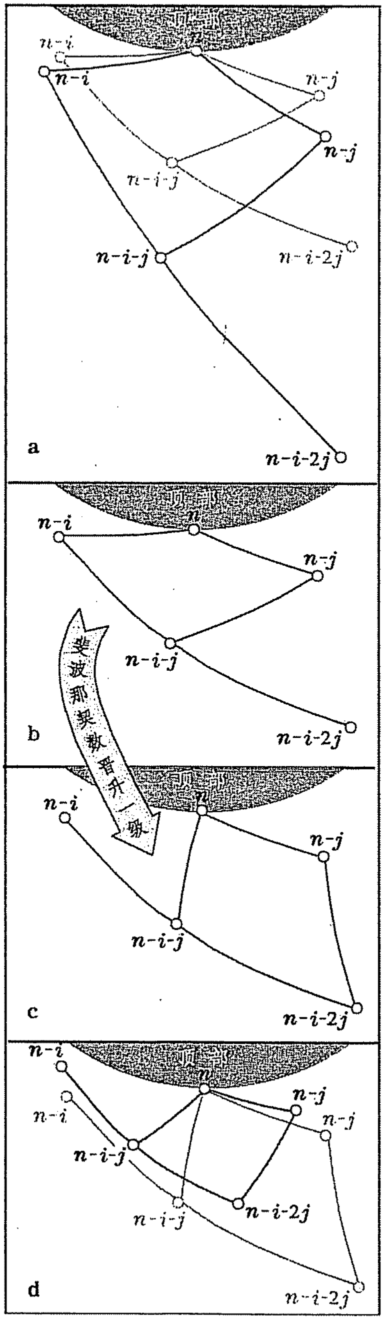


图 2–29 G 减小时双螺线数从 (i, j) 向 $(i+j, j)$ 过渡

(i,j)	G	φ
(1, 1)→(1, 2)	0.693	180.00°
(1, 2)→(2, 3)	0.347	135.00°
(2, 3)→(3, 5)	0.1852	141.15°
(3, 5)→(5, 8)	0.0832	136.17°
(5, 8)→(8, 13)	0.0352	138.09°

如图 2 – 28 所示，5 条顺时针螺线的端点降顶部圆周平分 5 等份，8 条顺时针螺线的端点降顶部圆周平分 8 等份，前者的第 2 条与后者的第 3 条的端点离得最近(1/40 周长)，它们的交点上是 $n-1$ 单元，它正好应该是除在圆周上的 n 外离圆周最近的单元。一般说来，如果 (i,j) 符合斐波那契数列，即 $i=F_n$，$j=F_{n+1}$，则前者的第 F_n-F_{n-1} 条与后者的第 $F_{n+1}-F_n$ 条的端点离得最近，这是因为

$$\frac{F_n-F_{n-1}}{F_n}-\frac{F_{n+1}-F_n}{F_{n+1}}=\frac{F_n^2-F_{n-1}F_{n+1}}{F_nF_{n+1}}=\frac{(-1)^{n+1}}{F_nF_{n+1}},$$

上式最后一步推导见(2.69)式。这是以 F_nF_{n+1} 为分母绝对值最小的非零分数。所以 $n-1$ 单元的角位置(即发散角 φ)介于 $360°\times(F_n-F_{n-1})/F_n$ 和 $360°\times(F_{n+1}-F_n)/F_{n+1}$ 之间 。在 $n\to\infty$ 时二者的极限都是

$$360°\times(1-0.618034)=137.508°,$$

故 φ 的极限值也是这个圆周的黄金分割角。

(3) 计算机模拟

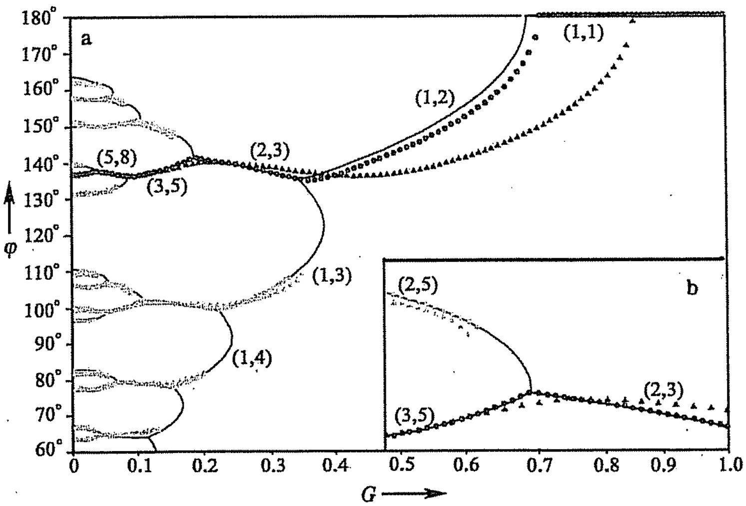


图 2–30 计算机模拟得到的 G、φ 关系

计算机模拟可以比物理实验在广泛的条件下来验证植物生长序的机理。这里每个出生的单元可看成是质点，它们之间的排斥力 f 可假定是质点间距 d 的任何递减函数，譬如

$$f(d) \propto \frac{1}{d},\ \frac{1}{d^4},\ \exp(-d/l),$$

上述物理实验中的磁偶极子属于反比于 d^4 情形。在计算机模拟中还可以预设某种 (i, j) 位形，它们不一定符合斐波那契数列，观察它们是否能够稳定存在和发展下去。图 2 – 30 所示为 $f(d) \propto 1/d^4$（数据用三角形表示）和 $\exp(-d/l)$（数据用圆形表示）两种情形的模拟结果。图中曲线是按几何模型的理论计算出来的，可以看出，指数形式是短程力，更符合硬盘模型的理论曲线。在 G 值较小的范围内二者很接近，区别主要反映在关节点附近曲线的过渡形式（见图 2 – 30b 中的特写）。

图 2 – 30 还显示，只有符合斐波那契数的主序列才能在 G 值从 1 减到 0 的全程下发展，其它片断都是在特定预设位形下开始发展的，在较大的 G 值下不能稳定存在。这就能说明，为什么我们在自然界看到的几乎都是符合斐波那契数的生长序列。

图 2 – 31a 所示为 G 从 1 减小到 0.01 的模拟情形，在最后达到的位形中 $i = 13$，$j = 21$。图 2 – 31b 为发散角 φ 在此过程中由 180° 趋于 137.47° 情形，它开始就急剧下降到 137° 附近，几经小幅的振荡，最后趋于稳定值。φ 值的每次振荡对应一次 (i, j) 数值的升级。

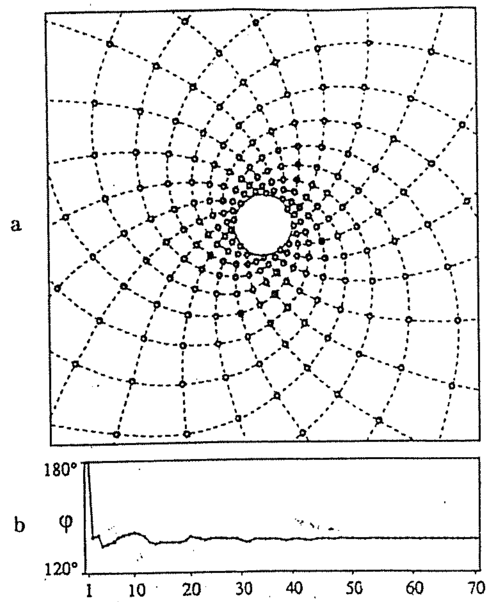
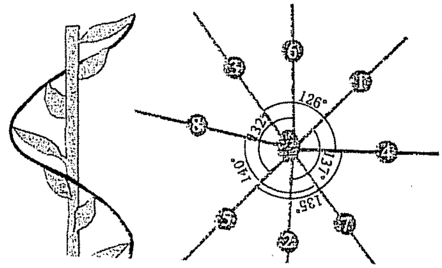


图 2–31　$(i, j) = (13, 21)$ 位形的计算机模拟

5.4 植物的叶序与黄金分割

有些植物的叶子是按螺旋状的叶序生长的（见图 2 – 32a）。以茎为轴，一片片叶子逐次朝不同方向伸展出去。投影到水平面上看，各叶片之间



环

难

理教师 J. C. Mattila 找到一颗孤立的蒿子，❶ 它远离其它障碍，他把这颗蒿子的叶子的角分布拍摄下来进行分析(见图2－32b)，结果发现，相继出生叶片之间的夹角接近 137.5°，即圆周的黄金分割角。如果此角是圆周的有理分数，例如 90°、120°、180°，则会发生叶片相互遮掩问题。只要此角是圆周的无理分数，就不会完全的遮掩。黄金分割是最无理的无理数，在这方面有什么优越性？图2－33按黄金分割角 137.5° 依次画出 21 片叶子的方向，看来角分布还是相当均匀的。但这还不是充分的理由。看来上节讨论的斐波那契数生长序列在这里也起作用。

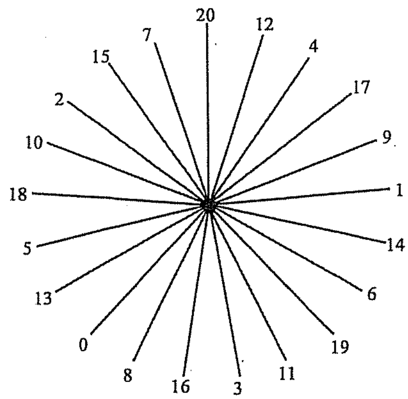


图 2－33 按黄金分割角分布的叶序

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译文见《物理教学》1986 年第 3 期，28 页。

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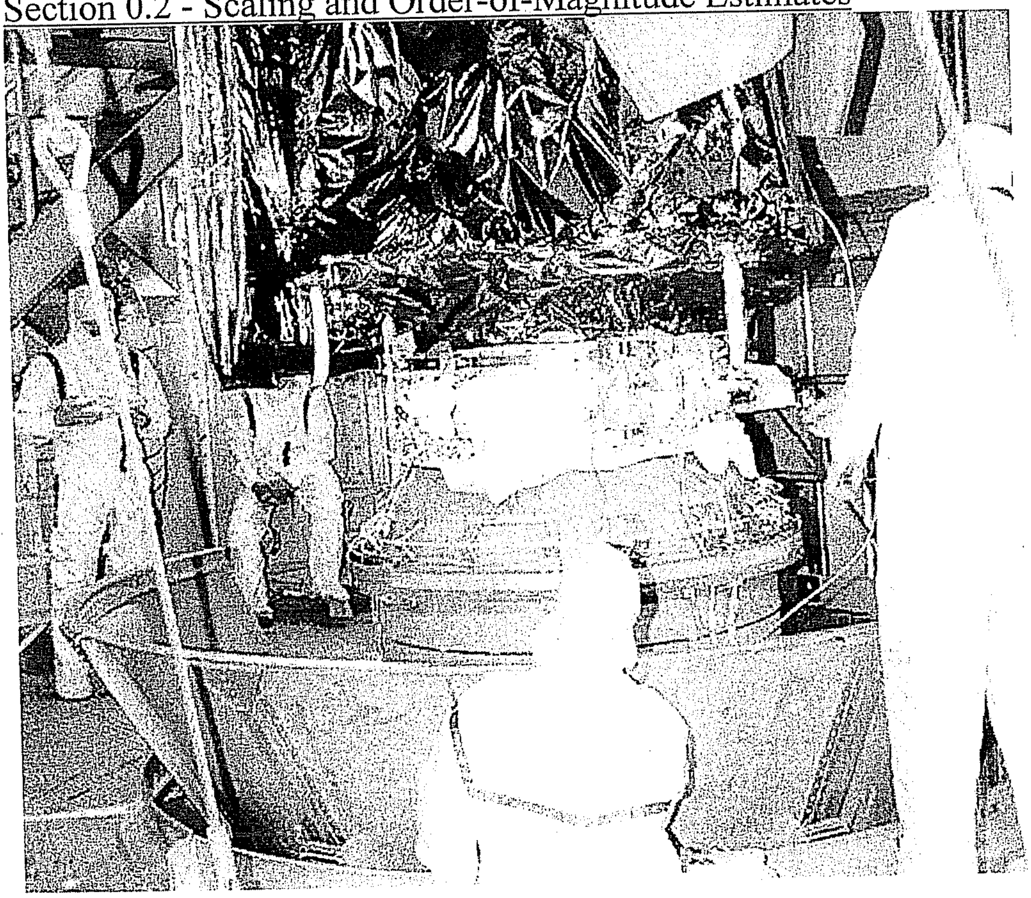


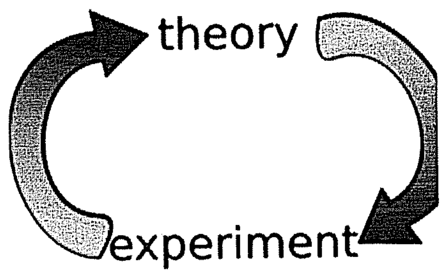
The Mars Climate Orbiter is prepared for its mission. The laws of physics are the same everywhere, even on Mars, so the probe could be designed based on the laws of physics as discovered on earth. There is unfortunately another eason why this spacecraft is relevant to the topics of this hapter: it was destroyed attempting to enter Mars' tmosphere because engineers at Lockheed Martin forgot to convert data on engine thrusts from pounds into the metric unit of force (newtons) before giving the information to NASA. Conversions are important!

Chapter 0. Introduction and Review

0.1 Introduction and Review

If you drop your shoe and a coin side by side, they hit the ground at the same time. Why doesn't the shoe get there first, since gravity is pulling harder on it? How does the lens of your eye work, and why do your eye's muscles need to squash its lens into different shapes in order to focus on objects nearby or far away? These are the kinds of questions that physics tries to answer about the behavior of light and matter, the two things that the universe is made of.

0.1.1 The scientific method



a / Science is a cycle theory and experiment.

b / A satirical drawing of a alchemist's laboratory. H Cock, after a drawing b Peter Brueghel the Eld (16th century).

Until very recently in history, no progress was made in answering questions like these. Worse than that, the *wrong* answers written by thinkers like the ancient Greek physicist Aristotle were accepted without question for thousands of years. Why is it that scientific knowledge has progressed more since the Renaissance than it had in all the preceding millennia since the beginning of recorded history? Undoubtedly the industrial revolution is part of the answer. Building its centerpiece, the steam engine, required improved techniques for precise construction and measurement. (Early on, it was considered a major advance when English machine shops learned to build pistons and cylinders that fit together with a gap narrower than the thickness of a penny.) But even before the industrial revolution, the pace of discovery had picked up, mainly because of the introduction of the modern scientific method. Although it evolved over time, most scientists today would agree on something like the following list of the basic principles of the scientific method:

(1) *Science is a cycle of theory and experiment.* Scientific theories are created to explain the results of experiments that were

created under certain conditions. A successful theory will also make new predictions about new experiments under new conditions. Eventually, though, it always seems to happen that a new experiment comes along, showing that under certain conditions the theory is not a good approximation or is not valid at all. The ball is then back in the theorists' court. If an experiment disagrees with the current theory, the theory has to be changed, not the experiment.

(2) *Theories should both predict and explain.* The requirement of predictive power means that a theory is only meaningful if it predicts something that can be checked against experimental measurements that the theorist did not already have at hand. That is, a theory should be testable. Explanatory value means that many phenomena should be accounted for with few basic principles. If you answer every "why" question with "because that's the way it is," then your theory has no explanatory value. Collecting lots of data without being able to find any basic underlying principles is not science.

(3) *Experiments should be reproducible.* An experiment should be treated with suspicion if it only works for one person, or only in one part of the world. Anyone with the necessary skills and equipment should be able to get the same results from the same experiment. This implies that science transcends national and ethnic boundaries; you can be sure that nobody is doing actual science who claims that their work is "Aryan, not Jewish," "Marxist, not bourgeois," or "Christian, not atheistic." An experiment cannot be reproduced if it is secret, so science is necessarily a public enterprise.

As an example of the cycle of theory and experiment, a vital step toward modern chemistry was the experimental observation that the chemical elements could not be transformed into each other, e.g., lead could not be turned into gold. This led to the theory that chemical reactions consisted of rearrangements of the elements in different combinations, without any change in the identities of the elements themselves. The theory worked for hundreds of years, and was confirmed experimentally over a wide range of pressures and temperatures and with many combinations of elements. Only in the twentieth century did we learn that one element could be trans-formed into one another under the conditions of extremely high pressure and temperature existing in a nuclear bomb or inside a star. That observation didn't completely invalidate the original theory of the immutability of the elements, but it showed that it was only an approximation, valid at ordinary temperatures and pressures.

self-check: A psychic conducts seances in which the spirits of the dead speak to the participants. He says he has special psychic powers not possessed by other people, which allow him to "channel" the communications with the spirits. What part of the scientific method is being violated here? (answer in the back of the PDF version of the book)

The scientific method as described here is an idealization, and should not be understood as a set procedure for doing science. Scientists have as many weaknesses and character flaws as any other group, and it is very common for scientists to try to discredit other people's experiments when the results run contrary to their own favored point of view. Successful science also has more to do with luck, intuition, and creativity than most people realize, and the restrictions of the scientific method do not stifle individuality and self-expression any more than the fugue and sonata forms stifled Bach and Haydn. There is a recent tendency among social scientists to go even further and to deny that the scientific method even exists, claiming that science is no more than an arbitrary social system that determines what ideas to accept based on an in-group's criteria. I think that's going too far. If science is an arbitrary social ritual, it would seem difficult to explain its effectiveness in building such useful items as airplanes, CD players, and sewers. If alchemy and astrology were no less scientific in their methods than chemistry and astronomy, what was it that kept them from producing anything useful?

Discussion Questions

withintro{Consider whether or not the scientific method is being applied in the following examples. If the scientific method is not being applied, are the people whose actions are being described performing a useful human activity, albeit an unscientific one? }

◊ Acupuncture is a traditional medical technique of Asian origin in which small needles are inserted in the patient's body to relieve pain. Many doctors trained in the west consider acupuncture unworthy of experimental study because if it had therapeutic effects, such effects could not be explained by their theories of the nervous system. Who is being more scientific, the western or eastern practitioners?

◊ Goethe, a German poet, is less well known for his theory of color. He published a book on the subject, in which he argued that scientific apparatus for measuring and quantifying color, such as prisms, lenses and colored filters, could not give us full insight into the ultimate meaning of color, for instance the cold feeling evoked by blue and green or the heroic sentiments inspired by red. Was his work scientific?

◊ A child asks why things fall down, and an adult answers “because of gravity.” The ancient Greek philosopher Aristotle explained that rocks fell because it was their nature to seek out their natural place, in contact with the earth. Are these explanations scientific?

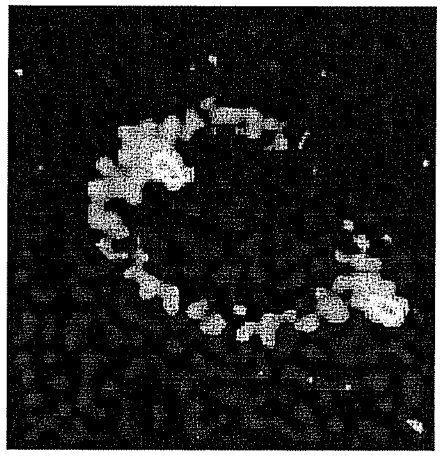
◊ Buddhism is partly a psychological explanation of human suffering, and psychology is of course a science. The Buddha could be said to have engaged in a cycle of theory and experiment, since he worked by trial and error, and even late in his life he asked his followers to challenge his ideas. Buddhism could also be considered reproducible, since the Buddha told his followers they could find enlightenment for themselves if they followed a certain course of study and discipline. Is Buddhism a scientific pursuit?

0.1.2 What is physics?

Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the things which compose it...nothing would be uncertain, and the future as the past would be laid out before its eyes. -- *Pierre Simon de Laplace*

√ Physics is the use of the scientific method to find out the basic principles governing light and matter, and to discover the implications of those laws. Part of what distinguishes the modern outlook from the ancient mind-set is the assumption that there are rules by which the universe functions, and that those laws can be at least partially understood by humans. From the Age of Reason through the nineteenth century, many scientists began to be convinced that the laws of nature not only could be known but, as claimed by Laplace, those laws could in principle be used to predict everything about the universe's future if complete information was available about the present state of all light and matter. In subsequent sections, I'll describe two general types of limitations on prediction using the laws of physics, which were only recognized in the twentieth century.

Matter can be defined as anything that is affected by gravity, i.e., that has weight or would have weight if it was near the Earth or another star or planet massive enough to produce measurable gravity. Light can be defined as anything that can travel from one place to another through empty space and can influence matter, but has no weight. For example, sunlight can influence your body by heating it or by damaging your DNA and giving you skin cancer. The physicist's definition of light includes a variety of phenomena that are not visible to the eye, including radio waves, microwaves, x-rays, and gamma rays. These are the "colors" of light that do not happen to fall within the narrow violet -to-red range of the rainbow that we can see.

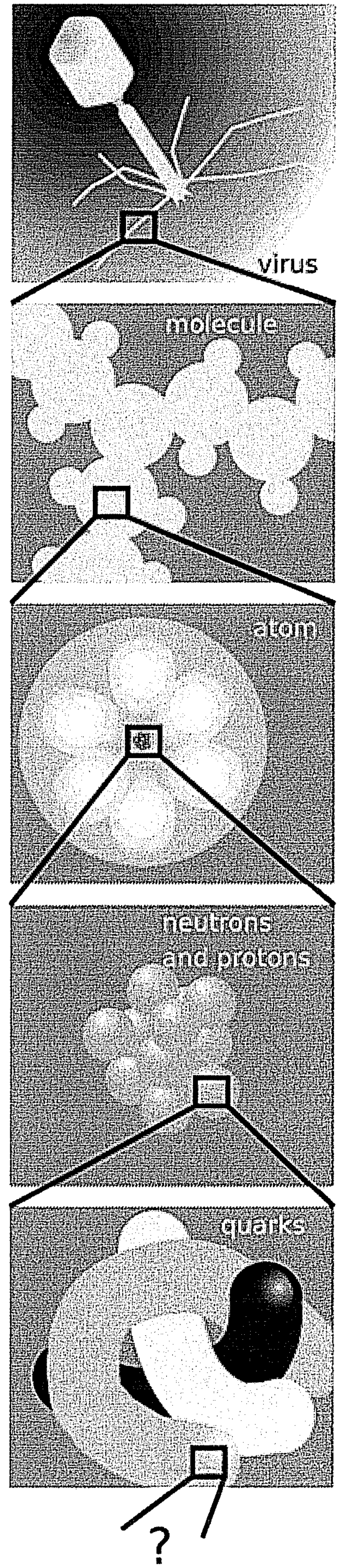


c / This telescope pictu shows two images of th same distant object, a exotic, very luminou object called a quasa This is interpreted a evidence that a massiv dark object, possibly black hole, happens to b between us and it. Ligl rays that would otherwis have missed the earth c either side have been be by the dark object's gravi so that they reach us. Th actual direction to th quasar is presumably the center of the imag but the light along th central line doesn't get t us because it is absorbe by the dark object. Th quasar is known by i catalog numbe MG1131+0456, or mo informally as Einstein Ring.

self-check: At the turn of the 20th century, a strange new phenomenon was discovered in vacuum tubes: mysterious rays of unknown origin and nature. These rays are the same as the ones that shoot from the back of your TV's picture tube and hit the front to make the picture. Physicists in 1895 didn't have the faintest idea what the rays were, so they simply named them "cathode rays," after the name for the electrical contact from which they sprang. A fierce debate raged, complete with nationalistic overtones, over whether the rays were a form of light or of matter. What would they have had to do in order to settle the issue? (answer in the back of the PDF version of the book)

Many physical phenomena are not themselves light or matter, but are properties of light or matter or interactions between light and matter. For instance, motion is a property of all light and some matter, but it is not itself light or matter. The pressure that keeps a bicycle tire blown up is an interaction between the air and the tire. Pressure is not a form of matter in and of itself. It is as much a property of the tire as of the air. Analogously, sisterhood and employment are relationships among people but are not people themselves.

Some things that appear weightless actually do have weight, and so qualify as matter. Air has weight, and is thus a form of matter even though a cubic inch of air weighs less than a grain of sand. A helium balloon has weight, but is kept from falling by the force of the surrounding more dense air, which pushes up on it. Astronauts in orbit around the Earth have weight, and are falling along a curved arc, but they are moving so fast that the curved arc of their fall is broad enough to carry them all the way around the Earth in a circle. They perceive themselves as being weightless because their space capsule is falling along with them, and the floor therefore does not push up on their feet.



d / Reductionism.

Optional topic: Optional Topic: Modern Changes in the Definition of Light and Matter

Einstein predicted as a consequence of his theory of relativity that light would after all be affected by gravity, although the effect would be extremely weak under normal conditions. His prediction was borne out by observations of the bending of light rays from stars as they passed close to the sun on their way to the Earth. Einstein's theory also implied the existence of black holes, stars so massive and compact that their intense gravity would not even allow light to escape. (These days there is strong evidence that black holes exist.)

Einstein's interpretation was that light doesn't really have mass, but that energy is affected by gravity just like mass is. The energy in a light beam is equivalent to a certain amount of mass, given by the famous equation $E=mc^2$, where c is the speed of light. Because the speed of light is such a big number, a large amount of energy is equivalent to only a very small amount of mass, so the gravitational force on a light ray can be ignored for most practical purposes.

There is however a more satisfactory and fundamental distinction between light and matter, which should be understandable to you if you have had a chemistry course. In chemistry, one learns that electrons obey the Pauli exclusion principle, which forbids more than one electron from occupying the same orbital if they have the same spin. The Pauli exclusion principle is obeyed by the subatomic particles of which matter is composed, but disobeyed by the particles, called photons, of which a beam of light is made.

Einstein's theory of relativity is discussed more fully in book 6 of this series.

The boundary between physics and the other sciences is not always clear. For instance, chemists study atoms and molecules, which are what matter is built from, and there are some scientists who would be equally willing to call themselves physical chemists or chemical physicists. It might seem that the distinction between physics and biology would be clearer, since physics seems to deal with inanimate objects. In fact, almost all physicists would agree that the basic laws of physics that apply to molecules in a test tube work equally well for the combination of molecules that constitutes a bacterium. (Some might believe that something more happens in the minds of humans, or even those of cats and dogs.) What differentiates physics from biology is that many of the scientific theories that describe living things, while ultimately resulting from the fundamental laws of physics, cannot be rigorously derived from physical principles.

✓Isolated systems and reductionism

To avoid having to study everything at once, scientists isolate the things they are trying to study. For instance, a physicist who wants to study the motion of a rotating gyroscope would probably prefer that it be isolated from vibrations and air currents. Even in biology, where field work is indispensable for understanding how living things relate to their entire environment, it is interesting to note the vital historical role played by Darwin's study of the Galápagos Islands, which were conveniently isolated from the rest of the world. Any part of the universe that is considered apart from the rest can be called a “system.”

Physics has had some of its greatest successes by carrying this process of isolation to extremes, subdividing the universe into smaller and smaller parts. Matter can be divided into atoms, and the behavior of individual atoms can be studied. Atoms can be split apart into their constituent neutrons, protons and electrons. Protons and neutrons appear to be made out of even smaller particles called quarks, and there have even been some claims of experimental evidence that quarks have smaller parts inside them. This method of splitting things into smaller and smaller parts and studying how those parts influence each other is called reductionism. The hope is that the seemingly complex rules governing the larger units can be better understood in terms of simpler rules governing the smaller units. To appreciate what reductionism has done for science, it is only necessary to examine a 19th-century chemistry textbook. At that time, the existence of atoms was still doubted by some, electrons were not even suspected to exist, and almost nothing was understood of what basic rules governed the way atoms interacted with each other in chemical reactions. Students had to memorize long lists of chemicals and their reactions, and there was no way to understand any of it systematically. Today, the student only needs to remember a small set of rules about how atoms interact, for instance that atoms of one element cannot be converted into another via chemical reactions, or that atoms from the right side of the periodic table tend to form strong bonds with atoms from the left side.

◊ I've suggested replacing the ordinary dictionary definition of light with a more technical, more precise one that involves weightlessness. It's still possible, though, that the stuff a lightbulb makes, ordinarily called "light," does have some small amount of weight. Suggest an experiment to attempt to measure whether it does.

◊ Heat is weightless (i.e., an object becomes no heavier when heated), and can travel across an empty room from the fireplace to your skin, where it influences you by heating you. Should heat therefore be considered a form of light by our definition? Why or why not?

◊ Similarly, should sound be considered a form of light?

0.1.3 How to learn physics

For as knowledges are now delivered, there is a kind of contract of error between the deliverer and the receiver; for he that delivereth knowledge desireth to deliver it in such a form as may be best believed, and not as may be best examined; and he that receiveth knowledge desireth rather present satisfaction than expectant inquiry. -- *Francis Bacon*

Many students approach a science course with the idea that they can succeed by memorizing the formulas, so that when a problem is assigned on the homework or an exam, they will be able to plug numbers in to the formula and get a numerical result on their calculator. Wrong! That's not what learning science is about! There is a big difference between memorizing formulas and understanding concepts. To start with, different formulas may apply in different situations. One equation might represent a definition, which is always true. Another might be a very specific equation for the speed of an object sliding down an inclined plane, which would not be true if the object was a rock drifting down to the bottom of the ocean. If you don't work to understand physics on a conceptual level, you won't know which formulas can be used when.

Most students taking college science courses for the first time also have very little experience with interpreting the meaning of an equation. Consider the equation $w=A/h$ relating the width of a rectangle to its height and area. A student who has not developed skill at interpretation might view this as yet another equation to memorize and plug in to when needed. A slightly more savvy student might realize that it is simply the familiar formula $A=wh$ in a different form. When asked whether a rectangle would have a greater or smaller width than another with the same area but a smaller height, the unsophisticated student might be at a loss, not having any numbers to plug in on a calculator. The more experienced student would know how to reason about an equation involving division --- if h is smaller, and A stays the same, then w must be bigger. Often, students fail to recognize a sequence of equations as a derivation leading to a final result, so they think all the intermediate steps are equally important formulas that they should memorize.

When learning any subject at all, it is important to become as actively involved as possible, rather than trying to read through all the information quickly without thinking about it. It is a good idea to read and think about the questions posed at the end of each section of these notes as you encounter them, so that you know you have understood what you were reading.

Many students' difficulties in physics boil down mainly to difficulties with math. Suppose you feel confident that you have enough mathematical preparation to succeed in this course, but you are having trouble with a few specific things. In some areas, the brief review given in this chapter may be sufficient, but in other areas it probably will not. Once you identify the areas of math in which you are having problems, get help in those areas. Don't limp along through the whole course with a vague feeling of dread about something like scientific notation. The problem will not go away if you ignore it. The same applies to essential mathematical skills that you are learning in this course for the first time, such as vector addition.

Sometimes students tell me they keep trying to understand a certain topic in the book, and it just doesn't make sense. The worst thing you can possibly do in that situation is to keep on staring at the same page. Every textbook explains certain things badly --- even mine! --- so the best thing to do in this situation is to look at a different book. Instead of college textbooks aimed at the same mathematical level as the course you're taking, you may in some cases find that high school books or books at a lower math level give clearer explanations.

Finally, when reviewing for an exam, don't simply read back over the text and your lecture notes. Instead, try to use an active method of reviewing, for instance by discussing some of the discussion questions with another student, or doing homework problems you hadn't done the first time.

0.1.4 Velocity and acceleration

Calculus was invented by a physicist, Isaac Newton, because he needed it as a tool for calculating velocity and acceleration; in your introductory calculus course, velocity and acceleration were probably presented as some of the first applications.

If an object's position as a function of time is given by the function $x(t)$, then its velocity and acceleration are given by the first and second derivatives with respect to time,

$$v = \frac{dx}{dt}$$

$$\text{and} a = \frac{d^2x}{dt^2} .$$

The notation relates in a logical way to the units of the quantities. Velocity has units of m/s, and that makes sense because d x is interpreted as an infinitesimally small distance, with units of meters, and d t as an infinitesimally small time, with units of seconds. The seemingly weird and inconsistent placement of the superscripted twos in the notation for the acceleration is likewise meant to suggest the units: something on top with units of meters, and something on the bottom with units of seconds squared.

Velocity and acceleration have completely different physical interpretations. Velocity is a matter of opinion. Right now as you sit in a chair and read this book, you could say that your velocity was zero, but an observer watching the Earth rotate would say that you had a velocity of hundreds of miles an hour. Acceleration represents a *change* in velocity, and it's not a matter of opinion. Accelerations produce physical effects, and don't occur unless there's a force to cause them. For example, gravitational forces on Earth cause falling objects to have an acceleration of 9.8 m/s^2.

Example 1: Constant acceleration

◊ How high does a diving board have to be above the water if the diver is to have as much as 1.0 s in the air?

◊ The diver starts at rest, and has an acceleration of 9.8 m/s^2. We need to find a connection between the distance she travels and time it takes. In other words, we're looking for information about the function *x*(*t*), given information about the acceleration. To go from acceleration to position, we need to integrate twice:

$$x = \int\int a\,\mathrm{d}t\,\mathrm{d}t$$

$$= \int (at + v_o)\,\mathrm{d}t \qquad [v_o \text{ is a constant of integration.}]$$

$$= \int at\,\mathrm{d}t \qquad [v_o \text{ is zero because she's dropping from rest.}]$$

$$= \frac{1}{2}at^2 + x_o \qquad [x_o \text{ is a constant of integration.}]$$

$$= \frac{1}{2}at^2 \qquad [x_o \text{ can be zero if we define it that way.}]$$

Note some of the good problem-solving habits demonstrated here. We solve the problem symbolically, and only plug in numbers at the very end, once all the algebra and calculus are done. One should also make a habit, after finding a symbolic result, of checking whether the dependence on the variables make sense. A greater value of *t* in this expression would lead to a greater value for *x*; that makes sense, because if you want more time in the air, you're going to have to jump from higher up. A greater acceleration also leads to a greater height; this also makes sense, because the stronger gravity is, the more height you'll need in order to stay in the air for a given amount of time. Now we plug in numbers.

$$x = \frac{1}{2}\left(9.8\ \mathrm{m/s^2}\right)(1.0\ \mathrm{s})^2$$

$= 4.9 \text{ m}$

Note that when we put in the numbers, we check that the units work out correctly, $(\text{m}/\text{s}^2)(\text{s})^2 = \text{m}$. We should also check that the result makes sense: 4.9 meters is pretty high, but not unreasonable.

The notation d q in calculus represents an infinitesimally small change in the variable q. The corresponding notation for a finite change in a variable is Δq. For example, if q represents the value of a certain stock on the stock market, and the value falls from q_o=5 dollars initially to q_f=3 dollars finally, then Δq=-2 dollars. When we study linear functions, whose slopes are constant, the derivative is synonymous with the slope of the line, and dy/dx is the same thing as $\Delta y/\Delta x$, the rise over the run.

Under conditions of constant acceleration, we can relate velocity and time,

$$a = \frac{\Delta v}{\Delta t} \quad ,$$

or, as in the example 1, position and time,

$$x = \frac{1}{2}at^2 + v_o t + x_o \quad .$$

It can also be handy to have a relation involving velocity and position, eliminating time. Straightforward algebra gives

$$v_f^2 = v_o^2 + 2\, a\, \Delta x \,,$$

where v_f is the final velocity, v_o the initial velocity, and Δx the distance traveled.

◊ Solved problem: Dropping a rock on Mars — problem 17

◊ Solved problem: The Dodge Viper — problem 19

0.1.5 Self-evaluation

The introductory part of a book like this is hard to write, because every student arrives at this starting point with a different preparation. One student may have grown up outside the U.S. and so may be completely comfortable with the metric system, but may have had an algebra course in which the instructor passed too quickly over scientific notation. Another student may have already taken vector calculus, but may have never learned the metric system. The following self-evaluation is a checklist to help you figure out what you need to study to be prepared for the rest of the course.

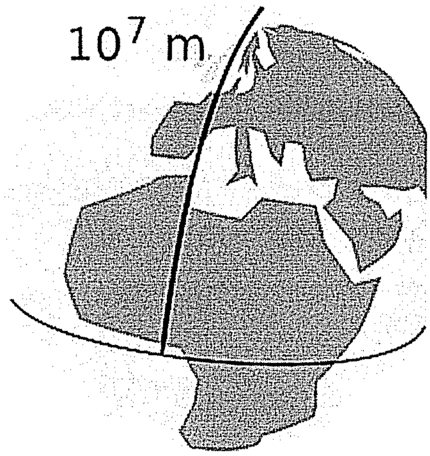
If you disagree with this statement...	you should study this section:
I am familiar with the basic metric units of meters, kilograms, and seconds, and the most common metric prefixes: milli- (m), kilo- (k), and centi- (c).	subsection ?? Basic of the Metric System
I am familiar with these less common metric prefixes: mega- (M), micro- μ), and nano- (n).	subsection ?? Less Common Metric Prefixes
I am comfortable with scientific notation.	subsection ?? Scientific Notation
I can confidently do metric conversions.	subsection ?? Conversions
I understand the purpose and use of significant figures.	subsection ?? Significant Figures

It wouldn't hurt you to skim the sections you think you already know about, and to do the self-checks in those sections.

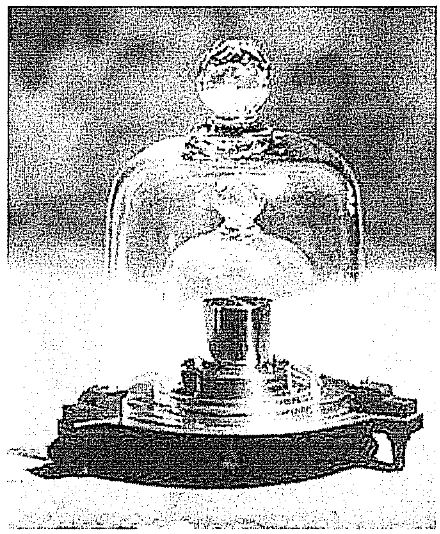
0.1.6 Basics of the metric system

The metric system

Units were not standardized until fairly recently in history, so when the physicist Isaac Newton gave the result of an experiment with a pendulum, he had to specify not just that the string was 37 $^7/_8$ inches long but that it was "37 $^7/_8$ London inches long." The inch as defined in Yorkshire would have been different. Even after the British Empire standardized its units, it was still very inconvenient to do calculations involving money, volume, distance, time, or weight, because of all the odd conversion factors, like 16 ounces in a pound, and 5280 feet in a mile. Through the nineteenth century, schoolchildren squandered most of their mathematical education in preparing to do calculations such as making change when a customer in a shop offered a one-crown note for a book costing two pounds, thirteen shillings and tuppence. The dollar has always been decimal, and British money went decimal decades ago, but the United States is still saddled with the antiquated system of feet, inches, pounds, ounces and so on.



e / The original definition of the meter.



f / A duplicate of the Paris kilogram, maintained at the Danish National Metrology Institute.

Every country in the world besides the U.S. has adopted a system of units known in English as the "metric system." This system is entirely decimal, thanks to the same eminently logical people who brought about the French Revolution. In deference to France, the system's official name is the Système International, or SI, meaning International System. (The phrase "SI system" is therefore redundant.)

The wonderful thing about the SI is that people who live in countries more modern than ours do not need to memorize how many ounces there are in a pound, how many cups in a pint, how many feet in a mile, etc. The whole system works with a single, consistent set of prefixes (derived from Greek) that modify the basic units. Each prefix stands for a power of ten, and has an abbreviation that can be combined with the symbol for the unit. For instance, the meter is a unit of distance. The prefix kilo- stands for 10^3, so a kilometer, 1 km, is a thousand meters.

The basic units of the metric system are the meter for distance, the second for time, and the gram for mass.

The following are the most common metric prefixes. You should memorize them.

prefix		meaning	example	
kilo-	k	10^3	60 kg	= a person's mass
centi-	c	10^{-2}	28 cm	= height of a piece of paper
milli-	m	10^{-3}	1 ms	= time for one vibration of a guitar string playing the note D

The prefix centi-, meaning 10^{-2}, is only used in the centimeter; a hundredth of a gram would not be written as 1 cg but as 10 mg. The centi- prefix can be easily remembered because a cent is 10^{-2} dollars. The official SI abbreviation for seconds is "s" (not "sec") and grams are "g" (not "gm").

The second

The sun stood still and the moon halted until the nation had taken vengeance on its enemies... -- *Joshua 10:12-14*

Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external... -- *Isaac Newton*

When I stated briefly above that the second was a unit of time, it may not have occurred to you that this was not really much of a definition. The two quotes above are meant to demonstrate how much room for confusion exists among people who seem to mean the same thing by a word such as "time." The first quote has been interpreted by some biblical scholars as indicating an ancient belief that the motion of the sun across the sky was not just something that occurred with the passage of time but that the sun actually caused time to pass by its motion, so that freezing it in the sky would have some kind of a supernatural decelerating effect on everyone except the Hebrew soldiers. Many ancient cultures also conceived of time as cyclical, rather than proceeding along a straight line as in 1998, 1999, 2000, 2001,... The second quote, from a relatively modern physicist, may sound a lot more scientific, but most physicists today would consider it useless as a definition of time. Today, the physical sciences are based on operational definitions, which means definitions that spell out the actual steps (operations) required to measure something numerically.

Now in an era when our toasters, pens, and coffee pots tell us the time, it is far from obvious to most people what is the fundamental operational definition of time. Until recently, the hour, minute, and second were defined operationally in terms of the time required for the earth to rotate about its axis. Unfortunately, the Earth's rotation is slowing down slightly, and by 1967 this was becoming an issue in scientific experiments requiring precise time measurements. The second was therefore redefined as the time required for a certain number of vibrations of the light waves emitted by a cesium atoms in a lamp constructed like a familiar neon sign but with the neon replaced by cesium. The new definition not only promises to stay constant indefinitely, but for scientists is a more convenient way of calibrating a clock than having to carry out astronomical measurements.

self-check: What is a possible operational definition of how strong a person is? (answer in the back of the PDF version of the book)

The meter

The French originally defined the meter as 10^{-7} times the distance from the equator to the north pole, as measured through Paris (of course). Even if the definition was operational, the operation of traveling to the north pole and laying a surveying chain behind you was not one that most working scientists wanted to carry out. Fairly soon, a standard was created in the form of a metal bar with two scratches on it. This was replaced by an atomic standard in 1960, and finally in 1983 by the current definition, which is that the meter is the distance traveled by light in a vacuum over a period of (1/299792458) seconds.

The kilogram

The third base unit of the SI is the kilogram, a unit of mass. Mass is intended to be a measure of the amount of a substance, but that is not an operational definition. Bathroom scales work by measuring our planet's gravitational attraction for the object being weighed, but using that type of scale to define mass operationally would be undesirable because gravity varies in strength from place to place on the earth.

There's a surprising amount of disagreement among physics textbooks about how mass should be defined, but here's how it's actually handled by the few working physicists who specialize in ultra-high-precision measurements. They maintain a physical object in Paris, which is the standard kilogram, a cylinder made of platinum-iridium alloy. Duplicates are checked against this mother of all kilograms by putting the original and the copy on the two opposite pans of a balance. Although this method of comparison depends on gravity, the problems associated with differences in gravity in different geographical locations are bypassed, because the two objects are being compared in the same place. The duplicates can then be removed from the Parisian kilogram shrine and transported elsewhere in the world. It would be desirable to replace this at some point with a universally accessible atomic standard rather than one based on a specific artifact, but as of 2010 the technology for automated counting of large numbers of atoms has not gotten good enough to make that work with the desired precision.

Combinations of metric units

Just about anything you want to measure can be measured with some combination of meters, kilograms, and seconds. Speed can be measured in m/s, volume in m^3, and density in kg/m^3. Part of what makes the SI great is this basic simplicity. No more funny units like a cord of wood, a bolt of cloth, or a jigger of whiskey. No more liquid and dry measure. Just a simple, consistent set of units. The SI measures put together from meters, kilograms, and seconds make up the mks system. For example, the mks unit of speed is m/s, not km/hr.

Checking units

A useful technique for finding mistakes in one's algebra is to analyze the units associated with the variables.

Example 2: Checking units

◊ Jae starts from the formula $V = \frac{1}{3}Ah$ for the volume of a cone, where *A* is the area of its base, and *h* is its height. He wants to find an equation that will tell him how tall a conical tent has to be in order to have a certain volume, given its radius. His algebra goes like this:

$$V = \frac{1}{3}Ah$$

$$A = \pi r^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$h = \frac{\pi r^2}{3V}$$

Is his algebra correct? If not, find the mistake.

◊ Line 4 is supposed to be an equation for the height, so the units of the expression on the right-hand side had better equal meters. The pi and the 3 are unitless, so we can ignore them. In terms of units, line 4 becomes

$$\mathrm{m} = \frac{\mathrm{m}^2}{\mathrm{m}^3} = \frac{1}{\mathrm{m}} \quad .$$

This is false, so there must be a mistake in the algebra. The units of lines 1, 2, and 3 check out, so the mistake must be in the step from line 3 to line 4. In fact the result should have been

$$h = \frac{3V}{\pi r^2} \quad .$$

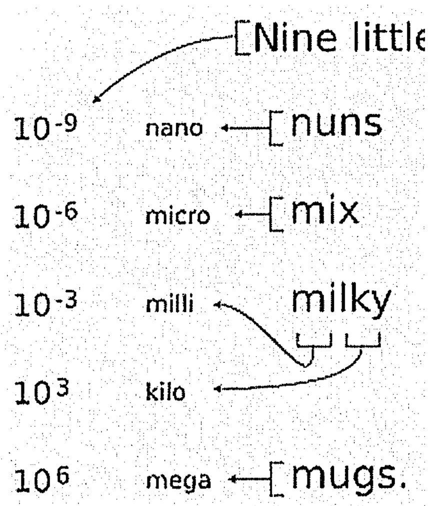
Now the units check: $\mathrm{m} = \mathrm{m}^3/\mathrm{m}^2$.

Discussion Question

◊ Isaac Newton wrote, "... the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time... It may be that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated or retarded..." Newton was right. Even the modern definition of the second in terms of light emitted by cesium atoms is subject to variation. For instance, magnetic fields could cause the cesium atoms to emit light with a slightly different rate of vibration. What makes us think, though, that a pendulum clock is more accurate than a sundial, or that a cesium atom is a more accurate timekeeper than a pendulum clock? That is, how can one test experimentally how the accuracies of different time standards compare?

0.1.7 Less common metric prefixes

The following are three metric prefixes which, while less common than the ones discussed previously, are well worth memorizing.



g / This is a mnemonic t help you remember th most important metr prefixes. The word "little" to remind you that the li starts with the prefixe used for small quantitie and builds upward. Th exponent changes by : except that of course th we do not need a speci prefix for 10^0, which equa one.

prefix	meaning		example
mega-	M 10^6	6.4 Mm	= radius of the earth
micro-	μ 10^{-6}	10mumunit	= size of a white blood cell
nano-	n 10^{-9}	0.154 nm	= distance between carbon nuclei in an ethane molecule

Note that the abbreviation for micro is the Greek letter mu, μ --- a common mistake is to confuse it with m (milli) or M (mega).

There are other prefixes even less common, used for extremely large and small quantities. For instance, 1 femtometer=10^{-15} m is a convenient unit of distance in nuclear physics, and 1 gigabyte=10^9 bytes is used for computers' hard disks. The international committee that makes decisions about the SI has recently even added some new prefixes that sound like jokes, e.g., 1 yoctogram = 10^{-24} g is about half the mass of a proton. In the immediate future, however, you're unlikely to see prefixes like "yocto-" and "zepto-" used except perhaps in trivia contests at science-fiction conventions or other geekfests.

self-check: Suppose you could slow down time so that according to your perception, a beam of light would move across a room at the speed of a slow walk. If you perceived a nanosecond as if it was a second, how would you perceive a microsecond? (answer in the back of the PDF version of the book)

0.1.8 Scientific notation

Most of the interesting phenomena in our universe are not on the human scale. It would take about 1,000,000,000,000,000,000,000 bacteria to equal the mass of a human body. When the physicist Thomas Young discovered that light was a wave, it was back in the bad old days before scientific notation, and he was obliged to write that the time required for one vibration of the wave was 1/500 of a millionth of a millionth of a second. Scientific notation is a less awkward way to write very large and very small numbers such as these. Here's a quick review.

Scientific notation means writing a number in terms of a product of something from 1 to 10 and something else that is a power of ten. For instance,

$$32 = 3.2 \times 10^1$$

$$320 = 3.2 \times 10^2$$

$$3200 = 3.2 \times 10^3 \quad \ldots$$

Each number is ten times bigger than the previous one.

Since 10^1 is ten times smaller than 10^2 , it makes sense to use the notation 10^0 to stand for one, the number that is in turn ten times smaller than 10^1 . Continuing on, we can write 10^{-1} to stand for 0.1, the number ten times smaller than 10^0 . Negative exponents are used for small numbers:

$$3.2 = 3.2 \times 10^0$$

$$0.32 = 3.2 \times 10^{-1}$$

$$0.032 = 3.2 \times 10^{-2} \quad \ldots$$

A common source of confusion is the notation used on the displays of many calculators. Examples:

3.2×10^6	(written notation)
3.2E+6	(notation on some calculators)
3.2^6	(notation on some other calculators)

The last example is particularly unfortunate, because 3.2^6 really stands for the number $3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2=$ 1074, a totally different number from 3.2×10^6=3200000. The calculator notation should never be used in writing. It's just a way for the manufacturer to save money by making a simpler display.

self-check: A student learns that 10^4 bacteria, standing in line to register for classes at Paramecium Community College, would form a queue of this size:

The student concludes that 10^2 bacteria would form a line of this length:

Why is the student incorrect? (answer in the back of the PDF version of the book)

0.1.9 Conversions

I suggest you avoid memorizing lots of conversion factors between SI units and U.S. units, but two that do come in handy are:

{}1 inch = 2.54 cm

An object with a weight on Earth of 2.2 pounds-force has a mass of 1 kg.

The first one is the present definition of the inch, so it's exact. The second one is not exact, but is good enough for most purposes. (U.S. units of force and mass are confusing, so it's a good thing they're not used in science. In U.S. units, the unit of force is the pound-force, and the best unit to use for mass is the slug, which is about 14.6 kg.)

More important than memorizing conversion factors is understanding the right method for doing conversions. Even within the SI, you may need to convert, say, from grams to kilograms. Different people have different ways of thinking about conversions, but the method I'll describe here is systematic and easy to understand. The idea is that if 1 kg and 1000 g represent the same mass, then we can consider a fraction like

$$\frac{10^3\ \text{g}}{1\ \text{kg}}$$

to be a way of expressing the number one. This may bother you. For instance, if you type 1000/1 into your calculator, you will get 1000, not one. Again, different people have different ways of thinking about it, but the justification is that it helps us to do conversions, and it works! Now if we want to convert 0.7 kg to units of grams, we can multiply kg by the number one:

$$0.7\ \text{kg} \times \frac{10^3\ \text{g}}{1\ \text{kg}}$$

If you're willing to treat symbols such as "kg" as if they were variables as used in algebra (which they're really not), you can then cancel the kg on top with the kg on the bottom, resulting in

$$0.7\ \cancel{\text{kg}} \times \frac{10^3\ \text{g}}{1\ \cancel{\text{kg}}} = 700\ \text{g} \quad .$$

To convert grams to kilograms, you would simply flip the fraction upside down.

One advantage of this method is that it can easily be applied to a series of conversions. For instance, to convert one year to units of seconds,

$$1 \; \cancel{\text{year}} \times \frac{365 \; \cancel{\text{days}}}{1 \; \cancel{\text{year}}} \times \frac{24 \; \cancel{\text{hours}}}{1 \; \cancel{\text{day}}} \times \frac{60 \; \cancel{\text{min}}}{1 \; \cancel{\text{hour}}} \times \frac{60 \text{ s}}{1 \; \cancel{\text{min}}} =$$

$$= 3.15 \times 10^7 \text{ s} .$$

Should that exponent be positive, or negative?

A common mistake is to write the conversion fraction incorrectly. For instance the fraction

$$\frac{10^3 \text{ kg}}{1 \text{ g}} \quad \text{(incorrect)}$$

does not equal one, because 10^3 kg is the mass of a car, and 1 g is the mass of a raisin. One correct way of setting up the conversion factor would be

$$\frac{10^{-3} \text{ kg}}{1 \text{ g}} \quad \text{(correct)} \quad .$$

You can usually detect such a mistake if you take the time to check your answer and see if it is reasonable.

If common sense doesn't rule out either a positive or a negative exponent, here's another way to make sure you get it right. There are big prefixes and small prefixes:

big prefixes:	k M
small prefixes:	m µ n

(It's not hard to keep straight which are which, since “mega” and “micro” are evocative, and it's easy to remember that a kilometer is bigger than a meter and a millimeter is smaller.) In the example above, we want the top of the fraction to be the same as the bottom. Since *k* is a big prefix, we need to *compensate* by putting a small number like 10^{-3} in front of it, not a big number like 10^3.

◊ Solved problem: a simple conversion — problem 6

◊ Solved problem: the geometric mean — problem 8

◊ Each of the following conversions contains an error. In each case, explain what the error is.

(a) $1000\ \text{kg} \times \dfrac{1\ \text{kg}}{1000\ \text{g}} = 1\ \text{g}$

(b) $50\ \text{m} \times \dfrac{1\ \text{cm}}{100\ \text{m}} = 0.5\ \text{cm}$

(c) “Nano” is 10^{-9}, so there are 10^{-9} nm in a meter.

(d) “Micro” is 10^{-6}, so 1 kg is 10^{6} *μg*.

0.1.10 Significant figures

An engineer is designing a car engine, and has been told that the diameter of the pistons (which are being designed by someone else) is 5 cm. He knows that 0.02 cm of clearance is required for a piston of this size, so he designs the cylinder to have an inside diameter of 5.04 cm. Luckily, his supervisor catches his mistake before the car goes into production. She explains his error to him, and mentally puts him in the “do not promote” category.

What was his mistake? The person who told him the pistons were 5 cm in diameter was wise to the ways of significant figures, as was his boss, who explained to him that he needed to go back and get a more accurate number for the diameter of the pistons. That person said “5 cm” rather than “5.00 cm” specifically to avoid creating the impression that the number was extremely accurate. In reality, the pistons' diameter was 5.13 cm. They would never have fit in the 5.04-cm cylinders.

The number of digits of accuracy in a number is referred to as the number of significant figures, or “sig figs” for short. As in the example above, sig figs provide a way of showing the accuracy of a number. In most cases, the result of a calculation involving several pieces of data can be no more accurate than the least accurate piece of data. In other words, “garbage in, garbage out.” Since the 5 cm diameter of the pistons was not very accurate, the result of the engineer's calculation, 5.04 cm, was really not as accurate as he thought. In general, your result should not have more than the number of sig figs in the least accurate piece of data you started with. The calculation above should have been done as follows:

5 cm (1 sig fig)

+0.04 cm (1 sig fig)

= 5 cm (rounded off to 1 sig fig)

The fact that the final result only has one significant figure then alerts you to the fact that the result is not very accurate, and would not be appropriate for use in designing the engine.

Note that the leading zeroes in the number 0.04 do not count as significant figures, because they are only placeholders. On the other hand, a number such as 50 cm is ambiguous --- the zero could be intended as a significant figure, or it might just be there as a placeholder. The ambiguity involving trailing zeroes can be avoided by using scientific notation, in which 5×10^1 cm would imply one sig fig of accuracy, while 5.0×10^1 cm would imply two sig figs.

self-check: The following quote is taken from an editorial by Norimitsu Onishi in the New York Times, August 18, 2002.

> Consider Nigeria. Everyone agrees it is Africa's most populous nation. But what is its population? The United Nations says 114 million; the State Department, 120 million. The World Bank says 126.9 million, while the Central Intelligence Agency puts it at 126,635,626.

What should bother you about this? (answer in the back of the PDF version of the book)

Dealing correctly with significant figures can save you time! Often, students copy down numbers from their calculators with eight significant figures of precision, then type them back in for a later calculation. That's a waste of time, unless your original data had that kind of incredible precision.

The rules about significant figures are only rules of thumb, and are not a substitute for careful thinking. For instance, \$20.00 + \$0.05 is \$20.05. It need not and should not be rounded off to \$20. In general, the sig fig rules work best for multiplication and division, and we also apply them when doing a complicated calculation that involves many types of operations. For simple addition and subtraction, it makes more sense to maintain a fixed number of digits after the decimal point.

When in doubt, don't use the sig fig rules at all. Instead, intentionally change one piece of your initial data by the maximum amount by which you think it could have been off, and recalculate the final result. The digits on the end that are completely reshuffled are the ones that are meaningless, and should be omitted.

self-check: How many significant figures are there in each of the following measurements?

(1) 9.937 m

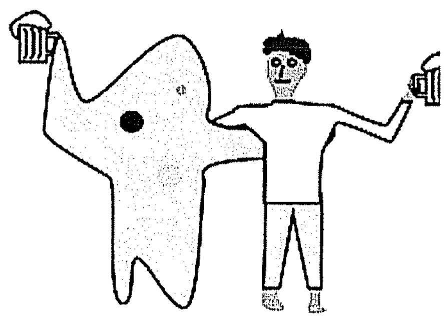
(2) 4.0 s

(3) 0.0000000000000037 kg (answer in the back of the PDF version of the book)

0.2 Scaling and Order-of-Magnitude Estimates

0.2.1 Introduction

Why can't an insect be the size of a dog? Some skinny stretched-out cells in your spinal cord are a meter tall --- why does nature display no single cells that are not just a meter tall, but a meter wide, and a meter thick as well? Believe it or not, these are questions that can be answered fairly easily without knowing much more about physics than you already do. The only mathematical technique you really need is the humble conversion, applied to area and volume.



a / Amoebas this size ar seldom encountered.

Area and volume

Area can be defined by saying that we can copy the shape of interest onto graph paper with 1 cm × 1 cm squares and count the number of squares inside. Fractions of squares can be estimated by eye. We then say the area equals the number of squares, in units of square cm. Although this might seem less "pure" than computing areas using formulae like $A=\pi r^2$ for a circle or $A=wh/2$ for a triangle, those formulae are not useful as definitions of area because they cannot be applied to irregularly shaped areas.

Units of square cm are more commonly written as cm^2 in science. Of course, the unit of measurement symbolized by "cm" is not an algebra symbol standing for a number that can be literally multiplied by itself. But it is advantageous to write the units of area that way and treat the units as if they were algebra symbols. For instance, if you have a rectangle with an area of 6 m^2 and a width of 2 m, then calculating its length as $(6\ \text{m}^2)/(2\ \text{m})=3\ \text{m}$ gives a result that makes sense both numerically and in terms of units. This algebra-style treatment of the units also ensures that our methods of converting units work out correctly. For instance, if we accept the fraction

$$\frac{100\ \text{cm}}{1\ \text{m}}$$

as a valid way of writing the number one, then one times one equals one, so we should also say that one can be represented by

$$\frac{100\ \text{cm}}{1\ \text{m}} \times \frac{100\ \text{cm}}{1\ \text{m}},$$

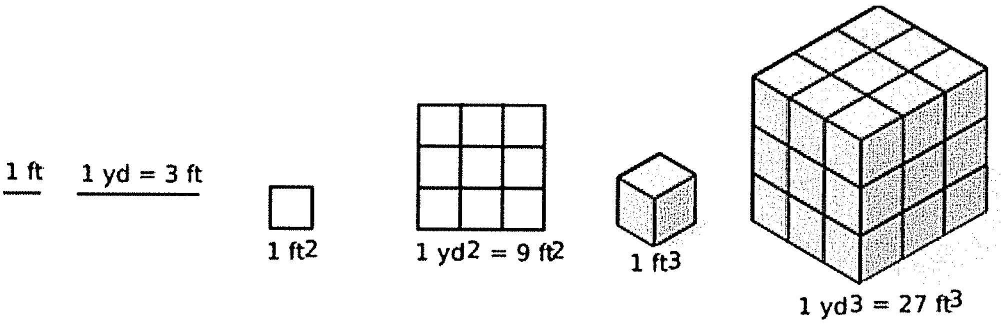
which is the same as

$$\frac{10000\ \text{cm}^2}{1\ \text{m}^2}.$$

That means the conversion factor from square meters to square centimeters is a factor of 10^4, i.e., a square meter has 10^4 square centimeters in it.

All of the above can be easily applied to volume as well, using one-cubic-centimeter blocks instead of squares on graph paper.

To many people, it seems hard to believe that a square meter equals 10000 square centimeters, or that a cubic meter equals a million cubic centimeters --- they think it would make more sense if there were 100 cm^2 in 1 m^2, and 100 cm^3 in 1 m^3, but that would be incorrect. The examples shown in figure b aim to make the correct answer more believable, using the traditional U.S. units of feet and yards. (One foot is 12 inches, and one yard is three feet.)



b / Visualizing conversions of area and volume using traditional U.S. units.

self-check: Based on figure b, convince yourself that there are 9 ft^2 in a square yard, and 27 ft^3 in a cubic yard, then demonstrate the same thing symbolically (i.e., with the method using fractions that equal one). (answer in the back of the PDF version of the book)

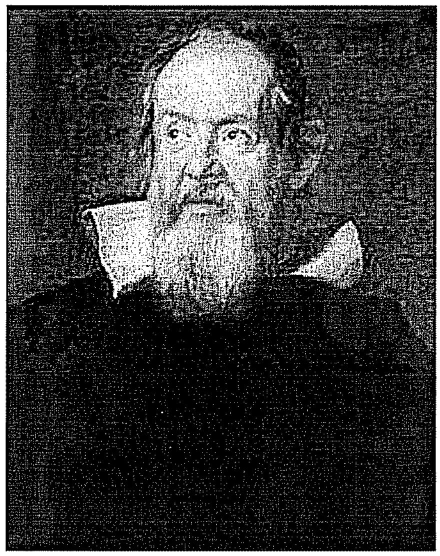
◊ Solved problem: converting mm^2 to cm^2 — problem 31

◊ Solved problem: scaling a liter — problem 40

Discussion Question

◊ How many square centimeters are there in a square inch? (1 inch = 2.54 cm) First find an approximate answer by making a drawing, then derive the conversion factor more accurately using the symbolic method.

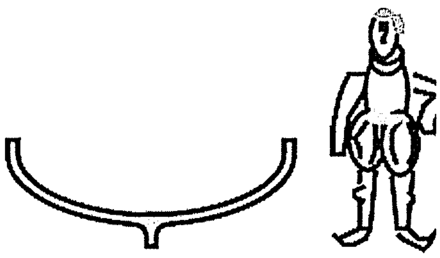
0.2.2 Scaling of area and volume



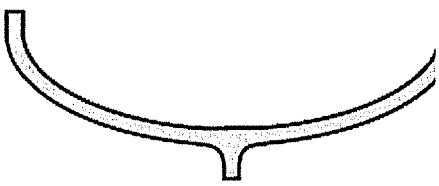
c / Galileo Galilei (156‹
1642).

Great fleas have lesser fleas
Upon their backs to bite 'em.
And lesser fleas have lesser still,
And so ad infinitum. -- *Jonathan Swift*

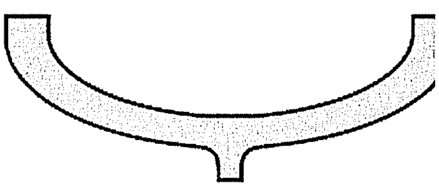
Now how do these conversions of area and volume relate to the questions I posed about sizes of living things? Well, imagine that you are shrunk like Alice in Wonderland to the size of an insect. One way of thinking about the change of scale is that what used to look like a centimeter now looks like perhaps a meter to you, because you're so much smaller. If area and volume scaled according to most people's intuitive, incorrect expectations, with 1 m^2 being the same as 100 cm^2, then there would be no particular reason why nature should behave any differently on your new, reduced scale. But nature does behave differently now that you're small. For instance, you will find that you can walk on water, and jump to many times your own height. The physicist Galileo Galilei had the basic insight that the scaling of area and volume determines how natural phenomena behave differently on different scales. He first reasoned about mechanical structures, but later extended his insights to living things, taking the then-radical point of view that at the fundamental level, a living organism should follow the same laws of nature as a machine. We will follow his lead by first discussing machines and then living things.



d / The small boat holds u
just fine.



e / A larger boat built wi
the same proportions a
the small one will collaps
under its own weight.

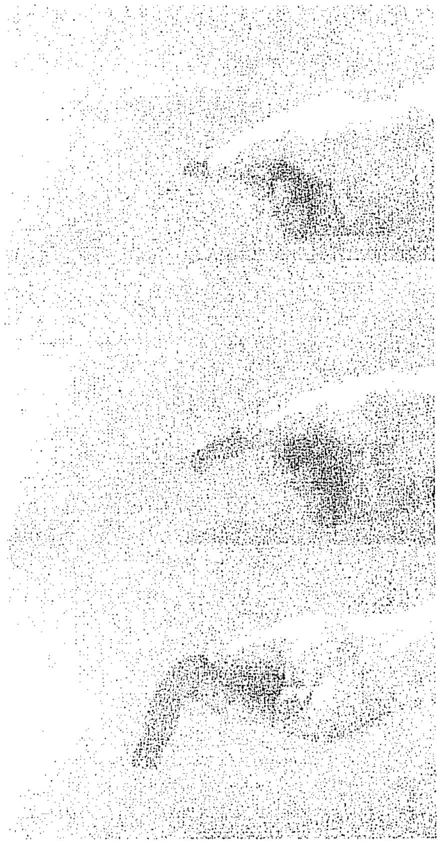


f / A boat this large neec
to have timbers that ai
thicker compared to i
size.

Galileo on the behavior of nature on large and small scales

One of the world's most famous pieces of scientific writing is Galileo's Dialogues Concerning the Two New Sciences. Galileo was an entertaining writer who wanted to explain things clearly to laypeople, and he livened up his work by casting it in the form of a dialogue among three people. Salviati is really Galileo's alter ego. Simplicio is the stupid character, and one of the reasons Galileo got in trouble with the Church was that there were rumors that Simplicio represented the Pope. Sagredo is the earnest and intelligent student, with whom the reader is supposed to identify. (The following excerpts are from the 1914 translation by Crew and de Salvio.)

Sagredo: Yes, that is what I mean; and I refer especially to his last assertion which I have always regarded as false...; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale. Now, since mechanics has its foundations in geometry, where mere size [is unimportant], I do not see that the properties of circles, triangles, cylinders, cones and other solid figures will change with their size. If, therefore, a large machine be constructed in such a way that its parts bear to one another the same ratio as in a smaller one, and if the smaller is sufficiently strong for the purpose for which it is designed, I do not see why the larger should not be able to withstand any severe and destructive tests to which it may be subjected.



h / Galileo discussed planks made of wood, but the concept may be easier to imagine with clay. All three clay rods in the figure were originally the same shape. The medium-sized one was twice the height, twice the length, and twice the width of the small one, and similarly the large one was twice as big as the medium one in all its linear dimensions. The big one has four times the linear dimensions of the small one, 16 times the cross-sectional area when cut perpendicular to the page, and 64 times the volume. That means that the big one has 64 times the weight to support, but only 16 times the strength compared to the smaller one.

Salviati contradicts Sagredo:

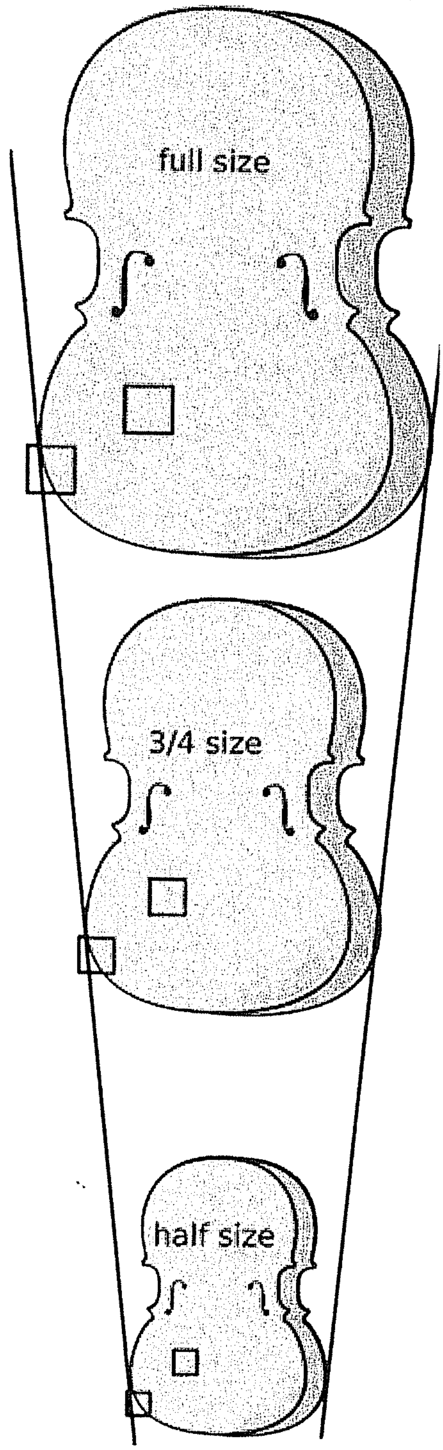
Salviati: ... Please observe, gentlemen, how facts which at first seem improbable will, even on scant explanation, drop the cloak which has hidden them and stand forth in naked and simple beauty. Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.

The point Galileo is making here is that small things are sturdier in proportion to their size. There are a lot of objections that could be raised, however. After all, what does it really mean for something to be "strong", to be "strong in proportion to its size," or to be strong "out of proportion to its size?" Galileo hasn't given operational definitions of things like "strength," i.e., definitions that spell out how to measure them numerically.

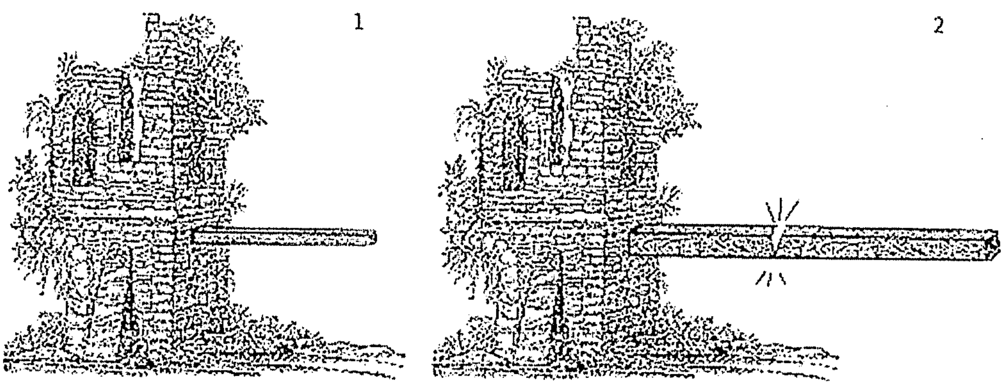
Also, a cat is shaped differently from a horse --- an enlarged photograph of a cat would not be mistaken for a horse, even if the photo-doctoring experts at the National Inquirer made it look like a person was riding on its back. A grasshopper is not even a mammal, and it has an exoskeleton instead of an internal skeleton. The whole argument would be a lot more convincing if we could do some isolation of variables, a scientific term that means to change only one thing at a time, isolating it from the other variables that might have an effect. If size is the variable whose effect we're interested in seeing, then we don't really want to compare things that are different in size but also different in other ways.

> Salviati: ... we asked the reason why [shipbuilders] employed stocks, scaffolding, and bracing of larger dimensions for launching a big vessel than they do for a small one; and [an old man] answered that they did this in order to avoid the danger of the ship parting under its own heavy weight, a danger to which small boats are not subject?

After this entertaining but not scientifically rigorous beginning, Galileo starts to do something worthwhile by modern standards. He simplifies everything by considering the strength of a wooden plank. The variables involved can then be narrowed down to the type of wood, the width, the thickness, and the length. He also gives an operational definition of what it means for the plank to have a certain strength "in proportion to its size," by introducing the concept of a plank that is the longest one that would not snap under its own weight if supported at one end. If you increased its length by the slightest amount, without increasing its width or thickness, it would break. He says that if one plank is the same shape as another but a different size, appearing like a reduced or enlarged photograph of the other, then the planks would be strong "in proportion to their sizes" if both were just barely able to support their own weight.



i / The area of a shape proportional to the squa of its linear dimension even if the shape irregular.



g / 1. This plank is as long as it can be without collapsing under its own weight. If it was a hundredth of an inch longer, it would collapse. 2. This plank is made out of the same kind of wood. It is twice as thick, twice as long, and twice as wide. It will collapse under its own weight.

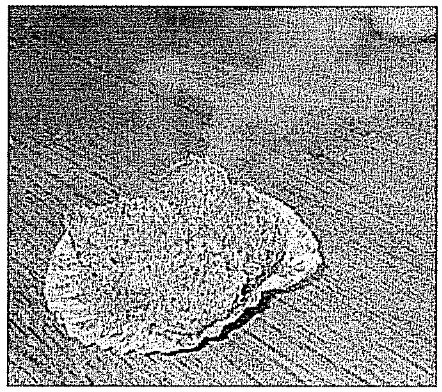
Also, Galileo is doing something that would be frowned on in modern science: he is mixing experiments whose results he has actually observed (building boats of different sizes), with experiments that he could not possibly have done (dropping an ant from the height of the moon). He now relates how he has done actual experiments with such planks, and found that, according to this operational definition, they are not strong in proportion to their sizes. The larger one breaks. He makes sure to tell the reader how important the result is, via Sagredo's astonished response:

> Sagredo: My brain already reels. My mind, like a cloud momentarily illuminated by a lightning flash, is for an instant filled with an unusual light, which now beckons to me and which now suddenly mingles and obscures strange, crude ideas. From what you have said it appears to me impossible to build two similar structures of the same material, but of different sizes and have them proportionately strong.

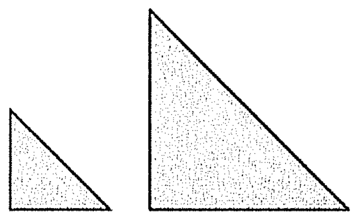
In other words, this specific experiment, using things like wooden planks that have no intrinsic scientific interest, has very wide implications because it points out a general principle, that nature acts differently on different scales.

To finish the discussion, Galileo gives an explanation. He says that the strength of a plank (defined as, say, the weight of the heaviest boulder you could put on the end without breaking it) is proportional to its cross-sectional area, that is, the surface area of the fresh wood that would be exposed if you sawed through it in the middle. Its weight, however, is proportional to its volume.[1]

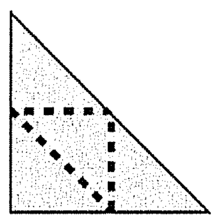
How do the volume and cross-sectional area of the longer plank compare with those of the shorter plank? We have already seen, while discussing conversions of the units of area and volume, that these quantities don't act the way most people naively expect. You might think that the volume and area of the longer plank would both be doubled compared to the shorter plank, so



j / The muffin comes out the oven too hot to ea Breaking it up into fo pieces increases i surface area while keepir the total volume the sam It cools faster because the greater surface-t volume ratio. In genera smaller things have great surface-to-volume ratio but in this example there no easy way to compu the effect exactly, becaus the small pieces aren't th same shape as the origin muffin.



k / Example 3. The b triangle has four time more area than the litt one.

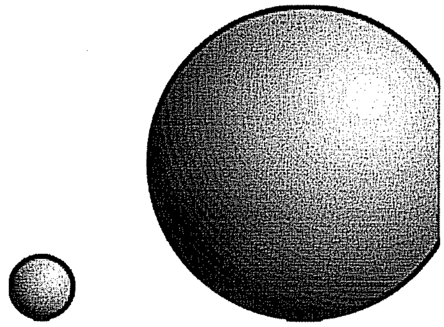


they would increase in proportion to each other, and the longer plank would be equally able to support its weight. You would be wrong, but Galileo knows that this is a common misconception, so he has Salviati address the point specifically:

> Salviati: ... Take, for example, a cube two inches on a side so that each face has an area of four square inches and the total area, i.e., the sum of the six faces, amounts to twenty-four square inches; now imagine this cube to be sawed through three times [with cuts in three perpendicular planes] so as to divide it into eight smaller cubes, each one inch on the side, each face one inch square, and the total surface of each cube six square inches instead of twenty-four in the case of the larger cube. It is evident therefore, that the surface of the little cube is only one-fourth that of the larger, namely, the ratio of six to twenty-four; but the volume of the solid cube itself is only one-eighth; the volume, and hence also the weight, diminishes therefore much more rapidly than the surface... You see, therefore, Simplicio, that I was not mistaken when ... I said that the surface of a small solid is comparatively greater than that of a large one.

The same reasoning applies to the planks. Even though they are not cubes, the large one could be sawed into eight small ones, each with half the length, half the thickness, and half the width. The small plank, therefore, has more surface area in proportion to its weight, and is therefore able to support its own weight while the large one breaks.

l / A tricky way of solvir example 3, explained solution #2.



m / Example 4. The b sphere has 125 times moı volume than the little one.

n / Example 5. The 4ℓ point "S" has 1.78 time more area than the 3(point "S."

Scaling of area and volume for irregularly shaped objects

You probably are not going to believe Galileo's claim that this has deep implications for all of nature unless you can be convinced that the same is true for any shape. Every drawing you've seen so far has been of squares, rectangles, and rectangular solids. Clearly the reasoning about sawing things up into smaller pieces would not prove anything about, say, an egg, which cannot be cut up into eight smaller egg-shaped objects with half the length.

Is it always true that something half the size has one quarter the surface area and one eighth the volume, even if it has an irregular shape? Take the example of a child's violin. Violins are made for small children in smaller size to accomodate their small bodies. Figure i shows a full-size violin, along with two violins made with half and 3/4 of the normal length.[2] Let's study the surface area of the front panels of the three violins.

Consider the square in the interior of the panel of the full-size violin. In the 3/4-size violin, its height and width are both smaller by a factor of 3/4, so the area of the corresponding, smaller square becomes 3/4×3/4=9/16 of the original area, not 3/4 of the original area. Similarly, the corresponding square on the smallest violin has half the height and half the width of the original one, so its area is 1/4 the original area, not half.

The same reasoning works for parts of the panel near the edge, such as the part that only partially fills in the other square. The entire square scales down the same as a square in the interior, and in each violin the same fraction (about 70%) of the square is full, so the contribution of this part to the total area scales down just the same.

Since any small square region or any small region covering part of a square scales down like a square object, the entire surface area of an irregularly shaped object changes in the same manner as the surface area of a square: scaling it down by 3/4 reduces the area by a factor of 9/16, and so on.

In general, we can see that any time there are two objects with the same shape, but different linear dimensions (i.e., one looks like a reduced photo of the other), the ratio of their areas equals the ratio of the squares of their linear dimensions:

$$\frac{A_1}{A_2} = \left(\frac{L_1}{L_2}\right)^2 \quad .$$

Note that it doesn't matter where we choose to measure the linear size, L, of an object. In the case of the violins, for instance, it could have been measured vertically, horizontally, diagonally, or even from the bottom of the left f-hole to the middle of the right f-hole. We just have to measure it in a consistent way on each violin. Since all the parts are assumed to shrink or expand in the same manner, the ratio L_1/L_2 is independent of the choice of measurement.

It is also important to realize that it is completely unnecessary to have a formula for the area of a violin. It is only possible to derive simple formulas for the areas of certain shapes like circles, rectangles, triangles and so on, but that is no impediment to the type of reasoning we are using.

Sometimes it is inconvenient to write all the equations in terms of ratios, especially when more than two objects are being compared. A more compact way of rewriting the previous equation is

$$A \propto L^2 .$$

The symbol "$\propto$" means "is proportional to." Scientists and engineers often speak about such relationships verbally using the phrases "scales like" or "goes like," for instance "area goes like length squared."

All of the above reasoning works just as well in the case of volume. Volume goes like length cubed:

$$V \propto L^3 .$$

If different objects are made of the same material with the same density, $\rho = m/V$, then their masses, $m = \rho V$, are proportional to L^3, and so are their weights. (The symbol for density is ρ, the lower-case Greek letter "rho.")

An important point is that all of the above reasoning about scaling only applies to objects that are the same shape. For instance, a piece of paper is larger than a pencil, but has a much greater surface-to-volume ratio.

One of the first things I learned as a teacher was that students were not very original about their mistakes. Every group of students tends to come up with the same goofs as the previous class. The following are some examples of correct and incorrect reasoning about proportionality.

Example 3: Scaling of the area of a triangle

◊ In figure k, the larger triangle has sides twice as long. How many times greater is its area?

Correct solution #1: Area scales in proportion to the square of the linear dimensions, so the larger triangle has four times more area (2^2=4).

Correct solution #2: You could cut the larger triangle into four of the smaller size, as shown in fig. (b), so its area is four times greater. (This solution is correct, but it would not work for a shape like a circle, which can't be cut up into smaller circles.)

Correct solution #3: The area of a triangle is given by

$A=bh/2$, where b is the base and h is the height. The areas of the triangles are

$$A_1 = b_1 h_1/2$$

$$A_2 = b_2 h_2/2$$

$$= (2b_1)(2h_1)/2$$

$$= 2b_1 h_1$$

$$A_2/A_1 = (2b_1 h_1)/(b_1 h_1/2)$$

$$= 4$$

(Although this solution is correct, it is a lot more work than solution #1, and it can only be used in this case because a triangle is a simple geometric shape, and we happen to know a formula for its area.)

Correct solution #4: The area of a triangle is $A= bh/2$. The comparison of the areas will come out the same as long as the ratios of the linear sizes of the triangles is as specified, so let's just say b_1=1.00 m and b_2=2.00 m. The heights are then also h_1=1.00 m and h_2=2.00 m, giving areas A_1=0.50 m^2 and A_2=2.00 m^2, so A_2/A_1=4.00.

(The solution is correct, but it wouldn't work with a shape for whose area we don't have a formula. Also, the numerical calculation might make the answer of 4.00 appear inexact, whereas solution #1 makes it clear that it is exactly 4.)

Incorrect solution: The area of a triangle is $A=bh/2$, and if you plug in b=2.00 m and h=2.00 m, you get A=2.00 m^2, so the bigger triangle has 2.00 times more area. (This solution is incorrect because no comparison has been made with the smaller triangle.)

Example 4: Scaling of the volume of a sphere

◊ In figure m, the larger sphere has a radius that is five times greater. How many times greater is its volume?

Correct solution #1: Volume scales like the third power of the linear size, so the larger sphere has a volume that is 125 times greater ($5^3=125$).

Correct solution #2: The volume of a sphere is $V=(4/3)\pi r^3$, so

$$V_1 = \frac{4}{3}\pi r_1^3$$

$$V_2 = \frac{4}{3}\pi r_2^3$$

$$= \frac{4}{3}\pi(5r_1)^3$$

$$= \frac{500}{3}\pi r_1^3$$

$$V_2/V_1 = \left(\frac{500}{3}\pi r_1^3\right) / \left(\frac{4}{3}\pi r_1^3\right) = 125$$

Incorrect solution: The volume of a sphere is $V=(4/3)\pi r^3$, so

$$V_1 = \frac{4}{3}\pi r_1^3$$

$$V_2 = \frac{4}{3}\pi r_2^3$$

$$= \frac{4}{3}\pi \cdot 5r_1^3$$

$$= \frac{20}{3}\pi r_1^3$$

$$V_2/V_1 = \left(\frac{20}{3}\pi r_1^3\right) / \left(\frac{4}{3}\pi r_1^3\right) = 5$$

(The solution is incorrect because $(5r_1)^3$ is not the same as $5r_1^3$.)

Example 5: Scaling of a more complex shape

◊ The first letter "S" in figure n is in a 36-point font, the second in 48-point. How many times more ink is required to make the larger "S"? (Points are a unit of length used in typography.)

Correct solution: The amount of ink depends on the area to be covered with ink, and area is proportional to the square of the linear dimensions, so the amount of ink required for the second "S" is greater by a factor of $(48/36)^2=1.78$.

Incorrect solution: The length of the curve of the second "S" is longer by a factor of 48/36=1.33, so 1.33 times more ink is required.

(The solution is wrong because it assumes incorrectly that the width of the curve is the same in both cases. Actually both the width and the length of the curve are greater by a factor of 48/36, so the area is greater by a factor of $(48/36)^2=1.78$.)

◊ Solved problem: a telescope gathers light — problem 32

◊ Solved problem: distance from an earthquake — problem 33

Discussion Questions

◊ A toy fire engine is 1/30 the size of the real one, but is constructed from the same metal with the same proportions. How many times smaller is its weight? How many times less red paint would be needed to paint it?

◊ Galileo spends a lot of time in his dialog discussing what really happens when things break. He discusses everything in terms of Aristotle's now-discredited explanation that things are hard to break, because if something breaks, there has to be a gap between the two halves with nothing in between, at least initially. Nature, according to Aristotle, "abhors a vacuum," i.e., nature doesn't "like" empty space to exist. Of course, air will rush into the gap immediately, but at the very moment of breaking, Aristotle imagined a vacuum in the gap. Is Aristotle's explanation of why it is hard to break things an experimentally testable statement? If so, how could it be tested experimentally?

0.2.3 Order-of-magnitude estimates

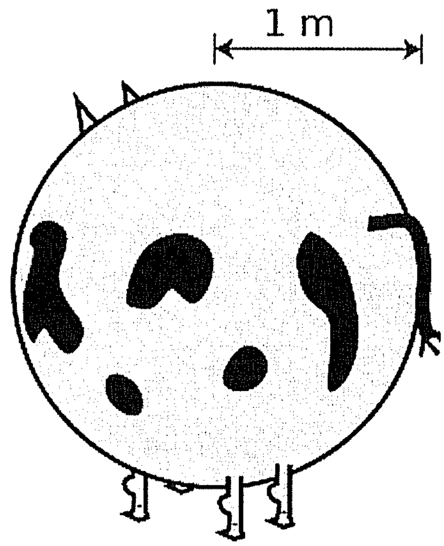
It is the mark of an instructed mind to rest satisfied with the degree of precision that the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible. -- *Aristotle*

It is a common misconception that science must be exact. For instance, in the Star Trek TV series, it would often happen that Captain Kirk would ask Mr. Spock, “Spock, we're in a pretty bad situation. What do you think are our chances of getting out of here?” The scientific Mr. Spock would answer with something like, “Captain, I estimate the odds as 237.345 to one.” In reality, he could not have estimated the odds with six significant figures of accuracy, but nevertheless one of the hallmarks of a person with a good education in science is the ability to make estimates that are likely to be at least somewhere in the right ballpark. In many such situations, it is often only necessary to get an answer that is off by no more than a factor of ten in either direction. Since things that differ by a factor of ten are said to differ by one order of magnitude, such an estimate is called an order-of-magnitude estimate. The tilde, ~, is used to indicate that things are only of the same order of magnitude, but not exactly equal, as in

$$\text{odds of survival} \sim 100 \text{ to one} \quad .$$

The tilde can also be used in front of an individual number to emphasize that the number is only of the right order of magnitude.

Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it's a mode of reasoning that is completely unfamiliar to most college students. Some of the typical mental steps can be illustrated in the following example.



o / Consider a spheric cow.

Example 6: Cost of transporting tomatoes

◊ Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?

◊ The following incorrect solution illustrates one of the main ways you can go wrong in order-of-magnitude estimates.

Incorrect solution: Let's say the trucker needs to make a \$400 profit on the trip. Taking into account her benefits, the cost of gas, and maintenance and payments on the truck, let's say the total cost is more like \$2000. I'd guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself. Transportation really adds a lot to the cost of produce, I guess.

The problem is that the human brain is not very good at estimating area or volume, so it turns out the estimate of 5000 tomatoes fitting in the truck is way off. That's why people have a hard time at those contests where you are supposed to estimate the number of jellybeans in a big jar. Another example is that most people think their families use about 10 gallons of water per day, but in reality the average is about 300 gallons per day. When estimating area or volume, you are much better off estimating linear dimensions, and computing volume from the linear dimensions. Here's a better solution:

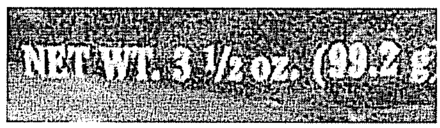
Better solution: As in the previous solution, say the cost of the trip is \$2000. The dimensions of the bin are probably 4 m × 2 m × 1 m, for a volume of 8 m^3. Since the whole thing is just an order-of-magnitude estimate, let's round that off to the nearest power of ten, 10 m^3. The shape of a tomato is complicated, and I don't know any formula for the volume of a tomato shape, but since this is just an estimate, let's pretend that a tomato is a cube, 0.05 m × 0.05 m × 0.05 m, for a volume of 1.25×10^{-4} m^3. Since this is just a rough estimate, let's round that to 10^{-4} m^3. We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato: 10 $m^3/10^{-4}$ $m^3=10^5$ tomatoes. The transportation cost per tomato is \$\$2000/10^5\$ tomatoes=\$0.02/tomato. That means that transportation really doesn't contribute very much to the cost of a tomato.

Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates. A similar situation would occur if you were trying to estimate how many m^2 of leather could be produced from a herd of ten thousand cattle. There is no point in trying to take into account the shape of the cows' bodies. A reasonable plan of attack might be to consider a spherical cow. Probably a cow has roughly the same surface area as a sphere with a radius of about 1 m, which would be $4\pi\ (1\ \text{m})^2$. Using the well-known facts that pi equals three, and four times three equals about ten, we can guess that a cow has a surface area of about 10 m^2, so the herd as a whole might yield $10^5\ \text{m}^2$ of leather.

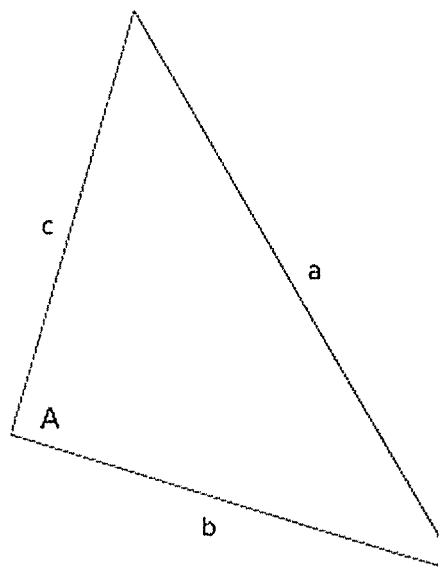
The following list summarizes the strategies for getting a good order-of-magnitude estimate.

1. Don't even attempt more than one significant figure of precision.
2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them.
3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.
4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield 0.01 m^2 of leather, then you have probably made a mistake with conversion factors somewhere.

Homework Problems



a / Problem 10.



b / Problem 12.

1. Correct use of a calculator: (a) Calculate $\frac{74658}{53222 + 97554}$ on a calculator. [Self-check: The most common mistake results in 97555.40.] (answer check available at lightandmatter.com)
(b) Which would be more like the price of a TV, and which would be more like the price of a house, $\$3.5\times10^5$ or $\$3.5^5$?

2. Compute the following things. If they don't make sense because of units, say so.
(a) 3 cm + 5 cm
(b) 1.11 m + 22 cm
(c) 120 miles + 2.0 hours
(d) 120 miles / 2.0 hours

3. Your backyard has brick walls on both ends. You measure a distance of 23.4 m from the inside of one wall to the inside of the other. Each wall is 29.4 cm thick. How far is it from the outside of one wall to the outside of the other? Pay attention to significant figures.

4. The speed of light is 3.0×10^8 m/s. Convert this to furlongs per fortnight. A furlong is 220 yards, and a fortnight is 14 days. An inch is 2.54 cm.(answer check available at lightandmatter.com)

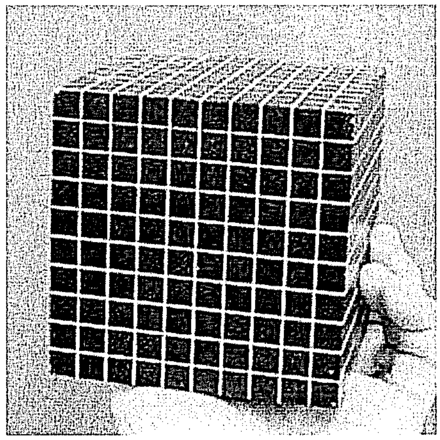
5. Express each of the following quantities in micrograms: (a) 10 mg, (b) 10^4 g, (c) 10 kg, (d) 100×10^3 g, (e) 1000 ng. (answer check available at lightandmatter.com)

6. (solution in the pdf version of the book) Convert 134 mg to units of kg, writing your answer in scientific notation.

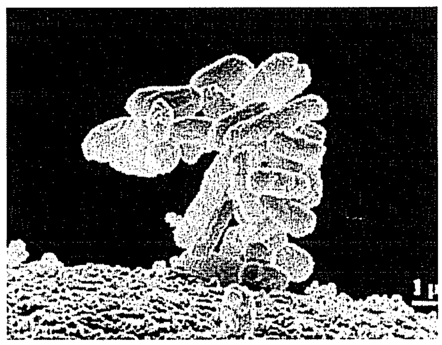
7. In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implanted steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas? (answer check available at lightandmatter.com)

8. (solution in the pdf version of the book) The usual definition of the mean (average) of two numbers a and b is $(a+b)/2$. This is called the arithmetic mean. The geometric mean, however, is defined as $(ab)^{1/2}$ (i.e., the square root of ab). For the sake of definiteness, let's say both numbers have units of mass. (a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If a and b both have units of grams, what should we call the units of ab? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as $(ab)^{1/3}$. Is this reasonable?

c / Albert Einstein, and h moustache, problem 35.



d / Problem 40.



e / Problem 42.

9. In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

10. The photo shows the corner of a bag of pretzels. What's wrong here?

11. The distance to the horizon is given by the expression $\sqrt{2rh}$, where r is the radius of the Earth, and h is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. (See example 2 on p. 27 for an example of how to do this.) Don't try to prove the result, just check its units.

12. (solution in the pdf version of the book) (a) Based on the definitions of the sine, cosine, and tangent, what units must they have? (b) A cute formula from trigonometry lets you find any angle of a triangle if you know the lengths of its sides. Using the notation shown in the figure, and letting $s=(a+b+c)/2$ be half the perimeter, we have

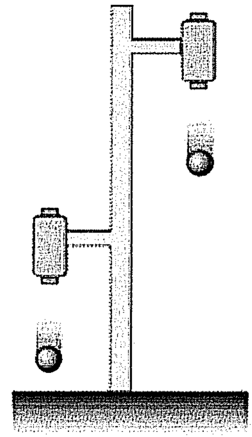
$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} .$$

Show that the units of this equation make sense. In other words, check that the units of the right-hand side are the same as your answer to part a of the question.

13. A physics homework question asks, "If you start from rest and accelerate at 1.54 m/s^2 for 3.29 s, how far do you travel by the end of that time?" A student answers as follows:

$$1.54 \times 3.29 = 5.07 \text{ m}$$

His Aunt Wanda is good with numbers, but has never taken physics. She doesn't know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?



f / Problem 43.

14. You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well? (answer check available at lightandmatter.com)

15. You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get half-way there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.
(a) Find a formula for the time, T, required for the round trip, in terms of d, the distance from our sun to the star, and a, the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.
(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of $d=4\times10^{16}$ m. Suppose you use an acceleration of $a=10$ m/s^2, just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?
(c) Using the same numbers for d and a, find your maximum speed. Compare this to the speed of light, which is 3.0×10^8 m/s. (Later in this course, you will learn that there are some new things going on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you've learned so far.) (answer check available at lightandmatter.com)

16. You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)

17. (solution in the pdf version of the book) If the acceleration of gravity on Mars is 1/3 that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance.

18. A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by an equation of the form

$$y = b - c\left(t + ke^{-t/k}\right) \quad ,$$

where e is the base of natural logarithms, and b, c, and k are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.
(a) What units would b, c, and k have to have for the equation to make sense?
(b) Find the person's velocity, v, as a function of time. [You will need to use the chain rule, and the fact that $d(e^x)/d\,x=e^x$.] (answer check available at lightandmatter.com)
(c) Use your answer from part (b) to get an interpretation of the constant c. [Hint: e^{-x} approaches zero for large values of x.]
(d) Find the person's acceleration, a, as a function of time.(answer check available at lightandmatter.com)
(e) Use your answer from part (b) to show that if she waits long enough to open her chute, her acceleration will become very small.

19. (solution in the pdf version of the book) In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car's top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with *constant* acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part *b* and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time?

20. The speed required for a low-earth orbit is 7.9×10^3 m/s (see ch. 10). When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to $3g$ to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn't necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth's surface.

21. Consider the following passage from Alice in Wonderland, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall *never* come to an end? “I wonder how many miles I've fallen by this time?” she said aloud. “I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think” (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a *very* good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)...

Alice doesn't know much physics, but let's try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of 10 m/s^2. This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, g is zero, because the earth is pulling you equally in every direction at once). (answer check available at lightandmatter.com)

22. How many cubic inches are there in a cubic foot? The answer is not 12.(answer check available at lightandmatter.com)

23. Assume a dog's brain is twice as great in diameter as a cat's, but each animal's brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

24. The population density of Los Angeles is about 4000 people/km^2. That of San Francisco is about 6000 people/km^2. How many times farther away is the average person's nearest neighbor in LA than in San Francisco? The answer is not 1.5. (answer check available at lightandmatter.com)

25. A hunting dog's nose has about 10 square inches of active surface. How is this possible, since the dog's nose is only about 1 in × 1 in × 1 in = 1 in^3? After all, 10 is greater than 1, so how can it fit?

26. Estimate the number of blades of grass on a football field.

27. In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

28. Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world's population to live.

29. A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

30. Estimate the volume of a human body, in cm^3.

31. (solution in the pdf version of the book) How many cm^2 is 1 mm^2?

32. (solution in the pdf version of the book) Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope.

33. (solution in the pdf version of the book) One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.

34. In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box? (answer check available at lightandmatter.com)

35. Estimate the mass of one of the hairs in Albert Einstein's moustache, in units of kg.

36. According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar's last words. Is this true? If so, how many?

37. The Earth's surface is about 70% water. Mars's diameter is about half the Earth's, but it has no surface water. Compare the land areas of the two planets.(answer check available at lightandmatter.com)

38. (solution in the pdf version of the book) The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

39. The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD's area is therefore lost? (answer check available at lightandmatter.com)

40. The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?(solution in the pdf version of the book)

41. Estimate the number of man-hours required for building the Great Wall of China. (solution in the pdf version of the book)

42. (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 μm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of 10^{13}. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?

43. The figure shows a practical, simple experiment for determining *g* to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights Δx_1 and Δx_2. A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity T defined as $T=\Delta t_2-\Delta t_1$, i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound when they are released, we have no way of measuring the individual times Δt_2 and Δt_1.

(a) Find an equation for g in terms of the measured quantities T, Δx_1 and Δx_2.(answer check available at lightandmatter.com)

(b) Check the units of your equation.

(c) Check that your equation gives the correct result in the case where Δx_1 is very close to zero. However, is this case realistic?

(d) What happens when $\Delta x_1=\Delta x_2$? Discuss this both mathematically and physically.

Footnotes

[1] Galileo makes a slightly more complicated argument, taking into account the effect of leverage (torque). The result I'm referring to comes out the same regardless of this effect.
[2] The customary terms "half-size" and "3/4-size" actually don't describe the sizes in any accurate way. They're really just standard, arbitrary marketing labels.

feature article

Dynamic similarity, the dimensionless science

Diogo Bolster, Robert E. Hershberger, and Russell J. Donnelly

Dimensional analysis, a framework for drawing physical parallels between systems of disparate scale, affords key insights into natural phenomena too expansive and too energetic to replicate in the lab.

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Many experiments seem daunting at first glance, owing to the sheer number of physical variables they involve. To design an apparatus that circulates fluid, for instance, one must know how the flow is affected by pressure, by the apparatus's dimensions, and by the fluid's density and viscosity. Complicating matters, those parameters may be temperature and pressure dependent. Understanding the role of each parameter in such a high-dimensional space can be elusive or prohibitively time consuming.

Dimensional analysis, a concept historically rooted in the field of fluid mechanics, can help to simplify such problems by reducing the number of system parameters. For example, in a fluid apparatus in which the flow is isothermal and incompressible, the number of relevant parameters can often be reduced to one. The rewards of such a reduction can be spectacular: It may allow a model the size of a children's toy to yield insight into the dynamics of a jet airplane, or a fluid-filled cylinder the size of a garbage can to elucidate the behavior of a stellar interior. (See box 1 for a brief history of dimensional analysis.)

Dimensional reasoning

Dimensional analysis comes in many forms. One of its simplest uses is to check the plausibility of theoretical results. For example, the displacement $\mathbf{x}(t)$ of a falling body having initial displacement $\mathbf{x}_0$ and initial velocity $\mathbf{u}_0$ is

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{u}_0 t + \frac{1}{2}\mathbf{g}t^2,$$

where $\mathbf{g}$ is its acceleration due to gravity. According to the principle of dimensional homogeneity, if the left- and right-hand sides of the equation are truly equal, they must share the same dimensions. Indeed, each term in the equation has dimensions of length. Despite the modesty of the dimensional-homogeneity requirement, it is violated by a number of equations often used in the hydraulics literature, such as the Manning formula for flow in an open channel and the Hazen–Williams formula, which describes flows of water through pipes.

Dimensional analysis can also help to supply a theoretical result. Consider the ray of light illustrated in figure 1, which, in accordance with general relativity, is deflected as it passes through the gravitational field of the Sun. Assuming the Sun can be treated as a point of mass m and that the ray of light passes the mass with a distance of closest approach r, dimensional reasoning can help predict the deflection angle θ.[1]

Expressed in terms of mass M, length L, and time T, the variables' dimensions—denoted with square brackets—are

Box 1. A brief history of dimensional analysis

Going back more than 300 years, discussions of dimensional analysis have appeared in scores of texts, often with different slants:

1687. Isaac Newton publishes the *Principia*, which, in book II, section 7, contains perhaps the earliest documented discussion of dimensional analysis.

1765. Leonhard Euler writes extensively about units and dimensional reasoning in *Theoria motus corporum solidorum seu rigidorum*, a comprehensive treatment of the mechanics of rigid bodies.

1822. Joseph Fourier employs concepts of dimensional analysis in his *Analytical Theory of Heat*.

1877. Lord Rayleigh outlines a "method of dimensions" in his *Theory of Sound*.

1908. At the 4th International Congress of Mathematicians in Rome, Arnold Sommerfeld introduces a dimensionless number that he calls the Reynolds number, in tribute to Osborne Reynolds. The Reynolds number, which appeared in what's now known as the Orr–Sommerfeld equation, is among the most famous of all dimensionless numbers.

1914. In what is generally regarded as the big breakthrough in dimensional analysis, physicist Edgar Buckingham introduces the theorem now known as the Buckingham Pi theorem. It is one of several methods of reducing a number of dimensional variables to a smaller number of dimensionless groups.

1922. In his influential book *Dimensional Analysis*, Percy Bridgman outlines a general theory of the subject.

1953. In his George Darwin lecture before the Royal Astronomical Society, Subrahmanyan Chandrasekhar names the Rayleigh number, a dimensionless temperature difference central to thermal convection.

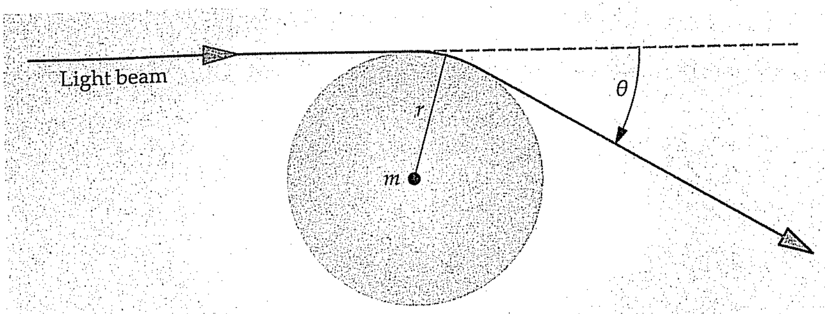


Figure 1. A light beam is deflected as it passes through the gravitational field of a star. Here, m is the star's mass and r is the distance of closest approach. Even without knowing the underlying physics, one can use dimensional reasoning to predict that the deflection angle θ scales as $Gmr^{-1}c^{-2}$, where G is the gravitational constant and c is the speed of light.

$[\theta] = L^0T^0M^0$, $[r] = L^1T^0M^0$, and $[m] = L^0T^0M^1$. It follows that no combination of powers of r and m can be dimensionally homogeneous with θ.

Adding the gravitational constant G, which has dimensions $L^3T^{-2}M^{-1}$, and the speed of light c, which has dimensions $L^1T^{-1}M^0$, seems sensible for a problem concerning gravity and light. Of the potential expressions containing m, r, c, and G, algebraic calculations reveal that dimensional homogeneity is achieved only with solutions of the form $m^{\kappa}r^{-\kappa}c^{-2\kappa}G^{\kappa}$, where κ is any real integer. (See reference 1 for a step-by-step treatment.)

If the light ray skims just across the Sun's surface, then $r = 6.96 \times 10^8$ m, $m = 1.99 \times 10^{30}$ kg, and the quantity $m^{\kappa}r^{-\kappa}c^{-2\kappa}G^{\kappa}$ will be small—on the order of 10^{-6} when $\kappa = 1$. The $\kappa = 1$ term will give the largest effect, and higher-order terms can be neglected. Arguments based purely on dimensional reasoning suggest, then, that

$$\theta = \alpha\left(\frac{Gm}{c^2r}\right),$$

where α is an unknown constant. When Isaac Newton considered the problem more than 300 years ago, he arrived at an identical expression, with $\alpha = 2$. General relativity predicts $\alpha = 4$, and the latest experiments agree with that result to within 0.02%.

Prototypes, models, and similitude

In fluid dynamics, dimensional analysis is used to reduce a large number of parameters to a small number of dimensionless groups, often in spectacular fashion. In addition to easing analysis, that reduction of variables gives rise to new classes of similarity.

Consider the simple example of flow around a prototype airfoil, p, and a much smaller model, m, as illustrated in figure 2. The model and prototype are geometrically similar if all of their corresponding length scales, including surface roughness, are proportionate. Likewise, flows in the two systems are kinematically similar if the velocity ratios $\mathbf{u}_p/\mathbf{u}_m$ are the same for all pairs of corresponding, or homologous, points.

Depending on what is to be learned from the model, kinematic similarity may be too lax a requirement. The stricter standard of dynamic similarity exists if the ratios of all forces acting on homologous fluid particles and boundary surfaces in the two systems are constant. Dynamically similar systems are by definition both geometrically and kinematically similar.

An important conclusion of fluid mechanics is that incompressible, isothermal flows in or around geometrically similar bodies are considered dynamically similar if they have the same Reynolds number Re, where Re is the ratio of inertial to viscous forces (see box 2). Consider the example of a typical attack submarine, 110 m long and capable of moving at 20 knots, or about 10 m/s. In water, that corresponds to $Re = 1.13 \times 10^9$. If all design tests must be conducted on a 6-m-long scale model that can be towed at a top speed of 10 knots, the highest achievable Re would be about 3×10^7, about 1/36 that of an actual submarine. The model would be a poor descriptor of large-Re effects such as turbulence.

If the same model were placed in a very large wind tunnel blowing air at, say, half the speed of sound, Re would be about 7×10^7—closer to, but still well short of, true submarine conditions. Further options are to cool the air, thus lowering its viscosity and increasing its density—and thereby reducing the kinematic viscosity—or to operate at high pressures, increasing density more still. Adopting just that strategy, engineers at the National Transonic Facility at NASA's Langley Research Center in Virginia reached Re of about 1×10^9 in a cryogenic wind tunnel.

Cryogenic tunnels are currently among the most advanced test facilities available. Remarkably, one of the smallest such tunnels, just 1.4 cm in diameter, is capable of reaching Re as high as 1.5 million.[2]

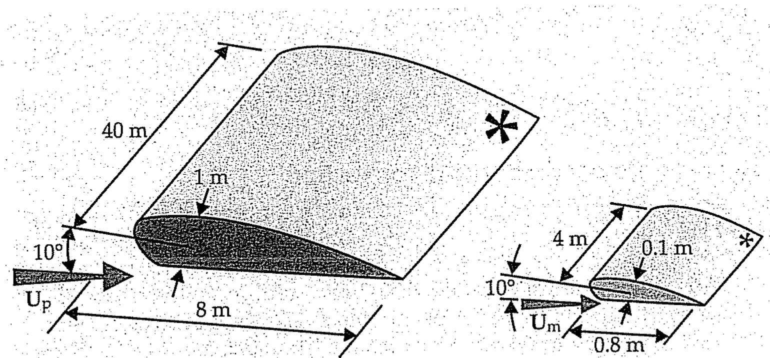


Figure 2. A prototype airfoil (left) can be tested with a much smaller model (right), provided the objects are geometrically similar and that the flows around them are dynamically similar. Dynamic similarity is achieved if the characteristic flow velocities $\mathbf{U}_p$ and $\mathbf{U}_m$ are such that the forces at all homologous points—such as the two marked by asterisks—are proportionate. (Adapted from ref. 15.)

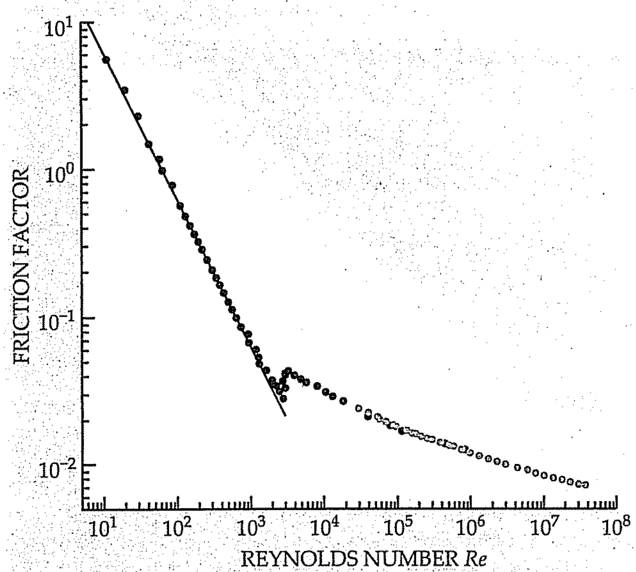


Figure 3. The friction factor, the dimensionless shear stress exerted by a fluid on a pipe, is plotted as a function of the Reynolds number *Re*. Blue symbols correspond to data obtained from a 12-cm-wide, 34-m-long "superpipe" operated at room temperature. Black symbols correspond to data collected in a pipe roughly 1/100 000 that size, operated at cryogenic temperatures. Where they overlap, the data sets agree to within about 2%, safely within the error margin of both experiments. The solid line is the theoretical result for laminar flow, and the discontinuity near $Re = 2 \times 10^3$ corresponds to the transition to turbulence. (Adapted from ref. 5.)

The friction factor

One consequence of dynamic similarity in pipe flows is that the so-called friction factor λ—the dimensionless shear stress exerted by the fluid on the pipe, and vice versa—is a function of *Re* only, provided entrance effects, surface roughness, and temperature variations are small. The friction factor is especially significant in engineering. The standard $\lambda(Re)$ plot, compiled from the results of eight papers published between 1914 and 1933, is reproduced in nearly all fluid dynamics texts and spans *Re* from 1×10^3 to 3×10^6. Two devices exploiting two different strategies have charted new territory.

In 1998 the Princeton University team of Mark Zagarola and Alexander Smits published the first results from their "superpipe,"[3] a closed-loop, 34-m-long pipe with a nominal diameter of 12 cm. Using room-temperature air compressed as high as 187 atmospheres, the pair measured $\lambda(Re)$ for *Re* up to 3.6×10^7.

Later, a University of Oregon group led by one of us (Donnelly) designed a device consisting of a 28-cm-long pipe, roughly a half-centimeter in diameter, housed in a tabletop helium cryostat.[4] Several room-temperature gases—helium, oxygen, nitrogen, carbon dioxide, and sulfur hexafluoride—were used to measure $\lambda(Re)$ for relatively small *Re*; liquid He was used to attain the highest *Re*, up to 1.1×10^6.

Figure 3 shows datasets from both experiments. Combined, the data span *Re* ranging from 11 to 37 million. Despite a dramatic difference in scale—the Princeton superpipe weighs about 25 tons, the Oregon tube about an ounce—the overlapping data sets agree[5] to within about 2%. It is a testament to the power of dynamic similarity.

Box 2. The origins of some dimensionless numbers in fluid mechanics

The Reynolds number. The most famous of the dimensionless numbers, the Reynolds number, can be derived from the Navier–Stokes equations for incompressible flows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is local velocity, p is pressure, ρ is the fluid density, and ν is the fluid's kinematic viscosity.

Choosing appropriate characteristic length and velocity scales, L and U, one can introduce a dimensionless displacement $\mathbf{x}' \equiv \mathbf{x}/L$, dimensionless time $t' \equiv tU/L$, dimensionless velocity $\mathbf{u}' \equiv \mathbf{u}/U$, and dimensionless pressure $p' \equiv p/\rho U^2$. The Navier–Stokes equations become

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' p' + \frac{1}{Re}\nabla'^2 \mathbf{u}'$$
$$\nabla' \cdot \mathbf{u}' = 0,$$

where $Re = UL/\nu$ is the Reynolds number.

The change to dimensionless variables is not just a superficial step; it greatly reduces the amount of work needed to study a given flow. Although it might seem that one would need to investigate separately the effects of varying ρ, L, U and ν, one needs to investigate only variations with *Re*.

The Rayleigh, Prandtl, and Nusselt numbers. Assuming density variations are small, thermal convection can be described by the Boussinesq equations,

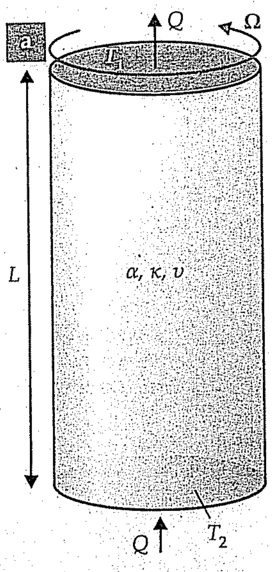
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -(\nabla p)/\rho + \nu \nabla^2 \mathbf{u} + \alpha \mathbf{g}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa \nabla^2 T,$$

where α and κ are, respectively, the fluid's thermal expansion coefficient and thermal diffusivity, $\mathbf{g}$ is the acceleration due to gravity, and T is temperature. Nondimensionalization of the Boussinesq equations yields two key parameters: the Rayleigh number $Ra \equiv g\alpha L/\kappa\nu$, the dimensionless temperature difference, and the Prandtl number $Pr \equiv \nu/\kappa$, the ratio of vorticity diffusivity to thermal diffusivity.

The heat transfer rate is usually described in terms of the Nusselt number *Nu*, the ratio of the actual heat transfer Q to the heat transfer Q_c that would result from conduction alone. For nonrotating Bénard cells having the same shape and boundary conditions, *Nu* is completely determined by *Ra* and *Pr*.

Rayleigh–Bénard convection

Thermally driven convection is a conceptually simple but experimentally challenging problem. In the lab it's typically carried out in a Bénard cell like that sketched in figure 4a, a container of fluid heated from below and cooled from above. The temperature difference ΔT gives rise to a density gradient; for typical fluids having a positive thermal expansion coefficient, the denser fluid will lie above the less dense fluid. If the den-



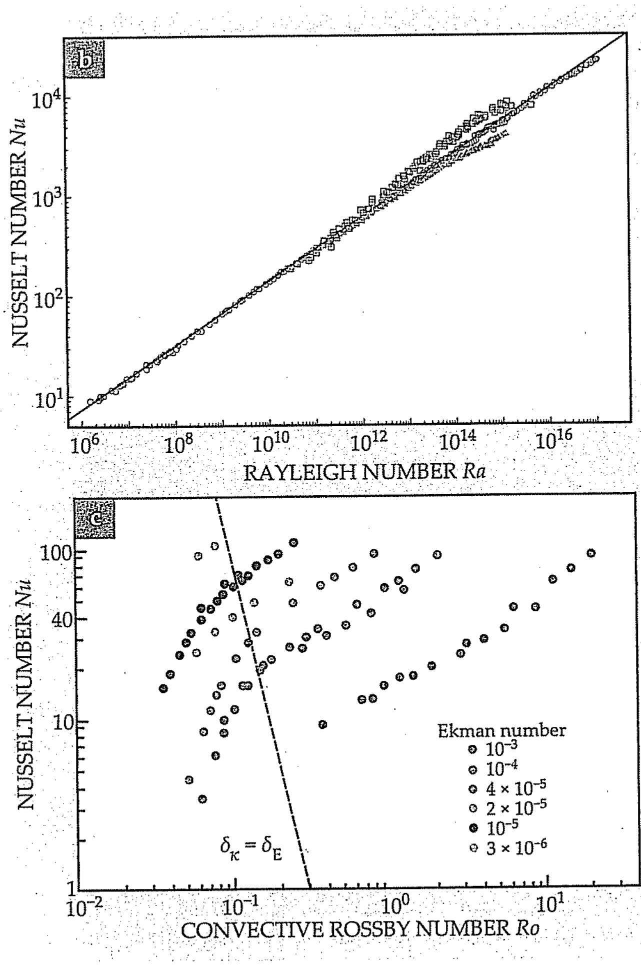


Figure 4. Rayleigh–Bénard convection. (a) A Bénard cell consists of a layer of fluid of thickness L that's heated from below and cooled from above. For normal fluids having a positive thermal expansion coefficient α, the configuration can be unstable—with denser fluid resting atop less dense fluid—and lead to convection. To study rotation effects, the cell can be rotated about its axis with angular velocity Ω. **(b)** Even in the nonrotating case, the relationship between the temperature difference $T_2 - T_1$ and the heat transfer rate (shown here in dimensionless form as the Rayleigh and Nusselt numbers, respectively) remains a matter of debate. Data obtained at the University of Oregon (red), Joseph Fourier University (green), and the University of Göttingen (blue) agree at moderately large Ra, but diverge for $Ra > 10^{13}$. (Data are from refs. 6, 7, and 9.) **(c)** In a rotated Bénard cell, the influence of rotation is less a function of the Rossby number Ro, the ratio of buoyant to Coriolis forces, than of the Ekman number E, the ratio of viscous to Coriolis forces. The dashed line corresponds to $Nu = 0.18\,E^{-1/2}$; it indicates where the thickness δ_E of the Ekman boundary layer equals the thickness δ_κ of the thermal boundary layer, and it marks the transition between rotationally dominated and nonrotationally dominated convection. (Adapted from ref. 10.)

sity gradients are large enough, the configuration destabilizes, leading to circulating flow known as Rayleigh–Bénard convection. That convection enhances the heat transfer from the hot lower boundary to the cool upper one.

As detailed in box 2, the character of a Rayleigh–Bénard flow can be described wholly in terms of dimensionless parameters. The Nusselt number Nu, the dimensionless heat-transfer rate, depends on the Rayleigh number Ra, the dimensionless temperature difference, and the Prandtl number Pr, the ratio of the diffusivity of vorticity to thermal diffusivity. (We've assumed that the system's geometry is fixed.) For small Ra, the fluid layer remains at rest, heat transfer is entirely conductive, and Nu is relatively small. But as Ra grows, the fluid begins to convect and Nu increases. A series of complicated flow transitions ensues, until eventually—roughly, around $Ra = 10^6$—the flow becomes turbulent. In that turbulent regime, dimensional arguments suggest that $Nu \propto Ra^\gamma$, where heuristic arguments suggest that γ should vary from around 2/7, or 1/3, to an asymptotic value of 1/2. (Other expressions for $Nu(Ra)$ have been hypothesized; see, for example, the article by Leo Kadanoff, PHYSICS TODAY, August 2001, page 34.)

The experimental challenge is to explore Nu in the highly turbulent regimes of large Ra. Dynamic similarity affords multiple ways to do so. One way to boost Ra—defined as $g\alpha L^3 \Delta T/\kappa\nu$, where α, κ, and ν are the thermal expansion coefficient, the vorticity diffusivity, and the thermal diffusivity, respectively—is to create large temperature gradients. A drawback of that approach, however, is that it can yield large density variations, which complicate theoretical modeling. When ΔT is large, a central assumption of the Boussinesq equations that describe thermal convection—namely, that density depends linearly on temperature—no longer holds true (see box 2).

Another strategy for obtaining large Ra is to use a thick fluid layer. Because Ra scales as L^3, modest increases in L can produce substantial gains in Ra. Alternatively, one can choose a fluid having large $\alpha/\kappa\nu$. By those measures, low-temperature He is, to our knowledge, the most ideal fluid available. Extreme increases in $\alpha/\kappa\nu$, however, can lead to undesirably large changes in Pr.

Several experiments adopt some combination of the above strategies to explore heat transfer at the higher reaches of Ra. A University of Oregon team led by Donnelly used low-temperature He to explore Ra spanning 11 orders of magnitude.[6] Strikingly, the data, shown in red in figure 4b, are described cleanly by a single power law, with $\gamma = 0.31$.

In 2001, about the same time as the Oregon data appeared, researchers at Joseph Fourier University in Grenoble, France, used a nearly identical cryogenic cell to obtain the results shown in green in figure 4b.[7] As Ra nears 2×10^{11}, the Grenoble data switch from a power law described by $\gamma = 0.31$ to one described by $\gamma = 0.39$. One potentially exciting interpretation is that the switch marks the transition to the "ultimate state," a state predicted by Robert Kraichnan for asymptotically large Ra, in which the viscous boundary layers at the ends of the Bénard cell become turbulent.[8]

The discrepancy between the Oregon and Grenoble data as plotted in figure 4b seems quite small. Is it of any real consequence? In geophysical and astrophysical fluid dynamics, the answer is yes. Natural phenomena such as mantle convection in Earth's outer core, atmospheric and oceanic winds, and flows in gas giants and stars are estimated to have Ra ranging from 10^{20} to 10^{30}, perhaps larger in stellar systems. Extrapolated to such geophysical and astrophysical proportions, a slight difference in scaling relationships could yield order-of-magnitude differences in Nu.

The ideas of dynamic similarity can help resolve the discrepancy. Guenter Ahlers of the University of California, Santa Barbara and colleagues at the University of Göttingen in Germany explored the range of Ra between 10^9 and 10^{15} using He, N_2, and SF_6 at ambient temperatures and pressures up to 15 atmospheres.[9] The high pressure, a feature absent from the Oregon and Grenoble experiments, limits changes in Pr.

Ahlers and company's data, shown in blue in figure 4b, agree closely with the Oregon results and contradict the ultimate-state interpretation of the Grenoble data. But as the figure shows, there is still no consensus for large-Ra behavior, and the story continues to unfold. It remains unclear whether Kraichnan's ultimate state exists, and if so, where it begins.

Rayleigh–Bénard convection, with rotation

Most convection systems of geophysical and astrophysical interest also involve rotation. The influence of rotation is studied in the lab by spinning a Bénard cell about its axis with some angular velocity Ω. Again, the flow behavior can be described in dimensionless terms. Except now, in addition to Ra and Pr, two new dimensionless numbers are also important.

The first is the convective Rossby number $Ro \equiv (g\alpha\Delta T/4\Omega^2 L)^{1/2}$, the ratio of temperature-induced buoyant forces to rotation-induced Coriolis forces. One might anticipate that the transition between rotationally dominated and nonrotationally dominated flow should occur somewhere near $Ro = 1$.

But there is also the Ekman number $E \equiv \nu(2\Omega L)^{-1}$, the ratio of viscous to Coriolis forces. Coriolis forces tend to sweep away the viscous boundary layer that exists near the container walls, and so the thickness δ_E of that boundary layer scales as $E^{1/2}$. A competing length scale is the thickness δ_κ of the thermal boundary layer, which scales as $Nu^{-1}L$. In general, communication between the container and the bulk fluid will be limited by the thinner of the two boundary layers. One might anticipate that the transition between rotationally dominated and nonrotationally dominated convection should occur when $\delta_E = \delta_\kappa$.

As shown in figure 4c, data from experiments at UCLA's simulated planetary interiors laboratory confirm that the condition $\delta_E = \delta_\kappa$, not $Ro = 1$, governs the transition from rotationally dominated to nonrotationally dominated convection.[10] When $\delta_E < \delta_\kappa$, rotation acts to prevent convection, and heat transfer is less efficient than in a nonrotating system. When $\delta_E > \delta_\kappa$, rotation effects are negligible, and Nu scales as it does in the nonrotating case. With that crucial observation, the UCLA researchers were able to estimate the temperature gradients in Earth's liquid-metal outer core as corresponding to $Ra = 7 \times 10^{24}$. Of course, the extrapolation of carefully controlled laboratory experiments to geophysical fluid mechanics carries caveats, several of which are detailed in reference 10.

A real-world pendulum

Among the first problems posed to undergraduate physics students is that of a simple pendulum: a point mass suspended in a vacuum, oscillating with small amplitude. A real pendulum oscillating in a viscous fluid, however, presents a greater challenge. Wilfried Schoepe and colleagues at the University of Regensburg in Germany studied the problem[11] using a 100-μm sphere immersed in liquid He. Their data, shown in figure 5, indicate a deviation from laminar flow at a critical Re near 700.

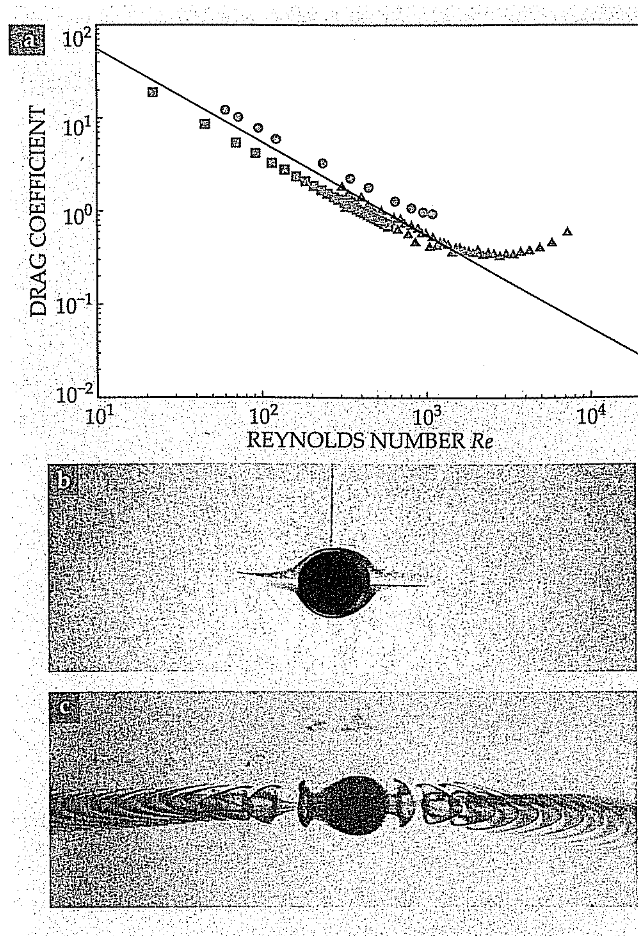


Figure 5. A real-world pendulum. (a) At low Reynolds number Re the drag force on a spherical pendulum agrees with the Stokes prediction for laminar flow (solid line). But two experiments—one with a 100-μm sphere immersed in liquid helium (green and blue symbols), another with a 1-inch steel bob in water (red)—demonstrate that the drag force deviates from laminar flow above a critical Re near 700. Photos of the steel bob show a laminar flow structure when **(b)** $Re < Re_c$ and show the bob shedding vortex rings when when **(c)** $Re > Re_c$. (Adapted from ref. 12.)

At the University of Oregon we duplicated the experiment with a 1-inch steel bob oscillating in water. The bob was 256 times as large as and 37 million times heavier than the Regensburg group's sphere. Yet our experiment yielded a nearly identical relationship between dimensionless drag and Re and showed a similar deviation from laminar flow at large Re. Photos revealed that the steel bob starts to shed vortex rings when Re surpasses the critical value.[12]

Beyond fluids

A quick check with an internet search engine reveals the ubiquity of dynamic similarity. Steven Vogel of Duke University has helped pioneer the use of dimensional analysis in biophysics.[13] He has used the concepts to highlight bounds on certain forms of physical behavior, such as the maximum height of a tree if getting sap to the leaves is the crucial factor (see PHYSICS TODAY, November 1998, page 22).

Principles of similarity also underlie key economic models, such as the debt-to-income ratio. For a long time now, economists have had a good understanding of what that ratio should be, regardless of total annual income, if the debt is to be manageable. A recent article, "Dimensions and Economics: Some Problems," suggests that many commonly used models are not dimensionally homogeneous, which could result in problems during application and analysis.[14]

As with all tools, it is important to be aware of the potential limitations of dynamic similarity. The principle of similarity could be crudely construed as follows: Two systems can be considered completely similar when all dimensionless numbers are the same. In practice, complete similarity is impossible to achieve unless the two systems are exactly the same.

For example, in the submarine problem, we ignored flow-compressibility effects, which become pronounced when flow speeds approach the speed of sound. Sound travels much more slowly in air than in water, so one must be cautious. It is typically assumed that as long as the flow speed is less than half that of sound, compressibility can be neglected. Similarly, Rayleigh–Bénard experiments hint that *Pr*, often assumed to be negligible, may play a more important role in heat transfer than once thought.

History demonstrates, however, that it is certainly possible to use principles of similarity to draw valuable parallels between systems that aren't entirely similar. Currently, dynamic similarity and dimensional analysis are topics that engineering students learn as part of their fluid mechanics course, typically in the second or third undergraduate year. We believe that emphasis on dimensional reasoning would be useful to students in many branches of physics as well.

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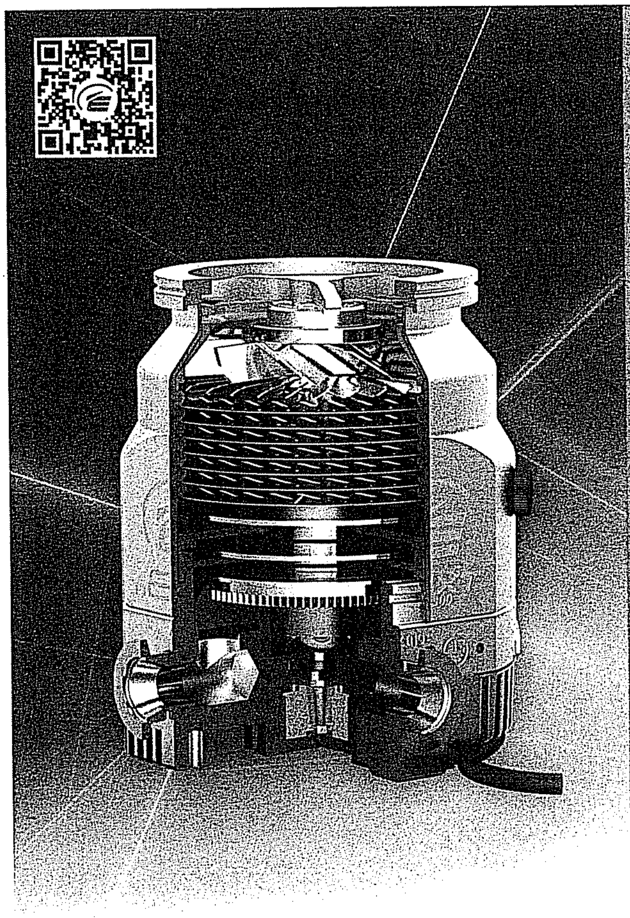

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5. Scaling

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