分類：
編號：
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第二章

直 線 運 動

除了 2.6 以外
高中都已學過

2.6 在第零章也已講過

⇒ 因此可以很快的講過去
鼓勵學生自修這一章

習題

外加之討論

強調自由度及起始條件

含圓周運動

分類：
編號： 1
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第二章 運動學

第一節 直綫運動

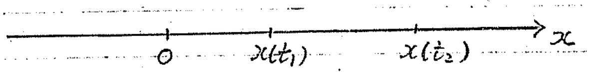
1. 簡介 質點運動學是描述一個質點在一固定坐標系統中的運動情況、一個質點在一固定坐標系統中之運動狀況可由此一質點在該坐標系統中之位置向量的時間函數 $\vec{r}(t)$ 所完全確定

該質點之速度及加速度可由 $\vec{r}(t)$ 之微分求得 同時當加速度為已知時我們也可用積分之方法來求 $\vec{r}(t)$, 在積分過程中所引進之積分常數則需用初位及初速來決定

在此節中我們將討論固定於一直線上之運動（直線運動）. 在此類運動一質點之運動情況可由 $x(t)$ 完全決定

2. 基本觀念

要描述直綫運動我們首先取一坐標系統[1], 在此坐標中我們取運動方向為 x 軸, 並定一點為原點. 質點在時間 t 時之坐標以 $x(t)$ 表示[2] $x(t)>0$ 表示該質點在時間 t 時位於原點之右. $x(t)<0$ 則表示該質點在 t 時位於原點之左



若在時間 t_1 時該質點位於 $x_1=x(t_1)$, 在時間 t_2 時, 該質點位於 $x_2=x(t_2)$[3]

則在 t_1 及 t_2 之間（此處我們假設 $t_2>t_1$）此質點所作之位移為

$$d \equiv x_2 - x_1 \tag{1}$$

而該質點在 t_1 及 t_2 間之平均速度[4]

$$\bar{v}_{t_1\to t_2} \equiv \frac{x_2-x_1}{t_2-t_1} = \frac{x(t_2)-x(t_1)}{t_2-t_1} \tag{2}$$

我們若取 $t_1=t$, $t_2=t+\Delta t$，則按上式之定義可知該質點在時間 t 与 $t+\Delta t$ 間的平均速度為

$$\bar{v}_{t\to t+\Delta t} = \frac{x(t+\Delta t)-x(t)}{\Delta t} \tag{3}$$

當 $\Delta t \to 0$ 時，我們得到的當然是該質點在時間 t 時之瞬時速度，$v(t)$. 由微分之定義，我們得知(5)(6)

$$v(t) = \frac{dx(t)}{dt} \tag{4}$$

若在時間 t_1 時，該質點之速度為 $v_1=v(t_1)$，在時間 t_2 時之速度為 $v_2=v(t_2)$ 則該質點在時間 t_1 及 t_2 間之平均加速度(7)為

$$\bar{a}_{t_1\to t_2} = \frac{v_2-v_1}{t_2-t_1} = \frac{v(t_2)-v(t_1)}{t_2-t_1} \tag{5}$$

我們若取 $t_1=t$, $t_2=t+\Delta t$，則按上式之定義，該質點在時間 t 与 $t+\Delta t$ 間之平均加速度為

$$\bar{a}_{t\to t+\Delta t} = \frac{v(t+\Delta t)-v(t)}{\Delta t} \tag{6}$$

當 $\Delta t \to 0$ 時，我們得到的當然是該質點在 t 時之瞬時加速度 $a(t)$ 由微分之定義，我們得知(8),(9)

$$a(t) = \frac{dv(t)}{dt} \tag{7}$$

由第 (4) 及第 (7) 式，我們得知

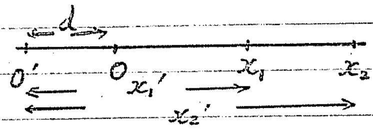
$$a(t) = \frac{d^2x(t)}{dt} \tag{8}$$

3. 討論

(1) 此一坐標系統並不必需是慣性坐標系統，因為在運動學中，我們只是描述質點之運動情況，而不討論其運動之原因，那是屬於動力學的範疇，在動力學中一坐標系統是否慣性坐標卻有極深遠的影响.

(2) 一質點在 x 軸之坐標完全決定了此質點之一度空間位置向量

(3) 位移與原點取於何點無關 例如若我們取 O' 為新原點，則在此一新的坐標系統中，$x_1' = x_1 + d$，$x_2' = x_2 + d$



因此其位移 $d' = x_2' - x_1' = x_2 - x_1 = d$

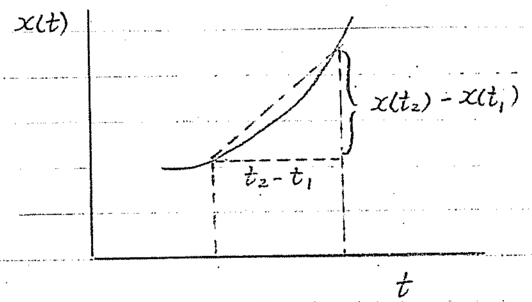
(4) 注意標明這是在 t_1 及 t_2 間之平均速度 此一定義與 $\frac{v(t_1)+v(t_2)}{2}$ 不同

只有在特殊情況下，例如等速度及等加速度時 $\overline{v}_{t_1\to t_2} = \frac{v(t_1)+v(t_2)}{2}$

(5) $v(t)>0$ 表示在 t 時 $x(t)$ 隨 t 增加而增大 因此在 t 時該質點沿 $+x$ 軸運動

$v(t)<0$ 表示在 t 時 $x(t)$ 隨 t 增加而減小 因此在 t 時該質點沿 $-x$ 軸運動

(6) 在下圖中我們將 $x(t)$ 對 t 之變化繪出

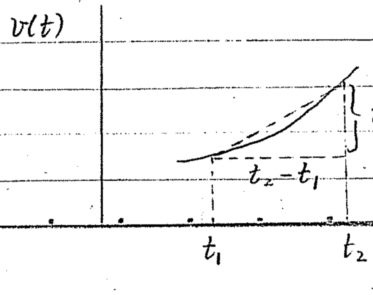


在圖中令 $t_2\to t_1$，我們可清楚地看出 $x(t)$ 曲線在 t 時其切線之斜率即為其瞬時速度 $v(t)$

(7) 注意標明為 t_1 與 t_2 間之平均加速度

(8) $a(t)>0$ 表示在 t 時 $v(t)$ 隨 t 增加而增大，因此在 t 時該質點沿 x 軸有加速度 $a(t)<0$ 表示在 t 時 $v(t)$ 隨 t 增加而減小 因此在 t 時該質點沿 x 軸有減速度

(9) 在下圖中我們將 $v(t)$ 對 t 之變化繪出



在圖中令 $t_2\to t_1$，我們可以清楚地看出 $v(t)$ 在 t 時其切線之斜率即為其瞬時加速度 $a(t)$

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4 應用

在質點之直線運動中有兩類基本問題 (一) 已知 $x(t)$，求 $v(t)$ 及 $a(t)$ (二) 已知加速度，其初位 $x(t_0)$ 及其初速 $v(t_0)$ 求 $x(t)$ 我們將順序舉例說明

(A) 已知 $x(t)$ 求 $v(t)$ 及 $a(t)$ 由 (4) 及 (7) 式中得知

$$v(t) = \frac{dx(t)}{dt} \tag{9}$$

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} \tag{10}$$

因此這一類問題主要的是求 $x(t)$ 之一度及二度微分

(1) $x(t) = c_2 t^2 + c_1 t + c_0$; c_0, c_1, c_2 均為常數

$$\left.\begin{aligned} v(t) &= 2c_2 t + c_1 \\ a(t) &= 2c_2 = \text{常數} \end{aligned}\right. \tag{11}$$

因此，$x(t)$ 描述一等加速度運動

$$\bar{v}_{t_1\to t_2} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{c_2 t_2^2 + c_1 t_2 + c_0 - c_2 t_1^2 - c_1 t_1 - c_0}{t_2 - t_1}$$

$$= c_2(t_2 + t_1) + c_1 \tag{12}$$

$$\frac{v(t_2) + v(t_1)}{2} = \frac{1}{2}\left[2c_2 t_2 + c_1 + 2c_2 t_1 + c_1\right] = c_2(t_2 + t_1) + c_1 \tag{13}$$

因此，在等速度運動中 $\bar{v}_{t_1\to t_2} = \dfrac{v(t_2) + v(t_1)}{2}$ (14)

若 $c_2 = 0$，則 $x(t) = c_1 t + c_0$，$v(t) = c_1$，$a(t) = 0$ 因此 $x(t)$ 描述一等速度運動

(2) $x(t) = c_3 t^3 + c_2 t^2 + c_1 t + c_0$; c_0，c_1，c_2，c_3 均為常數

$$v(t) = 3c_3 t^2 + 2c_2 t + c_1 \tag{15}$$

$$a(t) = 6c_3 t + 2c_2 \tag{16}$$

因此，其加速度並非一常數而為一時間之函數

$$\bar{v}_{t_1\to t_2} = \frac{x(t_2)-x(t_1)}{t_2-t_1} = \frac{c_3 t_2^3 + c_2 t_2^2 + c_1 t_2 + c_0 - c_3 t_1^3 - c_2 t_1^2 - c_1 t_1 - c_0}{t_2 - t_1}$$

$$= c_3(t_2^2 + t_1 t_2 + t_1^2) + c_2(t_2 + t_1) + c_1 \qquad (17)$$

$$\frac{v(t_2)+v(t_1)}{2} = \frac{1}{2}\left[3c_3(t_2^2 + t_1^2) + 2c_2(t_2+t_1) + 2c_1\right] \qquad (18)$$

因此除非 $c_3 = 0$（而可簡化成等速度運動），否則 $\bar{v}_{t_1\to t_2} \neq \frac{v(t_2)+v(t_1)}{2}$

(3) $x(t) = A\sin(\omega t + \alpha)$ ； A, ω, α 均為常數 (19)

$$v(t) = A\omega\cos(\omega t + \alpha) \qquad (20)$$

$$a(t) = -A\omega^2\sin(\omega t + \alpha) \qquad (21)$$

值得注意的是 $a(t) = -\omega^2 x(t)$. (22)

(B) 已知加速度 a，初位 $x(t_0)$ 及 $v(t_0)$ 求 $x(t)$. 在最普遍之情況下，a 加速度可以是 $x(t)$，$v(t)$ 及 t 之函數. 在此一部分中我們將只討論 a 是 t 之函數. 在第 7, 8 小節中我們會討論較複雜之情形

(1) 已知 $a(t) = a_0$ 　a_0 為一常數，並知在 $t = t_0$ 時，該質點位於 x_0，當時之速度為 v_0

將 $\frac{dv(t)}{dt} = a_0$ 積分，我們得到

$$v(t) = a_0 t + c_1 \qquad (23)$$

此公式在 $t = t_0$ 時也應滿足，所以

$$v_0 = a_0 t_1 + c_1 \Rightarrow c_1 = v_0 - a_0 t_1 \qquad (24)$$

代入 (23) 式，我們得到

$$v(t) = v_0 + a_0(t - t_1) \qquad (25)$$

此即是我們所求之 $v(t)$

將 $\frac{dx(t)}{dt} = v(t) = v_0 + a_0(t-t_0)$ 積分可得

$$x(t) = v_0 t + \frac{1}{2} a_0 t - a_0 t_0 t + c_0 \tag{26}$$

此公式於 $t=t_0$ 時也應滿足，所以

$$x_0 = v_0 t_0 + \frac{1}{2} a_0 t_0 - a_0 t_0 t_0 + c_0 \Rightarrow c_0 = \frac{1}{2} a t_0^2 - v_0 t_0 + x_0 \tag{27}$$

代入 (26) 式我們得到

$$x(t) = x_0 + v_0(t-t_0) + \frac{1}{2} a_0 (t-t_0)^2 \tag{28}$$

此即是我們所要求的 $x(t)$

由 (25) 式可得

$$t - t_0 = \frac{v - v_0}{a} \tag{29}$$

代入 (28) 式加以簡化得

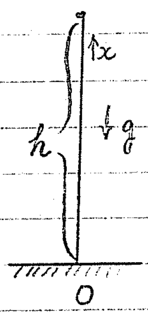
$$v^2 = v_0^2 + 2a_0(x - x_0) \tag{30}$$

這個公式直接表示出速度及位置間之關係

若 $a_0 = 0$ 時，則該質點在作等速運動而 (25) 及 (28) 式變成

$$v = v_0 \quad 及 \quad x(t) = x_0 + v_0(t-t_0) \tag{31}$$

(2) 若一自由落体由高 h 處下落，在最後一秒中其下落之距離為 $\frac{h}{2}$，求 h 及其下落所需之時間



定地面為原點，其垂直方向為 x 軸

因此 $t_0 = 0$，$x_0 = h$ 米，$v_0 = 0$　$g = -9.8$ 米/秒^2

令其整個下落所需之時間為 T 秒

$$0 = h - \frac{1}{2} g \cdot T^2 \tag{32}$$

在最後一秒所行之距離為 $x(T-1) - x(T) = \frac{h}{2}$

因此 $h-\frac{1}{2}g(T-1)^2-h+\frac{1}{2}gT^2 = \frac{h}{2}$

$$\Rightarrow \quad gT-\frac{1}{2}g = \frac{h}{2} \tag{33}$$

由 (32) 及 (33) 式得 $h=57.11\,m$ ， $T=3.414$ 秒

(3) 若 a 為 t 之函數，則由第(7)式得

$$\frac{dv}{dt} = a(t) \tag{34}$$

對 t 積分可得

$$v(t) = \int a(t)\,dt = f(t)+c_1 \tag{35}$$

此公式於 $t=t_0$ 時也應滿足，所以

$$v_0 = f(t_0)+c_1 \Rightarrow c_1 = v_0 - f(t_0) \tag{36}$$

代入 (35) 式得

$$v(t) = v_0 + f(t) - f(t_0) \tag{37}$$

由第(4)及(37)式可得

$$\frac{dx(t)}{dt} = v(t) = v_0 + f(t) - f(t_0)$$

對 t 積分可得

$$x(t) = v_0 t + g(t) - f(t_0)t + c_2 \tag{38}$$

此處 $\int f(t)\,dt = g(t)+C$ ， c_2 為一常數

此公式於 $t=t_0$ 時也應滿足，所以

$$x_0 = v_0 t_0 + g(t_0) - f(t_0)t_0 + c_2$$

$$\Rightarrow \quad c_2 = x_0 - g(t_0) + [f(t_0) - v_0]t_0 \tag{39}$$

代入 (38) 式則得

$$x(t) = x_0 + [f(t_0)-v_0](t-t_0) + g(t) - g(t_0) \tag{40}$$

(4) 若 $a(t) = e^{-\alpha t}$，在 $t = t_0 = 0$ 時 $x_0 = 0$, $v_0 = 0$ 求 $x(t)$

由 $f(t)$ 及 $g(t)$ 之定義可得

$$f(t) = -\frac{1}{\alpha} e^{-\alpha t}, \quad g(t) = \frac{1}{\alpha^2} e^{-\alpha t} \tag{41}$$

代入 (37) 式得

$$v(t) = v_0 - \frac{1}{\alpha} e^{-\alpha t} + \frac{1}{\alpha} \tag{42}$$

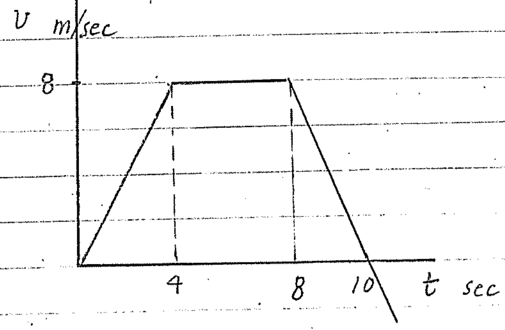
代入 (40) 式得

$$x(t) = \left[-\frac{1}{\alpha} - v_0\right] t + \frac{1}{\alpha^2} e^{-\alpha t} - \frac{1}{\alpha^2} \tag{43}$$

此即是我們所要求之 $x(t)$.

5. 習題

(1) 下圖中所繪乃是一沿 x 軸運動質點之速度的時間函數



(a) 求 $t = 2$ 秒時之瞬時加速度 [D]

(b) 求 $t = 6$ 秒時之瞬時加速度 [K]

(c) 求 $t = 10$ 秒時之瞬時加速度 [I]

(d) 在 $t=0$ 至 $t=4$ 秒之間該質點所行之距離為何? [P]

(e) 在 $t=4$ 至 $t=8$ 秒之間，該質點所行之距離為何? [M]

(f) 在何時該質點之瞬時速度為零? 求 $t=0$ 至此時間，該質點所行之距離 [S]

(2) 一個質點在 x 軸上運動，其 x 軸坐標之時間函數為

$$x(t) = \alpha t^4 + \beta t^2 + \gamma t + \delta$$

此處 $\alpha, \beta, \gamma, \delta$ 均為常數

(a) 此質點在時間 t 時之速度為何？ [A]

(b) 此質點在時間 t 時之加速度為何？ [L]

(c) 此質點之加速度是否為常數？ [N]

(d) 在何種情況之下，此質點之加速度為常數？ [Q]

(e) 若是 (d) 項之要求成立，該質點之加速度為何？ [E]

(3) 一個質點沿 x 軸運動，其速度之時間函數為

$v(t) = 9t^2 + 4t - 8$ 單位 m/sec

(a) 此質點在時間 $t=2$ 秒時其加速度為何？ [F]

(b) 此質點在時間 $t=4$ 秒時其加速度為何？ [O]

(c) 此質點之位移寫成時間之函數，其形狀為何？ [R]

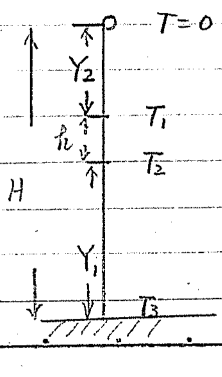
(d) 在此題中若在 $t=1$ 秒，其位移為 7米，決定此質點位移之時間函數 [G]

(4) 若一以 $v_{01} = 10$ m/sec 運動的汽車之駕駛看見在它車的同一線中前方距離 d 處有另一車以 $v_{02} = 4$ m/sec 進行，他馬上踩剎車而使他的車獲得一定值之減速度 $a = -2$ m/sec^2

(a) 令 x_1 為前車之位置，x_2 為後車之位置，求 $x_1(t)$ 及 $x_2(t)$ [B]

(b) 求兩車相撞之條件 [H]

(c) 若希望此兩車不相撞，d 最少為何？ [J]

(5) 

若一網球由一高樓之頂自由落下，在此樓一層樓中之住客看見此球由窗口之頂到窗口之底所費之時間為 $T=\frac{1}{4}$ 秒種，此窗之高度為 $h=2$米，此球經過窗口後蹈地，又完全彈性地反彈回來，在經过窗底 $T'=4$ 秒种以

後又再度出現於窗底，求此高樓之高度　[C]

6. 答案

[A] $4\alpha t^3 + 2\beta t + \gamma$　[B] $x_1(t) = d + v_{02} t$, $x_2(t) = v_{01} t + \frac{1}{2} a t^2$

[C] $Y_2 = \frac{1}{2} g T_1^2$, $Y_2 + h = \frac{1}{2} g T_2^2$　$T_3 - T_2 = \frac{1}{2}(T')$

$T_2 - T_1 = T$,　$H = \frac{1}{2} g T_3^2$　此處有五個公式，有五个未知數 Y_2, T_1, T_2, T_3, 及 H　所以可解 H.

[D] $2\,m/sec^2$　[E] 2β　[F] $40\ m/sec$　[G] $x = 3t^3 + 2t^2 - 8t + 10$

[H] $x_1(t) = x_2(t)$　[I] $-4\,m/sec^2$　[J] $9\,m$

[K] 0　[L] $12\alpha t^2 + 2\beta$　[M] $32\ m$　[N] 否

[O] $76\ m/sec^2$　[P] $16\,m$　[Q] $\alpha = 0$　[R] $x = 3t^2 + 2t^2 - 8t + c$

[S] $56\ m$

7. 其它問題

以上我們已提及 a 可能是 x, 及 v 之函數(A)我們首先來討論當 a 只是 v 之函數之情況

$$\frac{dv}{dt} = a(v) \qquad (44)$$

$$\frac{dv}{a(v)} = dt$$

將兩邊積分得

$$\int \frac{dv}{a(v)} = t + c \qquad (45)$$

由此可利用初速定出積分常數即可求得 $v(t) = \frac{dx}{dt}$ 再對 t 積分利用初位定出積分常數即可求得 $x(t)$

我們將舉例來說明

$a = \alpha - \beta v$　　$t=0$ 時之初速為 v_0，初位為 x_0

$$\frac{dv}{dt} = \alpha - \beta v \tag{46}$$

$$\frac{dv}{\alpha - \beta v} = dt$$

$$-\frac{1}{\beta}\ln(\alpha - \beta v) = t + C_1 \tag{47}$$

因此 $\alpha - \beta v = Ae^{-\beta t}$ 此處 $A = e^{-\beta C_1}$ 為一積分常數

$$v(t) = \frac{1}{\beta}[\alpha - Ae^{-\beta t}] \tag{48}$$

$t=0$，$v = v_0$ 代入上式得

$$v_0 = \frac{1}{\beta}[\alpha - A] \Rightarrow \beta v_0 = \alpha - A \Rightarrow A = \alpha - \beta v_0$$

$$x(t) = \frac{\alpha}{\beta}t + \frac{A}{\beta}e^{-\beta t} + C_2 \tag{49}$$

$t=0$，$x(t) = x_0$ 代入上式得

$$x_0 = \frac{1}{\beta}(\alpha - \beta v_0) + C_2 \Rightarrow C_2 = x_0 - \frac{\alpha}{\beta} + v_0 \tag{50}$$

因此

$$x(t) = \frac{\alpha}{\beta}t + \frac{\alpha}{\beta}e^{-\beta t} - v_0 e^{-\beta t} + x_0 - \frac{\alpha}{\beta} + v_0$$

$$= \frac{\alpha}{\beta}(t - 1 + e^{-\beta t}) + v_0(1 - e^{-\beta t}) \tag{51}$$

此即為所求之 $x(t)$

(B) 若是 a 只是 x 之函數時

$$\frac{dv}{dt} = a(x) \tag{52}$$

利用連鎖法則 $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ (53)

代入 (52) 式可得

$$v\,dv = a(x)\,dx \tag{54}$$

因此 $\frac{1}{2}v^2 = \int a(x)\,dx + c$，此處 c 為一常數 (55)

此式与以後將討論之能量守恒律有着密切之關係，我們將留待以後討論

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如果 a 同時為 v, x 或 t 之函數，則情況就極複雜，要解這類問題必需要用到微分方程，因此也就超出了本書之範圍。

§4. 直线运动

4.1 亚里士多德和伽利略的运动观

2000 多年前,古希腊人对现在物理学中研究的某些问题,如杠杆、简单机械、浮力、

光的某些性质,已有很好的了解,但运动的概念是混乱的。第一个对运动作过认真思考的人,大概要算伟大的思想家亚里士多德(Aristotle, 384–322 B. C.)了。

亚里士多德把运动分为两大类:自然运动和受迫运动。在亚里士多德看来,每个物体都有自己的固有位置,偏离固有位置的物体将趋向固有位置。地上物体的自然运动沿直线,轻者上升,重者下降;天体的自然运动永恒地沿圆周进行。受迫运动则是物体在推或拉的外力作用下发生的,没有外力,运动就会停止。例如箭是在弓弦的作用下飞出的。然而脱弦之后又是什么力在支持箭的飞行呢?对此的解释是,正像在浴缸里用手捏肥皂的一端,肥皂滑出后被推动在水中前进一样,周围的空气挤向被箭排开的尾部真空,推动着箭前进。

在欧洲中古漫长的黑暗世纪里,希腊典籍佚散殆尽。公元 10–12 世纪间,许多古籍在阿拉伯重新发现,并被译成拉丁文。在欧洲的一些大学里,又开始讲授起亚里士多德的译著来。起初,西欧的教会对新引进的古籍抱怀疑态度。经过基督教学者们的努力,将亚里士多德的学说与基督教义调和起来,亚里士多德的宇宙观成了基督救世福音的一部分。从此,谁反对亚里士多德就是反对教会本身,亚里士多德不幸地被奉做了神明。1543 年哥白尼发表的日心说如此尖锐地抵触了亚里士多德的宇宙观和基督教义,遭到教会的强烈反对。

历史上第一次用观测和实验决定性地驳倒亚里士多德观点的,是 16 世纪近代精密自然科学的创始人伽利略(Galileo,1564–1642)。他考察了自由落体的运动,由位移正比于时间的平方肯定了它是匀加速运动,并得到重力加速度与重量无关的结论。他从物体沿斜面的运动推论出惯性定律,即匀速直线运动是不要用力来支持的。为哥白尼的地动说辩护,他提出了力学相对性原理。伽利略的重大贡献还有很多,就不在此一一赘述了。单从运动学的角度看,是他首先提出了*加速度*的概念。这是人类认识史上最难建立的概念之一,也是每个初学物理的人最不易真正掌握的概念。在人们的日常生活中,对于运动的物体可以问它走了多远,这是距离的概念;可以问它走得多快,这是速度的概念。然而,在各国的生活语言中都没有与加速度对应的词儿。不学物理,在人们的头脑里是不会自发地形成加速度概念的。他们只有笼统的快和慢的概念,这有时指的是速度,有时模模糊糊指的是加速度。"加速度"的概念建立在瞬时速度和导数的基础上,下面我们将借助于函数的几何图解和初步的微积分知识来阐明这些概念。未学过微积分的读者,可参阅本书附录 A(微积分初步)。

4.2 平均速度和瞬时速度

物体(质点)轨迹是直线的运动,称为*直线运动*。直线运动可以用一维坐标描述。如图 1 – 7 所示,取 O 为坐标原点,物体在任一时刻 t 的位置可用函数 $s(t)$ 来描述。

大家知道,在匀速直线运动中,表征质点运动快慢的速度是一常量,其表达式为

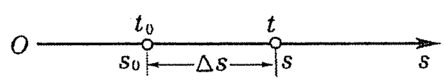

图 1 – 7 直线运动

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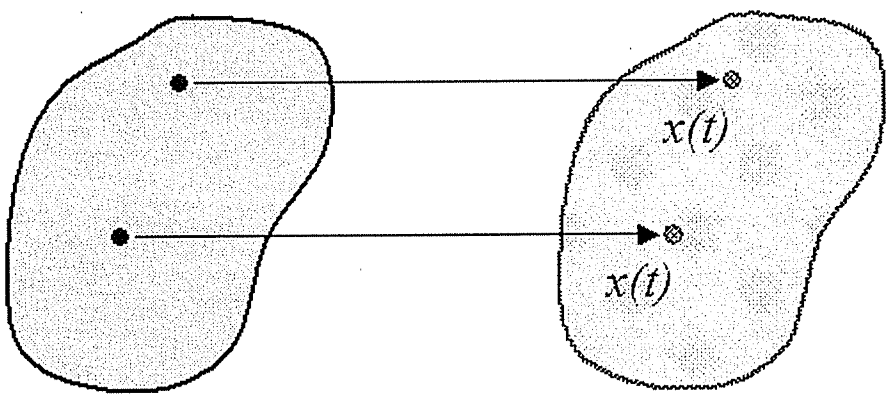
Rectilinear Motion

Rectilinear motion is another name for straight-line motion. This type of motion describes the movement of a particle or a body.

A body is said to experience rectilinear motion if any two particles of the body travel the same distance along two parallel straight lines. The figures below illustrate rectilinear motion for a particle and body.

Rectilinear motion for a particle:

Rectilinear motion for a body:



In the above figures, *x(t)* represents the position of the particles along the direction of motion, as a function of time *t*.

Given the position of the particles, *x(t)*, we can calculate the displacement, velocity, and acceleration. These are important quantities to consider when evaluating the kinematics of a problem.

A common assumption, which applies to numerous problems involving rectilinear motion, is that acceleration is constant. With acceleration as constant we can derive equations for the position, displacement, and velocity of a particle, or body experiencing rectilinear motion.

The easiest way to derive these equations is by using Calculus.

The acceleration is given by

$$\frac{d^2x}{dt^2} = a$$

where *a* is the acceleration, which we define as constant.

Integrate the above equation with respect to time, to obtain velocity. This gives us

$$v(t) = \int a\,dt = C_1 + at$$

where *v(t)* is the velocity and C_1 is a constant.

Integrate the above equation with respect to time, to obtain position. This gives us

$$x(t) = \int v(t)\,dt = C_2 + C_1 t + \frac{1}{2}at^2$$

where *x(t)* is the position and C_2 is a constant.

The constants C_1 and C_2 are determined by the initial conditions at time $t = 0$. The initial conditions are:

At time $t = 0$ the position is x_1.

At time $t = 0$ the velocity is v_1.

Substituting these two initial conditions into the above two equations we get

$$v(0) = v_1 = C_1$$

$$x(0) = x_1 = C_2$$

Therefore $C_1 = v_1$ and $C_2 = x_1$.

This gives us

$$x(t) = x_1 + v_1 t + \frac{1}{2}at^2$$

$$v(t) = v_1 + at$$

For convenience, set $x(t) = x_2$ and $v(t) = v_2$. As a result

$$x_2 = x_1 + v_1 t + \frac{1}{2}at^2 \quad \text{(1) – position equation}$$

$$v_2 = v_1 + at \quad \text{(2) – velocity equation}$$

$$v_2 = v_1 + at \qquad (2) - \text{velocity equation}$$

Displacement is defined as $\Delta d = x_2 - x_1$. Therefore, equation (1) becomes

$$\Delta d = v_1 t + \frac{1}{2}at^2 \qquad (3) - \text{displacement equation}$$

If we wish to find an equation that doesn't involve time t we can combine equations (2) and (3) to eliminate time as a variable. This gives us

$$v_2^2 = v_1^2 + 2a(\Delta d) \qquad (4) - 2^{\text{nd}} \text{ velocity equation}$$

Equations (1), (2), (3), and (4) fully describe the motion of particles, or bodies experiencing rectilinear (straight-line) motion, where acceleration a is constant.

For the cases where acceleration is not constant, new expressions have to be derived for the position, displacement, and velocity of a particle. If the acceleration is known as a function of time, we can use Calculus to find the position, displacement, and velocity, in the same manner as before.

Alternatively, if we are given the position $x(t)$ as a function of time, we determine the velocity by differentiating $x(t)$ once, and we determine the acceleration by differentiating $x(t)$ twice.

For example, let's say the position $x(t)$ of a particle is given by

$$x(t) = \cos(2t) + 4t^3$$

Thus, the velocity $v(t)$ is given by

$$v(t) = \frac{dx}{dt} = -2\sin(2t) + 12t^2$$

The acceleration $a(t)$ is given by

$$a(t) = \frac{d^2x}{dt^2} = -4\cos(2t) + 24t$$

Return from **Rectilinear Motion** to **Kinematics** page

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第三章

兩度及三度空間之運動

一般三度空間運動　見我的 Notes 第二節)

1. 位置与位移
2 速度 加速度 (切線及法線加速度)
3 等加速度運動　見我的 Notes 第三節
4. 拋物線運動　additional notes

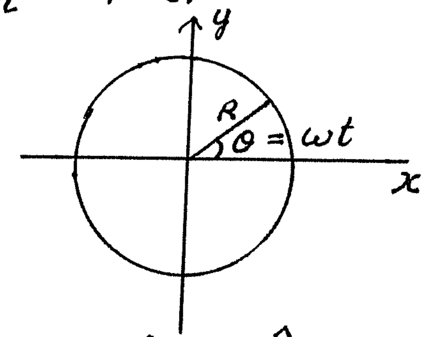
5 (均匀)圓周運動　見我 Notes 第四節
　及例題
　首先討論兩度空間的問題

　極坐標

6 相對運動　馮文蔚 "物理學
　相對直線運動　見我的 Notes
　相對轉動　(第四章)
　以後會作較仔細的討論

Circular Motion

Given $\vec{r}(t) \rightarrow \vec{v}(t) \longrightarrow \vec{a}(t)$



Uniform

$\theta = \omega t$

$\downarrow$

constant

$R = \text{constant}$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{r}(t) = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$$

$$= R\left[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right]$$

$$\vec{v}(t) = -R\omega\sin\omega t\,\hat{i} + R\omega\cos\omega t\,\hat{j}$$

$$= R\omega\left[-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right] \qquad v^2 = \vec{v}\cdot\vec{v} = R^2\omega^2$$

$$\vec{a}(t) = -R\omega^2\cos\omega t\,\hat{i} + R\omega^2\sin\omega t\,\hat{j}$$

$$= -R\omega^2\,[\underbrace{\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}}_{\hat{r}}]$$

$$= -\frac{R^2\omega^2}{R}\,\hat{r}$$

$$\Rightarrow \quad a_{\parallel} = 0$$

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Next, we consider the case $\theta \neq \omega t$
$\downarrow$
non-uniform circular motion

$$\vec{r} = R\cos\theta(t)\,\hat{i} + R\sin\theta(t)\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = -R\sin\theta\frac{d\theta}{dt}\hat{i} + R\cos\theta\frac{d\theta}{dt}\hat{j} \qquad \vec{r}\cdot\vec{v} = 0$$

$$\overset{\vec{v}}{\parallel} = R\frac{d\theta}{dt}[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}]$$

$$v = R\frac{d\theta}{dt}, \quad \frac{dv}{dt} = R\frac{d^2\theta}{dt}$$

$$\frac{d^2\vec{r}}{dt^2} = -R\sin\theta\frac{d^2\theta}{dt^2}\hat{i} - R\cos\theta\left(\frac{d\theta}{dt}\right)^2\hat{i}$$

$$+R\cos\theta\frac{d^2\theta}{dt}\hat{j} - R\sin\theta\left(\frac{d\theta}{dt}\right)^2\hat{j}$$

$$= \underbrace{R\frac{d^2\theta}{dt^2}}_{\frac{dv}{dt}}[\underbrace{-R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j}}_{\hat{v}}]$$

$$-R\left(\frac{d\theta}{dt}\right)^2[\underbrace{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}_{\hat{r}}]$$

$$\Rightarrow \quad a_{\parallel} = \frac{dv}{dt}$$

$$a_{\perp} = -R\left(\frac{d\theta}{dt}\right)^2 = -R^2\left(\frac{d\theta}{dt}\right)^2/R = v^2/R$$

1-2 圆 周 运 动

这一节讨论一种较为简单的曲线运动——圆周运动.

一、平面极坐标

设有一质点在如图 1-12 所示的 Oxy 平面内运动,某时刻它位于点 A. 它相对原点 O 的位矢 $\boldsymbol{r}$ 与 Ox 轴之间的夹角为 θ. 于是,质点在点 A 的位置可由(r,θ)来确定. 这种以(r,θ)为坐标的坐标系称为平面极坐标系. 而在平面直角坐标系内,点 A 的坐标则为(x,y). 这两种坐标系的坐标之间的变换关系即为 $x=r\cos\theta$ 和 $y=r\sin\theta$.

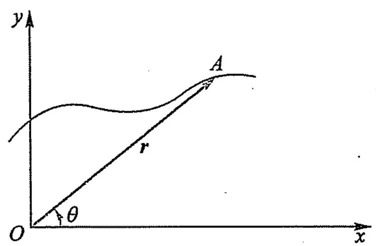

图 1-12　平面极坐标

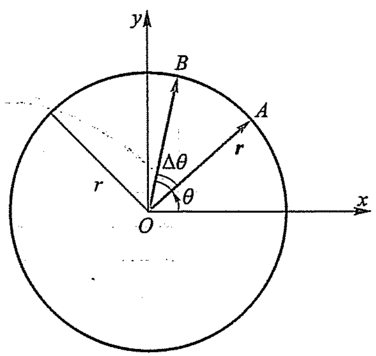

图 1-13　质点在平面上作圆周运动

二、圆周运动的角速度

如图 1-13 所示,一质点在 Oxy 平面上作半径为 r 的圆周运动,某时刻它位于点 A,位矢为 $\boldsymbol{r}$. 当质点在圆周上运动时,位矢 $\boldsymbol{r}$ 与 Ox 轴之间的夹角 θ 随时间而改变,即 θ 是时间的函数 $\theta(t)$.

我们定义:角坐标 $\theta(t)$ 随时间的变化率即 $\mathrm{d}\theta/\mathrm{d}t$,叫做角速度,用符号 ω 表示,则有

$$\omega=\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1-8}$$

通常用弧度(rad)来量度 θ,所以角速度 ω 的单位名称为弧度每秒,符号为 $\mathrm{rad}\cdot\mathrm{s}^{-1}$.

$r\Delta\theta$,$\Delta\theta$ 为时间 Δt 内,位矢 $\boldsymbol{r}$ 所转过的角度. 当 $\Delta t\to 0$ 时,$\Delta s/\Delta t$ 的极限值为

$$\frac{\mathrm{d}s}{\mathrm{d}t}=r\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

而 $\mathrm{d}s/\mathrm{d}t$ 为质点在点 A 的速率 v,$\mathrm{d}\theta/\mathrm{d}t$ 则为质点在点 A 的角速度 ω,故有

$$v=r\omega \tag{1-9}$$

式(1-9)是质点作圆周运动时速率和角速度之间的瞬时关系.

三、圆周运动的切向加速度和法向加速度　角加速度

如图 1-14 所示,质点在圆周上点 A 的速度为$\boldsymbol{v}$,它的数值为$|\boldsymbol{v}|=v$,方向与点 A 处圆的切线方向相同. 为了便于表示速度$\boldsymbol{v}$ 的方向,我们在点 A 处圆的切线方向上取一单位矢量 $\boldsymbol{e}_t$,叫做切向单位矢量,于是点 A 的速度$\boldsymbol{v}$ 可写为

$$\boldsymbol{v}=v\boldsymbol{e}_t \tag{1-10}$$

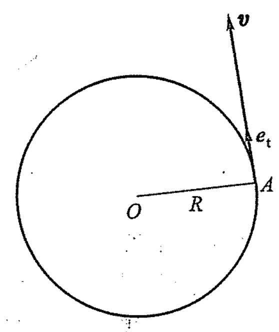

图 1-14　切向单位矢量 $\boldsymbol{e}_t$

一般来说,质点作圆周运动时,不仅速度的方向要改变,而且速度的值也会改变,即质点作变速率圆周运动. 由式(1-10)可得质点作变速率圆周运动时,它在圆周上任意点的加速度为

$$\boldsymbol{a}=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{e}_t+v\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}t} \tag{1-11}$$

从上式可以看出,加速度 $\boldsymbol{a}$ 具有两个分矢量,式中第一项$\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{e}_t$,是由于速度大小变化而引起的,其方向为 $\boldsymbol{e}_t$ 的方向,即与速度$\boldsymbol{v}$ 的方向相同. 因此,此项加速度分矢量称为切向加速度,用 $\boldsymbol{a}_t$ 表示,有

$$\boldsymbol{a}_t=\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{e}_t,\qquad |\boldsymbol{a}_t|=\frac{\mathrm{d}v}{\mathrm{d}t} \tag{1-12}$$

另外,由式(1-9),可得

$$\frac{\mathrm{d}v}{\mathrm{d}t}=r\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

式中 $\mathrm{d}\omega/\mathrm{d}t$ 为角速度随时间的变化率,叫做角加速度,用符号 α 表示,有

$$\alpha=\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{1-13}$$

角加速度 α 的单位名称是弧度每二次方秒,符号为 $\mathrm{rad}\cdot\mathrm{s}^{-2}$. 把上面两式代入式(1-12),可得

上式是质点作变速率圆周运动时，切向加速度与角加速度之间的瞬时关系.

至于式(1-11)中的第二项 $\mathrm{d}\boldsymbol{e}_t/\mathrm{d}t$，则表示切向单位矢量随时间的变化率. 这一点从图 1-15(a)中可以看出. 设在时刻 t，质点位于圆周上点 A，其速度为

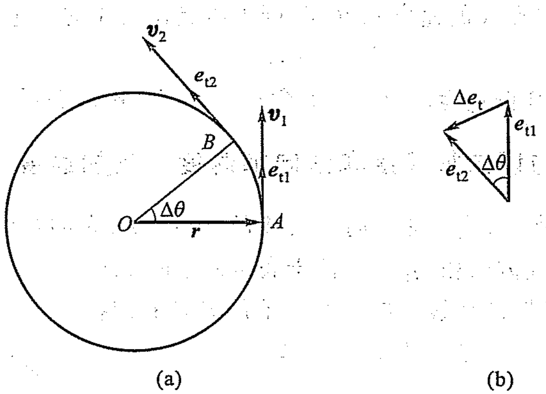

图 1-15　切向单位矢量随时间的变化率 $\mathrm{d}\boldsymbol{e}_t/\mathrm{d}t$

$\boldsymbol{v}_1$，切向单位矢量为 $\boldsymbol{e}_{t1}$；在时刻 $t+\Delta t$，质点位于点 B，速度为 $\boldsymbol{v}_2$，切向单位矢量为 $\boldsymbol{e}_{t2}$. 在时间间隔 Δt 内，$\boldsymbol{r}$ 转过的角度为 $\Delta\theta$，切向单位矢量的增量则为 $\Delta\boldsymbol{e}_t=\boldsymbol{e}_{t2}-\boldsymbol{e}_{t1}$. 由于切向单位矢量的值为 1，即 $|\boldsymbol{e}_{t1}|=|\boldsymbol{e}_{t2}|=1$，因而，从图 1-15(b)可以知道 $|\Delta\boldsymbol{e}_t|=\Delta\theta\times 1=\Delta\theta$. 当 $\Delta t\to 0$ 时，$\Delta\theta$ 亦趋于零，这时 $\Delta\boldsymbol{e}_t$ 的方向趋于与 $\boldsymbol{e}_{t1}$ 垂直，且趋于指向圆心. 如果我们在指向圆心的法线方向上取单位矢量 $\boldsymbol{e}_n$，称为法向单位矢量(图 1-16)，那么，在 $\Delta t\to 0$ 时，$\Delta\boldsymbol{e}_t/\Delta t$ 的极限值为

$$\lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{e}_t}{\Delta t}=\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}t}\boldsymbol{e}_n$$

这样，式(1-11)中第二项可以写成

$$v\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}t}=v\frac{\mathrm{d}\theta}{\mathrm{d}t}\boldsymbol{e}_n$$

这个加速度沿法线方向，故叫做法向加速度，用 $\boldsymbol{a}_n$ 表示，有

$$\boldsymbol{a}_n=v\frac{\mathrm{d}\theta}{\mathrm{d}t}\boldsymbol{e}_n \qquad (1-15\mathrm{a})$$

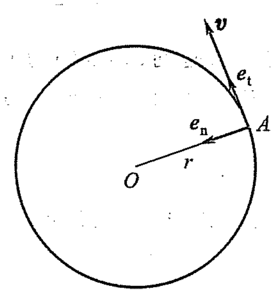

图 1-16　法向单位矢量 $\boldsymbol{e}_n$ 与切向单位矢量 $\boldsymbol{e}_t$ 相垂直

考虑到 $\omega=\mathrm{d}\theta/\mathrm{d}t$，$v=r\omega$，故上式为

$$\boldsymbol{a}_n=r\omega^2\boldsymbol{e}_n=\frac{v^2}{r}\boldsymbol{e}_n,\qquad |\boldsymbol{a}_n|=\frac{v^2}{r} \qquad (1-15\mathrm{b})$$

由式(1-12)和式(1-15)，可将质点作变速圆周运动时的加速度 $\boldsymbol{a}$ 的表示式

$$\boldsymbol{a}=\boldsymbol{a}_t+\boldsymbol{a}_n=\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{e}_t+\frac{v^2}{r}\boldsymbol{e}_n \qquad (1-16\mathrm{a})$$

或

$$\boldsymbol{a}=r\alpha\boldsymbol{e}_t+r\omega^2\boldsymbol{e}_n \qquad (1-16\mathrm{b})$$

其中切向加速度 $\boldsymbol{a}_t$ 是由于速度数值的变化而引起的，法向加速度 $\boldsymbol{a}_n$ 则是由于速度方向的变化而引起的.

在变速圆周运动中，速度的方向和大小都在变化，所以加速度 $\boldsymbol{a}$ 的方向不再指向圆心，其值和方向(图 1-17)为

$$a=(a_n^2+a_t^2)^{1/2},\qquad \tan\varphi=\frac{a_n}{a_t}$$

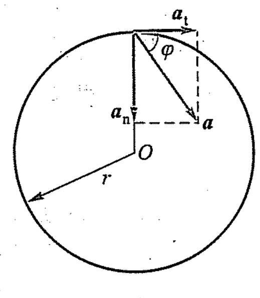

图 1-17　变速圆周运动的加速度

上述结果虽然是从变速圆周运动中得出的，但对于一般的曲线运动，式(1-1)、式(1-12)、(1-15)仍然适用. 此时可以把一段足够小的曲线看成是一段圆弧. 这样包含这段圆弧的圆周就被称为曲线在给定点的曲率圆，从而可用曲率半径 ρ 来替代圆的半径 r.

还必须指出一点，在图 1-16 中，这种以动点 A 为原点，以切向单位矢量 $\boldsymbol{e}_t$ 和法向单位矢量 $\boldsymbol{e}_n$ 建立的二维坐标系亦称为自然坐标系. 在讨论圆运动及曲线运动时，我们经常采用这种坐标系. 可参见第二章第 2-4 节例 2 和例 3.

四、匀速率圆周运动和匀变速率圆周运动

1. 匀速率圆周运动

质点作匀速率圆周运动时，其速率 v 和角速度 ω 都为常量，故角加速度 $\alpha=0$，切向加速度 $a_t=\mathrm{d}v/\mathrm{d}t=0$，而法向加速度的值 $a_n=r\omega^2=v^2/r$ 为常量. 于是匀速圆周运动的加速度为

$$\boldsymbol{a}=\boldsymbol{a}_n=r\omega^2\boldsymbol{e}_n$$

由式(1-8)可得

$$\mathrm{d}\theta=\omega\mathrm{d}t$$

如取 $t=0$ 时，$\theta=\theta_0$，则有

$$\theta=\theta_0+\omega t$$

2. 匀变速率圆周运动

质点作匀变速率圆周运动时，其角加速度 $\alpha=$ 常量，故圆周上某点的切向加速度的值为 $a_t=r\alpha=$ 常量，而法向加速度的值为 $a_n=r\omega^2=v^2/r$，但不为常量. 于是匀变速率圆周运动的加速度为

$$\boldsymbol{a}=\boldsymbol{a}_t+\boldsymbol{a}_n=r\alpha\boldsymbol{e}_t+r\omega^2\boldsymbol{e}_n \qquad (1-17)$$

$$
\left.\begin{aligned}
\omega &= \omega_0 + \alpha t \\
\theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\
\omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0)
\end{aligned}\right\} \qquad (1-18)
$$

这三个公式与在中学物理里已学过的匀变速直线运动的公式在形式上是相似的.

从以上对加速度的讨论中可以看出,速度的变化要用加速度来描述. 加速度也是可以变化的,为什么不用某个物理量来描述其变化呢? 这个问题单从质点运动学的角度是找不出答案的,学过了质点动力学,读者就会明白其中道理了.

例　如图 1－18 所示,一超音速歼击机在高空点 A 时的水平速率为 1 940 $\mathrm{km \cdot h^{-1}}$,沿近似于圆弧的曲线俯冲到点 B,其速率为 2 192 $\mathrm{km \cdot h^{-1}}$,所经历的时间为 3 s. 设圆弧$\overset{\frown}{AB}$的半径约为 3.5 km,且飞机从 A 到 B 的俯冲过程可视为匀变速率圆周运动. 若不计重力加速度的影响,求:(1) 飞机在点 B 的加速度;(2) 飞机由点 A 到达点 B 所经历的路程.

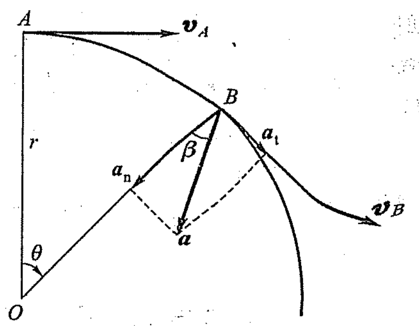

图 1－18

解　(1) 由于飞机在 AB 之间作匀变速率圆周运动,所以切向加速度 a_t 的值为常量,有

$$a_t = \frac{v_B - v_A}{t}$$

而在点 B 时的法向加速度为

$$a_n = \frac{v_B^2}{r}$$

由题意知,$v_A = 1\,940\ \mathrm{km \cdot h^{-1}} = 539\ \mathrm{m \cdot s^{-1}}$,$v_B = 2\,192\ \mathrm{km \cdot h^{-1}} = 609\ \mathrm{m \cdot s^{-1}}$,$t = 3$ s. $r = 3.5 \times 10^3$ m。将它们代入上两式,可得飞机在点 B 的切向加速度和法向加速度分别为

$$a_t = 23.3\ \mathrm{m \cdot s^{-2}}, \quad a_n = 106\ \mathrm{m \cdot s^{-2}}$$

故飞机在点 B 时的加速度的值为

$$a = (a_t^2 + a_n^2)^{1/2} = 109\ \mathrm{m \cdot s^{-2}}$$

而 $\boldsymbol{a}$ 与 $\boldsymbol{a}_n$ 之间夹角 β 为

$$\beta = \arctan \frac{a_t}{a_n} = 12.4°$$

(2) 在时间 t 内,$\boldsymbol{r}$ 所转过的角度 θ 为

$$\theta = \omega_A t + \frac{1}{2}\alpha t^2$$

其中 ω_A 是飞机在点 A 的角速度. 故在此时间内,飞机经过的路程为

$$s = v_A t + \frac{1}{2} a_t t^2$$

代入已知数据,有

$$s = (539 \times 3 + \frac{1}{2} \times 23.3 \times 3^2)\ \mathrm{m} = 1\,722\ \mathrm{m}$$

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Examples

1. Charged particle in the constant B-field

$$m \frac{d^2\vec{r}}{dt} = q\vec{V}\times\vec{B}$$

$$\vec{a} = \frac{q}{m}(\vec{V}\times\vec{B})$$

$$\vec{B} = B\hat{k}$$

$$\vec{V} = (V_x, V_y, V_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{V}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ 0 & 0 & B \end{vmatrix}$$

$$= V_y B\,\hat{i} - V_x B\,\hat{j}$$

$$\begin{cases} a_x = \dfrac{d^2x}{dt^2} = \dfrac{q}{m}B\,V_y = \dfrac{q}{m}B\dfrac{dy}{dt} \\ a_y = \dfrac{d^2y}{dt^2} = -\dfrac{q}{m}B\dfrac{dx}{dt} \end{cases}$$

$$\Rightarrow \quad a_z = 0 \qquad z = \underline{\underline{z_0}} + \underline{\underline{v_0}}\,t$$

solved

$$\frac{d^2x}{dt^2} = \frac{qB}{m}\frac{dy}{dt} \qquad \frac{d^2y}{dt^2} = -\frac{q}{m}B\frac{dx}{dt}$$

$$\downarrow$$

coupled

$$\boxed{\frac{dV_x}{dt} = \frac{qB}{m}V_y \,; \qquad \frac{dV_y}{dt} = -\frac{qB}{m}V_x}$$

Differential with to t

$$\frac{d^2V_x}{dt^2} = \frac{qB}{m}\frac{dV_y}{dt} = \frac{qB}{m}\left[-\frac{qB}{m}V_x\right]$$

$$= -\left(\frac{qB}{m}\right)^2 V_x$$

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$$\frac{d^2V_y}{dt^2} = -\frac{qB}{m}\frac{dV_x}{dt} = -\frac{qB}{m}\left(\frac{qB}{m}V_y\right)$$

$$= -\left(\frac{qB}{m}\right)^2 V_y$$

$$\omega^2 = \left(\frac{qB}{m}\right)^2 \iff \omega = \frac{qB}{m}$$

$$\frac{d^2V_x}{dt^2} = -\omega^2 V_x \qquad \frac{d^2V_y}{dt^2} = -\omega^2 V_y$$

$\downarrow$

$$V_x = A\sin\omega t + B\cos\omega t$$

$$V_y = C\sin\omega t + D\cos\omega t$$

4 constants to be determined

V_{x0}, V_{y0} two conditions

$$\underline{A\omega}\cos\omega t + B\omega(-\sin\omega t)$$

$$= \frac{qB}{m}(C\sin\omega t + \underline{D}\cos\omega t)$$

$$A\omega = \frac{qB}{m}D$$

$$-B\omega = \frac{qB}{m}C$$

two condition

V_x, V_y can be obtained by solving algebric equations

V_x, V_y are function of t

Integrate again $\Rightarrow$ $x(t)$, $y(t)$

$\downarrow$

with $x(0)$, $y(0)$ given

$x(t), y(t), z(t)$ can be obtain

$\vec{a}$ + initial condition $\Rightarrow$ $\vec{V}$ $\Rightarrow$ $\vec{x}(t)$

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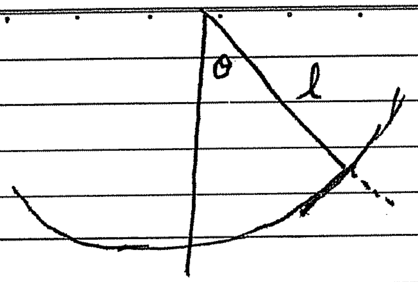
2 Simple Pendulum



Want to find $\theta(t)$

$$\theta(t=0) = \theta_0$$

$$v = R\frac{d\theta}{dt}$$

$$v(t=0)$$

$$a_{\parallel} = -g\sin\theta$$

$$\parallel$$

$$\frac{dV}{dt} = \frac{d}{dt}\overset{\ell}{\overset{\uparrow}{R}}\frac{d\theta}{dt}$$

$$= \overset{\ell}{\overset{\uparrow}{R}}\frac{d^2\theta}{dt^2}$$

$$R = \ell$$

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

$$= -\omega^2\sin\theta$$

$\downarrow$

equation for $\theta(t)$

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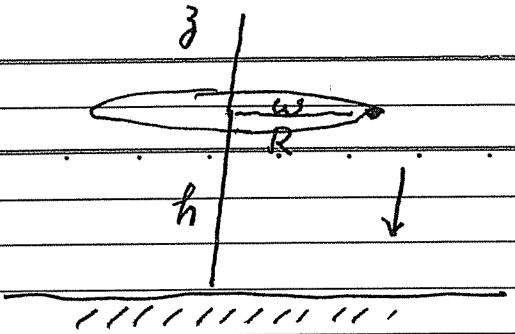
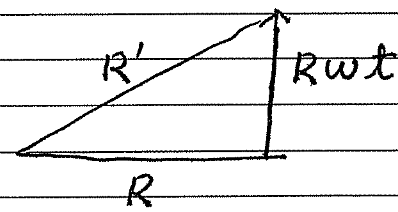
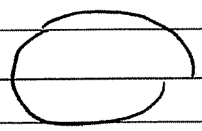
3



$v = R$



$$R'^2 = R^2\omega^2 t^2 + R^2$$



$$R'^2 = R^2 + (R\omega t)^2$$

在 z=h 的平面上，等速度運動

$$v_0 = R\omega$$

z direction $h = \frac{1}{2} g t^2$

$$t = \sqrt{\frac{2h}{g}}$$

$$R'^2 = R^2 + R^2\omega^2 \frac{2h}{g}$$

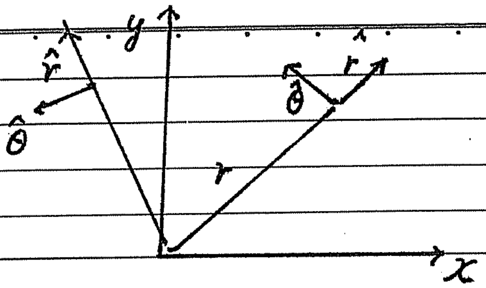
$$R = R\sqrt{1 + \omega^2 \frac{2h}{g}}$$

R, h, ω are parameters

for

灑水器

Polar Coordinates



(r, θ)

$x = r\cos\theta$
$y = r\sin\theta$

$$\vec{r} = x\hat{i} + y\hat{j} = r\hat{r}$$

$$\parallel$$

$$r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\parallel$$

$$r[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}]$$

$$\Rightarrow \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$\hat{r}\cdot\hat{\theta} = 0$ polar coordinate

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\cos\theta\,\hat{i} + r(-\sin\theta)\frac{d\theta}{dt}\hat{i} + \frac{dr}{dt}\sin\theta\,\hat{j} + r\cos\frac{d\theta}{dt}\hat{j}$$

$$= \frac{dr}{dt}[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}] + r\frac{d\theta}{dt}[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}]$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

Note $\hat{r}$, $\hat{\theta}$ are time-dependent

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\dot{\hat{r}} = \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta} = \dot{\theta}\hat{\theta}$$

$$\dot{\hat{\theta}} = \frac{d\hat{\theta}}{dt} = -\cos\theta\,\dot{\theta}\,\hat{i} + (-\sin\theta)\dot{\theta}\,\hat{j}$$
$$= -\dot{\theta}\hat{r}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad v^2 = (\dot{r})^2 + r^2\dot{\theta}^2$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + (r\dot{\theta})[-\dot{\theta}\hat{r}]$$

$$= \underbrace{(\ddot{r} - r(\dot{\theta})^2)}_{a_\perp}\hat{r} + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{a_\parallel}\hat{\theta}$$

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Useful in solving the planetary motion.

第二節 一般三度空間之運動

1. 簡介 一質點在一固定坐標中之運動情況可以由其位置向量之時間函數 $\vec{r}(t)$ 所完全決定 假如我們已知其加速度向量及在時間 t_0 之初位 $\vec{r}(t_0)$ 及初速 $v(t_0)$ 則可求 $\vec{r}(t)$.

2. 基本觀念

要描述一質點在三度空間之運動,我們首先取一坐標系統[1] 若質點在時間 t 時位於 P 點,則 $\overline{OP}$ （O 點為坐標系統之原點）向量即為 t 時該質點之位置向量 $\vec{r}(t)$

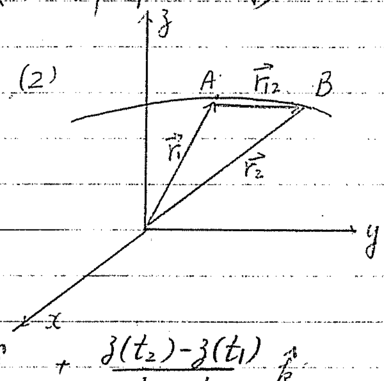
利用直角坐標, $\vec{r}(t)$ 可寫成

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \tag{1}$$

$x(t)$, $y(t)$, $z(t)$ 分別為 $\vec{r}(t)$ 在 x, y, z 軸方向之投影

若在 t_1 時,該質點之位置坐標為 $\vec{r}(t_1)$,在 t_2 時該質點位於 $\vec{r}(t_2)$ 則在 t_1 及 t_2 間（此處我們假設 $t_2 > t_1$）此質點所作之位移[2],[3] 為

$$\vec{r}_{12} \equiv \vec{r}(t_2) - \vec{r}(t_1) \equiv \vec{r}_2 - \vec{r}_1 \tag{2}$$



而該質點在 t_1 及 t_2 間之平均速度為[4]

$$\overline{\vec{v}}_{t_1\to t_2} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

$$= \frac{x(t_2) - x(t_1)}{t_2 - t_1}\,\hat{i} + \frac{y(t_2) - y(t_1)}{t_2 - t_1}\,\hat{j} + \frac{z(t_2) - z(t_1)}{t_2 - t_1}\,\hat{k}$$

$$= \overline{v}_{x,t_1\to t_2}\,\hat{i} + \overline{v}_{y,t_1\to t_2}\,\hat{j} + \overline{v}_{z,t_1\to t_2}\,\hat{k} \tag{3}$$

$\overline{v}_{x,t_1\to t_2}$, $\overline{v}_{y,t_1\to t_2}$, $\overline{v}_{z,t_1\to t_2}$ 分別為該質點在 t_1, t_2 間在 x, y, z 方向之平均速度

若取 $t_1 = t$, $t_2 = t + \Delta t$, 並令 $\Delta t \to 0$, 則如第一節中所討論之結果相同,我們得[4],[5]

$$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j} + v_z(t)\,\hat{k} = \frac{dx(t)}{dt}\,\hat{i} + \frac{dy(t)}{dt}\,\hat{j} + \frac{dz(t)}{dt}\,\hat{k} \tag{4}$$

此處 $\vec{v}(t)$ 是該質點在時間 t 時之瞬時速度[6]，$v_x(t)=\frac{dx(t)}{dt}$，$v_y(t)=\frac{dy(t)}{dt}$，$v_z=\frac{dz(t)}{dt}$

則是該質點在 x, y, z 方向於 t 時之瞬時速度

同樣的，若在時間 t_1，質點之速度為 $\vec{v}_1=\vec{v}(t_1)$，時間 t_2 時之速度為 $\vec{v}_2=\vec{v}(t_2)$

則該質點在 t_1 及 t_2 間之平均加速度為

$$
\begin{aligned}
\overline{\vec{a}}_{t_1\to t_2} &= \frac{\vec{v}_2-\vec{v}_1}{t_2-t_1} = \frac{\vec{v}(t_2)-\vec{v}(t_1)}{t_2-t_1} \\
&= \frac{v_x(t_2)-v_x(t_1)}{t_2-t_1}\hat{i} + \frac{v_y(t_2)-v_y(t_1)}{t_2-t_1}\hat{j} + \frac{v_z(t_2)-v_z(t_1)}{t_2-t_1}\hat{k} \\
&\equiv \overline{a}_{x,t_1\to t_2}(t)\,\hat{i} + \overline{a}_{y,t_1\to t_2}(t)\,\hat{j} + \overline{a}_{z,t_1\to t_2}(t)\,\hat{k} \qquad (5)
\end{aligned}
$$

$\overline{a}_{x,t_1\to t_2}$，$\overline{a}_{y,t_1\to t_2}$，$\overline{a}_{z,t_1\to t_2}$ 分別為該質點在 t_1 及 t_2 間在 x, y, z 方向之平均加速度

若取 $t_1=t$，$t_2=t+\Delta t$，並命 $\Delta t\to 0$，則如第一節所討論之結果相同，我們可

得[6,7]

$$
\begin{aligned}
\vec{a}(t) &= a_x(t)\,\hat{i} + a_y(t)\,\hat{j} + a_z(t)\,\hat{k} \\
&= \frac{dv_x(t)}{dt}\hat{i} + \frac{dv_y(t)}{dt}\hat{j} + \frac{dv_z(t)}{dt}\hat{k} \\
&= \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k} \qquad (6)
\end{aligned}
$$

3. 討論

(1) 此一坐標不必需是慣性坐標系統

(2) 位移與坐標系統原點取於何點無關

(3) $\vec{r}_{12}$ 之大小為 $[(x(t_2)-x(t_1))^2+(y(t_2)-y(t_1))^2+(z(t_2)-z(t_1))^2]^{1/2}$ 而非其所行曲線 AB 之長度

(4) 在 t 時速度之大小為 $|\vec{v}(t)|=(v_x^2(t)+v_y^2(t)+v_z^2(t))^{1/2}$

(5) $\vec{v}(t)$ 是沿該質點所行之軌道在 t 時所經之點的切線方向

(6) 在 t 時加速度之大小為 $|\vec{a}(t)|=(a_x^2(t)+a_y^2(t)+a_z^2(t))^{1/2}$

(7) 通常 $\vec{v}(t)$ 之方向與 $\vec{a}(t)$ 之方向不必相同

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4. 應用：質點在三度空間之基本問題是 (一) 已知 $\vec{r}(t)$，求 $\vec{v}(t)$ 及 $\vec{a}(t)$

(二) 已知加速度，其於 t_0 時之初位 $\vec{r}(t_0)$ 及初速 $\vec{v}(t_0)$ 求 $\vec{r}(t)$ 我們將分別說明

(A) 已知 $\vec{r}(t)$ 求 $\vec{v}(t)$ 及 $\vec{a}(t)$、已知 $\vec{r}(t)$ 表示 $x(t)$，$y(t)$ 及 $z(t)$ 為已知

$$\vec{v}(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

$$\vec{a}(t) = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k}$$

因此這一類問題就成為求 $x(t)$, $y(t)$, $z(t)$ 對 t 之一度及二度微分

例：若一質點對一坐標系統之位置向量之時間函數為

$$\begin{cases} x(t) = 5 - 8t + 3t^2 & (7a) \\ y(t) = 3t - 1 & (7b) \\ z(t) = 5 & (7c) \end{cases}$$

此處距離之單位為米，而時間之單位為秒

則 $v_x(t) = -8 + 6t$，$v_y(t) = 3$，$v_z(t) = 0$

$a_x(t) = -8$，$a_y(t) = 0$，$a_z(t) = 0$

所以該質點在 x 方向是一等加速度運動，在 y 方向是等速度運動，而在 z 方向是靜止不動的

注意 此運動之速度 $\vec{v}(t)$ 之方向是在 xy 平面上 而且其在 y 方面之投影為一常數，而此運動 $\vec{a}(t)$ 之方向是沿 $-x$ 軸。因此 $\vec{a}(t)$ 與 $\vec{v}(t)$ 之方向不同，因此此一運動不可能是直線運動（一質點之運動為直線運動之充要條件即 $\vec{v}(t)$ 與 $\vec{a}(t)$ 永遠平行）

由 (7c) 式中可知，該質點在 $z=5$ 之平面上運動，由 (7a) 及 (7b) 中將 t 消去得

$$x = \frac{8}{3} - 2y + \frac{y^2}{3} \tag{8}$$

這公式代表該質點在 $z=5$ 平面上之軌跡. 在此特殊情況下, 其軌跡為一拋物線

(B) 已知加速度, 其於 t_0 時之初位 $\vec{r}(t_0)$ 及初速 $\vec{v}(t_0)$ 求 $\vec{r}(t)$

基本上我們需將下列公式作兩式積分

$$\frac{d^2x}{dt^2} = a_x \quad , \quad \frac{d^2y^2}{dt^2} = a_y \quad , \quad \frac{dz^2}{dt^2} = a_z \tag{9}$$

(a) 首先我們將討論最簡單的情況, 也即是 a_x, a_y, a_z 均為常數的情況.

如同在第一節中一樣我們可以將第 (9) 式積分兩次而得

$$x(t) = x_0 + v_{0x}(t-t_0) + \frac{1}{2} a_x (t-t_0)^2 \tag{10a}$$

$$y(t) = y_0 + v_{0y}(t-t_0) + \frac{1}{2} a_y (t-t_0)^2 \tag{10b}$$

$$z(t) = z_0 + v_{0z}(t-t_0) + \frac{1}{2} a_z (t-t_0)^2 \tag{10C}$$

(x_0, y_0, z_0) 是該質點在 $t=t_0$ 之位置
(v_{0x}, v_{0y}, v_{0z}) 是該質點在 $t=t_0$ 之速度

由 (10b) 及 (10C) 中我們可得

$$a_z\, y(t) - a_y\, z(t) = (a_z y_0 - a_y z_0) + (a_z v_{0y} - a_y v_{0z})(t-t_0) \tag{11}$$

因此

$$(t-t_0) = \frac{a_z y(t) - a_y z(t) - (a_z y_0 - a_y z_0)}{(a_z v_{0y} - a_y v_{0z})} \tag{12}$$

將 (12) 式代入 (10a)

$$x(t) = x_0 + v_{0t} \frac{a_z y(t) - a_y z(t) - (a_z y_0 - a_y z_0)}{(a_z v_{0y} - a_y v_{0z})} + \frac{1}{2} a_x \left\{ \frac{(a_z y(t) - a_y z(t)) - (a_z y_0 - a_y z_0)}{(a_z v_{0y} - a_y v_{0z})} \right\}^2 \tag{13}$$

此式表示出 $x(t)$, $y(t)$, $z(t)$ 間之關係, 也即是該質點在空間運動之軌跡

例. 拋物体運動

(b) 其次我們討論 a_x, a_y, a_z 均為 t 之函數

$$\frac{d^2x}{dt^2} = a_x(t) \quad , \quad \frac{d^2y}{dt^2} = a_y(t) \quad , \quad \frac{d^2z}{dt^2} = a_z(t)$$

此時我們可對上式作兩次積分而求得 $x(t)$, $y(t)$ 及 $z(t)$

分類：
編號： 5
總號：

$v_x(t) = \int a_x(t)\,dt = A_x(t) + v_0$

$x(t) = \int (A_x(t) + v_0)\,dt = B_x(t) + v_0 t + x_0$

此處 $\frac{dA_x(t)}{dt} = a_x(t)$, $\frac{dB_x(t)}{dt} = A_x(t)$

用類似的方法可解得 $y(t)$ 及 $z(t)$

(c) 若 a_x, a_y, a_z 是 (x, y, z) 或 (v_x, v_y, v_z) 之函數,則

$m\frac{d\vec{v}}{dt} = \vec{a}$ 是一組微分方程式. 此類問題之討論是超出了本書的範圍

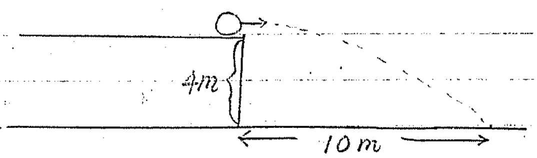
帶電粒子在
固定電磁場中之運動

5. 習題

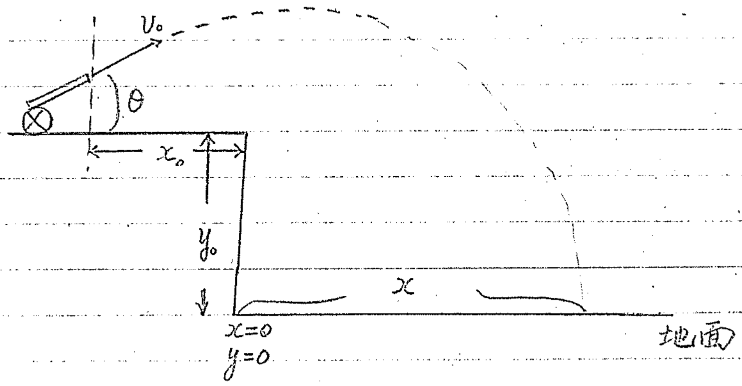
(1) 一個乒乓球由一高度為4m之桌面上滾落 此球之着地點距桌面為10m

求此球離開桌面時之速度 [D]

Example
$\vec{B} = (0, 0, B_0)$



(2)



在山坡上一尊巨炮以 98 m/sec 之初速及 $\theta = 37°$ 之角度發出一粒炮彈

此巨炮位於高 200 m之山坡,其離山腳之距離$|x_0|$為 49 m

(a) 何時此炮彈將恰好經過 $x=0$ 之上空? [G]

(b) 當該炮彈經過 $x=0$ 上空時之高度為何? [A]

(c) 何時此炮彈距地面之高度為最大? [L]

分類：	
編號：	6
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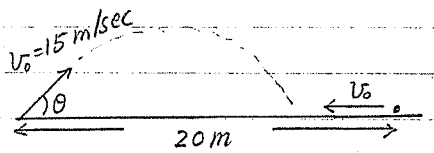
(c) 該炮彈離地之最高点之高度爲何? [K]

(d) 當該炮彈位於其最高点其 x 軸坐標爲何? [M]

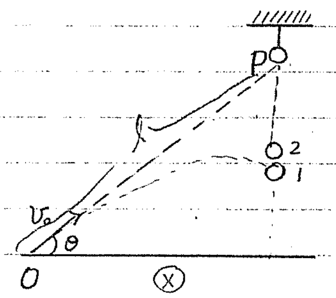
(e) 何時此炮彈落地? [J]

(f) 此炮彈落地点之 x 軸坐標爲何? [C]

(3) 對方以角度 $\theta = 37^\circ$ 及初速 $v_0 = 15\ m/sec$ 將球踢出, 一隊員位於球前 30 m 處. 當對方將球踢出時, 他立即以 v_0 之速度向前跑, 問該球員必須以何種速度奔跑方能在球將落地時將球接住? [F]



(4)



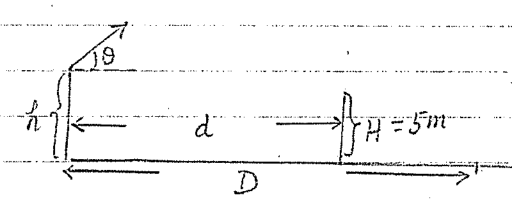
第一個球原来位於地上, 對掛着的第二球瞄準射擊 第一球之發射角爲 $\theta = 37^\circ$ 其初速爲 $v_0 = 5\ m/sec.$ OP 之距離爲 $l = 2.5\ m$

當第一球發射之同時將掛着第二球之繩索割斷, 因此第二球成爲自由落体

(a) 求此兩球相撞時離 P 点之垂直距離? [E]

(b) 求當兩球相撞時第一球之運動方向爲何? [H]

(5)



一小孩由一窗口以角度 $\theta = 37^\circ$ 將一球由一窗口拋出. 窗口之高度爲 10 m. 離屋 10 m 處有一門若門開時球落於離屋 D=26米

處

(a) 若大門是關着時，此球可否通過？ [B]

(b) 此球能超越之門的最大高度爲何？ [I]

6. 答案

(A) 234.8 米

(B) 是

(C) 1161.89 米

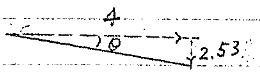
(D) 11.07 米/秒

(E) 離 P 點爲 1.01 米

(F) 4.34 米/秒

(G) $t = 0.625$ 秒

(H)



$\tan\theta = \dfrac{2.53}{4}$ $\theta = 32.3°$

(I) 16.25 米

(J) $t = 14.82$ 秒

(K) 376.4 米

(L) $t = 6$ 秒

(M) 470.4 米

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Chapter 3

Kinematics is the branch of classical mechanics
that describes the motion of bodies (objects)
and systems (group of objects) without consideration
of the forces that cause the motion.

Chapter 2 Straight line motion.

Chapter 3 Motion in two and three dimensions

extension of our description of 1- dimensional
motion.

One dimensional motion the directional aspects is
encapsulated in signs.

Two or three dimensions, we must use vectors
to describe the directional aspect properly

Particle kinematics is the study of a single particle.

- Position and Reference Frame

Specify a point in space

reference point (origin)

distance from the the reference point

direction in space of the straight line from the
reference point to the particle

$\Rightarrow$ vectors.

分類:	
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For measurement of distance and direction usually a coordinate system (most commonly Cartesian) are used with the origin coinciding with the reference point.

A coordinate system (whose origin coincides with the reference point) with some provision for time measurement is called a reference frame or frame of reference

Position Vector

$\vec{r}(t)$ instantaneous position vector of the particle

$\hat{i}$, $\hat{j}$, $\hat{k}$ unit vector ; time independent

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k}$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

$$\hat{i}\times\hat{j} = \hat{k}$$

Average velocity (Need to specify time interval.)

Instantaneous velocity

Speed.

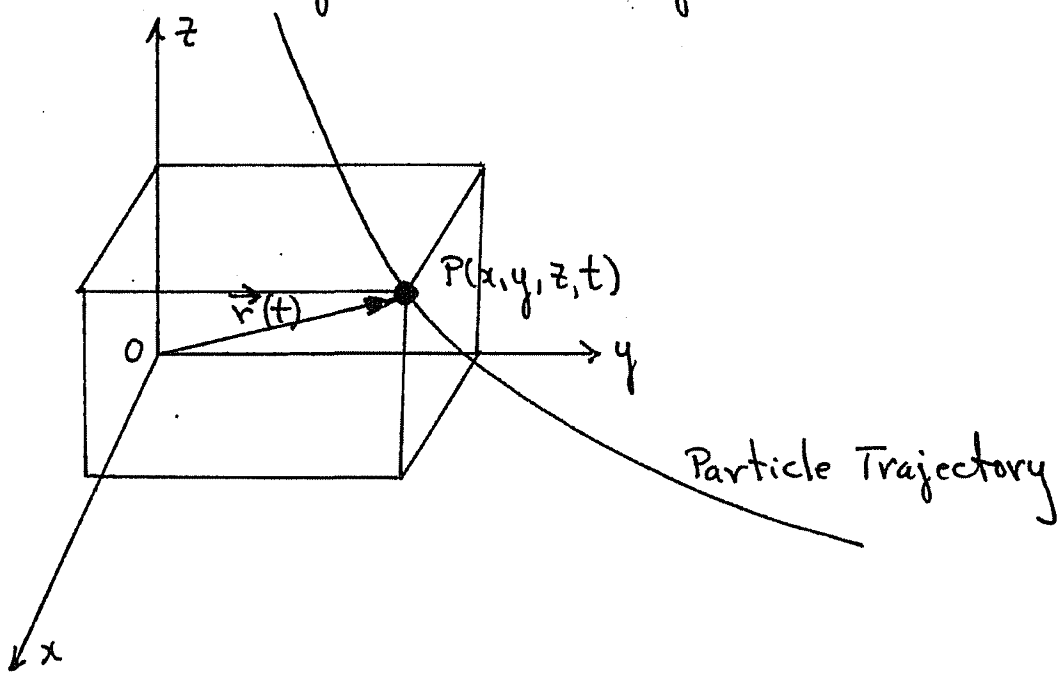
Comment on "definition"

Average acceleration

Instantaneous acceleration

Kinematics in Three Dimensions

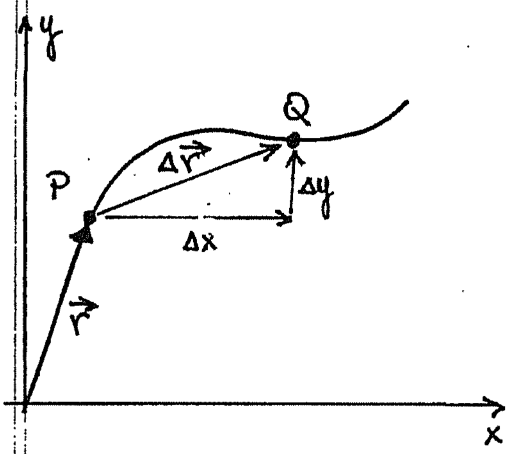
- extension of our description of 1-dimensional motion
- 3-dimensional motion is basically 3 - one dimensional motions occuring simultaneously.



$\vec{r}(t) \equiv$ instantaneous position vector of the particle.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Average velocity



Consider a particle in a plane moving from P to Q.
Position vector at P is $\vec{r}$.
The change in position vector is $\vec{\Delta r}$

$$\vec{\Delta r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

Let Δt be the time interval for the motion from P to Q. The average velocity of the particle is then defined as the vector quantity equal to the displacement divided by the time interval:

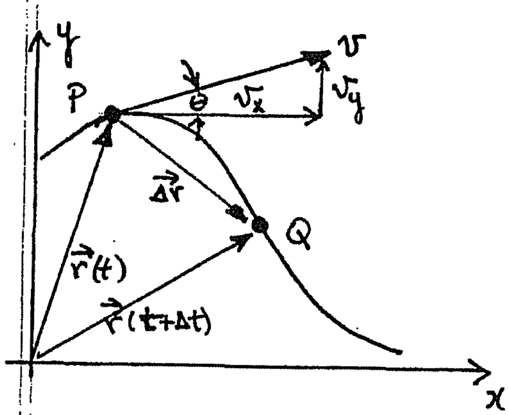
$$\vec{\bar{v}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\hat{i} + \frac{\Delta y}{\Delta t}\,\hat{j} + \frac{\Delta z}{\Delta t}\,\hat{k} \qquad [3\text{-D}]$$

- direction of vector $\vec{\Delta r}$
- magnitude $\Delta r/\Delta t$

Let $\bar{v}_x = \dfrac{\Delta x}{\Delta t}$, $\bar{v}_y = \dfrac{\Delta y}{\Delta t}$, $\bar{v}_z = \dfrac{\Delta z}{\Delta t}$

$$\bar{v} = \bar{v}_x\,\hat{i} + \bar{v}_y\,\hat{j} + \bar{v}_z\,\hat{k}$$

Instantaneous velocity, v, at the point P is defined in magnitude and direction as the limit approached by the average velocity when point Q is taken to be closer and closer to P (as $\Delta t \to 0$).



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- direction of $\vec{v}$ is tangent to path of particle at P.
- equal to time rate of change of position vector.
- describes motion at a particular point and particular instant of time

$$\vec{v} = \frac{d}{dt}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \qquad \text{since } \frac{d\hat{i}}{dt} = 0\text{, etc.}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Magnitude of $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ [rect. coord.]

$= \sqrt{\vec{v}\cdot\vec{v}}$

$\cancel{\tan\theta = v_y/v_x}$ [2-D]

Represent $\vec{v}$: components, or magnitude and direction

Speed

$$[\text{speed}] = v = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- instantaneous speed is magnitude of velocity vector.
- velocity has direction and magnitude.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time}}$$

Comments on "definition"

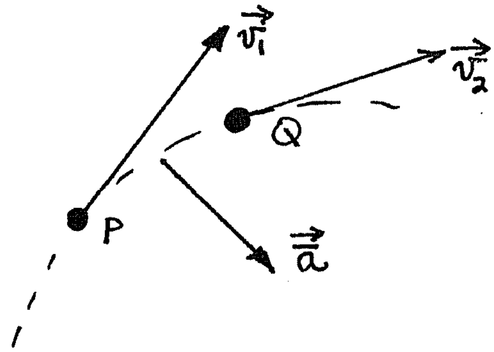
useful

concise

operatial

Acceleration

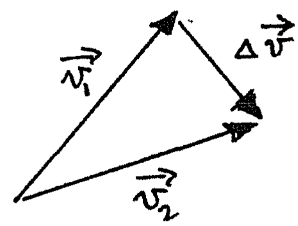
The <u>average</u> and <u>instantaneous</u> accelerations in 3-dimensions are straightforward generalizations of one-dimensional motion.



$\vec{v}_1$ and $\vec{v}_2$ are velocities for particle moving in a curved path at P and Q.

- different magnitudes
- different direction

$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$



The <u>average acceleration</u>, $\vec{\bar{a}}$, of the particle as it moves from P to Q is defined as the vector change in velocity, $\Delta\vec{v}$, divided by the time interval Δt.

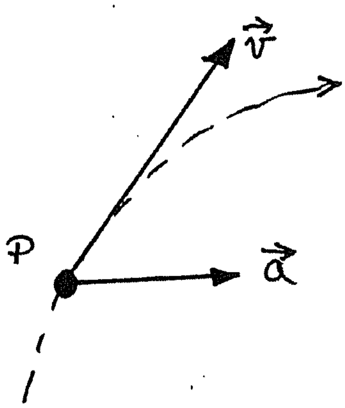
$$\vec{\bar{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} + \frac{\Delta v_z}{\Delta t}\hat{k}$$

$$= \bar{a}_x\hat{i} + \bar{a}_y\hat{j} + \bar{a}_z\hat{k}$$

$\vec{\bar{a}} \neq 0$ if velocity changes direction between P+Q or if velocity changes in magnitude.

The <u>instantaneous acceleration</u>, $\vec{a}$, at point P is defined in <u>magnitude</u> and <u>direction</u> as the limit approached by the average acceleration when point Q approaches point P and Δv and Δt both approach zero



$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$= a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

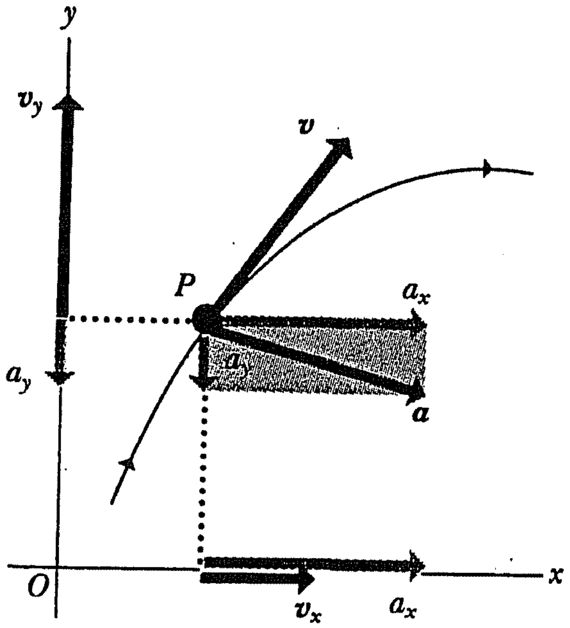
$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

- Equal to time rate of change of velocity
- $\neq 0$ if velocity changes in magnitude <u>or</u> direction
- It <u>does not</u> have same direction as velocity vector.
- Acceleration vector lies on concave side of curved path.

- average acceleration refers to finite time interval during which velocity changes.
- instantaneous acceleration is the rate of change of velocity at a specific point at a specific time.



The acceleration $\boldsymbol{a}$ is resolved into its components a_x and a_y.

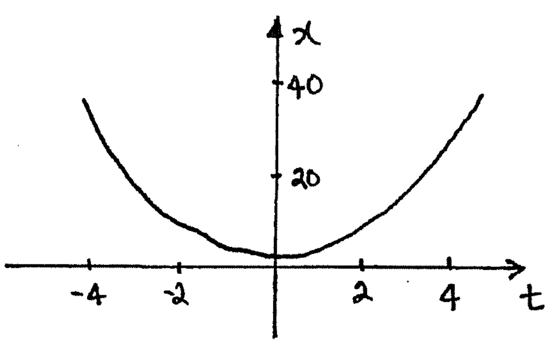
Example

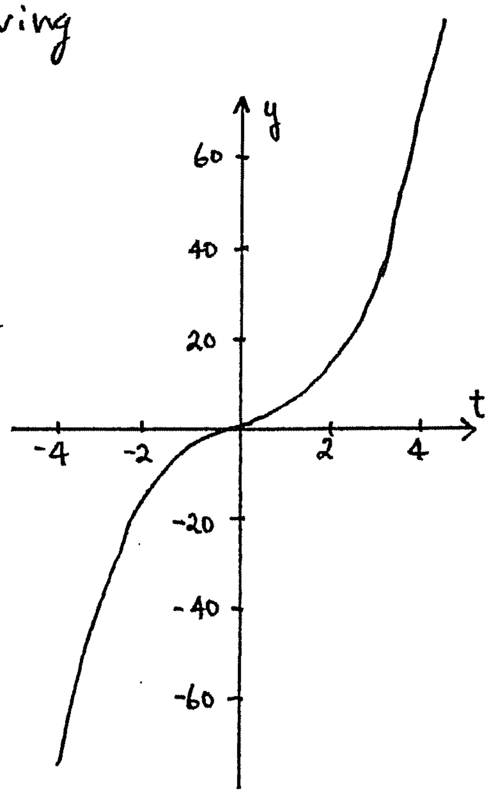
The coordinates of a particle moving in the xy-plane are given by

$x = 1 + 2t^2$ (m)

$y = 2t + t^3$ (m)

Find the particle's position, velocity and acceleration at time $t = 2$s.





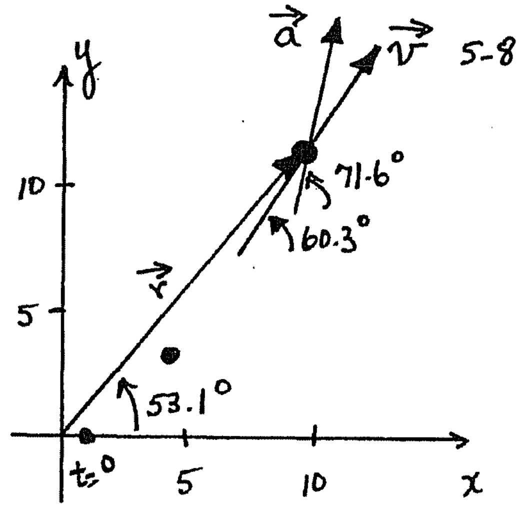
Position

$$x = 1 + 2(2)^2 = 9\,m$$
$$y = 2(2) + 1(2)^3 = 12\,m$$

$$\vec{r} = 9\hat{\imath} + 12\hat{\jmath}$$

Distance from origin

$$r = \sqrt{x^2+y^2} = \sqrt{9^2+12^2} = 15\,m$$
$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{12}{9} = 53.1°$$



Velocity

$$v_x = \frac{dx}{dt} = 4t \quad (m/s) \qquad v_y = \frac{dy}{dt} = 2 + 3t^2 \quad (m/s)$$

At $t = 2s$ $\quad v_x(2) = 8\ m/s \quad v_y(2) = 14\ m/s$

$$\vec{v}(t=2) = 8\hat{\imath} + 14\hat{\jmath}$$

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{8^2+14^2} = 16.1\,m/s$$
$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{14}{8} = 60.3°$$

Acceleration

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 4 \quad (m/s^2) \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 6t \quad (m/s^2)$$

At $t = 2s$ $\quad a_x = 4\ m/s^2 \quad a_y = 12\ m/s^2$

$$\vec{a} = 4\hat{\imath} + 12\hat{\jmath}$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2+12^2} = 12.6\,m/s^2$$
$$\theta_a = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{12}{4} = 71.6°$$

[Direction of $\vec{a} \neq \vec{v}$ and not tangent to path]

分類:
編號:
總號:

Velocity and Speed

Vector calculus
vector derivate

5-1 to 5-4 of M.I.T.

Acceleration

$$\vec{r}(t) \longrightarrow \vec{v}(t) \longrightarrow \vec{a}(t)$$

5-5 to 5-6

Example:

Two-dimensional $\quad \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$x(t) = 1 + 2t^2$

$y(t) = 2t + t^3$

$t = 2$ sec

$\vec{r} = 9\hat{i} + 12\hat{j}$

$\vec{v} = 8\hat{i} + 14\hat{j}$

$a = 4\hat{i} + 12\hat{j}$

Acceleration: $a_\perp$ and $a_\parallel$

圖 (a) on 5-9.

$\vec{a}_\perp$ normal to path ($\vec{v}$)

$\vec{a}_\parallel$ parallel to $\vec{v}$

$\vec{a} = \vec{a}_\parallel + \vec{a}_\perp \quad \cancel{\vec{a} \cdot \vec{v} = \vec{a}_\parallel \cdot \vec{v} + a_\perp}$

$a_\perp$

$\hat{v}$ = unit vector along the path

$= \dfrac{\vec{v}}{|\vec{v}\cdot\vec{v}|} \qquad \boxed{\hat{v}\cdot\hat{v} = 1}$

$\vec{a}_\parallel = a_\parallel \hat{v} \qquad \vec{a}_\perp \cdot \hat{v} = 0$

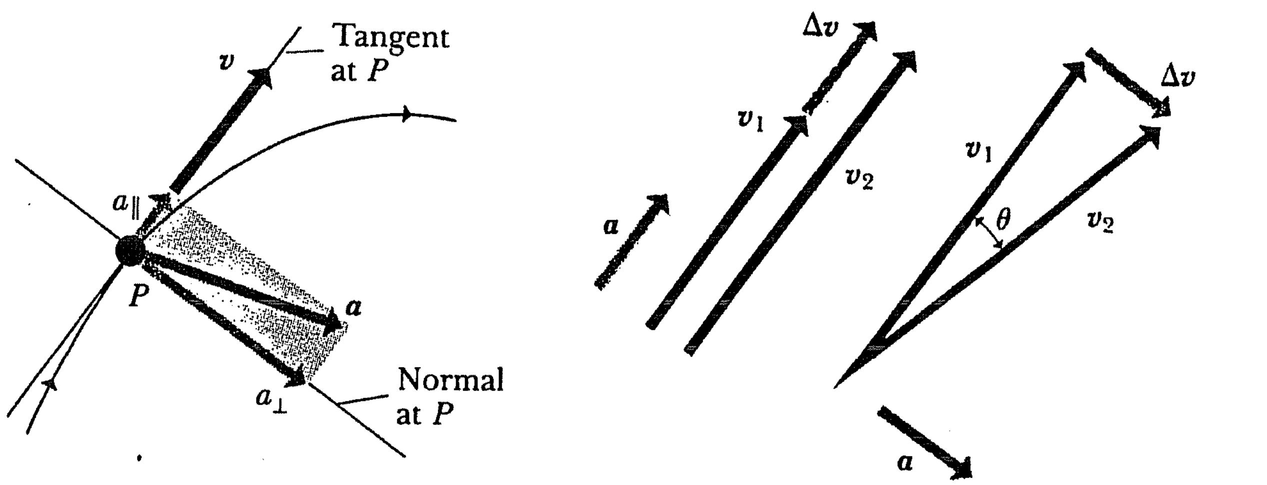
$\Rightarrow \; a_\parallel$

Acceleration : $a_\perp$ and $a_{\parallel}$

- Instructive to represent the acceleration of a particle moving along a curved path in terms of rectangular components:
  - $a_\perp$ - normal to path
    - associated with change in <u>direction</u> of $\vec{v}$
  - $a_{\parallel}$ - parallel to path
    - associated with change in <u>magnitude</u> of $\vec{v}$

$\vec{a}$ parallel to $\vec{v}$: Change in $\vec{v}_1$ during a small amount of time Δt is a vector $\Delta\vec{v}$ with same direction as $\vec{a}$ and same direction as $\vec{v}_1$. Velocity $\vec{v}_2$ at the end of Δt, is $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, same direction as $\vec{v}_1$ but somewhat greater in magnitude.

$\vec{a}$ perpindicular to $\vec{v}$: In time interval Δt the change $\Delta\vec{v}$ is $\perp$ to $\vec{v}_1$. $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$ but $\vec{v}_1$ and $\vec{v}_2$ differ in direction. As $\Delta t \to 0$, $\Delta\vec{v}$ is $\perp$ to $\vec{v}_1$ and $\vec{v}_2$. $\vec{v}_1$ and $\vec{v}_2$ have same magnitude.



(a) (b) (c)

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$d\vec{v} = \vec{a}\, dt$ change in velocity is in same direction as acceleration vector.

Using $\vec{a}_{\parallel}$ and $\vec{a}_{\perp}$ one easily sees that if

$\vec{a} \cdot \vec{v} > 0$ velocity increases

$\vec{a} \cdot \vec{v} < 0$ velocity decreases.

$\vec{a} \cdot \vec{v} \equiv 0$ velocity magnitude remains constant
velocity changes direction
$\Rightarrow$ circular motion !!

分類：
編號： 1
總號：

第三節 切線及法向加速度

一、簡介 在討論質點運動時，(例如圓周運動)，將其加速度分解為切線及法向加速度是相當有用的，因為我們在此節中將說明切線加速度是與質點運動速率(其速度之大小)變化有關而其法向加速度則與其方向之變化有關

二、基本觀念

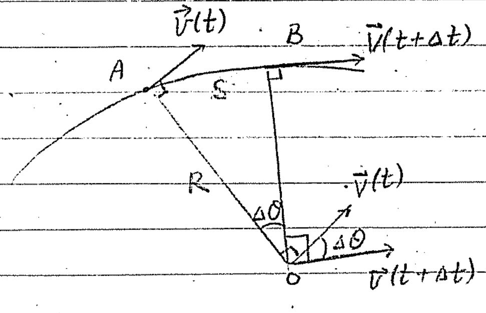
一質點作任意運動，其軌跡可由 $\vec{r}(t)$ 來表示，由上章中我們得知由 $\vec{r}(t)$ 吾人可求得 $\vec{v}(t)$ 及 $\vec{a}(t)$，若一質量在 t 時之位置為 $\vec{r}(t)$，其速度為 $\vec{v}(t)$ 其加速度為 $\vec{a}(t)$。我們將 $\vec{a}(t)$ 分解為切線加速度 $\vec{a}_T$ 及法線加速度 $\vec{a}_N$，而 $\vec{a}_T$ 及 $\vec{a}_N$ 分別為 $\vec{a}(t)$ 在平行於及垂直於 $\vec{v}(t)$ 方向之投影

$$\vec{a}(t) = \vec{a}_T(t) + \vec{a}_N(t) \qquad (1)$$

$$|\vec{a}_T| = \frac{\vec{a}(t)\cdot\vec{v}(t)}{|\vec{v}|} \text{ (1)} \qquad (2)$$

$$\text{and} \quad \vec{a}_N(t) = \vec{a}(t) - \vec{a}_T(t) \qquad (3)$$

我們現在將用幾何的方法來討論 $|\vec{a}_T|$，$|\vec{a}_N|$ 之大小及其物理意義



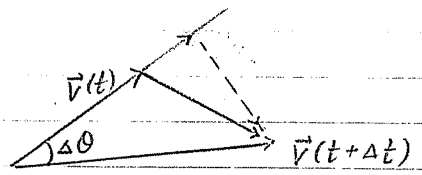
設在時間 t 時質點位於 A 點而其瞬時速度為 $\vec{v}(t)$，當時間為 $t+\Delta t$ 時該質點進行至 B 點而其瞬時速度為 $\vec{v}(t+\Delta t)$。

通過 A 點畫一直線垂直於 $\vec{v}(t)$，通過 B 點畫一垂直於 $\vec{v}(t+\Delta t)$ 之直線，其交點我們稱之為 O 點，如圖所示 (2)，令交角 $AOB = \Delta\theta$

由圖可知，$\vec{v}(t)$ 及 $\vec{v}(t+\Delta t)$ 間之交角亦為 $\Delta\theta$，因此我們有如下

分類：
編號： 2
總號：

之向量圖 $\vec{V}(t+\Delta t)-\vec{V}(t)$ 在 $\vec{V}(t)$ 方向之投影



稱之為 $(\Delta\vec{V})_T$，在垂直於 $\vec{V}(t)$ 方向之投影為 $(\Delta\vec{V})_N$

由圖可知[3]

$$|\Delta\vec{V}|_T = |\vec{V}(t+\Delta t)|\cos\Delta\theta - |\vec{V}(t)| \approx |\vec{V}(t+\Delta t)| - |\vec{V}(t)| \qquad (4)$$

$$(\Delta\vec{V})_N = |\vec{V}(t+\Delta t)|\sin\Delta\theta \approx |\vec{V}(t+\Delta t)|\Delta\theta \qquad (5)$$

$$|\vec{a}_T| = \lim_{\Delta t\to 0}\frac{|\Delta\vec{V}|_T}{\Delta t} = \lim_{\Delta t\to 0}\frac{|\vec{V}(t+\Delta t)|-|\vec{V}(t)|}{\Delta t}$$

$$= \frac{d|\vec{V}|}{dt} = \frac{dv}{dt} \qquad (6)$$

$$|\vec{a}_N| = \lim_{\Delta t\to 0}\frac{|\Delta\vec{V}|_N}{\Delta t} = \lim_{\Delta t\to 0}\frac{|\vec{V}(t+\Delta t)|\Delta\theta}{\Delta t} \qquad (7)$$

由圖一我們可看出 $R\Delta\theta=\Delta S$[4]. 所以

$$|\vec{a}_N| = \lim_{\Delta t\to 0}|\vec{V}(t+\Delta t)|\frac{\Delta S}{\Delta t}\frac{1}{R}$$

$$= \frac{v^2}{R} \qquad (8)$$

因此吾人導得 $|\vec{a}_T| = \frac{dV}{dt}$ 及 $|\vec{a}_N| = \frac{v^2}{R}$. (5),(6),(7),(8)

三. 討論

(1) $\frac{\vec{V}(t)}{|\vec{V}(t)|}$ 為一沿 $\vec{V}(t)$ 方向之單位向量

(2) 我們現在畫的是由 $\vec{V}(t)$ 及 $\vec{V}(t+\Delta t)$ 決定之平面. OA 之距離為 R 稱為此質量於 t 時之曲率半徑. R 為一時間之變數

(3) 當 x 小時, $\sin x$ 及 $\cos x$ 之展開式分別為 $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$ 及 $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$. 因此當 $x\ll 1$ 時而我們只要求準確至 x 時我們可用 $\sin x\simeq x$ 及 $\cos\theta\simeq 1$ 之近似公式. 此處我們用到 $\sin\Delta\theta\simeq\Delta\theta$ 及 $\cos\Delta\theta\doteq 1$. 此處 $\Delta\theta$ 是以弧度為單位.

(4) 記得 $\Delta\theta$ 是以弧度為單位，ΔS 是 AB 間之弧長因此 $\lim\limits_{\Delta t\to 0}\frac{\Delta S}{\Delta t}=|\vec{V}|$

(5) 此結果是對任何質點運動均成立，並非僅限於討論兩度空間或圓周運動

(6) 在等速圓周運動時 $|\vec{a}_T|=0$ 因此，$\vec{a}=\vec{a}_N$，加速度與速度垂直

(7) 在直線運動時 $R=\infty$，因此 $|\vec{a}_N|=0$，$\vec{a}=\vec{a}_T$，加速度與速度平行

(8) 通常 $\vec{a}_T$，$\vec{a}_N$，R，及 $|\vec{V}|$ 均為 t 之函數

四 目的：

若一質點之運動方程式已知，求 a_T，a_N，$\frac{d|\vec{V}|}{dt}$ 及 R

五 習題

若一質點之運動方程式為

$$\begin{cases} x(t) = 2t^2+7 \\ y(t) = 3t+8 \\ z(t) = 4 \end{cases}$$

求

(1) $\vec{V}(t)$ (F)

(2) $\vec{A}(t)$ (B)

(3) 求當 $t=1$ 時之 $\vec{V}$ 及 $\vec{a}$ (A)

(4) 求當 $t=1$ 時之 $\vec{a}_T$ (E)

(5) 求當 $t=1$ 時之 $\vec{a}_N$ (D)

(6) 求當 $t=1$ 時該質量運動之曲率半徑 (C)

六 答案

分類：
編號： 4
總號：

(A) $(4, 3, 0)$, $(4, 0, 0)$

(B) $a_x = 4$, $a_y = 0$, $a_z = 0$

(C) 1.92

(D) $(-\frac{44}{5}, -\frac{48}{5}, 0)$

(E) $(\frac{64}{5}, \frac{48}{5}, 0)$

(F) $(4t, 3, 0)$

分類：
編號： 1
總號：

第四節 圓周運動

一 簡介 在很多情況下，質點會沿圓周運動，在此小節中，我們將討論對此類運動之描述

二 基本觀念

一質點在一平面上作圓周運動，可由 $\theta(t)$ 完全決定，因此實質上是一度空間之問題

若於 t 時，其角位置為 $\theta(t)$，在 $t+\Delta t$ 時之角位置為 $\theta(t+\Delta t)$，則在 Δt 時間內其所作之角位移為 $\theta(t+\Delta t)-\theta(t)$ (1), (2)

我們可以定義在 t 至 $t+\Delta t$ 時間內之平均角速度為 $\dfrac{\theta(t+\Delta t)-\theta(t)}{\Delta t}$，

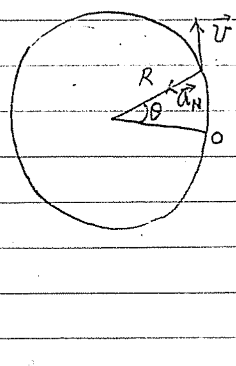
在 t 時之瞬時角速度為 $\omega(t)=\lim_{\Delta t\to 0}\dfrac{\theta(t+\Delta t)-\theta(t)}{\Delta t}$ (3)

若於 t 時其角速度為 $\omega(t)$，在 $t+\Delta t$ 時之角速度為 $\omega(t+\Delta t)$

則此質點在 t 至 $t+\Delta t$ 時間內之平均角加速度為 $\dfrac{\omega(t+\Delta t)-\omega(t)}{\Delta t}$.

我們同時可定義在 t 時之瞬時角加速度為 $\alpha(t)=\lim_{\Delta t\to 0}\dfrac{\omega(t+\Delta t)-\omega(t)}{\Delta t}$

若一質點在 t 至 $t+\Delta t$ 時間內之角位移為 $\Delta\theta$，而此圓周之半徑為 R，則在此時間內其所行之距離為 $R\Delta\theta$.



則此質點在 t 至 $t+\Delta t$ 之間之平均速率為 $R\dfrac{\Delta\theta}{\Delta t}$

在 t 時之瞬時速度之大小為 $|\vec{v}|=\lim_{\Delta t\to 0}R\dfrac{\Delta\theta}{\Delta t}$

$=R\omega$

在 t 時瞬時速度之方向為沿切線之方向.

因此我們可以定義此質點在 t 時之瞬時速度 $\vec{v}(t)$.

利用同樣之方法,我們可得質點在 $t+\Delta t$ 之瞬時速度 $\vec{v}(t+\Delta t)$

由此按照以前所討論之方法可以求得 $\vec{a}(t)$. 同時 $\vec{a}(t)$ 可分解為

$\vec{a}(t)=\vec{a}_T(t)+\vec{a}_N(t)$ 此處 $\vec{a}_T$ 為沿質點速度方向之加速度

也即是沿切線方向之加速度 $\vec{a}_N$ 是垂直於切線方向之加速度

由此節之討論可知 $|\vec{a}_T|=\frac{d|v|}{dt}=\frac{d}{dt}R\omega=R\frac{d\omega}{dt}=R\alpha$.

$|\vec{a}_N|=\frac{v^2}{R}$ (5),(6) 在此處 $\vec{a}_N$ 又稱為向心加速度 若此質點在一定

之時間間隔後通過圓上之一定點,則此一質點被稱為作週期運動.

一質點轉一圈所化之時間則稱為其運動之週期 P. (7),(8) 週期之倒數稱

為頻率 $\nu=\frac{1}{P}$ 也即是在單位時間所運轉之周數.

四. 討論

(1) 定 θ 時我們首先必須定一原點,然後沿反時鐘方向量度

(2) θ 之單位是弧度

(3) $\omega(t)$ 可以為正,也可以為負 其單位為弧度/秒

(4) 當 $\omega(t)=$ 常數時,稱之為等速圓周運動 此時質點之速度大小不變,因此 $\vec{a}_T=0$. 但是 $|\vec{a}_N|\neq 0$ 而為一常數 $\frac{v^2}{R}=R\omega^2$.

因此等速圓周運動時,其速度之大小不變,但這並不表示其加速度為零,因為其速度之方向在變 唯有一質點速度之大小及方向均不變其加速度方為 0.

(5) 由圖可知, $\vec{a}_N$ 之方向必須是沿向心方向. 因為若 $\vec{a}_N=0$ 則此質點將沿切線進行 $\vec{a}_N$ 必須存在方能使質點沿圓周運動

(6) 在等速圓周運動時 $\vec{a}_T=0$, 因此 $\vec{a}=\vec{a}_N$. 也即是說 $\vec{a}$ 與 $\vec{v}$

相垂直 $\vec{a}$ 与 $\vec{v}$ 均在圓周運動所進行之平面上. 此時我們可定義一向量 $\vec{\omega}$，其方向是沿 $\vec{v}\times\vec{a}$ 其大小則為 ω. 此一向量与 $\vec{v}$, $\vec{a}$ 之間有一簡單之関係式 $\vec{\omega}\times\vec{v}=\vec{a}$. $\vec{\omega}\times\vec{v}$ 及 $\vec{a}$ 之方向相同可以由 $\vec{v}$, $\vec{a}$ 及 $\vec{\omega}$ 形成一右手坐標系統而得 $|\vec{\omega}\times\vec{v}|=\frac{v}{R}\cdot v=\frac{v^2}{R}=|\vec{a}_N|=|\vec{a}|$

(此處記得 $\vec{\omega}\perp\vec{v}$). 我們強調此一関係式 $\vec{\omega}\times\vec{v}=\vec{a}$ 唯有在等速圓周運動時方成立.

(7) 只要是週期運動我們均可定義其週期及頻率 並不需要質點作等速圓周運動

(8) 在等速圓周運動時 運轉一周該質點所作之角位移為 2π, 因此所需之時間 $=\frac{2\pi}{\omega}$. 因此 $P=\frac{2\pi}{\omega}\Rightarrow\frac{1}{\nu}=\frac{2\pi}{\omega}\Rightarrow\omega=2\pi\nu$

(9) 頻率之單位是 每秒週数 又稱為赫茲.

五 應用

(1) 若一質點在一半徑為 R 之圓周上運動 其角位置 $\theta(t)$ 可以下式描述之.

$\theta(t)=at^2+bt+c$.

因為 θ 之單位為弧度, 所以 a, b, c 之單位分別為弧度/秒², 弧度/秒及弧度.

此質點在 t 時之角速度為 $\omega(t)=2at+b$ 其角加速度為 $\alpha(t)=2a$

此質點在 t 時瞬時速度之大小為 $R\,|2at+b|$. 此質點在 t 時其切線方面之加速度 $|\vec{a}_T|$ 為 $2aR$. 其向心加速度 $|\vec{a}_N|$ 則為 $(2at+b)^2R$.

其加速度之大小則為 $|\vec{a}_T|^2+|\vec{a}_N|^2=R\sqrt{(2at+b)^4+4a^2}$

(2) 地球表面上之點.

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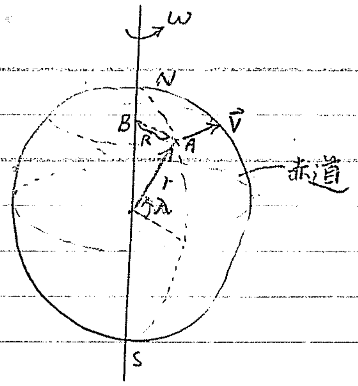


地球自轉之週期為一日 = 86400 秒

利用 $\omega = \frac{2\pi}{P} = \frac{2\pi}{86400}$ (秒)$^{-1}$

$= 7.27$ (秒)$^{-1}$

半徑 $R = r\cos\lambda$ 此處λ是該點之緯度，r是地球之半徑

其速度之大小為 $v = R\omega$

其切線方向之加速度為 0 因為其速度之大小不變，其法向方面之加速度

為 $R\omega^2 = r\cos\lambda\,\omega^2$

$r = 6.35\times 10^6$ m.

在赤道($\lambda=0$)所受之向心力加速度最大，$a = r\omega^2 = 3.34\times10^{-2}\ msec^{-2}$

六、習題

1. 一質點原來靜止($\theta(t=0)=0$, $\omega(t=0)=0$)，受一角加速度

$\alpha(t) = 120t^2 - 48t + 16$

在一半徑為 1.3 米之圓周上運動

(1) 求 $\theta(t)$ [N]

(2) 求 $\omega(t)$ [E]

(3) 求在 $t=5$ 秒時該質點沿切線方向之加速度 [P]

(4) 求在 $t=5$ 秒時該質點沿法線方向之加速度 [B]

2. 一輪子在6秒中內由靜止均勻加速至每分鐘轉 200 轉

(1)求其角加速度 [J] (2) 求在此6秒中內轉動之次數 [D]

當該輪子在此角速度下進行了 t 秒以後，以均勻之角加速度剎車，

而在5秒中內停住

分類：
編號：5
總號：

(3) 求在這段時間內之角加速度 [L]

(4) 求在此五秒中內所轉動之次數 [S]

若由始至終此輪子總共轉動之次數為 3100

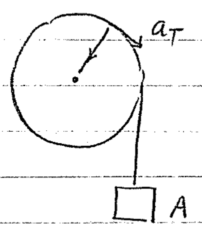
(5) 求 t. [J]

3. 一電子在氫原子中運轉，其半徑為 5×10^{-9} cm，其週期為 1.5×10^{-16} 秒

(1) 求電子之速度的大小 [A]

(2) 求其向心加速度 [X]

4.



若飛輪之半徑為 0.5 ft. 其角速度在 5 秒中內由 20 rad/sec 均勻地加速至 30 rad/sec.

一點 A 當開始加速時位於飛輪之頂點

(1) 求當加速 2.5 秒鐘以後該點之速度（大小及方向）為何 [V]

(2) 其時該點之切線加速度為何? [C]

(3) 其時該點之法線加速度為何? [M]

(4) 其時之總加速度為何? [Q]

5. 一個低飛 ($\underbrace{R_s}_{\text{衛星軌道之半徑}} \sim \overbrace{R_e}^{\text{地球軌道之半徑}}$) 它之切線速度之大小為

$v=(gR_e)^{1/2}$ 此處 g 是重力加速度，R_e 是地球之半徑

(1) 證明 $v=(gR_e)^{1/2}$ 式中兩邊單位相同

(2) 衛星之角速度為何? [G] ~~(由 (2) 至 (7) 答案以符号表出)~~

分類：
編號： 6
總號：

(3) 衛星之切線加速度為何? [I]

(4) 衛星之角加速度為何? [U]

(5) 衛星之向心加速度為何? [O]

(6) 此向心加速度由何而來? [F]

(7) 求此衛星之運轉週期 [K]

地球之半徑為 6.37×10^6 米

(8) 求此衛星沿切線速度大小 [W]

(9) 求此衛星角速度之大小 [H]

(10) 求此衛星之週期 [R]

七. 答案

[A] 2.094×10^8 cm/sec

[B] 2.61×10^7 cm/sec^2

[C] 1 ft/sec

[D] 10 轉

[E] $\omega(t) = 40t^3 - 24t^2 + 16t$

[F] 重力

[G] $\sqrt{g/R_e}$

[H] 1.24×10^{-3} sec^{-1}

[I] 0

[J] 922 sec

[K] $2\pi \sqrt{\frac{R_e}{g}}$

[L] $-\frac{4}{3}\pi$ sec^{-1}

[M] $312.5\ ft/sec^2$

[N] $\theta(t) = 10t^4 - 8t^3 + 8t^2$

[O] g

[P] $3608.8\ m/sec^2$

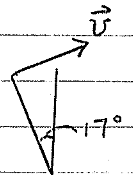
[Q] $a \sim a_N$

[R] $5.07 \times 10^3\ sec \sim 84.4\ min$

[S] $\frac{50}{3}$ 轉

[T] $\frac{10}{9}\pi\ sec^{-2}$

[U] 0

[V] 

$|\vec{v}| = 12.5\ ft/sec$

[W] $7.9 \times 10^3\ m/sec$

[X] $0.882 \times 10^{25}\ cm/sec$

1

Motion at Constant Acceleration

We had for the average acceleration

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

If initial velocity is $\vec{v}_0$, then the velocity after time t is

$$\vec{v} = \vec{v}_0 + \vec{a}\,t \qquad (1)$$

for constant acceleration.

The x, y, and z components are:

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ v_y &= v_{0y} + a_y t \\ v_z &= v_{0z} + a_z t \end{aligned} \qquad (2)$$

Using arguments as in the one-dimensional case the position vector becomes

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}\,t^2 \qquad (3)$$

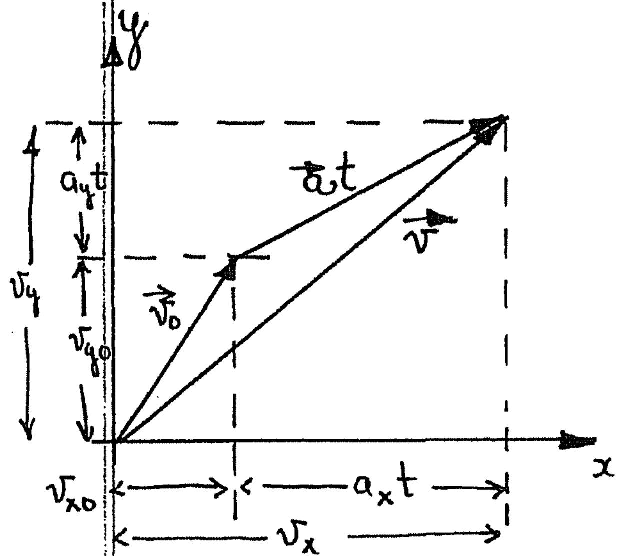
In components:

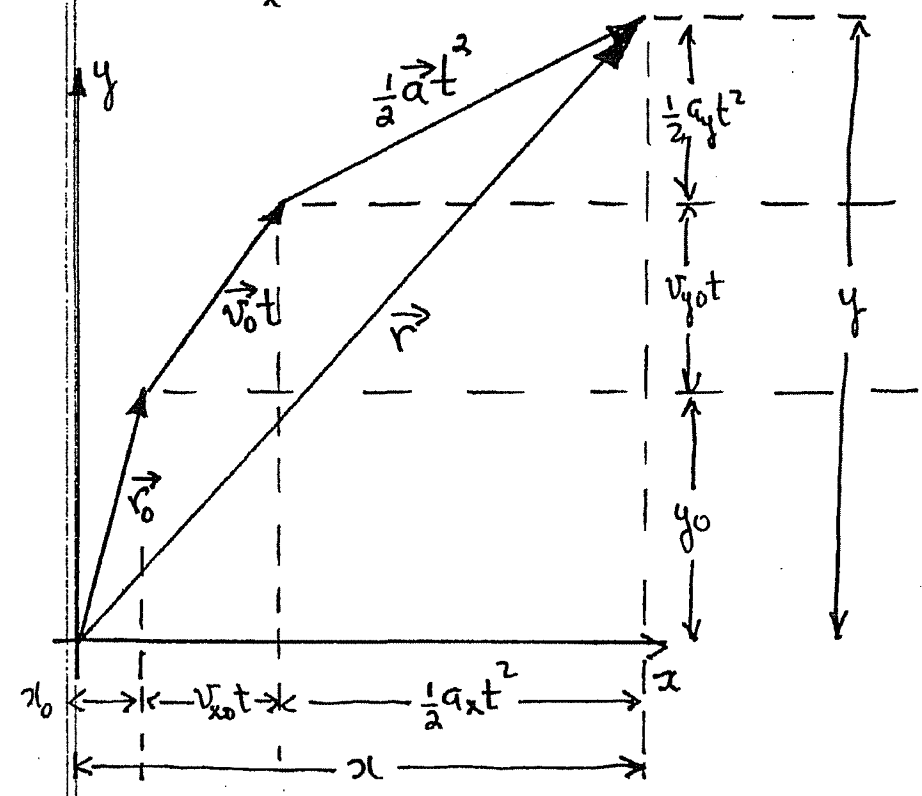
$$\begin{aligned} x(t) &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y(t) &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ z(t) &= z_0 + v_{0z}t + \frac{1}{2}a_z t^2 \end{aligned} \qquad (4)$$

In equations ② and ④ the various components of the motion proceed independent of each other. Time is common ~~to all~~ to all motions.

x-velocity affected by x-acceleration

x-position affected by initial x-velocity and x-acceleration





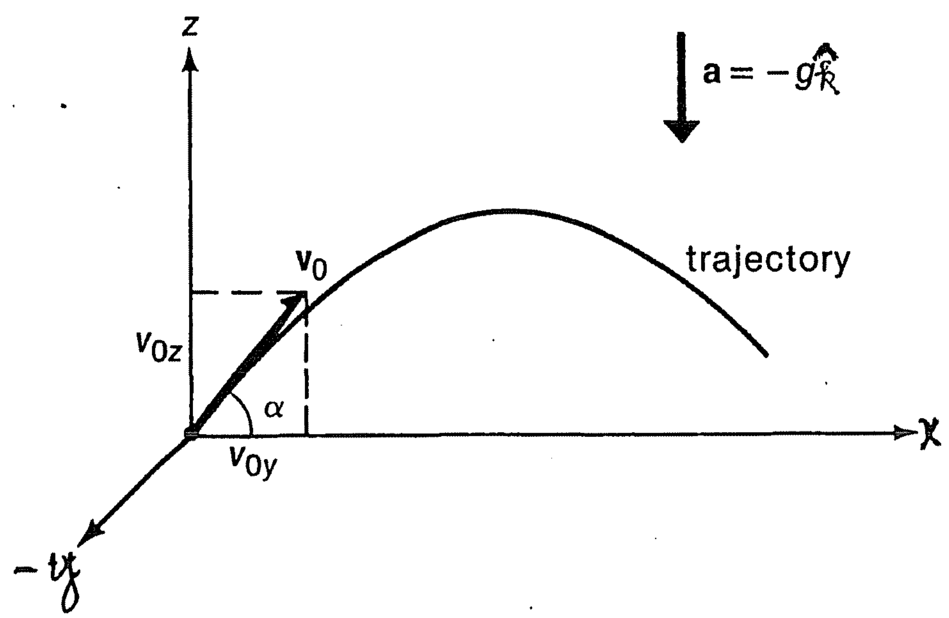
Motion of Projectiles

- Study of motion of a body which is given some initial velocity and starts from some initial position and follows a path determined by the effect of the gravitational acceleration and by air resistance. Projectile path is called its trajectory.
- Freely falling body near the earth's surface experiences downward acceleration with

$$g = 9.81 \ m/s^2$$

Ideal Model :

- Trajectories of short range so that $\vec{g}$, magnitude and direction are constant.
- Neglect air resistance
- Neglect effects due to rotation of earth
- Constant acceleration motion !!



Assume

- Motion is in the xz-plane
- z-axis is in the direction of the upward vertical
- x-axis is in the direction of the horizontal velocity

$a_x = 0$
$a_y = 0$
$a_z = -g = -9.81\ m/s^2$ [acceleration opposite to $+z$]

$v_{oy} = 0$

<u>Equations of Motion</u>:

$x(t) = x_o + v_{ox}t$ $\qquad$ $v_x(t) = v_{ox}$
$y(t) = 0$ $\qquad$ $v_y(t) = 0$
$z(t) = z_o + v_{oz}t - \frac{1}{2}gt^2$ $\qquad$ $v_z(t) = v_{oz} - gt$

- Motions are decoupled. Motion along each axis is independent of motions along other axes. (Experimental Fact). Can treat them separately.
- Chosen coordinates such that y, v_{oy} and a_y are initially zero and remain that way.
- 2-D motion.
- 1-D Vertical motion is a special case.

$x(t)$, $y(t)$, and $z(t)$ give complete <u>trajectory</u> of particle as a <u>function of time</u>. i.e. World lines along each axis.

What is the mathematical form of a ballistic trajectory?

$x(t) = x_0 + v_{0x} t$

Solve for t:

$$t = \frac{x - x_0}{v_{0x}}$$

Substitute for t in $z(t)$

$$z = z_0 + v_{0z}\left(\frac{x - x_0}{v_{0x}}\right) - \frac{1}{2} g \left(\frac{x - x_0}{v_{0x}}\right)^2$$

This can be written as

$$z = A + Bx + Cx^2$$

A, B, C constants

<u>"Eq. of a Parabola"</u>

Ballistic Motion

- Missile
- Bullet
- Ball
- Bomb

} Neglect air resistance.

- Independent of the initial conditions the trajectory will be part of a parabola.

Calculate:

- Maximum Height
- Time-of-Flight
- Range

Assume:

$z(0) = 0$
$x(0) = 0$ } choose coordinates so particle at origin at $t = 0$.

$v_z(0) = v_{z0}$ - Initial vertical velocity component
$v_x(0) = v_{x0}$ - Initial horizontal velocity component

- Motion in the xz-plane.

When projectile is at maximum height, $v_z = 0$. It is moving horizontally.

$$\therefore \quad 0 = v_{0z} - g t_{max}$$

$$t_{max} = \left(\frac{v_{0z}}{g}\right) \qquad \text{[Time to maximum height]}$$

$$z(t_{max}) = z_{max} = v_{0z}\, t_{max} - \frac{1}{2} g\, t_{max}^2$$

$$= v_{0z}\left(\frac{v_{0z}}{g}\right) - \frac{1}{2} g \left(\frac{v_{0z}}{g}\right)^2$$

$$z_{max} = \frac{1}{2} \frac{v_{0z}^2}{g}$$

<u>What is the Range?</u>

$$0 = v_{0z} t - \frac{1}{2} g t^2 \qquad [z = 0 \text{ at impact}]$$

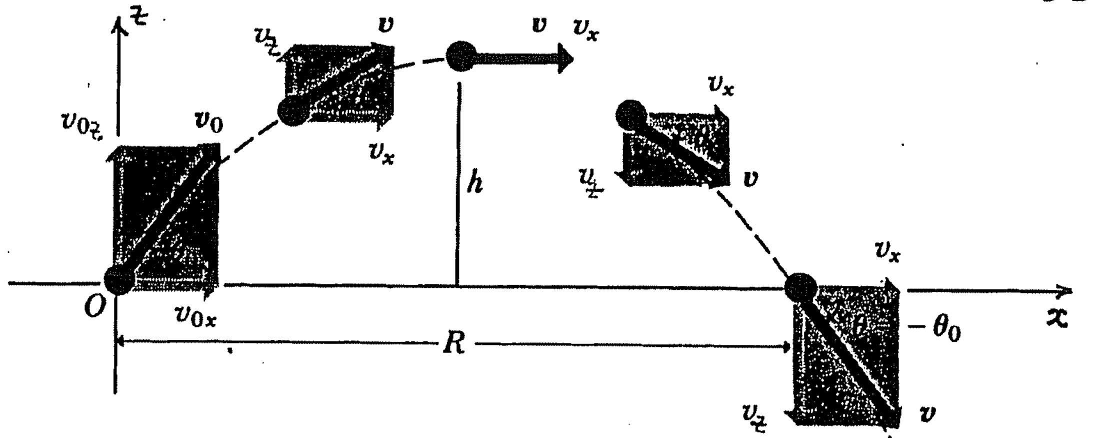
solving $\quad t = 0 \quad \leftarrow$ Starting Time

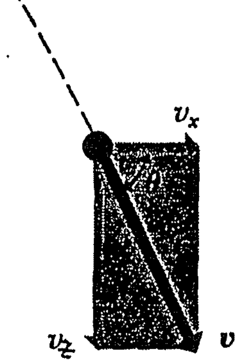
$\quad t_F = \dfrac{2 v_{0z}}{g} \quad \leftarrow$ Flight Time $= 2 t_{max}$

Range = Horizontal Velocity x Flight Time

$$R = v_{0x}\, t_F$$

$$= \frac{2 v_{0x} v_{0z}}{g}$$





What is the velocity, v, of the projectile at any point in the flight?

$$\vec{v}(t) = v_x(t)\,\hat{i} + v_z(t)\,\hat{k}$$

$$|v| = \sqrt{v_x^2 + v_z^2} \qquad \text{[Magnitude]}$$

$$\tan\theta = \frac{v_z(t)}{v_x(t)} \qquad \text{[Direction]}$$

<u>$\vec{v}$ at Maximum Height:</u>

$$v_x(t) = v_{ox} \qquad \text{[A constant during motion]}$$

$$v_z(t = t_{max}) = v_{oz} - g\left(\frac{v_{oz}}{g}\right)$$

$$= 0 \qquad \text{[Trajectory is horizontal]}$$

$$\vec{v} = v_{ox}\,\hat{i}$$

<u>$\vec{v}$ at Maximum Distance = Range:</u>

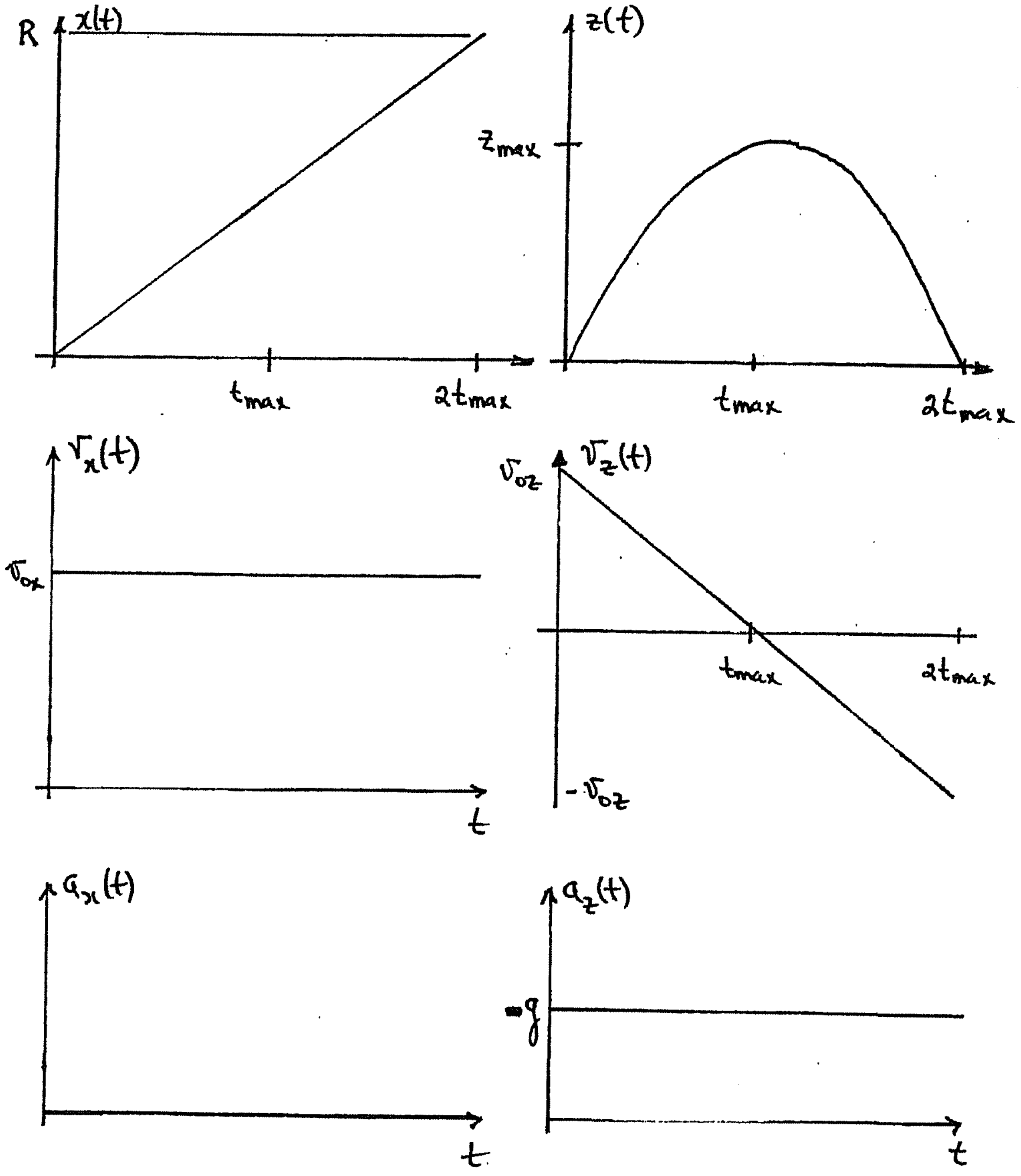
$$v_x(t) = v_{ox}$$

$$v_z\left(t = \frac{2v_{oz}}{g}\right) = v_{oz} - g\left(\frac{2v_{oz}}{g}\right) = -v_{oz}$$

$$\vec{v} = v_{ox}\,\hat{i} - v_{oz}\,\hat{j}$$

$|v| \equiv$ same as at the origin at $t=0$.

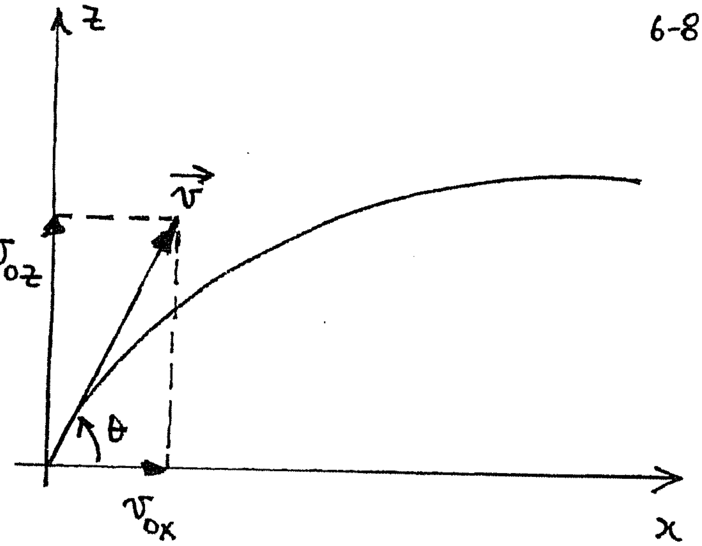
Direction $\quad \theta = -\theta_o$ instead of $+\theta_o$ at $t=0$



Where is $|v|$ the least during the motion?

Example

-Projectile launched
- Elevation angle θ
- Initial speed v_0



$$v_{0x} = v_0 \cos\theta$$
$$v_{0z} = v_0 \sin\theta$$

Ballistic Eq. of Motion:

$$x(t) = (v_0 \cos\theta)t + x_0$$
$$z(t) = z_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$v_x(t) = \frac{dx}{dt} = v_0 \cos\theta$$ [constant]

$$v_z(t) = \frac{dz}{dt} = v_0 \sin\theta - gt$$ [varies in flight]

$$z_{max} = \frac{v_0^2 \sin^2\theta}{2g}$$ Maximum Height

$$t_{Flight} = \frac{2v_0 \sin\theta}{g}$$ Flight-Time

$$x_{max} = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$ Range

- All these specific results (height, time, range) apply only if launch and impact points are at the same height, z.
- Special cases must be treated carefully.

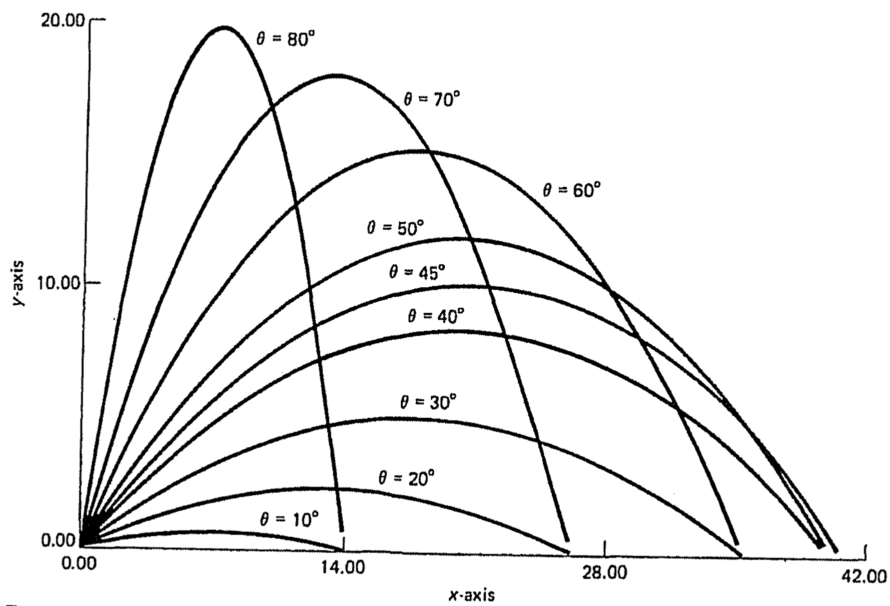
What angle θ gives maximum range?

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

Max value of $\sin 2\theta = 1$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$



Figure

These parabolic trajectories are for the same initial speed (20 m/s) and launch angles at 10° intervals.

Calculus: $\frac{dx_{max}}{d\theta} = \frac{v_0^2}{g} 2 \cos 2\theta = 0 \implies \theta = 45^\circ$

Look at $d^2x/d\theta^2$ to see if max. or min.

Example

What is the range of a 22-calibre rifle bullet?

$v_0 \cong 330$ m/s [Expt. in class]

$\theta = 45°$ [Maximizes range]

$$x_{max} = \frac{v_0^2 \sin 2\theta}{g} = \frac{330^2 \times 1}{9.81} = 11.1 \text{ km.}$$

- Air resistance neglected. Poor assumption for a light high speed projectile.

Example: Baseball

$v_0 \sim 90$ mi/h [60 mph = 88 ft/s]

$\sim \frac{90}{60} \times 88 = 132$ ft/s.

$\theta = 45°$

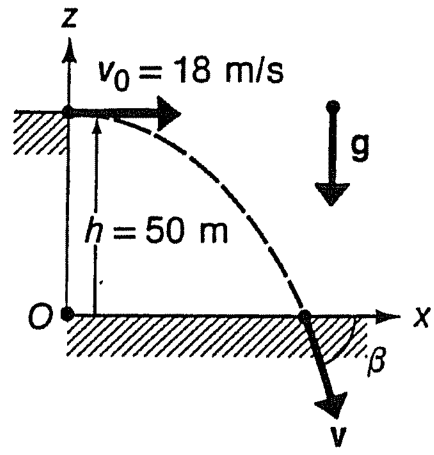
$$x_{max} = \frac{v_0^2 \sin 2\theta}{g} = \frac{132^2 \times 1}{32.2} = 541 \text{ ft.}$$

- Neglects air resistance.

Example

Ball kicked horizontally at 18 m/s off a 50 m high cliff.

- Time to impact?
- Speed at impact?
- Impact point?
- Angle at impact?



$x(t) = v_0 \cos\theta \, t$

$\quad = v_0 t \qquad ① \qquad \theta = 0^\circ$

$z(t) = z_0 + v_{0z} t - \frac{1}{2} g t^2 \qquad ②$

$\quad = H - \frac{1}{2} g t^2 \qquad ② \qquad v_{0z} = 0, \; z_0 = H$ (see coord. system)

At impact we must have $z = 0$.

Solving Eq. ② $\qquad T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 50}{9.81}} = 3.19\ s.$

$x(T) = 18 \times 3.19 = 57.42 \qquad$ [Eq. ①]

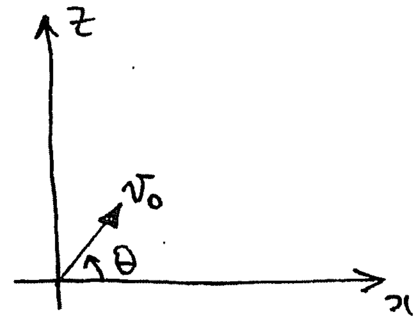
$v_x(T) = \frac{dx}{dt} = v_0 = 18\ m/s \qquad$ [Independent of Time]

$v_z(T) = \frac{dz}{dt} = -gT = -9.81 \times 3.19 = -31.26\ m/s.$

$\tan\beta = \frac{v_z}{v_x} = \frac{-31.26}{18.0} \qquad \beta = -60.1^\circ$

$|v| = \sqrt{v_x^2 + v_z^2} = \sqrt{18^2 + 31.26^2} = 36.1\ m/s \qquad$ [Speed]

Projectile Problem-Solving-Strategy



$$x(t) = x_0 + (v_0 \cos\theta_0)t$$
$$z(t) = z_0 + (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

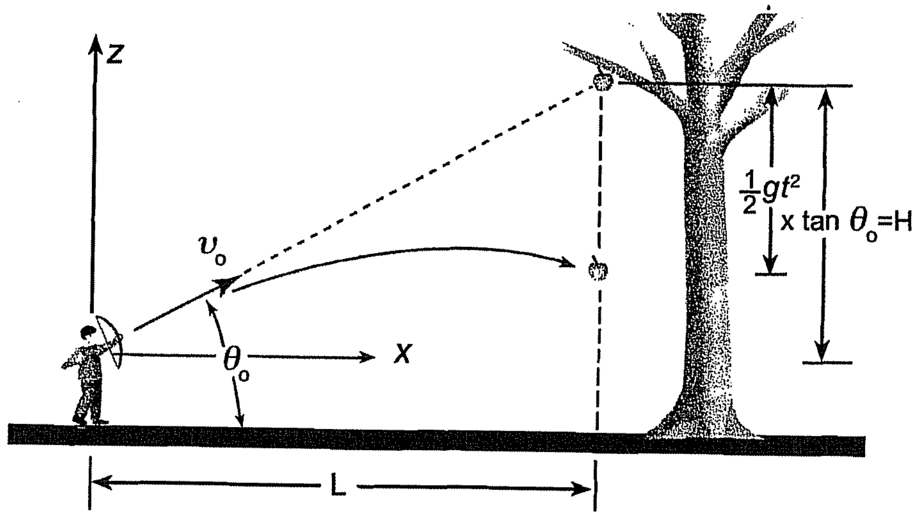
$$v_x(t) = v_0 \cos\theta_0$$
$$v_z(t) = v_0 \sin\theta_0 - gt$$

1. Define your coordinate axes. Sketch axes, label positive directions and show location of origin.
2. List known and onknown quantities. In some problems the initial velocity is given (mag. + dir.) and using Eq. o motion you can find coord. and velocity at any other time. In other problems you may know two points on the trajectory and asked to determine the initial velocity. Know what is given and what is onknown!!
3. Learn how to go from prose into symbols.
   When → Time, t
   Where → x, z
   Velocity → v_x, v_z, etc.
4. Have a picture of ballistic trajectory in mind.
   Highest point $v_z = 0$. → calculate t.
   Range $z = 0$ [Or some fixed value] calculate t.

5. Take a hard look at your results. Do they seem to make sense? Are they within the general range of magnitudes you expected? Is the sign correct? Are the resulting units correct?

Example

An arrow is fired at an apple at the same instant it drops from the tree. How must the arrow be aimed so as to hit the apple?



Trajectory of an arrow shot directly at a freely falling apple.

$\left.\begin{matrix} x_D(t) \\ z_D(t) \end{matrix}\right\}$ Trajectory of dart. $\vec{r}_D = x_D(t)\,\hat{i} + z_D(t)\,\hat{k}$

$\left.\begin{matrix} x_A(t) \\ z_A(t) \end{matrix}\right\}$ Trajectory of apple. $\vec{r}_A = L\,\hat{i} + z_A(t)\,\hat{k}$

Let impact occur at time $t = T$

Then for dart to strike apple we must have

$$\left.\begin{matrix} z_D(T) = z_A(T) \\ x_D(T) = x_A(T) \end{matrix}\right\} \text{Conditions}$$

Eq's of motion

Apple:

$$x_A(t) = L \qquad \text{[Constant x-location]}$$

$$z_A(t) = H - \frac{1}{2} g t^2$$

Dart:

$$x_D(t) = v_{0x} t$$

$$z_D(t) = v_{0z} t - \frac{1}{2} g t^2$$

At time $t = T$ we must have that

$$L = v_{0x} T \qquad \text{[x-motion]}$$

$$\therefore T = \frac{L}{v_{0x}} \qquad \text{[Time to impact]}$$

Also $v_{0z} T - \frac{1}{2} g T^2 = H - \frac{1}{2} g T^2$ [z-motion]

substituting for T:

$$v_{0z} \frac{L}{v_{0x}} = H$$

$$\boxed{\frac{v_{0z}}{v_{0x}} = \frac{H}{L}}$$

$$v_{0x} = v_0 \cos\theta$$

$$v_{0z} = v_0 \sin\theta$$

$$\therefore \frac{H}{L} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

- Aiming is independent of v_0! However v_0 must be great enough to reach target range.
- Angle θ represents sighting at target before it starts to fall.

分類:
編號:
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Uniform Circular Motion

$r^2 = R^2$

$x(t) = R\cos\omega t \quad , \quad y(t) = R\sin\omega t$

$\Leftarrow \dot{x}(t) = \underset{v_x}{\underset{\parallel}{-\omega R\sin\omega t}} \; , \quad \dot{y}(t) = \underset{v_y}{\underset{\parallel}{\omega R\cos\omega t}} \qquad v^2 = \omega^2 R^2$

$\ddot{x}(t) = -\omega^2 R\cos\omega t, \quad \ddot{y}(t) = -\omega^2 R\sin\omega t$

$\vec{a} = -\omega^2 R^2 \frac{1}{R}[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}]$

$= -\frac{v^2}{R}[\hat{r}]$

$\downarrow$

unit vector along

Polar coordinate $\quad \vec{x} \Rightarrow \overset{(x,y)}{(r,\theta)}$

$\hookrightarrow$ both

$x = r\cos\theta$

$y = r\sin\theta \qquad \hat{r} =$

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$\vec{r} = x\hat{i} + y\hat{j}$

$= r\underbrace{\cos\theta}\,\hat{i} + r\sin\theta\,\hat{j}$

$= r\hat{r} \qquad \boxed{\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}}$

$\boxed{\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}}$

$\hat{r}\cdot\hat{\theta} = 0$

$\boxed{\vec{r} = r\hat{r}}$

$\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}$

$= \dot{\theta}\hat{\theta}$

$\vec{v} = \boxed{\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}}}$

$\dot{\hat{\theta}} = -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j}$

$= -\dot{\theta}\hat{r}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\vec{a} = \ddot{r}\hat{r} + \dot{r}\underset{r\dot{\theta}\hat{\theta}}{\underset{\parallel}{\dot{\hat{r}}}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\underset{-\dot{\theta}\hat{r}}{\underset{\parallel}{\dot{\hat{\theta}}}}$

$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{0}\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

$R\dot{\theta}^2 \quad \hat{\theta}$

$r = R \qquad \underset{R}{\parallel}\,\omega^2\,\hat{\theta}$

Polar Coordinate

$\vec{r} = r\hat{r}$ both r, $\hat{r}$ are time-dependent

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Graph.

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\hat{i}$, $\hat{j}$ are time dependent

$$\vec{V} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}$$

$$= \dot{\theta}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right]$$

$$= \dot{\theta}\,\hat{\theta}$$

$$\underline{V^2 = (\dot{r})^2 + r^2\dot{\theta}^2}$$

$$\dot{\hat{\theta}} = -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j}$$

$$= -\dot{\theta}\,\hat{r}$$

$$\vec{a} = \dot{\vec{V}} = \ddot{r}\hat{r} + \underset{\substack{\parallel \\ \dot{r}\dot{\theta}\hat{\theta}}}{\dot{r}\dot{\hat{r}}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + \underset{\substack{\parallel \\ -r\dot{\theta}^2\hat{r}}}{r\dot{\theta}\dot{\hat{\theta}}}$$

$$= \hat{r}\left[\ddot{r} - r\dot{\theta}^2\right] + \hat{\theta}\left[\underbrace{\dot{r}\dot{\theta} + \dot{r}\dot{\theta}}_{2\dot{r}\dot{\theta}} + r\ddot{\theta}\right]$$

For $r = R$, $\theta = \omega t$ uniform circular motion

$$\vec{r} = R\hat{r}$$

$$\vec{V} = 0\hat{r} + \underbrace{R\omega}_{v}\hat{\theta}$$

$$\vec{a} = -R\omega^2\hat{r}$$

Check

If $r = R$ but $\theta(t)$, then

$$\vec{a}_N = -\hat{r}\,R\dot{\theta}^2 = -\frac{v^2}{R}\hat{r}$$

$$\vec{a}_T = R\ddot{\theta} = \frac{dv}{dt}$$

$$v = R\dot{\theta}$$

$$\frac{dv}{dt} = R\ddot{\theta}$$

分類:	
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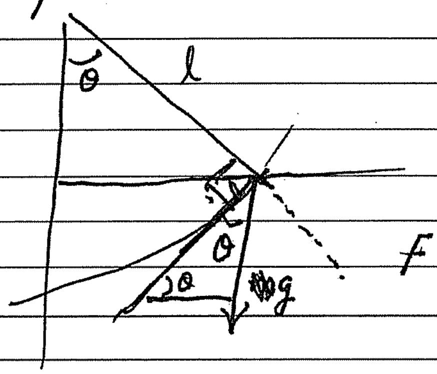
$$\vec{a} = a_T \hat{\theta} + a_N \hat{r}$$

$$a_T = \vec{a}\cdot\hat{\theta} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$a_N = \vec{a}\cdot\hat{r} = \ddot{r} - r\dot{\theta}^2$$

Example



$l = R$

$$\vec{a} = -g\sin\theta \quad \downarrow$$

沿切線方向

$$R\ddot{\theta} = -g\sin\theta$$

$$\cancel{l}\,\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta}$$

$$= -\omega^2\sin\theta$$

$$= -\omega^2\theta$$

$\downarrow$

simple harmonic

Kinematics

From Wikipedia, the free encyclopedia

Kinematics is the branch of classical mechanics that describes the motion of points, bodies (objects) and systems of bodies (groups of objects) without consideration of the causes of motion.[1][2][3] The term is the English version of A.M. Ampère's *cinématique,*[4] which he constructed from the Greek κίνημα, *kinema* (movement, motion), derived from κινεῖν, *kinein* (to move).[5] [6]

The study of *kinematics* is often referred to as the *geometry of motion.*[7] (See analytical dynamics for more detail on usage). To describe motion, kinematics studies the trajectories of points, lines and other geometric objects and their differential properties such as velocity and acceleration. Kinematics is used in astrophysics to describe the motion of celestial bodies and systems, and in mechanical engineering, robotics and biomechanics[8] to describe the motion of systems composed of joined parts (multi-link systems) such as an engine, a robotic arm or the skeleton of the human body.

The study of kinematics can be abstracted into purely mathematical expressions. For instance, rotation can be represented by elements of the unit circle in the complex plane. Other planar algebras are used to represent the shear mapping of classical motion in absolute time and space and to represent the Lorentz transformations of relativistic space and time. By using time as a parameter in geometry, mathematicians have developed a science of kinematic geometry.

The use of geometric transformations, also called rigid transformations, to describe the movement of components of a mechanical system simplifies the derivation of its equations of motion, and is central to dynamic analysis.

Kinematic analysis is the process of measuring the kinematic quantities used to describe motion. In engineering, for instance, kinematic analysis may be used to find the range of movement for a given mechanism, and, working in reverse, kinematic synthesis designs a mechanism for a desired range of motion.[9] In addition, *kinematics* applies algebraic geometry to the study of the mechanical advantage of a mechanical system, or mechanism.

Contents

Kinematics of a particle trajectory

Particle kinematics is the study of the properties of the trajectory of a particle. The position of a particle is defined to be the coordinate vector from the origin of a coordinate frame to the particle. For example, consider a tower 50 m south from your home, where the coordinate frame is located at your home, such that East is the x-direction and North is the y-direction, then the coordinate vector to the base of the tower is **r**=(0, -50, 0). If the tower is 50 m high, then the coordinate vector to the top of the tower is **r**=(0, -50, 50).

Usually a three dimensional coordinate systems is used to define the position of a particle. However if the particle is constrained to lie in a plane or on a sphere, a two dimensional coordinate system can be used. All observations in physics are incomplete without the reference frame being specified.

The position vector of a particle is a vector drawn from the origin of the reference frame to the particle. It expresses both the distance of the point from the origin and its direction from the origin. In three dimensions, the position of point *P* can be expressed as

$$\mathbf{P} = (x_P, y_P, z_P) = x_P\vec{i} + y_P\vec{j} + x_P\vec{k},$$

where x_P, y_P, and z_P are the Cartesian coordinates and i, j and k are the unit vectors along the x, y, and z coordinate axes, respectively. The magnitude of the position vector $|\mathbf{P}|$ gives the distance between the point P and the origin.

$$|\mathbf{P}| = \sqrt{x_P^{\ 2} + y_P^{\ 2} + z_P^{\ 2}}.$$

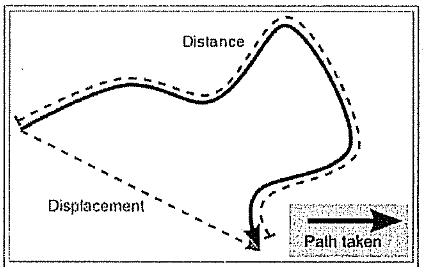
The direction cosines of the position vector provide a quantitative measure of direction. It is important to note that the position vector of a particle isn't unique. The position vector of a given particle is different relative to different frames of reference.

The *trajectory* of a particle is a vector function of time, $\mathbf{P}(t)$, which defines the curve traced by the moving particle, given by

$$\mathbf{P}(t) = x_P(t)\vec{i} + y_P(t)\vec{j} + z_P(t)\vec{k},$$

where the coordinates x_P, y_P, and z_P are each functions of time.

Velocity and speed



The distance traveled is always greater than or equal to the displacement.

The velocity of a particle is a vector that tells about the direction and magnitude of the rate of change of the position vector, that is, how the position of a point changes with each instant of time. Consider the ratio of the difference of two positions of a particle divided by the time interval, which is called the average velocity over that time interval. This average velocity is defined as

$$\overline{\mathbf{V}} = \frac{\Delta \mathbf{P}}{\Delta t} ,$$

where $\Delta\mathbf{P}$ is the difference in the position vector over the time interval Δt.

In the limit as the time interval Δt becomes smaller and smaller, the average velocity becomes the time derivative of the position vector,

$$\mathbf{V} = \lim_{\Delta t\to 0} \frac{\Delta \mathbf{P}}{\Delta t} = \frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{x}_p\vec{i} + \dot{y}_P\vec{j} + \dot{z}_P\vec{k}.$$

Thus, velocity is the time rate of change of position, and the dot denotes the derivative with respect to time. Furthermore, the velocity is tangent to the trajectory of the particle.

As a position vector itself is frame dependent, therefore its velocity is also dependent on the reference frame.

The speed of an object is the magnitude $|\mathbf{V}|$ of its velocity. It is a scalar quantity:

$$|\mathbf{V}| = |\dot{\mathbf{P}}| = \frac{ds}{dt},$$

where s is the arc-length measured along the trajectory of the particle. This arc-length traveled by a particle over time is a non-decreasing quantity. Hence, ds/dt is non-negative, which implies that speed is also non-negative.

Acceleration

The acceleration of a particle is the vector defined by the rate of change of the velocity vector. The average acceleration of a particle over a time interval is defined as the ratio

$$\overline{\mathbf{A}} = \frac{\Delta \mathbf{V}}{\Delta t} ,$$

where $\Delta\mathbf{V}$ is the difference in the velocity vector and Δt is the time interval.

The acceleration of the particle is the limit of the average acceleration as the time interval approaches zero, which is the time derivative,

$$\mathbf{A} = \lim_{\Delta t\to 0} \frac{\Delta \mathbf{V}}{\Delta t} = \frac{d\mathbf{V}}{dt} = \dot{\mathbf{V}} = \ddot{\mathbf{P}} = \ddot{x}_p\vec{i} + \ddot{y}_P\vec{j} + \ddot{z}_P\vec{k}.$$

Thus, acceleration is the second derivative of the position vector that defines the trajectory of a particle.

Relative position vector

A relative position vector is a vector that defines the position of a particle relative to another particle. It is the difference in position of the two particles.

If point A has position $\mathbf{P}_A = (x_A,y_A,z_A)$ and point B has position $\mathbf{P}_B = (x_B,y_B,z_B)$, the displacement $\mathbf{R}_{B/A}$ of B from A is given by

$$\mathbf{R}_{B/A} = \mathbf{P}_B - \mathbf{P}_A = (x_B - x_A, y_B - y_A, z_B - z_A).$$

Geometrically, the relative position vector $\mathbf{R}_{B/A}$ is the vector from point A to point B. The values of the coordinate vectors of points vary with the choice of coordinate frame, however the relative position vector between a pair of points has the same length no matter what coordinate frame is used and is said to be *frame invariant.*

To describe the motion of a particle B relative to another particle A, we notice that the position B can be formulated as the position of A plus the position of B relative to A, that is

$$\mathbf{P}_B = \mathbf{P}_A + (\mathbf{P}_B - \mathbf{P}_A) = \mathbf{P}_A + \mathbf{R}_{B/A}.$$

Relative velocity

Main article: Relative velocity

The relations between relative positions vectors become relations between relative velocities by computing the time-derivative. The second time derivative yields relations for relative accelerations.

For example, let the particle B move with velocity $\mathbf{V}_B$ and particle A move with velocity $\mathbf{V}_A$ in a given reference frame. Then the velocity of B relative to A is given by

$$\mathbf{V}_{B/A} = \mathbf{V}_B - \mathbf{V}_A.$$

This can be obtained by computing the time derivative of the relative position vector $\mathbf{R}_{B/A}$.

This equation provides a formula for the velocity of B in terms of the velocity of A and its relative velocity,

$$\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{B/A}.$$

With a large velocity $\mathbf{V}$, where the fraction $\mathbf{V}/c$ is significant, c being the speed of light, another scheme of relative velocity called rapidity, that depends on this ratio, is used in special relativity.

Particle trajectories under constant acceleration

Newton's laws state that a constant force acting on a particle generates a constant acceleration. For example, a particle in a parallel gravity field experiences a force acting downwards that is proportional to the constant acceleration of gravity, and no force in the horizontal direction. This is called projectile motion.

If the acceleration vector $\mathbf{A}$ of a particle P is constant in magnitude and direction, the particle is said to be undergoing *uniformly accelerated motion.* In this case, the trajectory $\mathbf{P}(t)$ of the particle can be obtained by integrating the acceleration $\mathbf{A}$ with respect to time.

The first integral yields the velocity of the particle,

$$\mathbf{V}(t) = \int_0^t \mathbf{A}dt = \mathbf{A}t + \mathbf{V}_0.$$

A second integration yields its trajectory,

$$\mathbf{P}(t) = \int_0^t \mathbf{V}(t)dt = \int (\mathbf{A}t + \mathbf{V}_0)dt = \tfrac{1}{2}\mathbf{A}t^2 + \mathbf{V}_0 t + \mathbf{P}_0.$$

Additional relations between displacement, velocity, acceleration, and time can be derived. Since $\mathbf{A} = (\mathbf{V} - \mathbf{V}_0)/t$,

$$\mathbf{P}(t) = \mathbf{P}_0 + \left(\frac{\mathbf{V} + \mathbf{V}_0}{2}\right) t.$$

By using the definition of an average, this equation states that when the acceleration is constant average velocity times time equals displacement.

A relationship without explicit time dependence may also be derived using the relation $\mathbf{A}t = \mathbf{V} - \mathbf{V}_0$,

$$(\mathbf{P} - \mathbf{P}_0) \cdot \mathbf{A}t = (\mathbf{V} - \mathbf{V}_0) \cdot \frac{\mathbf{V} + \mathbf{V}_0}{2}t \ ,$$

where $\cdot$ denotes the dot product. Divide both sides by t and expand the dot-products to obtain,

$$2(\mathbf{P} - \mathbf{P}_0) \cdot \mathbf{A} = |\mathbf{V}|^2 - |\mathbf{V}_0|^2.$$

In the case of straight-line motion, where $\mathbf{P}$ and $\mathbf{P}_0$ are parallel to $\mathbf{A}$, this equation becomes

$$|\mathbf{V}|^2 = |\mathbf{V}_0|^2 + 2|\mathbf{A}|(|\mathbf{P} - \mathbf{P}_0|).$$

This can be simplified using the notation $|\mathbf{A}|$=a, $|\mathbf{V}|$=v, and $|\mathbf{P}|$=r, so

$$v^2 = v_0^2 + 2a(r - r_0).$$

This relation is useful when time is not known explicitly.

Example: Rectilinear (1D) motion

Consider an object that is fired directly upwards and falls back to the ground so that its trajectory is contained in a straight line. If we adopt the convention that the upward direction is the positive direction, the object experiences a constant acceleration of approximately $-9.81\ \mathrm{m\ s^{-2}}$. Therefore, its motion can be modeled with the equations governing uniformly accelerated motion.

For the sake of example, assume the object has an initial velocity of +50 m s^{-1}. There are several interesting kinematic questions we can ask about the particle's motion:

How long will it be airborne?

To answer this question, we apply the formula

$$x_f - x_i = v_i t + \tfrac{1}{2}at^2.$$

Since the question asks for the length of time between the object leaving the ground and hitting the ground on its fall, the displacement is zero.

$$0 = v_i t + \tfrac{1}{2}at^2 = t\left(v_i + \tfrac{1}{2}at\right)$$

There are two solutions: the first, $t = 0$, is trivial. The solution of interest is

$$t = -\frac{2v_i}{a} = -\frac{2(50\ \mathrm{m\ s^{-1}})}{-9.81\ \mathrm{m\ s^{-2}}} = 10.2\ \mathrm{s}\,.$$

What altitude will it reach before it begins to fall?

In this case, we use the fact that the object has a velocity of zero at the apex of its trajectory. Therefore, the applicable equation is:

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

If the origin of our coordinate system is at the ground, then x_i is zero. Then we solve for x_f and substitute known values:

$$x_f = \frac{v_f^2 - v_i^2}{2a} + x_i = \frac{(0\ \mathrm{m\ s^{-1}})^2 - (50\ \mathrm{m\ s^{-1}})^2}{2(-9.81\ \mathrm{m\ s^{-2}})} + 0\ \mathrm{m} = 127.55\ \mathrm{m}\,.$$

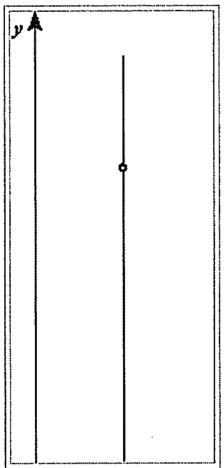


Figure A: An object is fired upwards, reaches its apex, and then begins its descent under a constant acceleration. Note: The equations described here holds for object fired from ground and should not be mistaken with this picture.

What will its final velocity be when it reaches the ground?

To answer this question, we use the fact that the object has an initial velocity of zero at the apex before it begins its descent. We can use the same equation we used for the last question, using the value of 127.55 m for x_i.

$$v_f = \sqrt{v_i^2 + 2a(x_f - x_i)} = \sqrt{(0\ \mathrm{m\ s^{-1}})^2 + 2(-9.81\ \mathrm{m\ s^{-2}})(0\ \mathrm{m} - 127.55\ \mathrm{m})} = 50\ \mathrm{m\ s^{-1}}\,.$$

Assuming this experiment were performed in a vacuum (negating drag effects), we find that the final and initial speeds are equal, a result which agrees with conservation of energy.

Example: Projectile (2D) motion

Suppose that an object is not fired vertically but is fired at an angle θ from the ground. The object will then follow a parabolic trajectory, and its horizontal motion can be modeled independently of its vertical motion. Assume that the object is fired at an initial velocity of 50 m s^{-1} and 30° from the horizontal.

How far will it travel before hitting the ground?

The object experiences an acceleration of −9.81 m s^{-2} in the vertical direction and no acceleration in the horizontal direction. Therefore, the horizontal displacement is

$$\Delta x = x_f - x_i = v_i \cos\theta\ t + \tfrac{1}{2}at^2 = v_i \cos\theta\ t$$

Solving the equation requires finding t. This can be done by analyzing the motion in the vertical direction. If we impose that the vertical displacement is zero, we can use the same procedure we did for rectilinear motion to find t.

$$0 = v_i \sin\theta\ t + \tfrac{1}{2}at^2 = t\left(v_i \sin\theta + \tfrac{1}{2}at\right)$$

We now solve for t and substitute this expression into the original expression for horizontal displacement.

$$\Delta x = v_i \cos\theta \left(\frac{-2v_i \sin\theta}{a}\right) = -\frac{v_i^2 \sin 2\theta}{a} = 220.93\ m$$

Note the use of the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$.

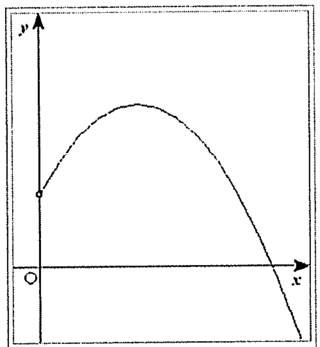


Figure B: An object fired at an angle θ from the ground follows a parabolic trajectory. Note: The equations described here holds for object fired from ground and should not be mistaken with this picture.

Particle trajectories in cylindrical-polar coordinates

See also: Generalized coordinates, Curvilinear coordinates, Orthogonal coordinates, and Frenet-Serret formulas

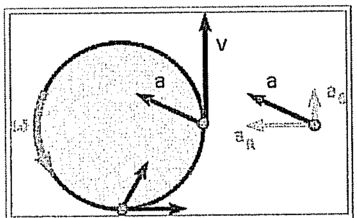


Figure 2: Velocity and acceleration for nonuniform circular motion: the velocity vector is tangential to the orbit, but the acceleration vector is not radially inward because of its tangential component $\mathbf{a}_\theta$ that increases the rate of rotation: $d\omega/dt = |\mathbf{a}_\theta|/R$.

It is often convenient to formulate the trajectory of a particle **P**(t) = (X(t), Y(t) and Z(t)) using polar coordinates in the *X*-*Y* plane. In this case, its velocity and acceleration take a convenient form.

Recall that the trajectory of a particle *P* is defined by its coordinate vector **P** measured in a fixed reference frame *F*. As the particle moves, its coordinate vector **P**(t) traces its trajectory, which is a curve in space, given by

$$\mathbf{P}(t) = X(t)\vec{i} + Y(t)\vec{j} + Z(t)\vec{k},$$

where *i*, *j*, and *k* are the unit vectors along the *X*, *Y* and *Z* axes of the reference frame *F*, respectively.

Consider a particle *P* that moves on the surface of a circular cylinder, it is possible to align the *Z* axis of the fixed frame *F* with the axis of the cylinder. Then, the angle θ around this axis in the *X*-*Y* plane can be used to define the trajectory as,

$$\mathbf{P}(t) = R\cos\theta(t)\vec{i} + R\sin\theta(t)\vec{j} + Z(t)\vec{k}.$$

The cylindrical coordinates for **P**(t) can be simplified by introducing the radial and tangential unit vectors,

$$\mathbf{e}_r = \cos\theta(t)\vec{i} + \sin\theta(t)\vec{j}, \quad \mathbf{e}_t = -\sin\theta(t)\vec{i} + \cos\theta(t)\vec{j}.$$

Using this notation, **P**(t) takes the form,

$$\mathbf{P}(t) = R\mathbf{e}_r + Z(t)\vec{k},$$

where *R* is constant.

Now, in general, the trajectory **P**(t) is not constrained to lie on a circular cylinder, so the radius *R* varies with time, and the trajectory in cylindrical-polor coordinates becomes

$$\mathbf{P}(t) = R(t)\mathbf{e}_r + Z(t)\vec{k}.$$

The velocity vector $\mathbf{V}_P$ is the time derivative of the trajectory **P**(t), which yields,

$$\mathbf{V}_P = \frac{d}{dt}(R(t)\mathbf{e}_r + Z(t)\vec{k}) = \dot{R}\mathbf{e}_r + R\dot{\theta}\mathbf{e}_t + \dot{Z}\vec{k},$$

where

$$\frac{d}{dt}\mathbf{e}_r = \dot{\theta}\mathbf{e}_t.$$

In this case, the acceleration $\mathbf{A}_P$, which is the time derivative of the velocity $\mathbf{V}_P$, is given by

$$\mathbf{A}_P = \frac{d}{dt}(\dot{R}\mathbf{e}_r + R\dot{\theta}\mathbf{e}_t + \dot{Z}(t)\vec{k}) = (\ddot{R} - R\dot{\theta}^2)\mathbf{e}_r + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\mathbf{e}_t + \ddot{Z}(t)\vec{k}.$$

If the radius is constant

If the trajectory of the particle is constrained to lie on a cylinder, then the radius *R* is constant and the velocity and acceleration vectors simplify. The velocity of $\mathbf{V}_P$ is the time derivative of the trajectory $\mathbf{P}(t)$,

$$\mathbf{V}_P = \frac{d}{dt}(R\mathbf{e}_r + Z(t)\vec{k}) = R\dot{\theta}\mathbf{e}_t + \dot{Z}\vec{k}.$$

The acceleration vector becomes

$$\mathbf{A}_P = \frac{d}{dt}(R\dot{\theta}\mathbf{e}_t + \dot{Z}\vec{k}) = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_t + \ddot{Z}\vec{k}.$$

Planar circular trajectories

A special case of a particle trajectory on a circular cylinder occurs when there is no movement along the Z axis, in which case

$$\mathbf{P}(t) = R\mathbf{e}_r + Z_0\vec{k},$$

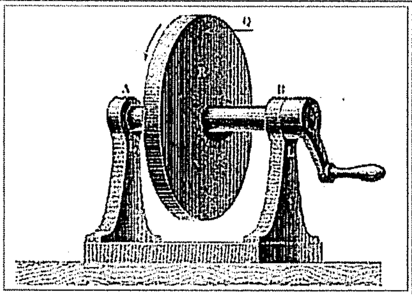
where R and Z_0 are constants. In this case, the velocity $\mathbf{V}_P$ is given by

$$\mathbf{V}_P = \frac{d}{dt}(R\mathbf{e}_r + Z_0\vec{k}) = R\dot{\theta}\mathbf{e}_t = R\omega\mathbf{e}_t,$$

where

$$\omega = \dot{\theta},$$

is the angular velocity of the unit vector $\mathbf{e}_t$ around the z axis of the cylinder.


Each particle on the wheel travels in a planar circular trajectory (Kinematics of Machinery, 1876).[10]

The acceleration $\mathbf{A}_P$ of the particle P is now given by

$$\mathbf{A}_P = \frac{d}{dt}(R\dot{\theta}\mathbf{e}_t) = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_t.$$

The components

$$a_r = -R\dot{\theta}^2, \quad a_t = R\ddot{\theta},$$

are called the *radial* and *tangential components* of acceleration, respectively.

The notation for angular velocity and angular acceleration is often defined as

$$\omega = \dot{\theta}, \quad \alpha = \ddot{\theta},$$

so the radial and tangential acceleration components for circular trajectories are also written as

$$a_r = -R\omega^2, \quad a_t = R\alpha.$$

Point trajectories in a body moving in the plane

The movement of components of a mechanical system is analyzed by attaching a reference frame to each part and determining how the reference frames move relative to each other. If the structural strength of the parts are sufficient then their deformation can be neglected and rigid transformations used to define this relative movement. This brings geometry into the study of mechanical movement.

Geometry is the study of the properties of figures that remain the same while the space is transformed in various ways---more technically, it is the study of invariants under a set of transformations.[11] Perhaps best known is high school Euclidean geometry where planar triangles are studied under congruent transformations, also called isometries or rigid transformations. These transformations displace the triangle in the plane without changing the angle at each vertex or the distances between vertices. Kinematics is often described as applied geometry, where the movement of a mechanical system is described using the rigid transformations of Euclidean geometry.

The coordinates of points in the plane are two dimensional vectors in $\mathbf{R}^2$, so rigid transformations are those that preserve the distance measured between any two points. The Euclidean distance formula is simply the Pythagorean theorem. The set of rigid transformations in an n-dimensional space is called the special Euclidean group on $\mathbf{R}^n$, and denoted *SE(n)*.

Displacements and motion

The position of one component of a mechanical system relative to another is defined by introducing a reference frame, say M, on one that moves relative to a fixed frame, F, on the other. The rigid transformation, or displacement, of M relative to F defines the relative position of the two components. A displacement consists of the combination of a rotation and a translation.

The set of all displacements of M relative to F is called the configuration space of M. A smooth curve from one position to another in this configuration space is a continuous set of displacements, called the motion of M relative to F. The motion of a body consists of a continuous set of rotations and translations.

Matrix representation

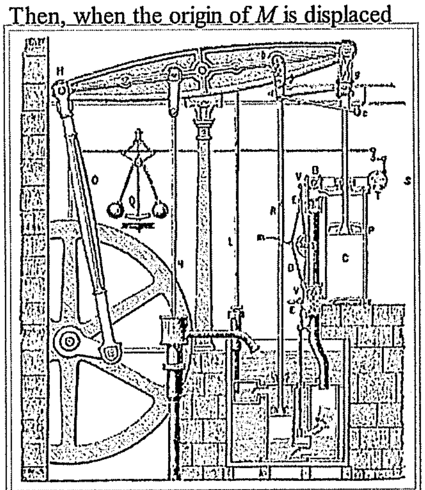
The combination of a rotation and translation in the plane $\mathbf{R}^2$ can be represented by a certain type of 3x3 matrix known as a homogeneous transform. The 3x3 homogenous transform is constructed from a 2x2 rotation matrix A(φ) and the 2x1 translation vector $\mathbf{d}=(d_x, d_y)$, as

$$[T(\phi,\mathbf{d})] = \begin{bmatrix} A(\phi) & \mathbf{d} \\ 0,0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & d_x \\ \sin\phi & \cos\phi & d_y \\ 0 & 0 & 1 \end{bmatrix}.$$

These homogeneous transforms perform rigid transformations on the points in the plane z=1, that is on points with coordinates **p**=(x, y, 1).

In particular, let **p** define the coordinates of points in a reference frame *M* coincident with a fixed frame *F*. Then, when the origin of *M* is displaced by the translation vector **d** relative to the origin of *F* and rotated by the angle φ relative to the x-axis of *F*, the new coordinates in *F* of points in *M* are given by

$$\mathbf{P} = [T(\phi,\mathbf{d})]\mathbf{p} = \begin{bmatrix} \cos\phi & -\sin\phi & d_x \\ \sin\phi & \cos\phi & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}.$$



The movement of each of the components of the Boulton & Watt Steam Engine (1784) is modeled by a continuous set of rigid displacements.

Homogeneous transforms represent affine transformations. This formulation is necessary because a translation is not a linear transformation of $\mathbf{R}^2$. However, using projective geometry, so that $\mathbf{R}^2$ is considered to be a subset of $\mathbf{R}^3$, translations become affine linear transformations.[12]

Pure translation

If a rigid body moves so that its reference frame *M* does not rotate relative to the fixed frame *F*, the motion is said to be pure translation. In this case, the trajectory of every point in the body is an offset of the trajectory **d**(t) of the origin of *M*, that is,

$$\mathbf{P}(t) = [T(0, \mathbf{d}(t)]\mathbf{p} = \mathbf{d}(t) + \mathbf{p}.$$

Thus, for bodies in pure translation the velocity and acceleration of every point *P* in the body are given by

$$\mathbf{V}_P = \dot{\mathbf{P}}(t) = \dot{\mathbf{d}}(t) = \mathbf{V}_O, \quad \mathbf{A}_P = \ddot{\mathbf{P}}(t) = \ddot{\mathbf{d}}(t) = \mathbf{A}_O,$$

where the dot denotes the derivative with respect to time and $\mathbf{V}_O$ and $\mathbf{A}_O$ are the velocity and acceleration, respectively, of the origin of the moving frame *M*. Recall the coordinate vector **p** in *M* is constant, so its derivative is zero.

Rotation of a body around a fixed axis

Main article: Circular motion

Rotational or angular kinematics is the description of the rotation of an object.[13] The description of rotation requires some method for describing orientation. Common descriptions include Euler angles and the kinematics of turns induced by algebraic products.

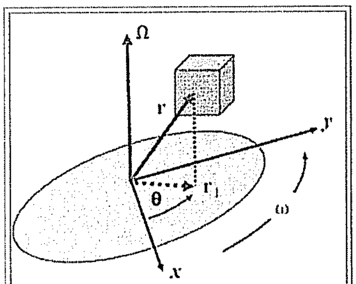


Figure 1: The angular velocity vector **Ω** points up for counterclockwise rotation and down for clockwise rotation, as specified by the right-hand rule. Angular position *θ*(*t*) changes with time at a rate *ω*(*t*) = d*θ*/d*t*.

In what follows, attention is restricted to simple rotation about an axis of fixed orientation. The *z*-axis has been chosen for convenience.

Position: This allows the description of a rotation as the angular position of a planar reference frame *M* relative to a fixed *F* about this shared *z*-axis. Coordinates **p**=(x, y) in *M* are related to coordinates **P**=(X, Y) in *F* by the matrix equation:

$$\mathbf{P}(t) = [A(t)]\mathbf{p},$$

where

$$[A(t)] = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

is the rotation matrix that defines the angular position of *M* relative to *F*.

Velocity: If the point **p** does not move in *M*, then its velocity in *F* is given by

$$\mathbf{V}_P = \dot{\mathbf{P}} = [\dot{A}(t)]\mathbf{p}.$$

It is convenient to eliminate the coordinates **p** and write this as an operation on the trajectory **P**(t),

$$\mathbf{V}_P = [\dot{A}(t)][A(t)^{-1}]\mathbf{P} = [\Omega]\mathbf{P},$$

where the matrix

$$[\Omega] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},$$

is known as the angular velocity matrix of *M* relative to *F*. The parameter ω is the time derivative of the angle θ, that is

$$\omega = \frac{d\theta}{dt}.$$

Acceleration: The acceleration of **P**(t) in *F* is obtained as the time derivative of the velocity,

$$\mathbf{A}_P = \ddot{P}(t) = [\dot{\Omega}]\mathbf{P} + [\Omega]\dot{\mathbf{P}},$$

which becomes

$$\mathbf{A}_P = [\dot{\Omega}]\mathbf{P} + [\Omega][\Omega]\mathbf{P},$$

where

$$[\dot{\Omega}] = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix},$$

is the angular acceleration matrix of *M* on *F*, and

$$\alpha = \frac{d^2\theta}{dt^2}.$$

Description of rotation then involves these three quantities:

- **Angular position**: The oriented distance from a selected origin on the rotational axis to a point of an object is a vector $\mathbf{r}(t)$ locating the point. The vector $\mathbf{r}(t)$ has some projection (or, equivalently, some component) $\mathbf{r}_\perp(t)$ on a plane perpendicular to the axis of rotation. Then the *angular position* of that point is the angle θ from a reference axis (typically the positive *x*-axis) to the vector $\mathbf{r}_\perp(t)$ in a known rotation sense (typically given by the right-hand rule).
- **Angular velocity**: The angular velocity ω is the rate at which the angular position θ changes with respect to time t:

$$\omega = \frac{d\theta}{dt}$$

The angular velocity is represented in Figure 1 by a vector **Ω** pointing along the axis of rotation with magnitude ω and sense determined by the direction of rotation as given by the right-hand rule.

- **Angular acceleration**: The magnitude of the angular acceleration α is the rate at which the angular velocity ω changes with respect to time t:

$$\alpha = \frac{d\omega}{dt}$$

The equations of translational kinematics can easily be extended to planar rotational kinematics for constant angular acceleration with simple variable exchanges:

$$\begin{aligned} \omega_{\mathrm{f}} &= \omega_{\mathrm{i}} + \alpha t \\ \theta_{\mathrm{f}} - \theta_{\mathrm{i}} &= \omega_{\mathrm{i}} t + \tfrac{1}{2}\alpha t^2 \\ \theta_{\mathrm{f}} - \theta_{\mathrm{i}} &= \tfrac{1}{2}(\omega_{\mathrm{f}} + \omega_{\mathrm{i}})t \\ \omega_{\mathrm{f}}^2 &= \omega_{\mathrm{i}}^2 + 2\alpha(\theta_{\mathrm{f}} - \theta_{\mathrm{i}}). \end{aligned}$$

Here θ_{i} and θ_{f} are, respectively, the initial and final angular positions, ω_{i} and ω_{f} are, respectively, the initial and final angular velocities, and α is the constant angular acceleration. Although position in space and velocity in space are both true vectors (in terms of their properties under rotation), as is angular velocity, angle itself is not a true vector.

Point trajectories in body moving in three dimensions

Important formulas in *kinematics* define the velocity and acceleration of points in a moving body as they trace trajectories in three dimensional space. This is particularly important for the center of mass of a body, which is used to derive equations of motion using either Newton's second law or Lagrange's equations.

Position

In order to define these formulas, the movement of a component *B* of a mechanical system is defined by the set of rotations [A(t)] and translations **d**(t) assembled into the homogenous transformation [T(t)]=[A(t), **d**(t)]. Let **p** be the coordinates of a point *P* in *B* measured in the moving reference frame *M*, then the trajectory of this point traced in *F* is given by

$$\mathbf{P}(t) = [T(t)]\mathbf{p} = \begin{Bmatrix} \mathbf{P} \\ 1 \end{Bmatrix} = \begin{bmatrix} A(t) & \mathbf{d}(t) \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix}.$$

This notation does not distinguish between **P** = (X, Y, Z, 1), and **P** = (X, Y, Z), which is hopefully clear in context.

This equation for the trajectory of *P* can be inverted to compute the coordinate vector **p** in *M* as,

$$\mathbf{p} = [T(t)]^{-1}\mathbf{P}(t) = \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix} = \begin{bmatrix} A(t)^T & -A(t)^T\mathbf{d}(t) \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{P}(t) \\ 1 \end{Bmatrix}.$$

This expression uses the fact that the transpose of a rotation matrix is also its inverse, that is

$$[A(t)]^T[A(t)] = I.$$

Velocity

The velocity of the point P along its trajectory **P**(t) is obtained as the time derivative of this position vector,

$$\mathbf{V}_P = [\dot{T}(t)]\mathbf{p} = \begin{Bmatrix} \mathbf{V}_P \\ 0 \end{Bmatrix} = \begin{bmatrix} \dot{A}(t) & \dot{\mathbf{d}}(t) \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix}.$$

The dot denotes the derivative with respect to time, and because **p** is constant its derivative is zero.

This formula can be modified to obtain the velocity of P by operating on its trajectory **P**(t) measured in the fixed frame F. Substitute the inverse transform for **p** into the velocity equation to obtain

$$\mathbf{V}_P = [\dot{T}(t)][T(t)]^{-1}\mathbf{P}(t) = \begin{Bmatrix} \mathbf{V}_P \\ 0 \end{Bmatrix} = \begin{bmatrix} \dot{A}A^T & -\dot{A}A^T\mathbf{d} + \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{P}(t) \\ 1 \end{Bmatrix} = [S]\mathbf{P}.$$

The matrix [S] is given by

$$[S] = \begin{bmatrix} \Omega & -\Omega\mathbf{d} + \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix}$$

where

$$[\Omega] = \dot{A}A^T,$$

is the angular velocity matrix.

Multiplying by the operator [S], the formula for the velocity $\mathbf{V}_P$ takes the form

$$\mathbf{V}_P = [\Omega](\mathbf{P} - \mathbf{d}) + \dot{\mathbf{d}} = \omega \times \mathbf{R}_{P/O} + \mathbf{V}_O,$$

where the vector ω is the angular velocity vector obtained from the components of the matrix [Ω], the vector

$$\mathbf{R}_{P/O} = \mathbf{P} - \mathbf{d},$$

is the position of P relative to the origin O of the moving frame M, and

$$\mathbf{V}_O = \dot{\mathbf{d}},$$

is the velocity of the origin O.

Acceleration

The acceleration of a point P in a moving body B is obtained as the time derivative of its velocity vector,

$$\mathbf{A}_P = \frac{d}{dt}\mathbf{V}_P = \frac{d}{dt}([S]\mathbf{P}) = [\dot{S}]\mathbf{P} + [S]\dot{\mathbf{P}} = [\dot{S}]\mathbf{P} + [S][S]\mathbf{P}.$$

This equation can be expanded by first computing

$$[\dot{S}] = \begin{bmatrix} \dot{\Omega} & -\dot{\Omega}\mathbf{d} - \Omega\dot{\mathbf{d}} + \ddot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{\Omega} & -\dot{\Omega}\mathbf{d} - \Omega\mathbf{V}_O + \mathbf{A}_O \\ 0 & 0 \end{bmatrix}$$

and

$$[S]^2 = \begin{bmatrix} \Omega & -\Omega\mathbf{d} + \mathbf{V}_O \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} \Omega^2 & -\Omega^2\mathbf{d} + \Omega\mathbf{V}_O \\ 0 & 0 \end{bmatrix}.$$

The formula for the acceleration $\mathbf{A}_P$ can now be obtained as

$$\mathbf{A}_P = \dot{\Omega}(\mathbf{P} - \mathbf{d}) + \mathbf{A}_O + \Omega^2(\mathbf{P} - \mathbf{d}),$$

or

$$\mathbf{A}_P = \alpha \times \mathbf{R}_{P/O} + \omega \times \omega \times \mathbf{R}_{P/O} + \mathbf{A}_O,$$

where α is the angular acceleration vector obtained from the derivative of the angular velocity matrix,

$$\mathbf{R}_{P/O} = \mathbf{P} - \mathbf{d},$$

is the relative position vector, and

$$\mathbf{A}_O = \ddot{\mathbf{d}}$$

is the acceleration of the origin of the moving frame *M*.

Kinematic constraints

Kinematic constraints are constraints on the movement of components of a mechanical system. Kinematic constraints can be considered to have two basic forms, (i) constraints that arise from hinges, sliders and cam joints that define the construction of the system, called holonomic constraints, and (ii) constraints imposed on the velocity of the system such as the knife-edge constraint of ice-skates on a flat plane, or rolling without slipping of a disc or sphere in contact with a plane, which are called non-holonomic constraints. Constraints can also arise from other interactions such as rolling without slipping, is any condition relating properties of a dynamic system that must hold true at all times. Below are some common examples:

Rolling without slipping

An object that rolls against a surface without slipping obeys the condition that the velocity of its center of mass is equal to the cross product of its angular velocity with a vector from the point of contact to the center of mass,

$$\boldsymbol{v}_G(t) = \boldsymbol{\Omega} \times \boldsymbol{r}_{G/O}.$$

For the case of an object that does not tip or turn, this reduces to v = R ω.

Inextensible cord

This is the case where bodies are connected by an idealized cord that remains in tension and cannot change length. The constraint is that the sum of lengths of all segments of the cord is the total length, and accordingly the time derivative of this sum is zero. See Kelvin and Tait[14][15] and Fogiel.[16] A dynamic problem of this type is the pendulum. Another example is a drum turned by the pull of gravity upon a falling weight attached to the rim by the inextensible cord.[17] An *equilibrium* problem (not kinematic) of this type is the catenary.[18]

Kinematic pairs

Reuleaux called the ideal connections between components that form a machine, kinematic pairs. He distinguished between higher pairs which were said to have line contact between the two links and lower pairs that have area contact between the links. J. Phillips[19] shows that there are many ways to construct pairs that do not fit this simple classification.

Lower pair: A lower pair is an ideal joint, or holonomic constraint, that maintains contact between a point, line or plane in a moving solid (three dimensional) body to a corresponding point line or plane in the fixed solid body. We have the following cases:

- A revolute pair, or hinged joint, requires a line, or axis, in the moving body to remain co-linear with a line in the fixed body, and a plane perpendicular to this line in the moving body maintain contact with a similar perpendicular plane in the fixed body. This imposes five constraints on the relative movement of the links, which therefore has one degree of freedom, which is pure rotation about the axis of the hinge.
- A prismatic joint, or slider, requires that a line, or axis, in the moving body remain co-linear with a line in the fixed body, and a plane parallel to this line in the moving body maintain contact with a similar parallel plan in the fixed body. This imposes five constraints on the relative movement of the links, which therefore has one degree of freedom. This degree of freedom is the distance of the slide along the line.
- A cylindrical joint requires that a line, or axis, in the moving body remain co-linear with a line in the fixed body. It is a combination of a revolute joint and a sliding joint. This joint has two degrees of freedom. The position of the moving body is defined by both the rotation about and slide along the axis.
- A spherical joint, or ball joint, requires that a point in the moving body maintain contact with a point in the fixed body. This joint has three degrees of freedom.
- A planar joint requires that a plane in the moving body maintain contact with a plane in fixed body. This joint has three degrees of freedom.

Higher pairs: Generally, a higher pair is a constraint that requires a curve or surface in the moving body to maintain contact with a curve or surface in the fixed body. For example, the contact between a cam and its follower is a higher pair called a *cam joint*. Similarly, the contact between the involute curves that form the meshing teeth of two gears are cam joints.

Kinematic chains

Rigid bodies, or links, connected by kinematic pairs, or joints, are called *kinematic chains*. Mechanisms and robots are examples of kinematic chains. The degree of freedom of a kinematic chain is computed from the number of links and the number and type of joints using the mobility formula. This formula can also be used to enumerate the topologies of kinematic chains that have a given degree of freedom, which is known as *type synthesis* in machine design.

Examples of kinematic chains: The planar one degree-of-freedom linkages assembled from *N* links and *j* hinged or sliding joints are:

- N=2, j=1: this is a two-bar linkage known as the lever;
- N=4, j=4: this is the four-bar linkage;
- N=6, j=7: this is a six-bar linkage. A six-bar linkage must have two links that support three joints, called ternary links. There are two distinct topologies that depend on how the two ternary linkages are connected. In the Watt topology, the two ternary links have a common joint. In the Stephenson topology the two ternary links do not have a common joint and are connected by binary links;[20]

- N=8, j=10: the eight-bar linkage has 16 different topologies;
- N=10, j=13: the 10-bar linkage has 230 different topologies,
- N=12, j=16: the 12-bar has 6856 topologies.

See Sunkari and Schmidt[21] for the number of 14- and 16-bar topologies, as well as the number of linkage topologies that have two, three and four degrees-of-freedom.

See also

- Motion
- Distance
- Velocity
- Acceleration
- Jerk (physics)
- Analytical mechanics
- Classical mechanics
- Applied mechanics
- Celestial mechanics
- Orbital mechanics
- Kepler's laws
- Statics
- Dynamics (physics)
- Kinetics (physics)
- Centripetal force
- Fictitious force
- Forward kinematics
- Inverse kinematics
- Kinematic coupling
- Four-bar linkage
- Chebychev–Grübler–Kutzbach criterion

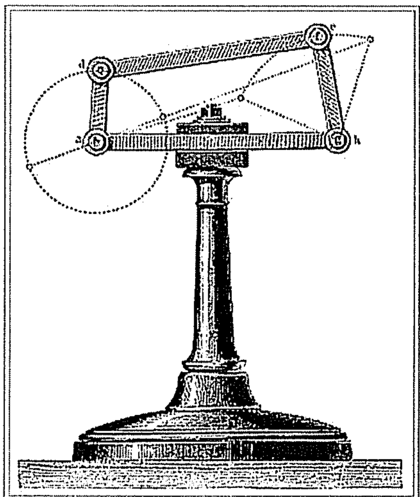


Illustration of a four-bar linkage from Kinematics of Machinery, 1876 (http://en.wikisource.org/wiki/The_Kinematics_of

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External links

- Java applet of 1D kinematics (http://www.phy.hk/wiki/englishhtm/Kinematics.htm)
- Physclips: Mechanics with animations and video clips (http://www.physclips.unsw.edu.au/) from the University of New South Wales
- Kinematic Models for Design Digital Library (KMODDL) (http://kmoddl.library.cornell.edu/index.php) Movies and photos of hundreds of working mechanical-systems models at Cornell University. Also includes an e-book library (http://kmoddl.library.cornell.edu/e-books.php) of classic texts on mechanical design and engineering.

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