分類:	
編號:	1
總號:	

Chapter 4 and 5
牛頓定律及其應用

- Newton's Law
  1. Newton's first law
  2. Newton's sceond law
  3. Newton's third law
  4. Relativity principle of Galileo
- Unit and dimension.
- Some Common Forces
- Application of Newton's Laws
- Noninertial frame of reference.

Newton's three laws and their applications

Kinematics
↓
mathematical description of motion

Dynamics

Why do bodies move as they do?
↓
we shall see that force are responsible for acceleration

The properties of force and the relationships between force and acceleration are given by Newton's three laws of motion

分類:	
編號:	3
總號:	

Idealization , Approximation.

Isolation system

To avoid having to study everything at once
scientists
isolate
the things they are trying to study.

Any part of the universe is considered
from the rest
of the world.

Physics has had some of its greatest successes
by carrying this process of isolation to extremes
subdividing the universe into smaller and smaller
parts.

Matter can be divided into atoms, the behavior
of individual atoms can be studied
Atoms can be split apart into their constituent
neutron, protons and electrons.

This method of splitting things into smaller and
smaller parts and studying how those parts
influence each other
is called
reductionism

分類:	
編號:	4
總號:	

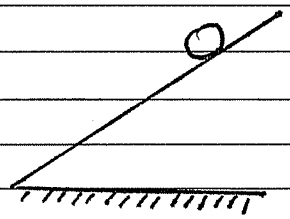
Newton's first law

Every particle continues in its state of rest or in uniform motion in a straight line unless it is compelled to change the state by forces impressed upon it.

$$\vec{F} = 0 \Rightarrow \vec{V} = \text{constant}$$

velocity

Galileo:



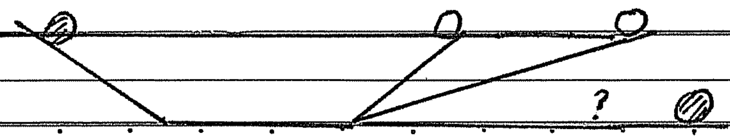
downward:
velocity increases
upward
velocity decreases

$\Rightarrow$ horizontal velocity $\Rightarrow$ constant

actually, it is slowed-down

eventually stop

Galileo's observation
presence of frictional force
(Smoother the surface farther it goes)



smaller the slope → further it goes

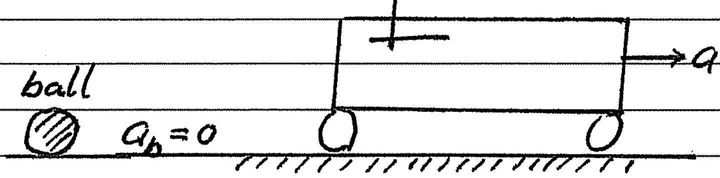
flat ⟶ keep moving

Newton's first law ⇒ Law of inertia.

express the tendency of bodies to maintain their original state of motion.

Reference frame

Law is not valid in all reference frame.



Ball fixed in ground
Train accelerates
Ball appears to be spontaneously accelerated even though it has no force acting

Test for inertial reference frame

take a free body (no forces acting)

if it persists in a state of uniform motion
⇒ inertial reference frame

Any other reference frame in uniform translational motion relative to the first is also an inertial frame.

分類:	
編號:	6
總號:	

A frame in accelerted motion relative to the first is not an inertial frame.

Earth based frame due to its rotation and revolution is not inertial frame.

however, the effects are small

In the first approximation, we shall take it as "inertial frame"

(Further discussion on the rotati frames of reference effects on the surface of the earth will be given later).

a (rotation at equator $\simeq 3.4 \cdot 10^{-2}$ m/sec^2

a (revolution) $\simeq 5.9 \cdot 10^{-3}$ m/sec^2

Earth-based reference frame may approximately taken to be an "inertial frame of reference"

Newton's Second Law

Established the relationship between force, acceleraction

$$\vec{F} = m\vec{a}$$

- A law of nature
- Valid only in inertial frame
  otherwise, it will violate the first law.
- $\vec{F}$ is a vector
  m is a scalar
  $\vec{a} \propto \vec{F}$, and in the same direction.

Operational definition of mass

- Common force acting on standard mass and unknown mass
- Under action of the force the bodies will be accelerated.
- Do the experiment in an inertial frame

$$F = ma = m_s a_s$$

$$\Rightarrow \frac{m}{m_s} = \frac{a_s}{a}$$

The procedure follow the one used in the textbook

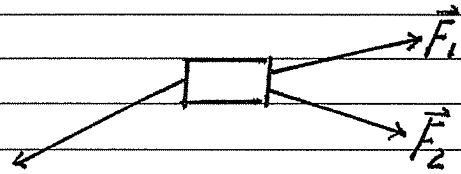
Mass ⇒ a measure of resistance a body offers to change in its velocity.

For a given force ⟶ larger masses have smaller acceleration.

· Mass is an additive property of matter

$$m = m_1 + m_2$$

Superposition of forces



Suppose several forces act on body, how does it move?

Principle of superposition

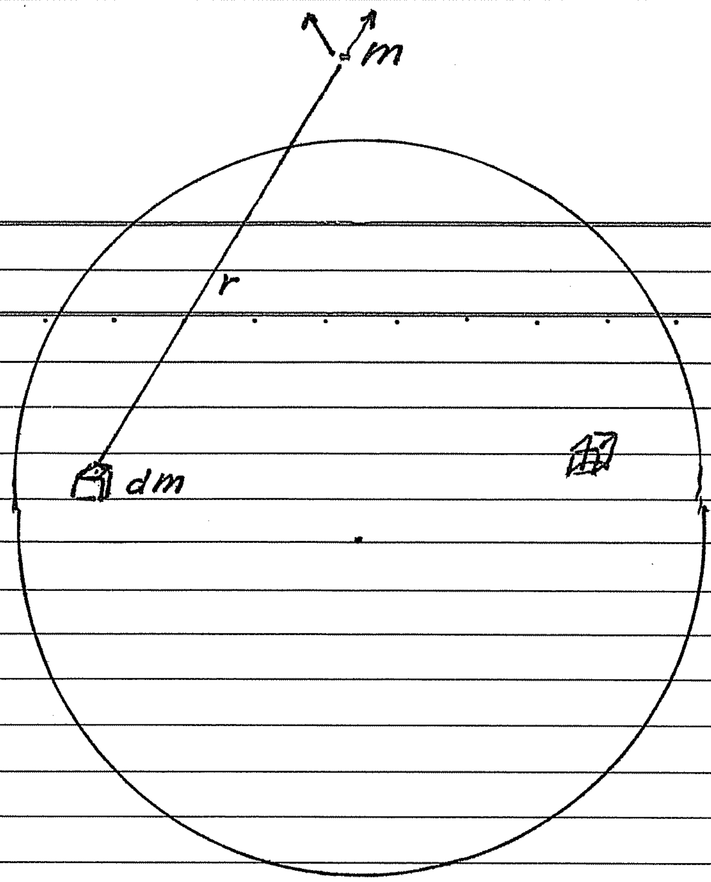
If several forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \cdots$ act simultaneously on a body then the acceleration is the same as that produced by the single force.

$$\underbrace{\vec{F}_{net}}_{\text{Net (Resultant) force}} = \underbrace{\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots}_{\text{vector sum of individual forces}}$$

Newton's Second Law

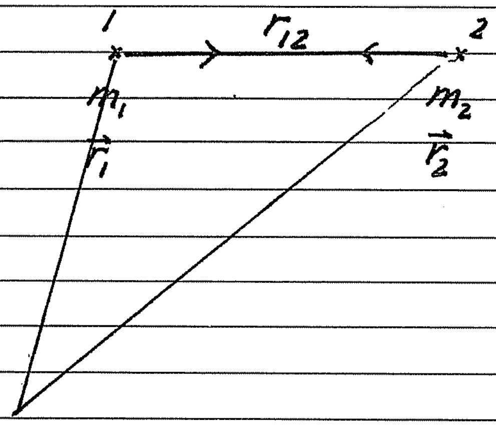
$$\vec{F}_{Net} = m\vec{a}$$

分類：	
編號：	9
總號：	



$$\vec{F}_{21} = \frac{G\, m_1\, m_2}{|\vec{r}_{12}|^2}\, \hat{r}_{12}$$

attraction

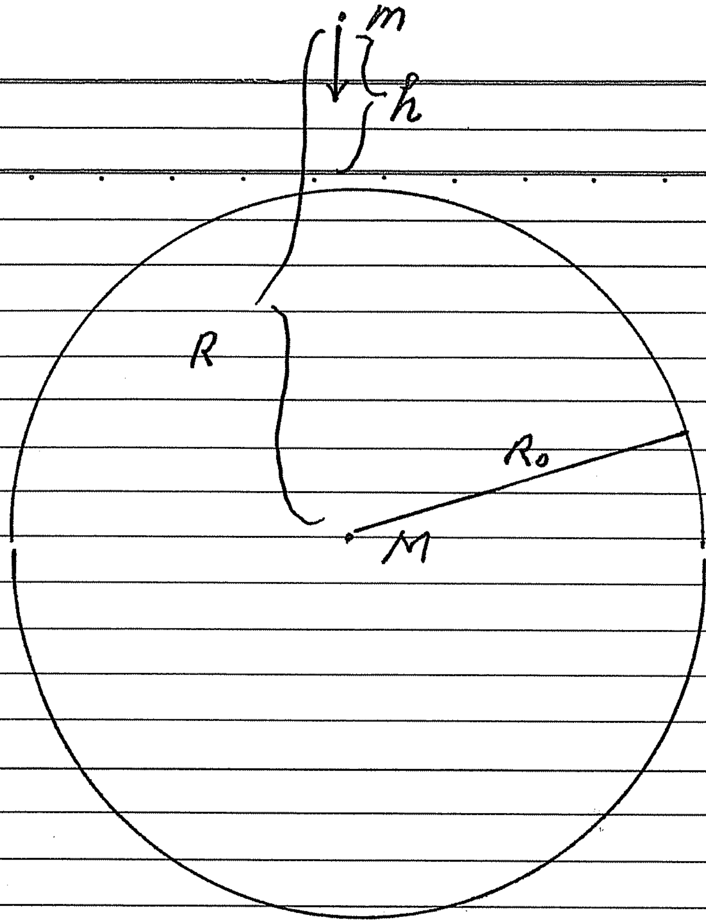


$\vec{F}_{12}$ force on 2 due to 1

F_{21} force on 1 due to 2

Use principle of superposition + calculus

integration



Symmetry

Near the earth surface

$$z \downarrow \quad m\downarrow \; -m\underbrace{\frac{GM}{R^2}}_{g}\hat{k} = \vec{F} = -mg\hat{k}$$

$\vec{a} = -g\hat{k}$

$\vec{F}$: act on the particle with mass

Free fall problem, projectile problem.

$\vec{a}$ is given, initial condition specified.

kinematics

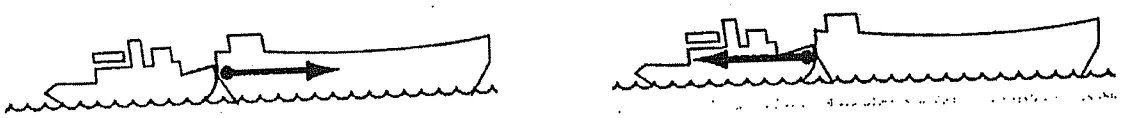
Newton's Third Law

Gravitional force

$$\vec{F}_{12} = -\vec{F}_{21}$$

From observations, Newton concluded that all force in nature act in pairs.

Impossible to have a single isolated force

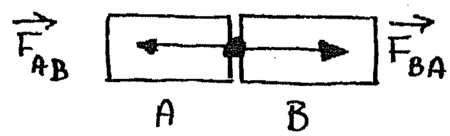


Action ⟷ Reaction

(a) tugboat pushes on barge (action)

(b) barge pushes on tugboat (reaction)

Newton : " To every action (force) there is always opposed an equal reaction (force); or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.



$$\vec{F}_{AB} = -F_{BA}$$ [action - reaction pair]

Note.

- The two forces in an action-reaction pair act on different objects
- A body is accelerated only by the forces acting on it. Forces it exerts on another body do not affect its motion.
- Object m resting on surface of the earth

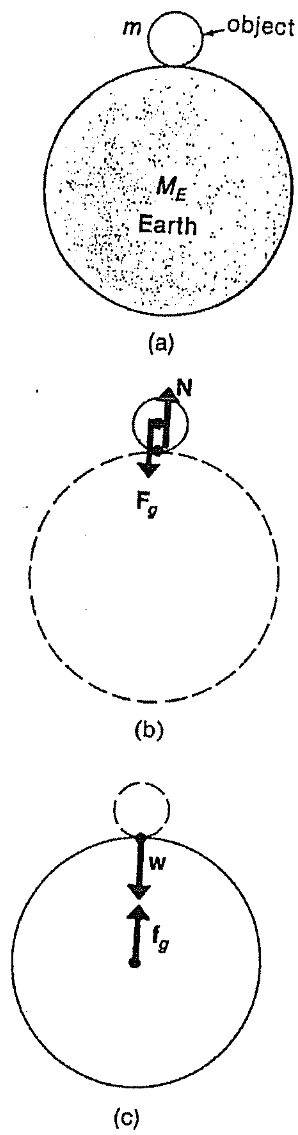


Fig. (a) An object at rest on the Earth's surface. (b) The forces acting on the *object*. (c) The forces acting on the *Earth*. The forces shown together in (b) and (c) are *not* action-reaction pairs.

Object. Two forces acting

Earth's attractive gravitational force $\vec{F}_g$

Contact force $\vec{N}$ with ground.

Since object is at rest

$$\vec{N} + \vec{F}_g = 0$$
$$\vec{N} = -\vec{F}_g$$

Earth Two forces acting

Attractive gravitational force $\vec{f}_g$ due to object

Contact force $\vec{W}$ with object

No relative motion

$$\vec{W} = -\vec{f}_g$$

Action - Reaction Force Pairs

$$\vec{F_g} = -\vec{f_g}$$

$$\vec{N} = -\vec{W}$$

All four forces have equal magnitude

$$F_g = f_g = N = W = mg$$

第五章 牛頓定律

在本章我們將討論牛頓定律 這是古典力學的基礎

第一節 慣性,質量,動量守恆及力的定義

1. 簡介

在此節中我們將引進動量的觀念並給它下一個質量之定義. 利用這個定義,我們可以引進動量守恆定則 這個定則是由觀察許多現象歸納而成的. 我們在此接受它為一基本物理公理. 我們然後討論此一定則的種種結果. 基於動量守恆定則,力和質量的觀點可以有一較精確的定義. 在此節中,我們並將強調慣性座標之重要性。

2. 基本觀念

由日常經驗中得知一個質點之運動是有與其周圍之其他物體之交互作用所引起的. 而此交互作用常常是以力來表示. 當一個質點不受任何交互作用時,則稱之為自由質點。[1]

牛頓的第一定律(又叫慣性定律). 一個自由質點作等速度運動.[2,3]

若一不受外力的觀察者選擇一座標系統,則在系統中牛頓第一定律成立.這樣的座標系統被稱為慣性座標系統。[4]

我們將先將質量作一個操作型的粗略定義. 我們首先假定質量是質點的基本特性。它是一純量,同時我們也假定他與質點之運動情況無關. 它可用一標準質量及一平衡天平來定量.

有了質量之定義以後,我們可以定義一質點之動量

$$\vec{p} = m\vec{v} \tag{1}$$

因為我們假設質量與其運動情況無關,牛頓之第一定律可重寫成. 一個自由質量在慣性座標中之動量不變.

分類：	
編號：	2
總號：	

現在，我們將討論一個由兩個質點所組成的獨立系統。由實驗中我們發現此一系統之總動量保持不變。此一定律即是動量守恒定律，它是物理中最基本之定律之一。若當時間為 t 時此兩質點之動量分別為 $\vec{p}_1$, $\vec{p}_2$，在 t' 時此兩質量之動量分別為 $\vec{p}_1{}'$, $\vec{p}_2{}'$，則動量守恒定律可寫成

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}' \qquad (2)$$

由此可得 $\vec{p}_1{}' - \vec{p}_1 = -(\vec{p}_2{}' - \vec{p}_2)$ 也即是

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 \qquad (3)$$

$\Delta\vec{p}_1 = \vec{p}_1{}' - \vec{p}_1$ 是在時間 t 與 t' 間，第一個質點動量的變化。$\Delta\vec{p}_2 = \vec{p}_2{}' - \vec{p}_2$ 是在 t 與 t' 間，第二個質點動量的變化。

因為質量與質點之運動情況無關，第(3)式可寫成

$$m_1\Delta\vec{v}_1 = -m_2\Delta\vec{v}_2 \qquad (4)$$

也即是

$$\frac{m_1}{m_2} = \frac{|\Delta\vec{v}_2|}{|\Delta\vec{v}_1|} \qquad (5)$$

若取 m_1 為標準質量則我們可以由量度 $|\Delta\vec{v}_1|$ 及 $|\Delta\vec{v}_2|$ 來決定 m_2

討論

(1) 嚴格講起來，自由質點根本不存在。但在以下兩種情況以下一個質點可大約的被當作自由質點。(一) 它與其他物体甚遠，因此與其他物体之交互作用甚小，而可略去不計。(二) 其他物体對此質點之交互作用相互抵消。

(2) 此一定律是由伽里略所首先提出。

(3) 牛頓假設有一絕對靜止座標，質點的運動是以此一座標來描述。同時，我們再度強調速度是一向量，因此等速要求速度的大小及方向均保持不變。

分類：
編號： 3
總號：

) 由於地球有自轉且繞著太陽旋轉，因此它嚴格的來講並非一慣性座標系統。但一般而言，其影響不大，因此可輕易地將它當作一慣性座標系統。太陽及很遠的星（因其運動更接近等速運動）當然是比較好的慣性座標系統。

(5) 我們假設此兩質量除了彼此之交互作用外，與其他物体無任何交互作用。

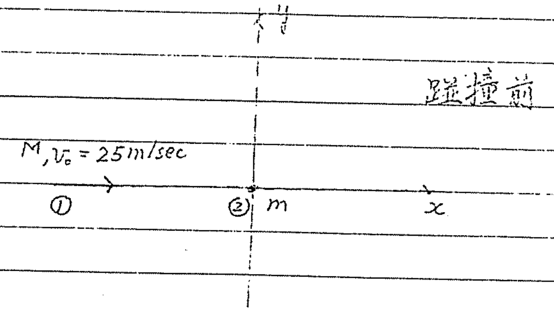
(6) 此一守恒律是由實驗結果得來。我們強調動量是向量，其大小及方向均不改變。

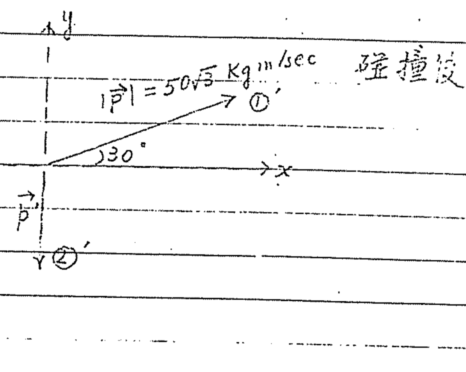
(7) 此一結果可推廣至由 N 個質點所組成之獨立系統。

用

我們經常利用動量守恒定律來計算有關質點之動量、速度、質量等。我們現在舉例來說明。

若質點 1 沿 x 軸以 $v_0 = 25\ m/sec$ 與一靜止之質量 2 相撞，碰撞後質點 2 與 x 軸成 $30°$ 進行，其動量之大小為 $50\sqrt{3}\ Kg\ m/sec$，質量 1 則沿 y 軸進行。我們現在將此碰撞情形繪於圖一。





沿 x 軸之動量必須守恒，因此

$$M v_0 = |\vec{P}| \cos 30° \qquad (6)$$

由此式中，我們馬上可求出 $M = 3\ Kg$。

沿 y 軸之動量必須守恒，因此

$$|\vec{P}'| = M v' = |\vec{P}| \sin 30° \qquad (7)$$

分類：
編號：4
總號：

(2)式，我們馬上可求出 $|\vec{P}'| = 25$ [illegible] kg m/sec，此處 $|\vec{P}'|$ 為碰撞後質點動量之大小，其方向為 $-y$ 軸

球之質量為 $m=0.05$ kg，以 $v=20$ m/sec 之速度沿水平方向向左進行，而一豎直牆以 $V=20$ m/sec 之速度沿水平方向向右進行。我們同時利用高速照相機定出牆接觸的時間為 0.02 sec

碰撞前球之動量向量為何？ [K]

碰撞後球之動量向量為何？ [E]

球的動量守恆嗎？ [P]

假若動量守恒成立，如何將它應用到球的上面？ [R]

若是想應用動量守恒定律，必須考慮些什麼因素？ [A]

球動量改變之大小為多少？ [F]

球動量改變之方向為何？ [N]

在球與牆接觸的時間內，球上所受之平均力為何？ [I]

在接觸期內，力是固定的嗎？ [Q]

在接觸期內，球上所受之平均力方向為何？ [C]

在接觸期間，牆上所受之平均力大小為何？ [B]

在接觸期間，牆上所受之平均力方向為何？ [J]

為何你可利用動量守恒定律得到 (k)、(l) 之解答？ [L]

兩個質量相同的小孩乘坐一雪橇在一平滑之湖面上以 $V_0 = 15$ m/sec 沿 x 軸進行，突然雪橇裂為相同質量之兩半（此二半仍沿原方向進行）。坐在後面之小

分類：	
編號：	5
總號：	

想以手去抓住前面的小孩但反而把他推開了。在後面的小孩看見前面的小

以 $u = 8$ m/sec 之速度離開。 M 是雪橇及兩個小孩之總質量

在意外發生前之總動量為何？ [D]

在意外發生後，前面的一半對地面以 V_1' 前進，後面的一半以 V_2' 前進；

則此時之總動量和為何？ [O]

c) V_1'，V_2' 及 u 之關係為何？ [H]

d) 動量守恆在此成立與否？寫出動量守恆定律公式。[G]

e) V_1' 為何？ V_2' 為何？ [M]

案

[A] 必須將球及牆合成一系統，此一系統方為獨立系統 [B] 100 Newton

[C] 向左 [D] MV_0 [E] 1 Kg m/sec 向左 [F] 2 Kg m/sec

[G] 成立。 $M \cdot V_0 = \frac{1}{2}MV_1' + \frac{1}{2}MV_2'$ [H] $V_1' = V_2' + u$ [I] 100 Newton

[J] 向右 [K] 1 Kg m/sec^2 向右 [L] 由於此一獨立系統不受外力，所以

球所受牆之力與牆受球之力是作用力及反作用。因此它們大小相等方向相反。

[M] $V_1' = 19$ m/sec， $V_2' = 11$ m/sec [N] 向左 [O] $\frac{1}{2}MV_1' + \frac{1}{2}MV_2'$

[P] 不守恆 [Q] 不一定 [R] 動量守恆定律只在該系統不受外力時才成立；而

把球當成一系統時，它受有牆對它施的力，所以動量守恆定律不成立。

分類：	
編號：	1
總號：	

第二節 牛頓第二、第三定律

簡介

我們在此節中將討論牛頓第二定律及牛頓第三定律。由第二定律我們得知一個質點之運動取決於其所受之力及其原來之位置及速度。因為由其所受之外力和，我們可以求得該質點之加速度。當加速度為已知時，經過兩次積分即可求得 $\vec{r}(t)$。因此 [illegible] 作用於物體上所有之力。我們在此節中強調處理此一問題之方法。

基本觀念

一個質點在 t 時之動量為 $\vec{p}$，在 t' 時之動量為 $\vec{p}'$，則此質點在 t 及 t' 之間動量之變化為 $\vec{p}'-\vec{p}$。在一慣性座標中，此一量與該質點所受之平均力間之關係為

$$\bar{\vec{F}}_{t\to t'}(t'-t) = \vec{p}'-\vec{p} \qquad (1)$$

令 $t=t,\ t'=t+\Delta t$

$$\bar{\vec{F}}_{t\to t+\Delta t}\,\Delta t = \vec{p}(t+\Delta t)-\vec{p}(t) \qquad (2)$$

當 $\Delta t\to 0$ 時，我們即得

$$\vec{F}(t) = \lim_{\Delta t\to 0}\frac{\vec{p}(t+\Delta t)-\vec{p}(t)}{\Delta t} \qquad (3)$$

$$= \frac{d\vec{p}}{dt} \qquad (4)$$

此即是牛頓第二定律。

若 m 並非時間之函數，則

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad (5)$$

由上一節中，我們得知當兩質點作用時

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 \qquad (6)$$

$\Delta \vec{p}_1$ 是質點1在 t 與 t' 之間動量之變化，$\Delta \vec{p}_2$ 是質點2在 t 與 t' 之間動量之變化。

令 $t=t$，$t'=t+\Delta t$，然後取 $\Delta t \to 0$，我們得到

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \qquad (7)$$

利用牛頓第二定律，上式可寫成

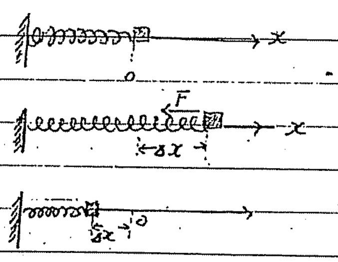
$$\vec{F}_1 = -\vec{F}_2 \qquad (8)$$

此處 $\vec{F}_1$ 是質點2對質點1所施之力，$\vec{F}_2$ 是質點1對質點2所施之力。也即是說若一物体 A 對 B 施一力 $\vec{F}_{AB}$ 則物体 B 對 A 也施一力 $\vec{F}_{BA}$ 其大小相等方向相反。[3,4,5]此為牛頓第三定律，也叫做反作用定律。而 $\vec{F}_{AB}$ 及 $\vec{F}_{BA}$ 構成一作用一反作用偶。

由第(5)式中我們可以看出質點的運動是取決於它所受的力與其初速及最初的位置

我們現在來討論一般常見的力。

(i) 彈簧力（虎克定律）



平衡狀態

拉長之情況

壓縮之情況

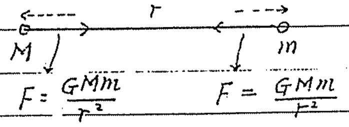
$$F = -k\Delta x \qquad (9)$$ [6]

此處 x 是離平衡狀況之距離，k 是一常數，与此彈簧之硬度有關，k 叫做彈簧之力常數。負號顯示當 $x>0$（拉長）時，其力之方向為沿 $-x$ 軸，當 $x<0$（壓縮）時，其力之方向為沿 x 軸。

分類：	
編號：	3
總號：	

法向接觸力 N[7] 是當物体接觸時,能自行調整的力。它的方向是沿防止穿透,垂於接觸面。它的大小,除了在題意明白顯示以外,通常不能預知,而是從解一組公式得來。

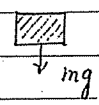
牛頓之萬有引力定律：兩個質點之間的引力。它与質量之乘積成正比,而与其間之距離成反比



$$F = G\frac{Mm}{r^2} \qquad (10)$$

此處 G 是重力常數

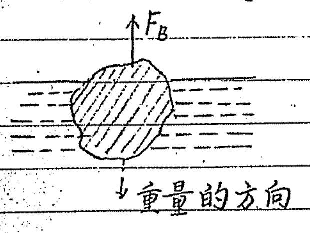
(iv) 重量[9] 是(iii)的一個特例,它是地球對地球表面物体之吸引力。其方向是指向地心,對一質量為 m 之物体,其大小為



$$\therefore F = W = m\frac{GM}{R^2} = mg \qquad (11)$$

此處 M 是地球之質量, R 是地球之半徑, g 是由重力加速度。

(v) 浮力(又稱亞幾米德原理)：當一物体浸於一流体中所受之力。其方向与重量之方向相反,其大小是該物体所排出流体之重量。



$$F_B = \rho_f V g \qquad (12)$$

此處 ρ_f 是流体之密度, V 是該物体排出液体之体積, g 是重力加速度。

(vi) 摩擦力

(a) 靜摩擦力[10]：當一物体在一不為完全平滑之表面上,當外加之力小於某一值時

分類：	
編號：	4
總號：	

該物体仍保持其靜止之狀態。因此，必須有一能自行調整之力存在。這個力被稱為靜摩擦力。當外加之力小於最大之靜摩擦力時，靜摩擦力之大小與外加之力相同，但方向相反。因此，此兩力相消而使該物体保持靜止。最大之靜摩擦力與法向接觸力成正比，也即是

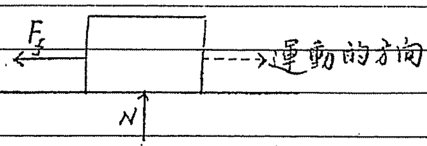
$$|F_{f,max}| = \mu_s N \qquad (13)$$

此處 μ_s 為靜摩擦係數。

(b) 動摩擦力" 當物体在一不平滑之面上運動時，有動摩擦力之存在，其方向與物体對表面之相對速度相反。其大小與法向接觸力成正比。

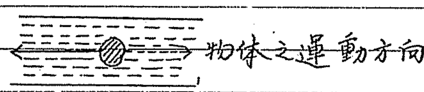
$$F_f = \mu_k N \qquad (14)$$

此處 μ_k 是動摩擦係數。



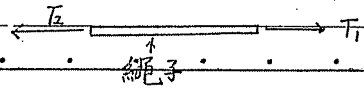
(c) 在流体中之黏滯摩擦力。當物體在流体中運動時，有黏滯摩擦力之存在，其方向與運動之方向相反。其大小與物体在流体中之相對速度之大小成正比。也即是

$$F_v = -kv \qquad (15)$$ [12,13]



此處 v 是物体與流体之相對速度，k 是一比例常數，此常數與物体之形狀及流体之性質有關。

(vii) 繩上之張力。我們在很多情形下，用繩子來連接物体。在一般情況之下，我們假設繩子之質量可略去不計。同時繩子是繃緊的，則繩子所受之張力如圖所



分類：
編號：5
總號：

$|\vec{T}_1|=|\vec{T}_2|$。這點可由下面的討論得知

應用牛頓第二定律至繩子上，則得 $\vec{T}_1+\vec{T}_2=m_r\vec{a}_r=0$ （m_r=繩子的質量，我們在此假設其為0）。因此 $\vec{T}_1=-\vec{T}_2$，$|\vec{T}_1|=|\vec{T}_2|$

討論

(1) 只有在慣性座標系統中，此公式方成立。

(2) 若質量是時間的函數時，我們在利用牛頓第二定律時，必須十分小心，我們將在第4小節中舉例說明此點。

(3) 若是我們稱 $\vec{F}_{AB}$ 為力則 $\vec{F}_{BA}$ 即是反作用力，若是稱 $\vec{F}_{BA}$ 為作用力則 $\vec{F}_{AB}$ 即是反作用力

(4) 作用力及反作用力永遠不能作用於同一物体上。✓

(5) 並非所有大小相同，反向相反之力均為作用－反作用力偶。✓

(6) 此一公式只在拉長或壓縮的長度均不大時，才成立。此一公式是由彈性學中得到，但虎克是由實驗中得到此公式。✓

(7) 兩個相接觸的物体與很強的彈簧相似，✓因此，接觸力的原因與彈簧力相同。唯一不同的是其變形的程度甚小可以略去不計

(8) 這個公式的適用範圍與物体之性質有關，當在表面上之重量太大時，則表面會破和被穿透。

(9) 注意重量是地球對一物体的吸引力。因此其大小與其在地球表面的位置及高度而略有變化。✓

(10) 此一結果是由實驗中所得來。

(11) 此一公式只在速度較小的時候成立。此一公式也是由實驗中所得到的。

分類：	
編號：	6
總號：	

此一公式只有在低速度時才成立。

半徑為r之

此一公式可由流体動力學中導得，當物体為圓球形時 $k = 6\pi\eta r$ 此處 η 是流体之黏滞係數。

Stoke's law

find the demensional of η

$F = -kv$ $\qquad ML/T^2 = -k\frac{L}{T}$

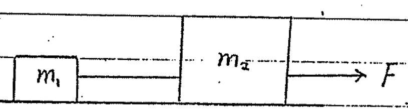
$k = \frac{M}{T}$ $\qquad \frac{M}{T} = \eta L \Rightarrow \eta = \frac{M}{LT}$

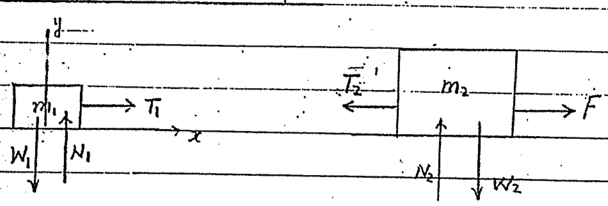
用牛頓定律之步驟大致如下：

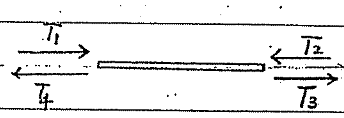
(1) 決定我們注意的系統，而將它標出。
(2) 在系統的界限上，繪出外界環境對此系統所施的所有之力。這個圖形稱為自由力圖
(3) 選擇適合之座標軸
(4) 將牛頓第二定律沿各座標軸方向寫出
(5) 獲得更多的公式：
這些公式通常來自
(a) 力定律：如 (α) $W = mg$；或 (β) $F_s = -kx$（彈簧力）
(b) 動力關係：如 (α) 繩子是繃緊而不帶質量的，$T_1 = T_2$，(β) 繩子在變鬆之邊緣，$T = 0$
(γ) 物体在方表面脫離接觸的邊緣 $N = 0$
(c) 運動關係 如 (α) 物体是靜止或以等速運動則 $\vec{a} = 0$ (β) 若物体沿 x 軸平移則 $v_y = a_y = 0$
(d) 力与反作用力關係。
(6) 解方程式

我們將舉例說明如何應用牛頓定律

(1) 求在繩上之張力







T_1 作用於 m_1（由繩子而來的）
T_2 作用於 m_2（由繩子而來的）
T_3 作用於繩子（由 m_2 而來的）
T_4 作用於繩子（由 m_1 而來的）

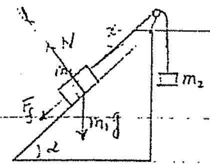
T_1，T_4 是作用力及反作用力偶
T_2，T_3 是作用力及反作用力偶
$T_3 = T_4$ 是由於我們假設繩子沒有質量

$$|T_1| = |T_2| = |T_3| = |T_4| = T \qquad (16)$$

分類：
編號： 7
總號：

$$\left.\begin{aligned} F - T &= m_2 a \\ T &= m_1 a \end{aligned}\right\} \Rightarrow T = \left(\frac{m_1}{m_1+m_2}\right) F \qquad (17)$$

2)



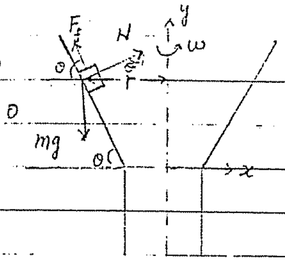
$$m_2 g - T = m_2 a$$

$$T - m_1 g \sin\alpha - \mu_k N = m_1 a \quad (沿\ x\ 軸) \qquad (18)$$

$$N - m_1 g \cos\alpha = 0 \quad (沿\ y\ 軸)$$

三個公式，三個未知數 T, N, a 可解以上三式得到答案。

(3)



若此物恣向下滑時

$$N\sin\theta - \mu N\cos\theta = m\frac{v^2}{r} = mr\omega^2 \quad (沿\ x\ 軸) \qquad (19A)$$

$$N\cos\theta + \mu N\sin\theta - mg = 0 \quad (沿\ y\ 軸) \qquad (19B)$$

$$\Rightarrow \omega_{min} = \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{r(\cos\theta + \mu\sin\theta)}} \quad 最小之\ \omega\ 使\ m\ 不向下滑 \qquad (20)$$

系統的選擇

若此物体向上升時 F_f 之方向與以上考慮之相反，也即是

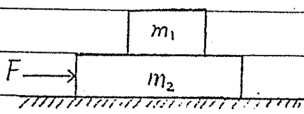
$$N\sin\theta + \mu N\cos\theta = m\frac{v^2}{r} = mr\omega^2 \quad (沿\ x\ 軸) \qquad (21A)$$

$$N\cos\theta - \mu N\sin\theta - mg = 0 \quad (沿\ y\ 軸) \qquad (21B)$$

$$\Rightarrow \omega = \sqrt{\frac{g}{r}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)} \quad 最大之\ \omega\ 使\ m\ 不向上升 \qquad (22)$$

在 ω_{min} 至 ω_{max} 則由於 F_f 能自行調整而使 m 保持不動。

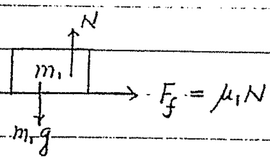
(4)



m_1, m_2 間之靜摩擦係數為 $\mu_s = \mu_1$

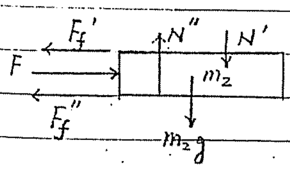
m_2 與桌面間之動摩擦係數為 $\mu_k = \mu_2$

求 m_1, m_2 不分開之最大加速度，及該時之外力為何？



$$N = m_1 g \quad 沿\ y\ 軸 \qquad (23)$$

$$\mu_1 N = m_1 a_{max} \quad 沿\ x\ 軸 \Rightarrow a_{max} = \mu_1 g$$



F_f, F_f' 是作用力及反作用力偶

N, N' 是作力及反作用力偶

$$N'' - N' - m_2 g = 0 \quad 沿\ y\ 軸 \qquad (24)$$

$$N'' = (m_1 + m_2) g$$

$$F - F_f' - F_f'' = m_2 a \qquad F_f'' = \mu_2 N''$$

$$F - \mu_1 m_1 g - \mu_2 (m_1 + m_2) g = m_2 a_{max} \qquad (25)$$

$$F = \mu_1 (m_1 + m_2) g + \mu_2 (m_1 + m_2) g = (\mu_1 + \mu_2)(m_1 + m_2) g \qquad (26)$$

若將 $m_1 + m_2$ 看成一單一系統，則沿 x 軸之公式

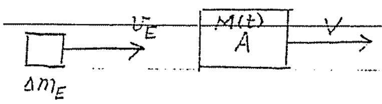
$F - F_f'' = (m_1 + m_2) a_{max} \Rightarrow F = (\mu_1 + \mu_2)(m_1 + m_2) g$，得到同樣的結果 (27)

分類：
編號： 8
總號：

我們現在來討論質量是時間的函數。我們將只討論一度空間的運動，此一結果能很容易地推廣至三度空間的情況。

在一慣性座標中一系統由 (i) 系統 A 其質量為 $M(t)$，$V(t)$，在 Δt 時間內 Δm_L 將離開系統 (A)，(ii) Δm_E（在系統 A 以外）其速度為 v_E

（$V(t)$，v_E 均是在慣性座標所度量的）所組成的，我們將稱此系統為 I.

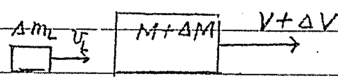


在 Δt 時間內 $M(t)$ 中將有 Δm_L 離開系統 A

在 t 時，系統 I 之總動量 $= \Delta m_E \vec{v}_E + M\vec{V}$ (28)

在 t 時，系統 I 之總質量 $= \Delta m_E + M$ (29)

在 $t+\Delta t$ 時，系統 I 是由 (i) 系統 A 其質量變成 $M+\Delta M$，速度變成 $V+\Delta V$ (ii) Δm_L（在系統外）其速度為 $\vec{v}_L$（$\vec{V}+\Delta\vec{V}$，$\vec{v}_L$ 均是在原來的慣性系統中所度量的）



在 $t+\Delta t$ 時系統 I 之總動量 $= (M+\Delta M)(\vec{V}+\Delta\vec{V}) + (\Delta m_L)\vec{v}_L$ (30)

在 $t+\Delta t$ 時系統 I 之總質量 $= (M+\Delta M) + \Delta m_L$ (31)

若系統 I 不受外力，而且是一封閉系統，則動量及質量均守恆，因此

$$\Delta m_E v_E + MV = M\vec{V} + (\Delta M)\vec{V} + M\Delta\vec{V} + (\Delta M)(\Delta\vec{V}) + (\Delta m_L)\vec{v}_L \quad (32)$$

$$\Delta m_E + M = M + \Delta M + \Delta m_L \quad (33)$$

將以上兩式左右方各除以 Δt，然後令 $\Delta t \to 0$（$(\Delta M)(\Delta V)$ 將趨近於 0），我們獲得

$$\frac{dm_E}{dt}\vec{v}_E = \frac{dM}{dt}\vec{V} + M\frac{d\vec{V}}{dt} + \frac{dm_L}{dt}\vec{v}_L \quad (34)$$

$$\frac{dm_E}{dt} = \frac{dM}{dt} + \frac{dm_L}{dt} \quad (35)$$

$$M\frac{d\vec{V}}{dt} = \frac{dm_E}{dt}\vec{v}_E - \frac{dm_L}{dt}\vec{v}_L - \left(\frac{dm_E}{dt} - \frac{dm_L}{dt}\right)\vec{V} \quad (36)$$

分類：	
編號：	9
總號：	

為討論質量為時間函数之基本公式。

$\dot{m}_E = 0$，則 (36) 式變成

$$M\frac{d\vec{V}}{dt} = -\frac{dm_L}{dt}\vec{v}_L + \frac{dm_L}{dt}\vec{V} \qquad \frac{dm_L}{dt} = -\frac{dM}{dt}$$

$$= \frac{dM}{dt}\vec{v}_L - \frac{dM}{dt}\vec{V} \qquad (37)$$

系統 I 受外力，則

$$\frac{d\vec{P}}{dt} = \frac{dM}{dt}\vec{V} + M\frac{d\vec{V}}{dt} + \frac{dm_L}{dt}\vec{v}_L - \frac{dm_E}{dt}\vec{v}_E \qquad (38)$$

$$\frac{dM}{dt} = \frac{dm_E}{dt} - \frac{dm_L}{dt} \qquad (39)$$

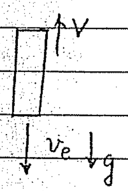
若 $m_E = 0$，則

$$\frac{d\vec{P}}{dt} = \frac{dM}{dt}\vec{V} + M\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{v}_L \qquad \frac{dM}{dt} = -\frac{dm_L}{dt} \qquad (40)$$

$\vec{v}_e = \vec{v}_L - \vec{V} =$ m_L 與 M 之相對速度

$$\vec{F} = M\frac{d\vec{V}}{dt} - \vec{v}_e\frac{dM}{dt} \qquad \text{（第 7.24 式）} \qquad (41)$$

應用至火箭時，$\vec{V}$ 向上，$\vec{v}_e$ 向下，$\vec{F}$ 向下 $|\vec{F}| = Mg$（我們不考慮 g 隨高度之變化，同時假設 v_e 是常数，空氣阻力也略去不計）



因此，

$$M\frac{dV}{dt} + v_e\frac{dM}{dt} = -Mg \qquad (42)$$

$$\frac{dV}{dt} + \frac{v_e}{M}\frac{dM}{dt} = -g \qquad (43)$$

$$\int_{V_0}^{V} dV + v_e\int_{M_0}^{M}\frac{dM}{M} = -g\int_{t_0}^{t} dt$$

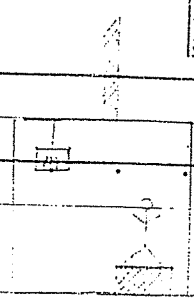
$t = t_0$ 時，火箭之速度為 V_0，質量為 M_0

$t = t$ 時，火箭之速度為 V，質量為 M

$$(V - V_0) + v_e \ln\frac{M}{M_0} = -g(t - t_0)$$

因此 $$V = V_0 + V_e \ln\frac{M_0}{M} - g(t - t_0) \qquad (44)$$

一個電梯其質量為 200 kg，其中有一 75 kg 之人站於磅秤之上，同時在天花板用繩子掛着一質量為 25 kg 之物体，整個電梯是掛在一纜线上可使電梯上下運動。



(a) 繪出整個電梯之中[illegible]東西之系統的自由力圖

並標出[illegible]有[illegible]系統所受之力 [[illegible]]

(b) 繪出電梯之人的自由力圖，並標出[illegible]人

所受之所有之力 [Y]

(c) 繪出掛著的物体的自由力圖，並標出所有作用於此物体之力。[F]

(d) 假若電梯以 5 m/sec 等速上昇時，纜[illegible]上之張力為何？[U]

(e) 假若電梯以均勻加速上昇，並[illegible]秒中由靜止達到 [illegible] m/sec 之速度，則纜繩上之張力為何？[C]

(f) 假若以如 (e) 項中同樣之加速度下降，此時纜繩上之張力為何？[BB]

(g) 假若以如 (d) 項中之等速上昇時，磅秤對人所施之力為何？[J]

(h) 如 (g) 項之情形，人對磅秤所施之力為何？[B]

(i) 此時，磅秤上所量的結果為何（假設它自動將所受之力除以 g）？[X]

(j) 當電梯在 (e) 項的情況時，磅秤上所量之結果為何？[I]

(k) 當電梯在 (f) 項的情況時，磅秤上所量之結果為何？[N]

(l) 當電梯在 (d) 項的情況時，掛著 25 Kg 物体之繩上所受之張力為何？[L]

(m) 當電梯在 (e) 項的情況時，掛著 25 Kg 物体之繩上所受之張力為何？[CC]

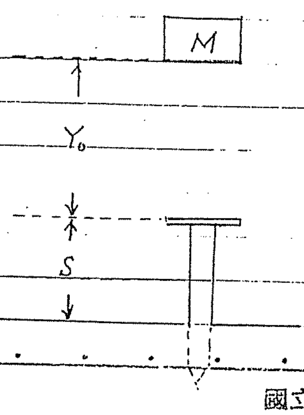
(n) 當電梯在 (f) 項的情況時，掛著 25 Kg 物体之繩上所受之張力為何？[A]

2) 一根部分插入木塊之釘子，利用

一個質量下落將其更敲深。

質點 M 之物体由距釘頭 Y_0

處下落而將釘敲深 s 之距離



分類：	
編號：	11
總號：	

(a) 在質量 M 與釘子接觸前一刻，質量 M 之動量為何？ [W]

(b) 假設在釘釘子的期間中，作用於 M 上之力為常數，繪出在此段時間內 M 系統之自由力圖，並標出它所受之所有的力。 [B]

(c) 在此段時間內，M 之加速度之方向為何？ [H]

(d) 算出此一加速度之大小。 [K]

(e) 算出作用於 M 上淨力之大小及方向。 [P]

(f) 在此段時間內，M 施於釘子之力為何？ [E]

(g) 假如需要平均為 80 (lbf) 之力才能將釘子釘深 ¼ 吋，問若用 4 (lbf) 的物体達成此目的，問需由多高處拋下此物体？ [Z]

(h) 將釘子釘深距離 S 中，質量 M 之動量變化為何？ [T]

(i) 算將釘子釘深距離 S 所需之時間為何？ 利用運動學中等加速運動的公式來計算所需之時間。兩者結果相同嗎？

一個救火水管其橫切面之面積為 A，密度為 ρ 之水由此水管射出

(a) 在一秒中射出水之体積為何？在 T 秒中射出之水的体積為何？ [M]

(b) 在 T 秒中，由水管中射出液体之質量為何？ [D]

(c) 在 T 秒中，由水管中射出液体所帶之動量為何？ [DD]

(d) 在 T 秒中，水管所得到之反彈動量為何？ [G]

(e) 若要使水管固定，你需要加的平均外力為何？ [R]

(f) 水之密度為 $10^3\ kg/m^3$，水管每分鐘噴出 1000 立分的液体，其速度為 90 m/sec，求要使水管在噴射過程中固定所需之力為何？ [V]

(g) 4.45 Newton 等於 1 lbf，將 (f) 項之結果以 lbf 之單位表出。 [AA]

答案

[A] 205 Newtons [B] 重量向下，由釘頭產生之力向上 [C] 3420 Newtons

[D] ρVAT [E] $Mg[1+\frac{Y_0}{s}]$ 向下 [F] 25g 向下，繩上之張力向上

[G] ρV^2AT，与液体之流向相反 [H] 向上 [I] 87.2 Kg [J] 735 Newtons 向上 [K] $\frac{gY_0}{s}$ [L] 245 Newtons [M] VA, VAT [N] 62.8 Kg

[O] $(2s^2g^{-1}Y_0^{-1})^{\frac{1}{2}}$ [P] MgY_0/s 向上 [Q] 由纜線上來的向上之力，一個 300×9.8 Newtons 向下之力（來自電梯及其中所有東西之重量） [R] ρV^2A

[S] 735 Newtons 向下 [T] $M\sqrt{2gY_0}$ 向上 [U] 2940 Newtons

[V] 1.5×10^3 Newtons [W] $M\sqrt{2gY_0}$ 向下 [X] 75 Kg

[Y] 75g 向下，磅秤對其施一向上之力 [Z] 19 吋 [AA] 337 lbf

[BB] 2460 Newtons [CC] 285 Newtons [DD] ρV^2AT

Friction

Surfaces in contact exert two forces on each other

- Normal: force $\perp$ to surfaces
- Parallel: friction

Force of friction always opposes relative motion
or potential relative motion of the surfaces

- Frictional forces

Frictional forces play an important role in the
motion of real objects

Arise from adhesion between atoms in the two
surface

Microscopic level description is very complicated

Macroscopic level description is purely empirical.

- proportional to normal force
- independent of area of area of contact
- independent of speed.

分類：
編號：
總號：

Kinetic Friction

· Surfaces in relative motion

$$f_k = \mu_k N$$

μ_k = coefficient of kinetic friction ($0 < \mu_k < 1$)

N = contact, normal force

· proportional to N

· $\vec{f}_k$ parallel to the surface of contact

opposite to the direction of motion.

· law is approximate and empirical

· μ_k depends on the nature of the materials

μ_k independent of v over wide range

The friction force on each interacting body is opposite in direction to the motion of that body relative to the other.

Some Common Force

Gravity

Normal force

Tension

Friction

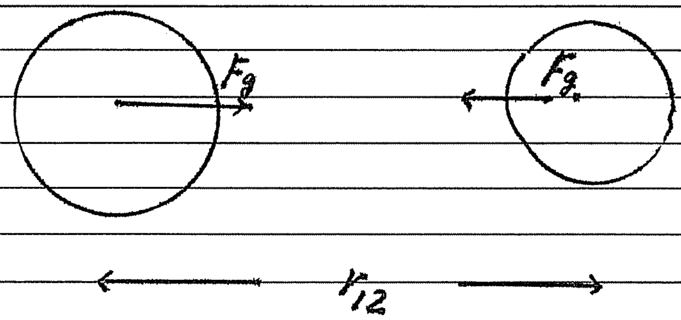
Spring

Gravitational Force

Newton's Postulate: Every pair of particles in the universe exerts on one another a mutual gravitational force of attraction.

This force is proportional to the product of the masses and inversely proportional to the square of the distance between them

$$F_g = \frac{G m_1 m_2}{r_{12}^2}$$



$= 6.673 \times 10^{-11}\ N \cdot m^2/kg^2$

universal gravitational constant

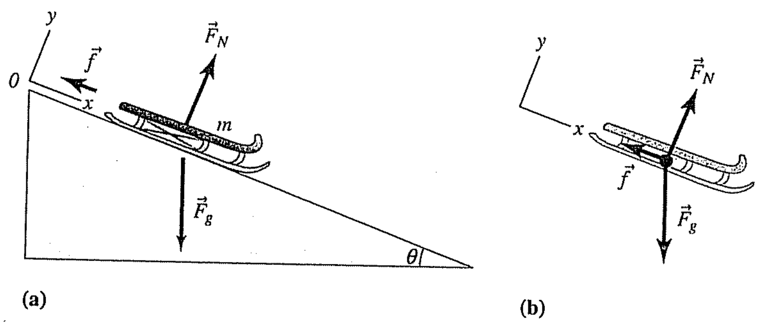
Consider a mass m at the surface of the earth which has a mass M_E and radius R_E

$$F_g = m\left(\frac{M_E G}{R_E^2}\right) = mg$$

pointing toward the center of the earth

Gravitional constant G can be determined by measuring the force with known masses that are seperated by a known distance [Cavendish Experiment]

Normal Force $\vec{F}_N$

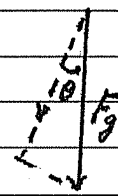


▶ **FIGURE 4–22** (a) A sled on an inclined plane, with the forces acting on it. (b) Free-body diagram for the sled. (c) The force of gravity is decomposed into components perpendicular and parallel to the plane.

Example

A sled of mass m moving down a snow covered hill.

Three forces acting on the sled



$\vec{F}_g$ gravitional force

$\vec{f}$ friction force

$\vec{F}_N$ normal force

exerted by the hill on the sled

If $\vec{F}_N$ is absent, the force of gravity would cause the sled the sled to accelerate down into the hill.

Choose the coordinates as shown

In the x direction

$$F_g \sin\theta - f = ma_x$$

In the y direction

$$-F_g \cos\theta + F_N = 0 = ma_y$$

分類:
編號:
總號:

$$\Rightarrow \quad F_N = F_g \cos\theta$$

The direction of the normal force is normal to the surface.

Hooke's Law - Springs

All bodies are elastic to varying degrees. When a stretching or compressing force is applied they deform.

- steel balls
- rubber bands
- springs

A body resists deformation by means of a <u>restoring force</u>.

Pull on a spring - it pulls back.

In many cases the relationship between the resulting restoring force and the deformation obeys a simple empirical law known as <u>Hooke's Law.</u>

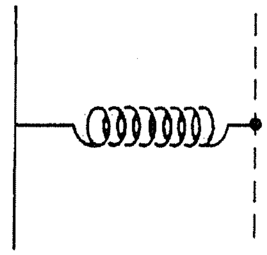
"The magnitude of the restoring force is directly proportional to the deformation"

- approximate
- empirical description
- good for small deformations

Coil Spring

$+x$ - spring stretched

$-x$ - spring compressed



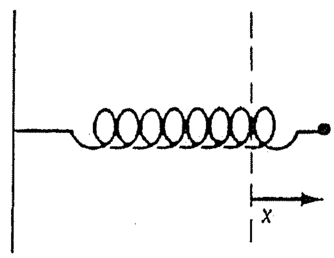
Spring, relaxed.

Hooke's Law:

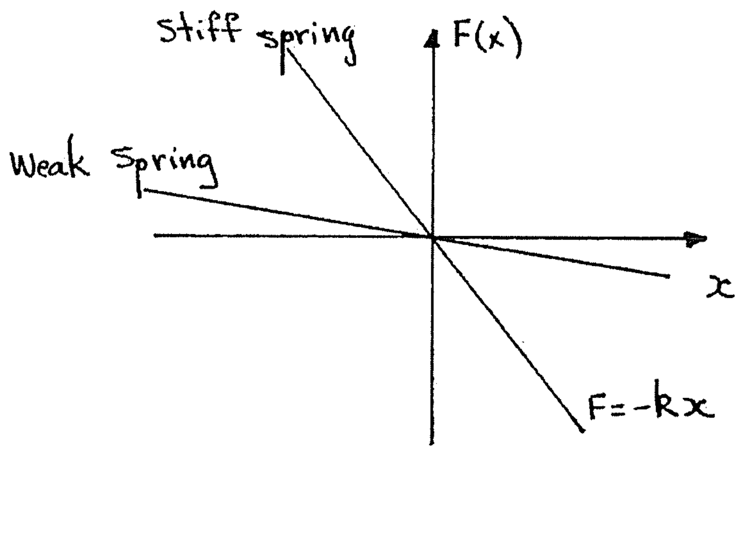
$$F = -kx$$

spring constant

Negative sign means the restoring force opposes the deformation.



Spring, stretched by a length x.

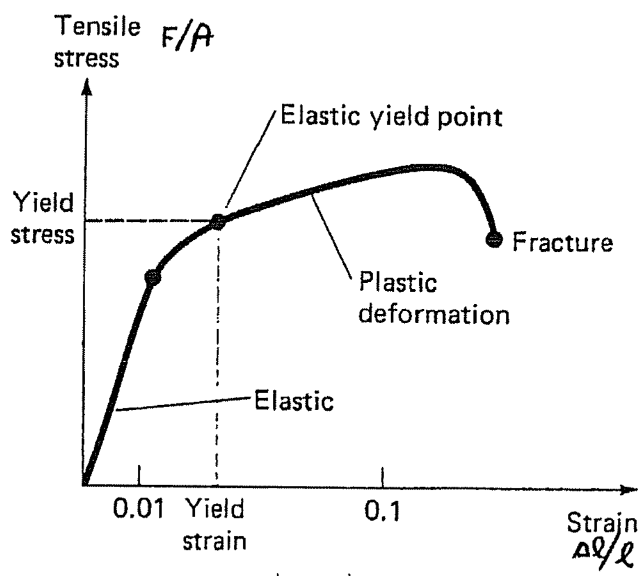


$[k] = N/m$

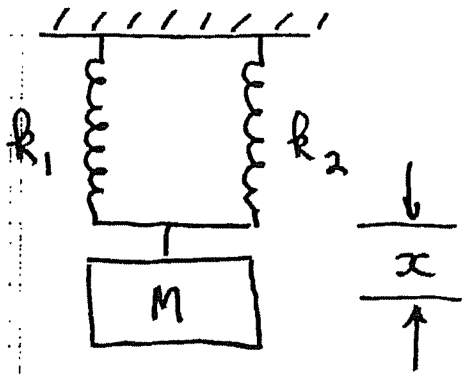
k large: stiff spring
large force / unit displacement

k small: soft spring
small force / unit displacement

+x -elongation ; Force negative opposes stretching
-x -compression ; Force positive opposes compression



Parallel Springs



$$-k_1 x - k_2 x + Mg = 0$$

$$x(k_1 + k_2) = Mg$$

$$x = \frac{Mg}{(k_1 + k_2)} = \frac{Mg}{k_{eff}} \qquad k_{eff} = k_1 + k_2$$

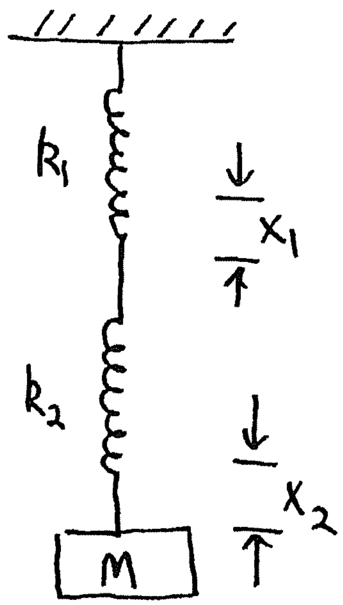
$k_{eff} \Rightarrow$ Stiffer Spring

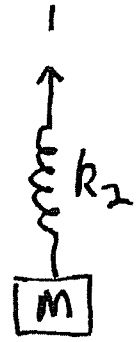
If $k_1 = k_2 = k$

$$k_{eff} = 2k$$

Twice as stiff

Series Springs





$T = Mg$

$\therefore k_2 X_2 = Mg$

Also $k_1 x_1 = Mg$

$$x = x_1 + x_2 = \frac{Mg}{k_1} + \frac{Mg}{k_2}$$

$$x = Mg\left[\frac{1}{k_1} + \frac{1}{k_2}\right]$$

Let $\dfrac{1}{k_{eff}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$ Weaker Spring

If $k_1 = k_2 = k$

$k_{eff} = \frac{1}{2} k$ weaker

Friction

Surfaces in contact exert two forces on each other:

i) Normal; force $\perp$ to surfaces.

ii) Parallel; friction

- Force of friction always opposes relative motion or potential relative motion of the surfaces.

Frictional Forces

- Frictional forces play an important role in the motion of real objects
- Arise from adhesion between atoms in the two surfaces.
- Microscopic level description is very complicated.
- Macroscopic level description is purely empirical (L. da. Vinci)
  - proportional to normal force between surfaces
  - independent of area of contact
  - independent of speed

Kinetic Friction

- Surfaces in relative motion

$$f_k = \mu_k N$$

μ_k = coefficient of kinetic friction $(0 < \mu_k < 1)$
N = contact (normal) force

- proportional to N
- $\vec{f}_k$ parallel to the surface of contact
- opposite to the direction of motion
- law is approximate and empirical.
- μ_k depends on the nature of the materials.
- μ_k independent of v over wide range

The friction force on each interacting body is opposite in direction to the motion of that body relative to the other.

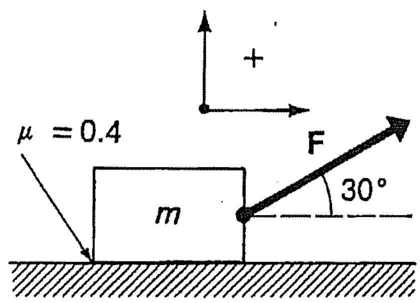
Example

$m = 100\,kg$

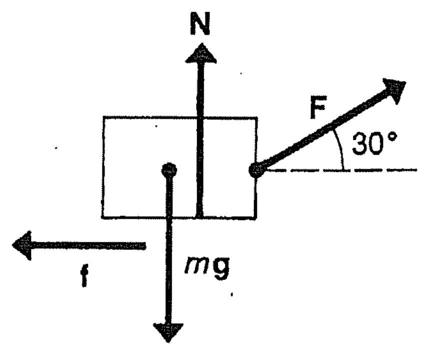
$\mu_k = 0.40$

Crate is moved forward at constant speed.

$\vec{a} \equiv 0.$



What is F ?



Vertical force components:

$$N + F \sin 30^\circ - mg = 0 \qquad (1)$$

Horizontal force components

$$F \cos 30^\circ - f_k = 0 \qquad (2)$$

$$f_k = \mu_k N \qquad (3)$$

$$F \cos 30^\circ - \mu_k \left[mg - F \sin 30^\circ \right] = 0$$

$$F = \frac{\mu_k mg}{\cos 30^\circ + \mu_k \sin 30^\circ}$$

$$F = \frac{0.40 \times 100 \times 9.81}{0.866 + 0.40 \times 0.50} = 368\ N$$

$\theta = 0^\circ \quad F = 392\ N$

$\theta = 45^\circ \quad F = 396\ N$

$\theta = 90^\circ \quad F = 981\ N$

Static Friction

- Frictional forces also act between surfaces that are at rest (no relative motion)
- Objects at rest require a non-zero force to start them moving.

$$f_s \leq \mu_s N$$

μ_s = coefficient of static friction
N = contact (normal) force

- Force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) up to a maximum value of $\mu_s N$. Equality sign holds when motion is about to start.

  - proportional to normal force
  - independent of area
  - empirical law
  - opposite to lateral push that tries to move the body.
  - Usually $\mu_s > \mu_k$, so that once block starts moving it will take less force to keep it from accelerating.
  - μ_s depends on nature and condition of surfaces

Example: Block on Surface.

a) No motion

$$f_1 < \mu_s N$$

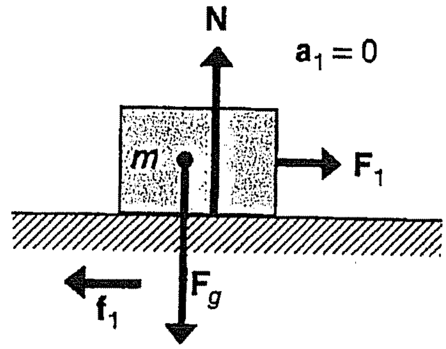
b) Motion impends

$$f_2 = \mu_s N$$

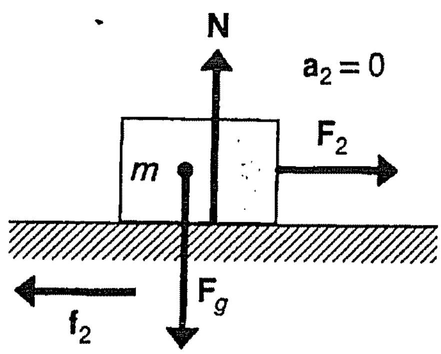
c) Motion exists

$$f_3 = \mu_k N$$

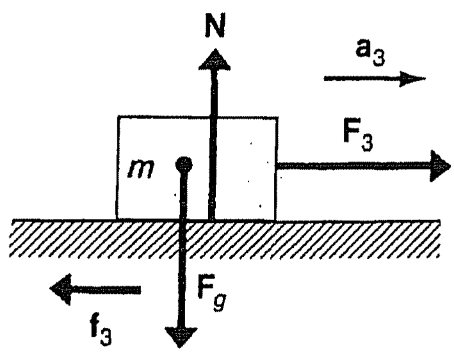
Fig. (a, b) For $f_s < f_{s,max}$, the frictional force exactly balances the applied force; then, there is no acceleration. (c) When a force sufficient to cause motion is applied, the frictional force is equal to $\mu_k N$ and the acceleration is $(F - \mu_k N)/m$.



(a) $\mathbf{f}_1 = -\mathbf{F}_1$

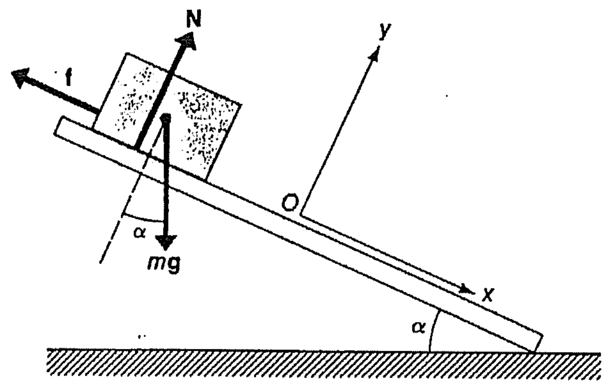


(b) $\mathbf{f}_2 = -\mathbf{F}_2$



(c) $f_3 = \mu_k N$

Example : Block-on-Plane



Block starts to slip at $\alpha = 23°$, what is coefficient of static friction, μ_s?

(y-axis) $\quad N - mg\cos\alpha = 0$

(x-axis) $\quad mg\sin\alpha - f = 0$

$$\left.\begin{array}{l} N - mg\cos\alpha = 0 \\ mg\sin\alpha - f = 0 \end{array}\right\} \vec{a} = 0$$

$$\frac{f}{N} = \frac{mg\sin\alpha}{mg\cos\alpha} = \tan\alpha$$

$$\left(\frac{f}{N}\right)_{Max} = \mu_s = \tan 23° = 0.424$$

Maximum angle is called the angle of repose.
It is independent of the mass of the block.

分類：
編號：
總號：

3. 摩擦力

两个相互接触的物体(指固体)沿着接触面的方向有相对滑动时(图 2.4)，在各自的接触面上都受到阻止相对滑动的力。这种力叫**滑动摩擦力**，它的方向总是与相对滑动的方向相反。实验证明当相对滑动的速度不是太大或太小时；滑动摩擦力 f_k 的大小和滑动速度无关而和正压力 N 成正比，即

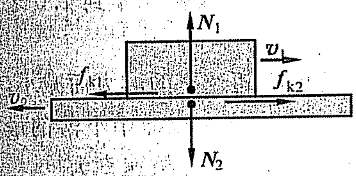


图 2.4　滑动摩擦力

$$f_k = \mu_k N \tag{2.12}$$

式中 μ_k 为**滑动摩擦系数**，它与接触面的材料和表面的状态(如光滑与否)有关。一些典型情况的 μ_k 的数值列在表 2.2 中，它们都只是粗略的数值。

表 2.2　一些典型情况的摩擦系数

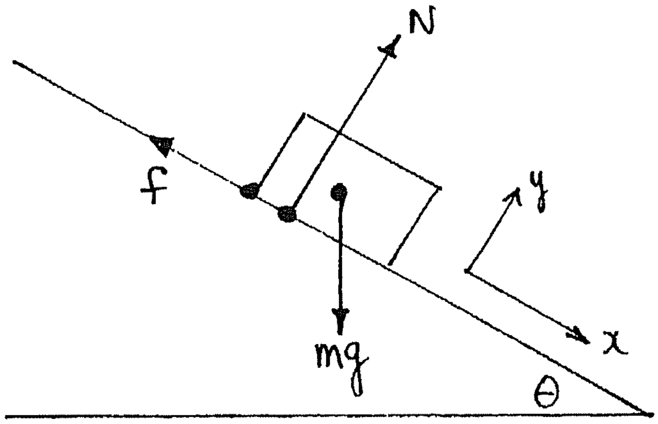
接触面材料	μ_k	μ_s
钢—钢(干净表面)	0.6	0.7
钢—钢(加润滑剂)	0.05	0.09
铜—钢	0.4	0.5
铜—铸铁	0.3	1.0
玻璃—玻璃	0.4	0.9～1.0
橡胶—水泥路面	0.8	1.0
特氟隆—特氟隆(聚四氟乙烯)	0.04	0.04
涂蜡木滑雪板—干雪面	0.04	0.04

当有接触面的两个物体相对静止但有相对滑动的趋势时，它们之间产生的阻碍相对滑动的摩擦力叫**静摩擦力**。静摩擦力的大小是可以改变的。例如人推木箱，推力不大时，木箱不动。木箱所受的静摩擦力 f_s 一定等于人的推力 f。当人的推力大到一定程度时，木箱就要被推动了。这说明静摩擦力有一定限度，叫做**最大静摩擦力**。实验证明，最大静摩擦力 $f_{s\max}$ 与两物体之间的正压力 N 成正比，即

$$f_{s\max} = \mu_s N \tag{2.13}$$

式中 μ_s 叫**静摩擦系数**，它也取决于接触面的材料与表面的状态。对同样的两个接触面，静摩擦系数 μ_s 总是大于滑动摩擦系数 μ_k。一些典型情况的静摩擦系数也列在表 2.2 中，它们也都只是粗略的数值。

Example: Block-down-Plane



$$f = \mu_k N$$

(x-Axis) $\quad mg\sin\theta - f = m a_x$

(y-Axis) $\quad N - mg\cos\theta = m a_y = 0$

$$\therefore\ N = mg\cos\theta$$

$$\cancel{mg}(\sin\theta - \mu_k\cos\theta) = \cancel{m}a_x$$

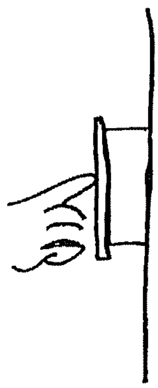
$$a_x = (\sin\theta - \mu_k\cos\theta)\, g$$

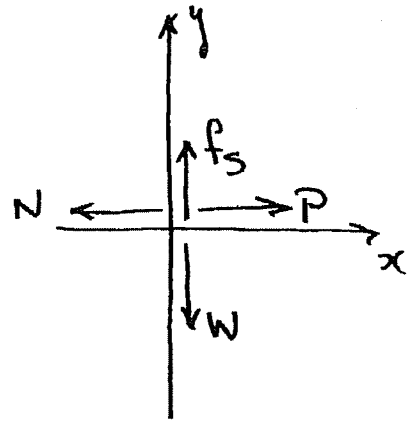
If $\quad a_x = 0$

$$\mu_k = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\Rightarrow$ angle at which object will move down the plane at constant velocity for a given μ_k. Measure of μ_k.

Example





$$\sum F_x = P - N = 0 \qquad (1)$$

$$\sum F_y = f_s - W = 0 \qquad (2)$$

From (2) $\quad f_s = W$

(1) $\quad P = N$

$$f_s \le \mu_s P$$

For no slipping $\quad f_s \ge W$

$$\therefore \mu_s P \ge W$$

$$P \ge \frac{W}{\mu_s}$$

$$P = \frac{W}{\mu_s} \equiv \text{ minimom push needed.}$$

三、摩擦力

两个互相接触的物体间有相对滑动的趋势但尚未相对滑动时，在接触面上便产生阻碍发生相对滑动的力，这个力称为静摩擦力．把物体放在一水平面上，有一外力 $\boldsymbol{F}$ 沿水平面作用在物体上，若外力 $\boldsymbol{F}$ 较小，物体尚未滑动，这时静摩擦力 $\boldsymbol{F}_{f0}$ 与外力 $\boldsymbol{F}$ 在数值上相等，方向则与 $\boldsymbol{F}$ 相反．随着 $\boldsymbol{F}$ 的增大静摩擦力 $\boldsymbol{F}_{f0}$ 也相应增大，直到 $\boldsymbol{F}$ 增大到某一定数值时，物体即将滑动，静摩擦力达到最大值，称为最大静摩擦力 F_{f0m}．实验表明，最大静摩擦力的值与物体的正压力 F_N 成正比，即

$$F_{f0m}=\mu_0 F_N$$

μ_0 叫做静摩擦因数．静摩擦因数与两接触物体的材料性质以及接触面的情况有关，而与接触面的大小无关．应强调指出，在一般情况下，静摩擦力总是满足下述关系的：

$$F_{f0}\leqslant F_{f0m}$$

物体在平面上滑动时所受摩擦力叫做滑动摩擦力 F_f，其方向总是与物体相对平面的运动方向相反，其大小也是与物体的正压力 F_N 成正比，即

$$F_f=\mu F_N$$

μ 叫做滑动摩擦因数．μ 与两接触物体的材料性质、接触表面的情况、温度、干湿度等有关，还与两接触物体的相对速度有关．在相对速度不太大时，为计算简单起见，可以认为滑动摩擦因数 μ 略小于静摩擦因数 μ_0；在一般计算时，除非特别指明，可认为它们是近似相等的，即 $\mu\approx\mu_0$．

摩擦产生的影响有利弊两个方面．所有机器的运动部分都有摩擦，它既磨损机器又浪费大量能量，而且由于摩擦会使机器局部温度升高，从而降低机器的精度，这是摩擦有害的一面．为此，必须设法减少摩擦，通常是在产生有害摩擦的部

位涂以润滑油，或者以滚动摩擦替代滑动摩擦，或者改变摩擦材料的性能等．此外，摩擦也是生产和生活中所必需的．很难想像，没有摩擦的自然界会是什么情况，人的行走，车轮的滚动，货物借助皮带输送等等，都是依赖于摩擦才能进行的．下面所举的例 2 中，绳索与圆柱体之间的摩擦在日常生活和生产中是经常遇到的．

(3) 摩擦力

行驶着的汽车，当发动机关闭后，走一段距离就会停下来。我们推桌子时，如果用力较小就推不动。这些现象说明，当互相接触的物体作相对运动或有相对运动的趋势时，它们之间就有摩擦力。

摩擦有干摩擦和湿摩擦两种。干摩擦是固体表面之间的摩擦，又叫外摩擦；湿摩擦是液体内部或液体和固体的摩擦，又叫内摩擦。此外干摩擦又分静摩擦和滑动摩擦、滚动摩擦[1]。现在我们只介绍静摩擦和滑动摩擦。

静摩擦：设有两个物体 A 和 B（如货物和地板）相互接触，如图 2－25 所示。我们推货物时如果用力 F 较小就推不动。A 不动的事实表明，B 对 A 的摩擦力和外力 F 大小相等，方向相反，这种摩擦力是在 A 和 B 相对静止但却具有相对运动趋势

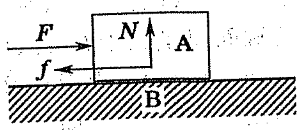


图 2－25 摩擦力

[1] 刚体的滚动靠静摩擦（见第四章），在这里静摩擦不作功，没有阻力，不耗散能量。所谓"滚动摩擦"，是指因形变（轮胎瘪，地面下陷等）造成的阻力和耗散。

的情况下发生的，称为静摩擦力。当外力逐渐增大时，静摩擦力也增大。但当外力达到某一数值时，A 开始移动。可见静摩擦力增到一定数值后就不能再增大了，这一数值的静摩擦力叫做*最大静摩擦力*。实验证明：*最大静摩擦力 f_0 与接触面间的正压力 N 成正比*，即

$$f_0 = \mu N, \tag{2.35}$$

式中的 μ 叫做*静摩擦系数*，[1]它由相互接触物体的质料和表面情况（如粗糙程度、干湿程度）决定。表 2－1 列举了某些 μ 的数值。

表 2－1 静摩擦系数 μ

相互接触的物体对	μ
钢－钢（干面）	0.15
钢－钢（涂油面）	0.12
金属－木材（干面）	0.5
金属－木材（涂油面）	0.1
金属－皮带（干面）	0.56

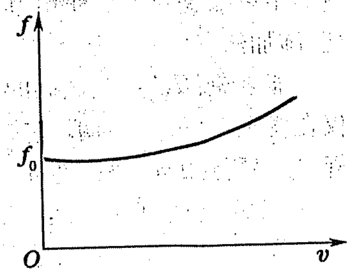


图 2－26 摩擦力与速度的关系

必须注意，静摩擦力的大小由外力的大小决定，可随外力的增大取 0 到 f_0 之间的各个数值。当外力 $F \geqslant f_0$ 时，物体 A 相对于 B 发生运动。

滑动摩擦：当物体间有相对滑动时，出现一种阻止物体间相对运动的表面接触力，这个力和相对运动速度方向相反，叫做*滑动摩擦力*。滑动摩擦力不但与物体的质料、表面情况以及正压力有关，一般还和相对速度 v 有关。在滑动刚开始发生时，滑动摩擦力比最大静摩擦力小，而且随着相对速度的增大而继续减少，以后又随着相对速度的增大而增加。滑动摩擦力 f 随相对速度 v 的变化关系可粗略用图 2－26 表示。

实验证明，*滑动摩擦力 f 也和正压力 N 成正比*，即

$$f = \mu' N, \tag{2.36}$$

式中 μ' 称为*滑动摩擦系数*，它和相对滑动速度 v 有关。

对于两个给定表面，滑动摩擦实际上与接触表面面积的大小无关，对这个事实也许有人觉得奇怪。按照目前较流行的一种理论认为，这是因为实际接触面积是属于原子尺度的，它只占总的几何接触面积的一个极微小的部分，而摩擦力的出现是由于在原子接触的这些微小区域内原子之间的相互作用力。原子接触面积占几何接触面积的比例，正比于法向力除以几何接触面积。因此，当法向力增大一倍，原子接触面积也增大一倍，摩擦力便增大一

[1] 本书中今后不特别指明为滑动摩擦力或滑动摩擦系数时，"摩擦力"和"摩擦系数"均指静摩擦力和静摩擦系数。

倍，这就是摩擦力正比于正压力的原因。但是，如果几何接触面积增加一倍，而法向力保持不变，则原子接触面积占几何接触面积的比例减小一半，即原子接触面积的实际面积不变，因而摩擦力也不变。对于非刚性的物体，例如汽车轮胎等，摩擦力的情况更为复杂。有关摩擦力的起因及微观机理，尚有许多未知领域，有待进一步探讨。

摩擦在实际中具有很重要的意义。摩擦的害处主要是消耗大量有用的能源，使机器的运转部件发热，甚至烧毁，因而不得不进行冷却。减少摩擦的主要方法是化滑动为滚动，例如在机器中尽量使用滚珠轴承，另外是变干摩擦为湿摩擦，例如加润滑油。近年来已愈来愈多采用气垫悬浮和磁悬浮的先进技术来减少摩擦。另一方面，在许多场合下摩擦是必要的。例如人的行走，任何车辆的开动与制动，机器的传动（皮带轮），弦乐器（二胡、提琴等）的演奏……，没有摩擦或摩擦过小都不行，这时往往要想办法增大摩擦，例如在鞋底和轮胎上弄上些花纹。在失重状态下悬浮在飞船舱内的宇航员，因完全受不到摩擦力，他们的那种奇妙感受，是我们这些平常人从来也没有经历过的。如果不事先把自己的身体固定在舱壁的某件东西上，当他想开抽屉时，不但抽屉未被拉开，自己反而被拉过去；当他想拧紧螺丝钉时，螺丝钉未被拧动，自己的身体反而朝反方向旋转起来。

现在我们可以来裁决某甲和某乙围绕那场拔河比赛的争论了，胜负的关键在于脚下的摩擦力。❶如果两人脚下地面的摩擦系数一样，大力士可能得到的最大静摩擦力 f_2 比瘦子的 f_1 大（图 2－27a）。尽管两人拉绳子两端的力 f 总是大小相等，方向相反，当 $f_2 > f > f_1$ 时，大力士就能把瘦子拉过去。若让大力士站在光滑的冰上，而瘦子站在粗糙的土地上，大力士有再大的力气也赢不了。所以，某甲错了，某乙是对的。

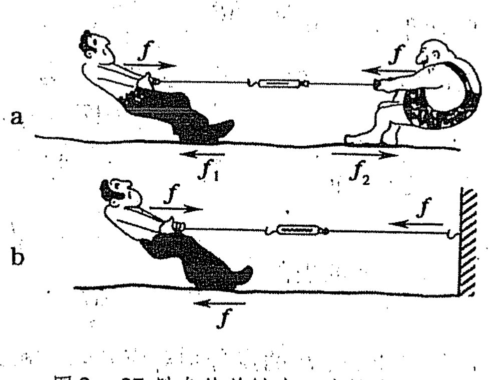


图 2－27 胜负的关键在于摩擦力

最后，我们看看摩擦力对绳子中张力分布的影响。一种称为绞盘的装

❶ 这里推论的是拔河的简化模型，把人体看成没有其它躯体动作（如下蹲、用力后仰）的刚体。实际情况当然要复杂得多。

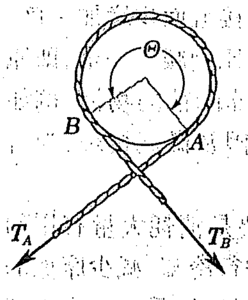


图2－28 绞盘装置

置，绳索绕在绞盘的固定圆柱上，当绳子承受负荷巨大的拉力 T_A，人可以用小得多的力 T_B 拽住绳子。设绳与圆柱的摩擦系数为 μ，绳子绕圆柱的张角为 Θ（见图2－28）。下面分析这个问题。

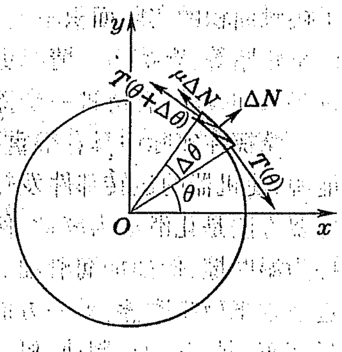


图2－29 摩擦力对张力分布的影响

如图2－29所示，用隔离体法，考虑在 θ 处对圆心张角 $\Delta\theta$ 的一段线元，分析它受力的情况。略去绳索质量，该线元受四个力的作用：两端张力 $T(\theta)$，$T(\theta+\Delta\theta)$，法向力 ΔN，和摩擦力 $\mu\Delta N$. 在无加速度的情况下四力的合成为0. 分为切向和法向分量，则有

$$\begin{cases}\text{切向：}\quad [T(\theta+\Delta\theta)-T(\theta)]\cos\dfrac{\Delta\theta}{2}=-\mu\Delta N,\\ \text{法向：}\quad [T(\theta+\Delta\theta)+T(\theta)]\sin\dfrac{\Delta\theta}{2}=\Delta N.\end{cases}$$

因 $\Delta\theta$ 很小，$\sin\dfrac{\Delta\theta}{2}\approx\dfrac{\Delta\theta}{2}$，$\cos\dfrac{\Delta\theta}{2}\approx1$，$T(\theta+\Delta\theta)-T(\theta)\approx\Delta T$（$T$ 的微分增量），$T(\theta+\Delta\theta)+T(\theta)\approx2T$，故上式可写为

$$\begin{cases}\mathrm{d}T=-\mu\Delta N,\\ T\,\Delta\theta=\Delta N.\end{cases}$$

消去 ΔN 可得

$$\frac{\Delta T}{T}=-\mu\Delta\theta,$$

取 $\Delta\theta\to0$ 的极限，

$$\frac{\mathrm{d}T}{T}=-\mu\,\mathrm{d}\theta,$$

设绞盘上 A、B 两点分别对应 $\theta=\theta_A$ 和 θ_B，对上式积分：

$$\int_{T_A}^{T_B}\frac{\mathrm{d}T}{T}=-\mu\int_{\theta_A}^{\theta_B}\mathrm{d}\theta,\quad\text{得}\quad\ln\frac{T_B}{T_A}=-\mu(\theta_B-\theta_A),$$

或

$$T_B=T_A\,\mathrm{e}^{-\mu\Theta},\tag{2.37}$$

式中 $\Theta=\theta_B-\theta_A$. 此式表明，张力随 θ 按指数减小，故很容易做到让 $T_B\ll T_A$.

若摩擦力可忽略，$\mu\to0$，$T_B\approx T_A$，即两端绳的张力相等。这便是轻绳跨过无摩擦滑轮的情况。

4. 流体曳力

一个物体在流体(液体或气体)中和流体有相对运动时,物体会受到流体的阻力,这种阻力称为流体曳力。这曳力的方向和物体相对于流体的速度方向相反,其大小和相对速度的大小有关。在相对速率较小,流体可以从物体周围平顺地流过时,曳力 f_d 的大小和相对速率 v 成正比,即

$$f_d = kv \tag{2.14}$$

式中比例系数 k 决定于物体的大小和形状以及流体的性质(如黏性、密度等)。在相对速率较大以致在物体的后方出现流体旋涡时(一般情形多是这样),曳力的大小将和相对速率的平方成正比。对于物体在空气中运动的情况,曳力的大小可以表示为

$$f_d = \frac{1}{2}C\rho A v^2 \tag{2.15}$$

其中,ρ 是空气的密度;A 是物体的有效横截面积;C 为曳引系数,一般在 0.4 到 1.0 之间(也随速率而变化)。相对速率很大时,曳力还会急剧增大。

由于流体曳力和速率有关,物体在流体中下落时的加速度将随速率的增大而减小,以致当速率足够大时,曳力会和重力平衡而物体将以匀速下落。物体在流体中下落的最大速率叫终极速率。对于在空气中下落的物体,利用式(2.15)可以求得终极速率为

$$v_t = \sqrt{\frac{2mg}{C\rho A}} \tag{2.16}$$

其中 m 为下落物体的质量。

按上式计算,半径为 1.5 mm 的雨滴在空气中下落的终极速率为 7.4 m/s,大约在下落 10 m 时就会达到这个速率。跳伞者,由于伞的面积 A 较大,所以其终极速率也较小,通常为 5 m/s 左右,而且在伞张开后下降几米就会达到这一速率。

浮力 ⇒ 阿基米德原理

5. 表面张力

拿一根缝衣针放到一片薄棉纸上，小心地把它们平放到碗内的水面上。再小心地用细棍把已浸湿的纸按到水下面。你就会看到缝衣针漂在水面上(图 2.5)。这种漂浮并不是水对针的浮力(遵守阿基米德定律)作用的结果，针实际上是躺在已被它压陷了的水面上，是水面兜住了针使之静止的。这说明水面有一种绷紧的力，在水面凹陷处这种绷紧的力 F 抬起了缝衣针。旅游寺庙里盛水的大水缸里常见到落到水底的许多硬币，这都是那些想使自己的硬币漂在水面上(而得到降福?)的游客操作不当的结果。有些昆虫能在水面上行走，也是靠了这种沿水面作用的绷紧的力(图 2.6)。

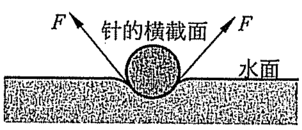


图 2.5 缝衣针漂在水面上

液体表面总处于一种绷紧的状态。这归因于液面各部分之间存在着相互拉紧的力。

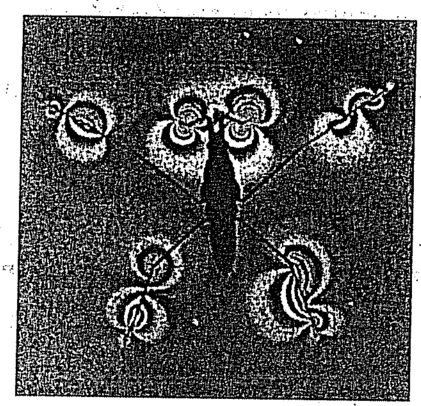


图 2.6 昆虫“水黾”(学名 Hygrotrechus Conformis)在水面上行走以及引起的水面波纹(R. L. Reese)

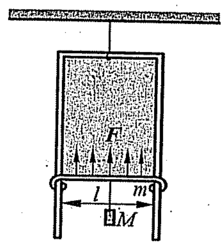


图 2.7 液膜的表面张力

这种力叫**表面张力**。它的方向沿着液面(或其“切面”)并垂直于液面的边界线。它的大小和边界线的长度成正比。以 F 表示在长为 l 的边界线上作用的表面张力，则应有

$$F=\gamma l \tag{2.17}$$

式中 γ(N/m)叫做**表面张力系数**，它的大小由液体的种类及其温度决定。例如在 20℃时，乙醇的 γ 为 0.0223 N/m，水银的为 0.465 N/m，水的为 0.0728 N/m，肥皂液的约为 0.025 N/m 等。

表面张力系数 γ 可用下述方法粗略地测定。用金属细棍做一个一边可以滑动的矩形框(图 2.7)，将框没入液体。当向上缓慢把框提出时，框上就会蒙上一片液膜。这时拉动下侧可动框边再松手时，膜的面积将缩小，这就是膜的表面张力作用的表现。在这一可动框边上挂上适当的砝码，则可以使这一边保持不动，这时应该有

$$F=(m+M)g \tag{2.18}$$

式中 m 和 M 分别表示可动框边和砝码的质量。由于膜有两个表面，所以其下方在两条边线上都有向上的表面张力。以 l 表示膜的宽度，则由式(2.17)，在式(2.18)中应有 $F=2\gamma l$。代入式(2.18)可得

$$\gamma=(m+M)g/2l \tag{2.19}$$

一个液滴由于表面张力，其表面有收缩趋势，这就使得秋天的露珠，夏天荷叶上的小水珠以及肥皂泡都呈球形。天体一般也是球形，这也是在其长期演变过程中表面张力作用的结果。

2.3 基本的自然力

2.2节介绍了几种力的特征，实际上，在日常生活和工程技术中，遇到的力还有很多种。例如皮球内空气对球胆的压力，江河海水对大船的浮力，胶水使两块木板固结在一起的黏结力，两个带电小球之间的吸力或斥力，两个磁铁之间的吸力或斥力等。除了这些宏观世界我们能观察到的力以外，在微观世界中也存在这样或那样的力。例如分子或原子之间的引力或斥力，原子内的电子和核之间的引力，核内粒子和粒子之间的斥力和引力等。尽管力的种类看来如此复杂，但近代科学已经证明，自然界中只存在4种基本的力（或称相互作用），其他的力都是这4种力的不同表现。这4种力是引力、电磁力、强力、弱力，下面分别作一简单介绍。

1. 引力(或万有引力)

引力指存在于任何两个物质质点之间的吸引力。它的规律首先由牛顿发现，称之为引力定律，这个定律说：**任何两个质点都互相吸引，这引力的大小与它们的质量的乘积成正比，和它们的距离的平方成反比。**用 m_1 和 m_2 分别表示两个质点的质量，以 r 表示它们的距离，则引力大小的数学表示式是

$$f = \frac{Gm_1m_2}{r^2} \tag{2.20}$$

式中，f 是两个质点的相互吸引力；G 是一个比例系数，叫**引力常量**，在国际单位制中它的值为

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2} \tag{2.21}$$

式(2.20)中的质量反映了物体的引力性质，是物体与其他物体相互吸引的性质的量度，因此又叫**引力质量**。它和反映物体抵抗运动变化这一性质的惯性质量在意义上是不同的。但是任何物体的重力加速度都相等的实验表明，同一个物体的这两个质量是相等的，因此可以说它们是同一质量的两种表现，也就不必加以区分了。

根据现在尚待证实的物理理论，物体间的引力是以一种叫做"引力子"的粒子作为传递媒介的。

2. 电磁力

电磁力指带电的粒子或带电的宏观物体间的作用力。两个静止的带电粒子之间的作用力由一个类似于引力定律的库仑定律支配着。库仑定律说，两个静止的点电荷相斥或相吸，这斥力或吸力的大小 f 与两个点电荷的电量 q_1 和 q_2 的乘积成正比，而与两电荷的距离 r 的平方成反比，写成公式

$$f = \frac{kq_1q_2}{r^2} \tag{2.22}$$

式中比例系数 k 在国际单位制中的值为

$$k = 9 \times 10^9\ \mathrm{N \cdot m^2 \cdot C^{-2}}$$

这种力比万有引力要大得多。例如两个相邻质子之间的电力按上式计算可以达到 10^2 N，是它们之间的万有引力(10^{-34} N)的 10^{36} 倍。

运动的电荷相互间除了有电力作用外，还有磁力相互作用。磁力实际上是电力的一种表现，或者说，磁力和电力具有同一本源。(关于这一点，本书第3篇电磁学有较详细的讨论。)因此**电力和磁力统称电磁力**。

电荷之间的电磁力是以**光子**作为传递媒介的。

由于分子或原子都是由电荷组成的系统，所以它们之间的作用力就是电磁力。中性分子或原子间也有相互作用力，这是因为虽然每个中性分子或原子的正负电荷数值相等，但在它们内部正负电荷有一定的分布，对外部电荷的作用并没有完全抵消，所以仍显示出

有电磁力的作用。中性分子或原子间的电磁力可以说是一种残余电磁力。2.2 节提到的相互接触的物体之间的弹力、摩擦力、流体阻力、表面张力以及气体压力、浮力、黏结力等都是相互靠近的原子或分子之间的作用力的宏观表现，因而从根本上说也是电磁力。

3. 强力

我们知道，在绝大多数原子核内有不止一个质子。质子之间的电磁力是排斥力，但事实上核的各部分并没有自动飞离，这说明在质子之间还存在一种比电磁力还要强的自然力，正是这种力把原子核内的质子以及中子紧紧地束缚在一起。这种存在于质子、中子、介子等强子之间的作用力称做**强力**。强力是夸克所带的"色荷"之间的作用力——色力——的表现。色力是以**胶子**作为传递媒介的。两个相邻质子之间的强力可以达到 10^4 N。强力的力程，即作用可及的范围非常短。强子之间的距离超过约 10^{-15} m 时，强力就变得很小而可以忽略不计；小于 10^{-15} m 时，强力占主要的支配地位，而且直到距离减小到大约 0.4×10^{-15} m 时，它都表现为吸引力，距离再减小，则强力就表现为斥力。

4. 弱力

弱力也是各种粒子之间的一种相互作用，但仅在粒子间的某些反应(如 β 衰变)中才显示出它的重要性。弱力是以 W^+，W^-，Z^0 等叫做**中间玻色子**的粒子作为传递媒介的。它的力程比强力还要短，而且力很弱。两个相邻的质子之间的弱力大约仅有 10^{-2} N。

表 2.3 中列出了 4 种基本力的特征，其中力的强度是指两个质子中心的距离等于它们直径时的相互作用力。

表 2.3 4 种基本自然力的特征

力的种类	相互作用的物体	力的强度	力 程
万有引力	一切质点	10^{-34} N	无限远
弱力	大多数粒子	10^{-2} N	小于 10^{-17} m
电磁力	电荷	10^2 N	无限远
强力	核子、介子等	10^4 N	10^{-15} m

从复杂纷纭、多种多样的力中，人们认识到基本的自然力只有 4 种，这是 20 世纪 30 年代物理学取得的很大成就。此后，人们就企图发现这 4 种力之间的联系。爱因斯坦就曾企图把万有引力和电磁力统一起来，但没有成功。20 世纪 60 年代，温伯格和萨拉姆在杨振宁等提出的理论基础上，提出了一个把电磁力和弱力统一起来的理论——电弱统一理论。这种理论指出在高能范围内，电磁相互作用和弱相互作用本是同一性质的相互作用，称做**电弱相互作用**。在低于 250 GeV 的能量范围内，由于"对称性的自发破缺"，统一的电弱相互作用分解成了性质极不相同的电磁相互作用和弱相互作用。这种理论已在 20 世纪 70 年代和 80 年代初期被实验证实了。电弱统一理论的成功使人类在对自然界的统一性的认识上又前进了一大步。现在，物理学家正在努力，以期建立起总括电弱色相互作用的"大统一理论"(它管辖的能量尺度为 10^{15} GeV，目前有些预言已被用实验"间接地探索过了")。人们还期望，有朝一日，能最后(?)建立起把 4 种基本相互作用都统一起来的……"超统一理论"。

2.4 应用牛顿定律解题

利用牛顿定律求解力学问题时，最好按下述“三字经”所设计的思路分析。

1. 认物体

在有关问题中选定一个物体(当成质点)作为分析对象。如果问题涉及几个物体，那就一个一个地作为对象进行分析，认出每个物体的质量。

2. 看运动

分析所认定的物体的运动状态，包括它的轨道、速度和加速度。问题涉及几个物体时，还要找出它们之间运动的联系，即它们的速度或加速度之间的关系。

3. 查受力

找出被认定的物体所受的所有外力。画简单的示意图表示物体受力情况与运动情况，这种图叫示力图。

4. 列方程

把上面分析出的质量、加速度和力用牛顿第二定律联系起来列出方程式。利用直角坐标系的分量式(式(2.5))列式时，在图中应注明坐标轴方向。在方程式足够的情况下就可以求解未知量了。

动力学问题一般有两类，一类是已知力的作用情况求运动；另一类是已知运动情况求力。这两类问题的分析方法都是一样的，都可以按上面的步骤进行，只是未知数不同罢了。

Problem-Solving Strategy

1. Draw a diagram indicating all key features in the problem.
2. Draw one or more free-body diagrams for the objects. For the chosen object include all the forces acting on it. Do not include any internal forces. Do not include any forces exerted by the body on some other body.

3. Select a coordinate system and show it in the free-body diagram. Determine components of the forces with reference to these axes. When the direction of acceleration is known in advance — choose that direction as $+x$-axis. Can choose different reference frame for each body. All must be 'inertial'.

4. If there are geometrical relationships between two or more bodies — relate these algebraically!!

5. Write down Newton's Eq. of Motion for each body and solve for unknowns.

$$\vec{F} = m\vec{a}$$

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

$$\Sigma F_z = ma_z$$

6. Check special cases and extreme values of quantities, compare with intuitive expectations. Does the result make sense?

Examples of Solving the Mechanical Problems

例 2.1

用皮带运输机向上运送砖块。设砖块与皮带间的静摩擦系数为 μ_s，砖块的质量为 m，皮带的倾斜角为 α。求皮带向上匀速输送砖块时，它对砖块的静摩擦力多大？

解 认定砖块进行分析。它向上匀速运动，因而加速度为零。在上升过程中，它受力情况如图 2.8 所示。

选 x 轴沿着皮带方向，则对砖块用牛顿第二定律，可得 x 方向的分量式为

$$-mg\sin\alpha + f_s = ma_x = 0$$

由此得砖块受的静摩擦力为

$$f_s = mg\sin\alpha$$

注意，此题不能用公式 $f_s=\mu_s N$ 求静摩擦力，因为这一公式只对最大静摩擦力才适用。在静摩擦力不是最大的情况下，只能根据牛顿定律的要求求出静摩擦力。

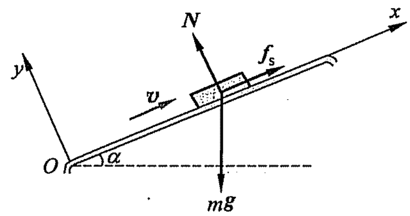


图 2.8 例 2.1 用图

例 2.2

在光滑桌面上放置一质量 $m_1=5.0$ kg 的物块，用绳通过一无摩擦滑轮将它和另一质

量为 $m_2=2.0$ kg 的物块相连。(1)保持两物块静止，需用多大的水平力 $\boldsymbol{F}$ 拉住桌上的物块？(2)换用 $F=30$ N 的水平力向左拉 m_1 时，两物块的加速度和绳中张力 T 的大小各如何？(3)怎样的水平力 $\boldsymbol{F}$ 会使绳中张力为零？

解 如图 2.9 所示，设两物块的加速度分别为 a_1 和 a_2。参照如图所示的坐标方向。

(1) 如两物体均静止，则 $a_1=a_2=0$，用牛顿第二定律，对 m_1，

$$-F+T=m_1a_1=0$$

对 m_2，

$$T-m_2g=m_2a_2=0$$

此二式联立给出

$$F=m_2g=2.0\times 9.8=19.6\ (\mathrm{N})$$

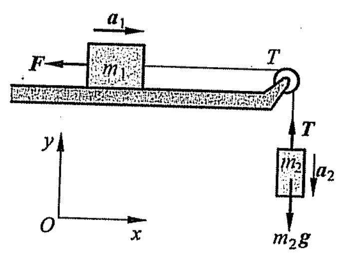


图 2.9 例 2.2 用图

(2) 当 $F=30$ N 时，则用牛顿第二定律，对 m_1，沿 x 方向，有

$$-F+T=m_1a_1 \tag{2.23}$$

对 m_2，沿 y 方向，有

$$T-m_2g=m_2a_2 \tag{2.24}$$

由于 m_1 和 m_2 用绳联结着，所以有 $a_1=a_2$，令其为 a。

联立解式(2.23)和式(2.24)，可得两物块的加速度为

$$a=\frac{m_2g-F}{m_1+m_2}=\frac{2\times 9.8-30}{5.0+2.0}=-1.49\ (\mathrm{m/s^2})$$

和图 2.9 所设 $\boldsymbol{a}_1$ 和 $\boldsymbol{a}_2$ 的方向相比，此结果的负号表示，两物块的加速度均与所设方向相反，即 m_1 将向左而 m_2 将向上以 1.49 m/s^2 的加速度运动。

由上面式(2.24)可得此时绳中张力为

$$T = m_2(g - a_2) = 2.0 \times [9.8 - (-1.49)] = 22.6\ (\mathrm{N})$$

(3) 若绳中张力 $T=0$，则由式(2.24)知，$a_2=g$，即 m_2 自由下落，这时由式(2.23)可得

$$F = -m_1 a_1 = -m_1 a_2 = -m_1 g = -5.0 \times 9.8 = -49\ (\mathrm{N})$$

负号表示力 $\boldsymbol{F}$ 的方向应与图 2.9 所示方向相反，即需用 49 N 的水平力向右推桌上的物块，才能使绳中张力为零。

例 2.3

一个质量为 m 的珠子系在线的一端，线的另一端绑在墙上的钉子上，线长为 l。先拉动珠子使线保持水平静止，然后松手使珠子下落。求线摆下至 θ 角时这个珠子的速率和线的张力。

解 这是一个变加速问题，求解要用到微积分，但物理概念并没有什么特殊。如图 2.10 所示，珠子受的力有线对它的拉力 $\boldsymbol{T}$ 和重力 $m\boldsymbol{g}$。由于珠子沿圆周运动，所以我们按切向和法向来列牛顿第二定律分量式。

对珠子，在任意时刻，当摆下角度为 α 时，牛顿第二定律的切向分量式为

$$mg\cos\alpha = ma_t = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

以 $\mathrm{d}s$ 乘以此式两侧，可得

$$mg\cos\alpha\ \mathrm{d}s = m\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}s = m\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}v$$

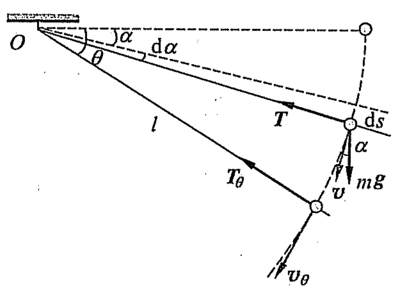


图 2.10 例 2.3 用图

由于 $\mathrm{d}s = l\mathrm{d}\alpha$，$\frac{\mathrm{d}s}{\mathrm{d}t} = v$，所以上式可写成

$$gl\cos\alpha\ \mathrm{d}\alpha = v\mathrm{d}v$$

两侧同时积分，由于摆角从 0 增大到 θ 时，速率从 0 增大到 v_θ，所以有

$$\int_0^\theta gl\cos\alpha \cdot \mathrm{d}\alpha = \int_0^{v_\theta} v\mathrm{d}v$$

由此得

$$gl\sin\theta = \frac{1}{2}v_\theta^2$$

从而

$$v_\theta = \sqrt{2gl\sin\theta}$$

对珠子，在摆下 θ 角时，牛顿第二定律的法向分量式为

$$T_\theta - mg\sin\theta = ma_n = m\frac{v_\theta^2}{l}$$

将上面 v_θ 值代入此式，可得线对珠子的拉力为

$$T_\theta = 3mg\sin\theta$$

这也就等于线中的张力。

例 2.4

一跳伞运动员质量为 80 kg，一次从 4000 m 高空的飞机上跳出，以雄鹰展翅的姿势下落（图 2.11），有效横截面积为 0.6 m²。以空气密度为 1.2 kg/m³ 和曳引系数 $C=0.6$ 计算，他下落的终极速率多大？

图 2.11　2007 年 11 月 10 日，美国得克萨斯州，现年 83 岁的美国前总统老布什（下）通过跳伞庆祝其个人博物馆开馆

解　空气曳力用式(2.15)计算，终极速率出现在此曳力等于运动员所受重力的时候。由此可得终极速率为

$$v_t=\sqrt{\frac{2mg}{C\rho A}}=\sqrt{\frac{2\times 80\times 9.8}{0.6\times 1.2\times 0.6}}=60\ (\mathrm{m/s})$$

这一速率比从 4000 m 高空“自由下落”的速率（280 m/s）小得多，但运动员以这一速率触地还是很危险的，所以他在接近地面时要打开降落伞。

例 2.5

一个水平的木制圆盘绕其中心竖直轴匀速转动（图 2.12）。在盘上离中心 $r=20$ cm 处放一小铁块，如果铁块与木板间的静摩擦系数 $\mu_s=0.4$，求圆盘转速增大到多少（以 r/min 表示）时，铁块开始在圆盘上移动？

解　对铁块进行分析。它在盘上不动时，是作半径为 r 的匀速圆周运动，具有法向加速度 $a_n=r\omega^2$。图 2.12 中示出铁块受力情况，f_s 为静摩擦力。

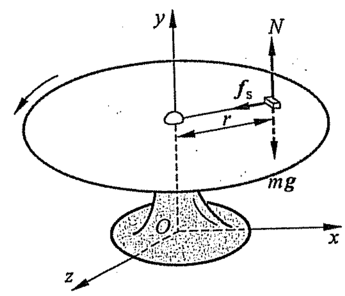


图 2.12　转动圆盘

对铁块用牛顿第二定律，得法向分量式为

$$f_s=ma_n=mr\omega^2$$

由于

$$f_s\leqslant \mu_s N=\mu_s mg$$

所以

$$\mu_s mg\geqslant mr\omega^2$$

即

$$\omega\leqslant\sqrt{\frac{\mu_s g}{r}}=\sqrt{\frac{0.4\times 9.8}{0.2}}=4.43\ (\mathrm{rad/s})$$

由此得

$$n=\frac{\omega}{2\pi}\leqslant 42.3\ (\mathrm{r/min})$$

这一结果说明，圆盘转速达到 42.3 r/min 时，铁块开始在盘上移动。

例 2.6

开普勒第三定律。谷神星(最大的小行星,直径约 960 km)的公转周期为 1.67×10^3 d。试以地球公转为参考,求谷神星公转的轨道半径。

解 以 r 表示某一行星轨道的半径,T 为其公转周期。按匀加速圆周运动计算,该行星的法向加速度为 $4\pi^2 r/T^2$。以 M 表示太阳的质量,m 表示行星的质量,并忽略其他行星的影响,则由引力定律和牛顿第二定律可得

$$G\frac{Mm}{r^2}=m\frac{4\pi^2 r}{T^2}$$

由此得

$$\frac{T^2}{r^3}=\frac{4\pi^2}{GM}$$

由于此式右侧是与行星无关的常量,所以此结果即说明行星公转周期的平方和它的轨道半径的立方成正比。(由于行星轨道是椭圆,所以,严格地说,上式中的 r 应是轨道的半长轴。)这一结果称为关于行星运动的开普勒第三定律。

以 r_1,T_1 表示地球的轨道半径和公转周期,以 r_2,T_2 表示谷神星的轨道半径和公转周期,则

$$\frac{r_2^3}{r_1^3}=\frac{T_2^2}{T_1^2}$$

由此得

$$r_2=r_1\left(\frac{T_2}{T_1}\right)^{2/3}=1.50\times10^{11}\times\left(\frac{1.67\times10^3}{365}\right)^{2/3}=4.13\times10^{11}\ (\mathrm{m})$$

这一数值在火星和木星的轨道半径之间。实际上,在火星和木星间存在一个小行星带。

例 2.7

直径为 2.0 cm 的球形肥皂泡内部气体的压强 p_{in} 比外部大气压强 p_0 大多少?肥皂液的表面张力系数按 0.025 N/m 计。

解 肥皂泡形成后,其肥皂膜内外表面的表面张力要使肥皂泡缩小。当其大小稳定时,其内部空气的压强 p_{int} 要大于外部的大气压强 p_0,以抵消这一收缩趋势。为了求泡内外的压强差,可考虑半个肥皂泡,如图 2.13 中肥皂泡的右半个。泡内压强对这半个肥皂泡的合力应垂直于半球截面,即水平向右,大小为 $F_{in}=p_{in}\cdot\pi R^2$,$R$ 为泡的半径。大气压强对这半个泡的合力应为 $F_{ext}=p_0\cdot\pi R^2$,方向水平向左。与受到此二力的同时,这半个泡还在其边界上受左半个泡的表面张力,边界各处的表面张力方向沿着球面的切面并与边界垂直,即都水平向左。其大小由式(2.16)求得 $F_{sur}=2\cdot\gamma\cdot 2\pi r$,其中的 2 倍是由于肥皂膜有内外两个表面。对右半个泡的力的平衡要求 $F_{in}=F_{ext}+F_{sur}$,即

$$p_0\pi R^2=2\cdot\gamma\cdot 2\pi R+p_{in}\pi R^2$$

由此得 $p_{in}-p_0=\dfrac{4\gamma}{R}=\dfrac{4\times0.025}{1.0\times10^{-2}}=10.0\ (\mathrm{Pa})$

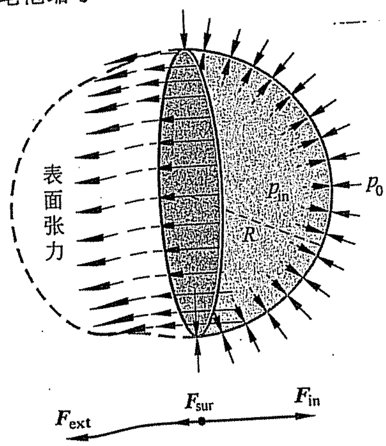
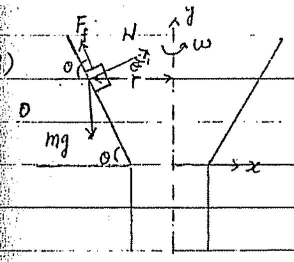


图 2-13 肥皂泡受力分析

3)



若此物体向下滑時

$$N\sin\theta - \mu N\cos\theta = m\frac{v^2}{r} = mr\omega^2 \quad (\text{沿 } x \text{ 軸}) \qquad (19A)$$

$$N\cos\theta + \mu N\sin\theta - mg = 0 \quad (\text{沿 } y \text{ 軸}) \qquad (19B)$$

$$\Rightarrow \quad \omega_{min} = \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{r(\cos\theta + \mu\sin\theta)}} \quad \text{最小之 } \omega \text{ 使 } m \text{ 不向下滑} \qquad (20)$$

系統的選擇

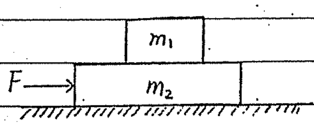
若此物体向上升時 F_f 之方向与以上考慮之相反，也即是

$$N\sin\theta + \mu N\cos\theta = m\frac{v^2}{r} = mr\omega^2 \quad (\text{沿 } x \text{ 軸}) \qquad (21A)$$

$$N\cos\theta - \mu N\sin\theta - mg = 0 \quad (\text{沿 } y \text{ 軸}) \qquad (21B)$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g}{r}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)} \quad \text{最大之 } \omega \text{ 使 } m \text{ 不向上升} \qquad (22)$$

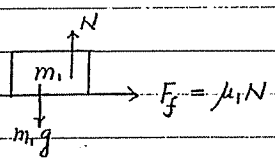
在 ω_{min} 至 ω_{max} 則由於 F_f 能自行調整而使 m 保持不動



m_1, m_2 間之靜摩擦係數為 $\mu_s = \mu_1$

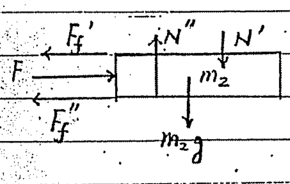
m_2 与桌面間之動摩擦係數為 $\mu_k = \mu_2$

求 m_1, m_2 不分開之最大加速度，及該時之外力為何？



$$N = m_1 g \quad \text{沿 } y \text{ 軸} \qquad (23)$$

$$\mu_1 N = m_1 a_{max} \quad \text{沿 } x \text{ 軸} \quad \Rightarrow \quad a_{max} = \mu_1 g$$



F_f, F_f' 是作用力及反作用力偶

N, N' 是作力及反作用力偶

$$N'' - N' - m_2 g = 0 \quad \text{沿 } y \text{ 軸} \qquad (24)$$

$$N'' = (m_1 + m_2)g$$

$$F - F_f' - F_f'' = m_2 a \qquad F_f'' = \mu_2 N''$$

$$F - \mu_1 m_1 g - \mu_2(m_1 + m_2)g = m_2 a_{max} \qquad (25)$$

$$F = \mu_1(m_1 + m_2)g + \mu_2(m_1 + m_2)g = (\mu_1 + \mu_2)(m_1 + m_2)g \qquad (26)$$

若將 $m_1 + m_2$ 看成一單一系統，則沿 x 軸之公式

$F - F_f'' = (m_1 + m_2)a_{max} \Rightarrow F = (\mu_1 + \mu_2)(m_1 + m_2)g$ 得到同樣的結果 (27)

例题 10 质量为 m_2 的木块放在质量为 m_1 的斜面体的斜面上，假定摩擦可以忽略，问当 $\boldsymbol{F}$ 多大时，恰使 m_2 相对 m_1 静止不动？

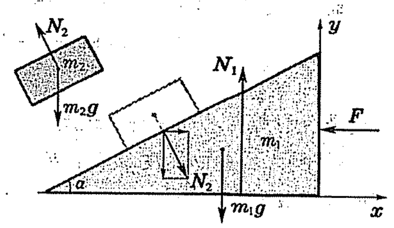


图 2－33 例题 10——斜面上的木块

解：选坐标如图 2－33 所示，用隔离体法分析作用在 m_1 和 m_2 上的力：

对 m_1： $m_1a_{1x}=N_2\sin\alpha+F,$ (a)

$m_1a_{1y}=N_1-N_2\cos\alpha-m_1g.$ (b)

对 m_2： $m_2a_{2x}=-N_2\sin\alpha,$ (c)

$m_2a_{2y}=N_2\cos\alpha-m_2g.$ (d)

且有 $\boldsymbol{a}_1=\boldsymbol{a}_2$（即 $a_{1x}=a_{2x}$；$a_{1y}=a_{2y}=0$），由此解得

$$N_2=\frac{m_2g}{\cos\alpha},$$

$$a_{1x}=a_{2x}=\frac{-N_2\sin\alpha}{m_2}=-g\tan\alpha,$$

$$N=m_1a_{1x}-N_2\sin\alpha=-m_1g\tan\alpha-\frac{m_2g}{\cos\alpha}\sin\alpha.$$

即

$$F=-(m_1+m_2)g\tan\alpha\quad（方向从右到左）.$$ ▌

例题 11 一质量为 $m_2=5.0\times10^2$ g 的夹子，以压力 $P=12$ kgf 夹着质量为 $m_1=1.0$ kg 的木板，已知夹子与木板间的摩擦系数 $\mu=0.2$，问以多大的力 F 竖直往上拉时，才会使木板脱离夹子（图 2－34）。

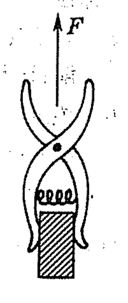


图 2－34 例题 11——夹子中的木块

解：设木板 m_1 的加速度为 a_1，夹子 m_2 的加速度为 a_2. 根据牛顿第二定律写出方程：

对夹子： $F-2\mu P-m_2g=m_2a_2,$

对木板： $2\mu P-m_1g=m_1a_1.$

木板脱离夹子的条件是 $a_2>a_1$，即

$$\frac{F-2\mu Pm_2g}{m_2}>\frac{2\mu P-m_1g}{m_1},$$

故

$$F>\frac{2\mu P(m_1+m_2)}{m_1}=7.2\,\text{kgf}.$$ ▌

在生产实际中起重机的爪钩利用摩擦力吊起物体时要考虑这问题，提升速度过快时物体有脱落的危险。

例 2 如图 2－4(a) 所示，有一绳索围绕在圆柱上，绳索绕圆柱的张角为 θ，绳与圆柱间的静摩擦因数为 μ. 求绳索处于滑动的边缘时，绳两端的张力 F_{TA} 和 F_{TB} 间的关系. 设绳索的质量略去不计.

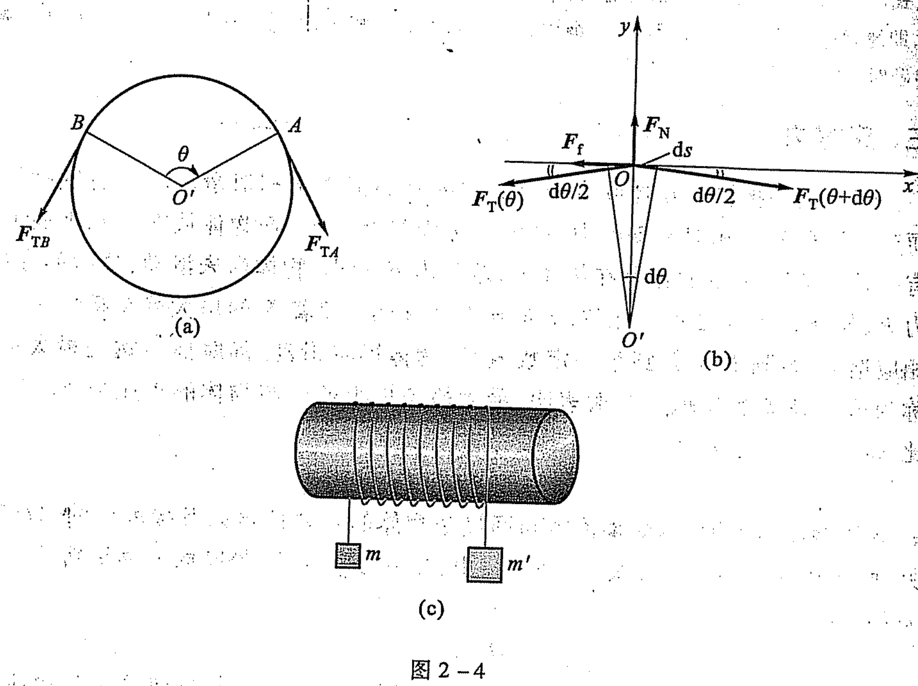


图 2－4

解　如图 2-4(b)所示，在绕于圆柱的绳索 AB 上，取一微小段绳索 $\mathrm{d}s$，其相对圆心的张角为 $\mathrm{d}\theta$. 设 $\mathrm{d}s$ 两端的张力分别为 $\boldsymbol{F}_T(\theta)$ 和 $\boldsymbol{F}_T(\theta+\mathrm{d}\theta)$，圆柱对 $\mathrm{d}s$ 的支持力为 $\boldsymbol{F}_N$. 当圆柱有顺时针旋转的趋势时，圆柱对 $\mathrm{d}s$ 的摩擦力为 $\boldsymbol{F}_f$. 由于绳索的质量略去不计，故 $\mathrm{d}s$ 所受重力亦不予考虑.

由题意知，绳索处于滑动边缘，所以绳索的加速度 $\boldsymbol{a}=0$. 取如图 2-4(b)所示的 Ox 轴和 Oy 轴，根据牛顿第二定律，微小段绳索 $\mathrm{d}s$ 在 Ox 轴和 Oy 轴上的分量式分别为

$$F_T(\theta+\mathrm{d}\theta)\cos\frac{\mathrm{d}\theta}{2}-F_T(\theta)\cos\frac{\mathrm{d}\theta}{2}-F_f=0 \tag{1}$$

$$-F_T(\theta+\mathrm{d}\theta)\sin\frac{\mathrm{d}\theta}{2}-F_T(\theta)\sin\frac{\mathrm{d}\theta}{2}+F_N=0 \tag{2}$$

此外，由摩擦力定义有

$$F_f=\mu F_N \tag{3}$$

考虑到 $\mathrm{d}s$ 相对圆心 O' 的张角 $\mathrm{d}\theta$ 很小，即 $\sin\frac{\mathrm{d}\theta}{2}\approx\frac{\mathrm{d}\theta}{2}$，$\cos\frac{\mathrm{d}\theta}{2}\approx 1$，以及 $F_T(\theta+\mathrm{d}\theta)-F_T(\theta)=\mathrm{d}F_T$，式(1)和式(2)分别为

$$\mathrm{d}F_T=F_f=\mu F_N \tag{4}$$

$$\frac{1}{2}\mathrm{d}F_T\mathrm{d}\theta+F_T\mathrm{d}\theta=F_N \tag{5}$$

上式中略去二阶无限小量 $\mathrm{d}\theta\mathrm{d}F_T$，那么由式(4)和式(5)得

$$\int_{F_{TB}}^{F_{TA}}\frac{\mathrm{d}F_T}{F_T}=\mu\int_0^{\theta}\mathrm{d}\theta$$

得

$$F_{TB}=F_{TA}\mathrm{e}^{-\mu\theta} \tag{6}$$

上式表明，由于绳索与圆柱间存在摩擦力，所以，绳索两端的张力之比 $\frac{F_{TB}}{F_{TA}}$ 是随张角 θ 按指数规律而变化的. 对于绳索与圆柱间的摩擦因数 $\mu=0.25$ 来说，当绳索绕半圈时 $(\theta=\pi)$，$\frac{F_{TB}}{F_{TA}}=\mathrm{e}^{-0.25\pi}=0.46$；当绳索绕 1 圈时 $(\theta=2\pi)$，$\frac{F_{TB}}{F_{TA}}=\mathrm{e}^{-0.25\times2\pi}=0.21$；当绳索绕 5 圈时 $(\theta=10\pi)$，$\frac{F_{TB}}{F_{TA}}=\mathrm{e}^{-0.25\times10\pi}=0.000\,39$. 如果把绳端点 A 与一负荷相连接，F_{TA} 为负荷所引起的张力，而绳端点 B 与拉力相连接，F_{TB} 为拉力所引起的张力，那么，由上述数据可以看出，绳索绕在圆柱上的圈数越多，F_{TB} 比 F_{TA} 就小得越多. 人们常将这个道理用于工农业生产和日常生活之中. 例如，为了使轮船平稳地停靠在码头上，人们常将缆绳在桩柱上多绕几圈；又如，欲把重物挂在屋内的梁柱的钉子上，有经验的人总是把系有重物的绳索先在梁柱上绕几圈，等等. 你能举几个这方面的例子吗？在如图 2-4(c)所示的圆柱上绕有 n 圈绳索. 绳与圆柱之间的摩擦系数仍为 0.25. 如果我们在绳索的两端分别悬挂质量分别为 $m'=1\,000$ g 和 $m=10$ g 的两个物体，并使之平衡. 你知道 n 至少为多少吗？（n 大约是 3 圈，你算算看.）

从式(6)还可以看出，如果绳索与圆柱间的摩擦可略去不计，即 $\mu=0$，那么 $F_{TB}=F_{TA}$. 这时跨过光滑圆柱上的轻绳中各处的张力均相等. 如不特别指明，本章所讨论的有关绳索跨过滑轮的问题，都不计及绳索与滑轮间的相对滑动.

* Non-Inertial Frame of Reference

We shall illustrate the basic concepts with (uniform) rotational frame of reference

1. Outline
2. 我的 Notes.
3. Qualitative Discussion of the Motion Relative to the Rotating Earth
4. Free Falling Body
5. Foucault Pendulum.

* Optional

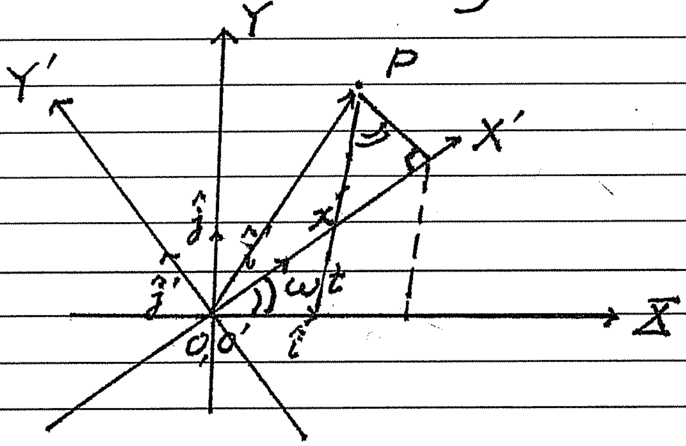
Rotational Frame of Reference

S reference frame
$O \longrightarrow$ origin
$\hat{i}, \hat{j}, \hat{k}$

S' reference frame
O' origin
$\hat{i}', \hat{j}', \hat{k}'$

O, O' coincides, rotating around z direction

with constant angular velocity ω



z, z' axes are perpendicular to xy plane pointing toward the reader

At time t

x, y, z
$\downarrow$
$\vec{OP} = \vec{r}$
$= x\hat{i} + y\hat{j} + z\hat{k}$
$\downarrow$
position vector in the S' system

x', y', z'
$\downarrow$
$\vec{OP} = \vec{r} = \vec{r}'$
$= x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$
$\downarrow$
position vector in the S' system

$$x = x'\cos\omega t - y'\sin\omega t$$

$$y = x'\sin\omega t + y'\cos\omega t$$

$$z = z'$$

分類:	
編號:	
總號:	

$$\dot{x} = \dot{x}'\cos\omega t - \dot{y}'\sin\omega t - \omega x'\sin\omega t - \omega y'\cos\omega t$$

$$\dot{y} = \dot{x}'\sin\omega t + \dot{y}'\cos\omega t + \omega x'\cos\omega t - \omega y'\sin\omega t$$

$$\dot{z} = \dot{z}'$$

Differential respect to t

$$\ddot{x} = \ddot{x}'\cos\omega t - \ddot{y}'\sin\omega t - 2\omega\dot{x}'\sin\omega t - 2\omega\dot{y}'\cos\omega t - \omega^2 x'\cos\omega t + \omega^2 y'\sin\omega t$$

$$\ddot{y} = \ddot{x}'\sin\omega t + \ddot{y}'\cos\omega t + 2\omega\dot{x}'\cos\omega t - 2\omega\dot{y}'\sin\omega t - \omega^2 x'\sin\omega t - \omega^2 y'\cos\omega t$$

$$\ddot{z} = \ddot{z}'$$

S system: $\hat{i}, \hat{j}, \hat{k}$

S' system: $\hat{i}', \hat{j}', \hat{k}'$

$$\hat{i}' = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$$

$$\hat{j}' = -\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}$$

$$\hat{k}' = \hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{V} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Acceleration vector at point P with respect to S' system

$$\vec{r}' = x'\hat{i}' + y'\hat{j}' + k'\hat{k}'$$

$\vec{r}''$

V

$$\vec{v}' = \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'$$

$$\vec{a}'' = \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}'$$

$\vec{a}'$, $\vec{v}'$, $\vec{r}'$ is the acceleration, velocity and position vector measured in S' system

分類：
編號：
總號：

In vector notation

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}'$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$= [\dot{x}'\cos\omega t - \dot{y}'\sin\omega t - \omega x'\sin\omega t - \omega y'\cos\omega t]\hat{i}$$

$$+ [\dot{x}'\sin\omega t + \dot{y}'\cos\omega t + \omega x'\cos\omega t - \omega y'\sin\omega t]\hat{j}$$

$$+ \dot{z}'\hat{k}$$

$$= \dot{x}'[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}] + \dot{y}'[-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}]$$

$$+ \dot{z}'\hat{k} + \omega x'[\cos\omega t\,\hat{j} - \sin\omega t\,\hat{i}] - \omega y'(\cos\omega t\,\hat{i}$$

$$+ \sin\omega t\,\hat{j}]$$

$$= \underbrace{\dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'}_{\vec{v}'} \underbrace{- \omega y'\hat{i}' + \omega x'\hat{j}'}_{\vec{\omega}\times\vec{r}'}$$

$$\vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix}$$

$$= \hat{i}'(-\omega y') + \hat{j}'\omega x'$$

check

分類:
編號:
總號:

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$= [\ddot{x}'\cos\omega t - \ddot{y}'\sin\omega t - 2\omega\dot{x}'\sin\omega t - 2\omega\dot{y}'\cos\omega t - \omega^2 x'\cos\omega t + \omega^2 y'\sin\omega t]\hat{i}$$
$$+ [\ddot{x}'\sin\omega t + \ddot{y}'\cos\omega t + 2\omega\dot{x}'\cos\omega t - 2\omega\dot{y}'\sin\omega t - \omega^2 x'\sin\omega t - \omega^2 y'\cos\omega t]\hat{j}$$
$$+ \ddot{z}'\hat{k}$$

$$= \ddot{x}'[\overbrace{\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}}^{\hat{i}'}] + \ddot{y}'[\overbrace{-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}}^{\hat{j}'}]$$
$$+ \ddot{z}'\overset{\hat{k}'}{\overset{\shortparallel}{\hat{k}}} + 2\omega\dot{x}'[\overbrace{-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}}^{\hat{j}'}] - 2\omega\dot{y}'[\overbrace{\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}}^{\hat{i}'}]$$
$$- \omega^2 x'[\overbrace{\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}}^{\hat{i}'}]$$
$$- \omega^2 y'[\overbrace{-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}}^{\hat{j}'}]$$

$$\Rightarrow \vec{a} = \underbrace{\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}}_{\vec{a}'} - 2\omega\dot{y}'\hat{i}' + 2\omega\dot{x}'\hat{j}'$$
$$- \omega^2 x'\hat{i}' - \omega^2 y'\hat{j}' \qquad (B)$$

$\downarrow$

This is the special case for $\vec{\omega} = \omega\hat{k}$

For the general case

$$\vec{a} = \vec{a}' + 2\vec{\omega}\times\vec{V}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}') \qquad (A)$$

We shall first show that (A) $\Rightarrow$ B when $\vec{\omega} = \omega\hat{k}$

First order in ω

$$\vec{\omega}\times\vec{V}' = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ 0 & 0 & \omega \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= \hat{i}'(-\omega\dot{y}') + \hat{j}'(\omega\dot{x}')$$

$$2\vec{\omega}\times\vec{V}' = -2\omega\dot{y}'\hat{i}' + 2\omega\dot{x}'\hat{j}'$$

$\downarrow$

check

分類：
編號：
總號：

$$\vec{\omega} \times \vec{r}' = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix}$$

$$= \hat{i}'(-\omega y') + \hat{j}'(x'\omega)$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ 0 & 0 & \omega \\ -\omega y' & x'\omega & 0 \end{vmatrix}$$

$$= \hat{i}'(-x'\omega^2) + \hat{j}'(-\omega^2 y')$$

$$= -\omega^2 x' \hat{i}' - \omega^2 y' \hat{j}$$

check

Note

No dynamics is involved.

only kinematics is involved

The general case can be proved only the mathematics (the algebra) is more involved

We shall now discuss the application of equation (A)

S $\qquad\qquad\qquad$ S'

$\vec{V}$ $\qquad\qquad\qquad$ $\vec{V}'$

$\vec{a}$ $\qquad\qquad\qquad$ $\vec{a}'$

$\vec{V}'=0$, $\vec{a}'=0$ $\qquad$ $\vec{x}'=\vec{x}'_0$

↓

at rest in S' system

$$\vec{a} = \vec{a}' + 2\vec{\omega}\times\vec{V}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$= \underset{\downarrow}{0} \qquad \underset{\downarrow}{0} \qquad \underbrace{(\vec{\omega}\cdot\vec{r}')}_{0}\vec{\omega} - (\vec{\omega}\cdot\vec{\omega})\vec{r}'$$

$\vec{\omega}\perp\vec{r}'$

[Note $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})C$

$$= -\omega^2\vec{r}'$$

S is an inertial frame

$\vec{F} = m\vec{a}$

$$\frac{\vec{F}}{m} = -\omega^2\vec{r}$$

$$\vec{F} = -m\omega^2\vec{r}$$

(See 1)

Cylinding problem

In S system

The body is moving in the z = constant plane

It is in uniform circular motion (with radius R angular velocity ω)

$$\vec{a} = -R\omega^2 \hat{r}$$

$\downarrow$

the body has a force

$$\vec{F} = -mR\omega^2 \hat{r}$$

acting on it

$\hat{r}$ is a unit vector in the z plane pointing toward the origin

the force diagram

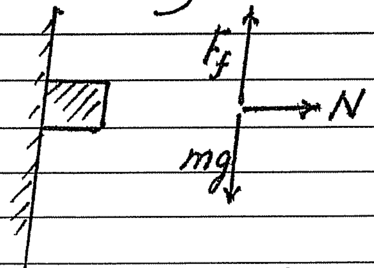


Fig. 2

In S' system

The body is at rest

$$\vec{a} = \vec{a}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{a} = -\omega^2 \vec{r}, \quad \vec{a}' = 0, \quad 2\vec{\omega}\times\vec{v}' = 0, \quad \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\omega^2\vec{r}_{\parallel}', \quad \vec{r}_{\parallel}' = \vec{r}$$

$$\vec{a}' = \vec{a} - 2\vec{\omega}\times\vec{v}' - \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$m\vec{a}' = m\vec{a} - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$m\vec{a}' = \vec{F}_{eff} = \vec{F} - \underbrace{2m\vec{\omega}\times\vec{v}'}_{\text{Coriolis force}} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r}')}_{\text{Centrifugal force}}$$

分類：
編號：1
總號：

第二節 均勻相對轉動運動

簡介：討論兩個坐標系統作均勻相對轉動之特殊情形

基本觀念：設 S 與 S' 為兩個坐標系統，其原點 O 與 O' 位於一點。在 S 坐標中其坐標單位向量 $\hat{i},\hat{j},\hat{k}$ 不為時間之函數。S' 中 $\hat{i}',\hat{j}',\hat{k}'$ 則對以一固定方向 $\vec{\omega}$ 為軸，作等速轉動，其角速度之大小為 $|\vec{\omega}|$。一物体 A 之位置向量在 S 坐標系統中可寫成

$$\vec{r}_{AO}(t) = x_{AO}(t)\,\hat{i} + y_{AO}(t)\,\hat{j} + z_{AO}(t)\,\hat{k} \qquad (1)$$

上式中 $x_{AO}(t)$，$y_{AO}(t)$，$z_{AO}(t)$ 均為時間之函數，但是 $\hat{i},\hat{j},\hat{k}$ 卻不是時間之函數。令 $\vec{r} = (x_{AO}(t)=x, y_{AO}(t)=y, z_{AO}(t)=z)$

物体 A 在 S' 坐標系統中之位置向量可寫成

$$\vec{r}_{AO'}(t) = x_{AO'}(t)\,\hat{i}' + y_{AO'}(t)\,\hat{j}' + z_{AO'}(t)\,\hat{k}' \qquad (2)$$

上式中 $x_{AO'}(t)$，$y_{AO'}(t)$，$z_{AO'}(t)$ 及 $\hat{i}',\hat{j}',\hat{k}'$ 均為時間之函數

令 $\vec{r}'(t) = (x_{AO'}(t)=x', y_{AO'}(t)=y', z_{AO'}(t)=z')$ 是 A 在 S' 系統中之位置向量[1]

將 (1) 及 (2) 式對 t 微分，吾人得

$$\frac{d\vec{r}_{AO}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \vec{V} \qquad (3)$$

$$\frac{d\vec{r}_{AO'}}{dt} = \underbrace{\frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}'}_{\text{在 } S' \text{ 系統中隨 } \hat{i}',\hat{j}',\hat{k}' \text{ 轉動座標中所見之速度}}$$

$$+ \underbrace{x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}}_{\text{由於座標隨時間變化而引起之結果}}$$

$\hat{i}',\hat{j}',\hat{k}'$ 對一方向 $\vec{\omega}$ 作等速轉動，其角速度為 $|\vec{\omega}|$，則[2],[3]

$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}', \quad \frac{d\hat{j}'}{dt} = \vec{\omega}\times\hat{j}', \quad \frac{d\hat{k}'}{dt} = \vec{\omega}\times\hat{k}' \qquad (4)$$

因此 $\frac{d\vec{r}_{AO'}}{dt} = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}' + x'\,\vec{\omega}\times\hat{i}' + y'\,\vec{\omega}\times\hat{j}'$

分類：
編號： 2
總號：

$+ z' \vec{\omega} \times \vec{j}'$ (5)

$\vec{V}' = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}'$ 是物体在 S' 坐標系統中之速度

$$\frac{d\vec{r}_{AO'}}{dt} = \vec{V}' + \vec{\omega} \times \vec{r}_{AO'} \qquad (6)$$

由上節中我們得知當 O 與 O' 位於一点時 $\vec{r}_{AO} = \vec{r}_{AO'}$ $(\vec{r} = \vec{r}')$

因此 $\vec{V} = \vec{V}' + \vec{\omega} \times \vec{r}_{AO}$ (7)

此式代表在兩個坐標系統中所觀察之速度間之關係

將(7)式再對時間之微分可得

$$\vec{a} = \frac{d\vec{V}'}{dt} + \vec{\omega} \times \frac{d\vec{r}}{dt} \qquad (8)$$

$$\vec{V}' = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}' \qquad (9)$$

將第(9)式微分可得

$$\frac{d\vec{V}'}{dt} = \frac{d^2x'}{dt^2}\hat{i}' + \frac{d^2y'}{dt^2}\hat{j}' + \frac{d^2z'}{dt^2}\hat{k}'$$

$$+ \frac{dx'}{dt}\frac{d\hat{i}'}{dt} + \frac{dy'}{dt}\frac{d\hat{j}'}{dt} + \frac{dz'}{dt}\frac{d\hat{k}'}{dt}$$

$$= \vec{a}' + \vec{\omega} \times \vec{V}' \qquad (10)$$

此式中 $\vec{a}' = \frac{d^2x'}{dt^2}\hat{i}' + \frac{d^2y'}{dt^2}\hat{j}' + \frac{d^2z'}{dt^2}\hat{k}'$ 為此物体在 S' 坐標系統中所觀察之加速度

將第(8)及第(10)式合併可得

$$\vec{a} = \vec{a}' + \vec{\omega} \times \vec{V}' + \vec{\omega} \times \vec{V} \quad \left(\vec{V}' + (\vec{\omega} \times \vec{r}_{AO})\right) \qquad (11)$$

將上式與第(7)式合併可得

$$\vec{a} = \vec{a}' + \vec{\omega} \times \vec{V}' + \vec{\omega} \times (\vec{V}' + \vec{\omega} \times \vec{r})$$

$$= \vec{a}' + 2\vec{\omega} \times \vec{V}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad (12)$$

此式表示出在兩個不同坐標系統中所觀察之加速度間之關係

第(12)式可改寫成

$$\vec{a}' = \vec{a} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\vec{v}' \qquad (13)$$

$-\vec{\omega}\times(\vec{\omega}\times\vec{r})$ 稱為離心加速度，$-2\vec{\omega}\times\vec{v}'$ 稱為克督雷加速度(6)，(7)

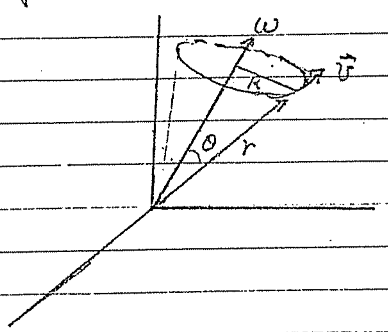
3. 討論：

(1) 在此節中必需小心了解每一項之物理意義 同時必需了解每一物理量倒底是在那一個坐標系統所量度的 $\vec{r}'(t) = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$

$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$. $\vec{r}'(t)$ 是在時間 t 時由 O' 點至物體 P 之向量. $\vec{r}(t)$ 是在時間 t 時由 O 點至物體 P 之向量. 因為 O 點與 O' 點為同一點，所以 $\vec{r}(t) = \vec{r}'(t)$. 但是由於 $\hat{i}', \hat{j}', \hat{k}'$ 通常與 $\hat{i}, \hat{j}, \hat{k}$ 不同所以 x', y', z' 及 x, y, z 通常也不相同 要想找出 x, y, z 與 x', y', z' 間之關係必須首先找出 $\hat{i}, \hat{j}, \hat{k}$ 與 $\hat{i}', \hat{j}', \hat{k}'$ 間之關係

(2) $\vec{\omega}$ 是一固定向量與時間無關，通常是將它寫在 S 坐標系統中，在此系統中其分量 $\omega_x, \omega_y, \omega_z$ 均不是時間之函數

(3) 任一通過原點之向量以原點為軸心對一固定方向 $\vec{\omega}$ 以等加速度 $|\omega|$ 轉動則其頂點之速度 $\vec{v} = \vec{\omega}\times\vec{r}$



此點之軌跡位於一⊥於 $\vec{\omega}$ 方向之平面上

因此 $\underset{\frac{d\vec{r}}{dt}}{\vec{v}} \perp \vec{\omega}$

因為此一向量之大小不變

$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 0 \Rightarrow 2\vec{r}\cdot\vec{v} = 0$

$\Rightarrow |\vec{r}| = 0$ 或是 $\vec{r}\cdot\vec{v} = 0$，$|\vec{r}| \neq 0$ (否則

以上之公式顯然不需要證明 所以 $\vec{r}$ 必須垂直於 $\vec{v}$

所以 $\vec{v}$ 垂直於 $\vec{\omega}$ 及 $\vec{r}$，因此 $\vec{v}$ 之方向是平行或反平行於 $\vec{\omega}\times\vec{r}$

我們採用右手定則，由圖中我們可看出 $\vec{v}$ 是平行於 $\vec{\omega}\times\vec{r}$

因為其角速度為 $|\vec{\omega}|$ 所以在 Δt 時間內所行之距離為 $R\omega\Delta t$

此處 $R=|\vec{r}|\sin\theta$，θ 為 $\vec{\omega}$ 及 $\vec{r}$ 間之夾角，所以 $|\vec{v}|=R\omega$

$=|\vec{r}||\omega|\sin\theta=|\vec{\omega}\times\vec{r}|$

所以 $\vec{v}$ 與 $\vec{\omega}\times\vec{r}$ 之方向與大小均相同，因此 $\underset{\frac{d\vec{r}}{dt}}{\underset{\parallel}{\vec{v}}}=\vec{\omega}\times\vec{r}$

以上之公式是對任何經原點對 $\vec{\omega}$ 方向作等角速度 $|\vec{\omega}|$ 轉動之向量均成立，$\hat{i}'$，$\hat{j}'$，$\hat{k}'$ 也屬此類向量，應用以上公式得

$\frac{d\hat{i}'}{dt}=\vec{\omega}\times\hat{i}'$，$\frac{d\hat{j}'}{dt}=\vec{\omega}\times\hat{j}'$，及 $\frac{d\hat{k}'}{dt}=\vec{\omega}\times\hat{k}'$

(4) 注意這是一向量公式，$\vec{v}'$ 是在 S' 坐標系統中之速度向量 $\vec{v}'=\vec{v}-\vec{\omega}\times\vec{r}$，若 $\vec{v}$，$\vec{\omega}$，$\vec{r}$ 均以單位向量 $\hat{i},\hat{j},\hat{k}$ 表出，則由上式中得來的 $\vec{v}'$ 也是 $\vec{v}'$ 向量以 $\hat{i},\hat{j},\hat{k}$ 表出，若要以 $\hat{i}',\hat{j}',\hat{k}'$ 表出，而求得 $\frac{dx'}{dt}$，$\frac{dy'}{dt}$，$\frac{dz'}{dt}$ 則必須找出 $\hat{i},\hat{j},\hat{k}$ 及 $\hat{i}',\hat{j}',\hat{k}'$ 間之關係

(5) 注意第 (12)，(13) 式均為向量公式

(6) 離心加速度及克勞雷加速度均是由觀察者間之轉動運動而產生，而非由任何外力加於此粒子上所引起之加速度，因此有些書上用離心力或克勞雷力這些名詞嚴格來說，並非正確，我們將避免這些名詞

(7) 當 $\vec{v}'=0$ 或 $\vec{v}'\parallel\vec{\omega}$ 時，克勞雷加速度不存在；當 $\vec{r}\parallel\vec{\omega}$ 時離心加速度不存在。

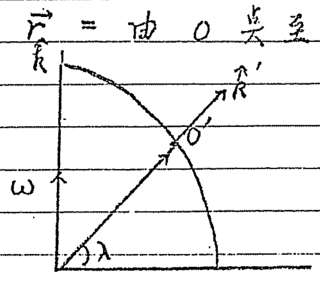
分類：
編號：
總號：

三、應用

在此節中最重要的應用是將地球轉動的效應考慮進去，此處我們嚴格的來說有三個坐標系統 S, S' 及 S''。S 坐標系統位於地球之質心 O，坐標之單位向量 $\hat{i}, \hat{j}, \hat{k}$ 是固定的向量不是時間之函數。S' 坐標系統之原點是地球表面之一點 O'，其單位坐標向量 $\hat{k}'$ 是沿 $\overrightarrow{OO'}$ 方向，$\hat{i}'$ 是沿北的方向，$\hat{j}'$ 是向西之方向，由於地球自轉的關係 $\hat{i}', \hat{j}', \hat{k}'$ 是時間之函數。S'' 坐標系統之原點位於地心 O，而其單位坐標向量 $\hat{i}'', \hat{j}'', \hat{k}''$ 與 $\hat{i}', \hat{j}', \hat{k}'$ 相同。由第一節中之結果得知 $\vec{V}' = \vec{V}''$ 及 $\vec{a}' = \vec{a}''$，所以當我們討論加速度時 S', S'' 坐標系統結果相同。S, S'' 兩坐標系統則原點位於同一點，只是兩個坐標系統間之等角速度轉動，因此可直接利用此節之結果。

在 S 坐標系統中我們取地心 O 及北極之方向為 $\hat{z}$ 軸，因為地球是繞此軸以角速度 $\omega = \dfrac{2\pi}{24\times60\times60}$ rad/sec $= 7.292\times10^{-5}$ rad/sec 轉動，因此 $\vec{\omega} = \omega\hat{k}$

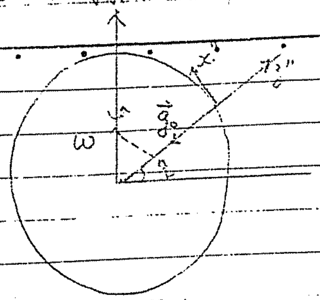
$\vec{r}$ = 由 O 點至 O' 點之向量



$$\vec{a}' = \vec{a} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\vec{V}'$$

此處未加 ′ 的物理量是在 S 坐標系統中所量度，加 ′ 的則是在 S' 坐標系統所量度者

取 S'' 坐標系統中之坐標系統如下



$\hat{k}''$ 是沿地心至地面之向量

取地心至北極之向量位於 $\hat{i}''$, $\hat{k}''$ 平面上

則 $\hat{i}''$ 可取指向北極, 用右手定則 $\hat{j}''$ 是指向西方

$\hat{i}'=\hat{i}''$, $\hat{j}'=\hat{j}''$, $\hat{k}'=\hat{k}''$, $\vec{\omega}=\omega\cos\lambda\,\hat{i}'+\omega\sin\lambda\,\hat{k}'$　λ = 緯度

離心加速度　$-\vec{\omega}\times(\vec{\omega}\times\underset{\hat{k}'}{\vec{r}'})$

$\vec{\omega}\times\vec{r}' = -r\omega\cos\lambda\,\hat{j}'$

$\vec{\omega}\times(\vec{\omega}\times\vec{r}') = \omega^2 r\cos\lambda\sin\lambda\,\hat{i}' + (-\omega^2 r\cos^2\lambda)\,\hat{k}'$

$\vec{a}' = \overbrace{-g_0\hat{k}'}^{\vec{a}} - [\omega^2 r\cos\lambda\sin\lambda\,\hat{i}' + \omega^2 r\cos^2\lambda]\,\hat{k}'$

↳ 在隨地球轉動坐標中測得之加速度

在 S 坐標系統中　$\vec{a} = -g_0\hat{r}$　$\hat{r}$ 是沿地心至地面之向量

$\vec{a}' = -(g_0-\omega^2 r\cos^2\lambda)\,\hat{k}' - \omega^2 r\cos\lambda\sin\lambda\,\hat{i}'$

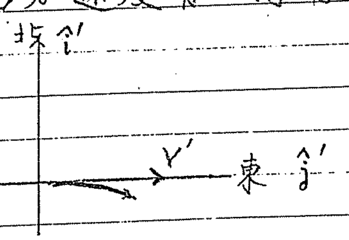
克勞雷加速度　$-2(\vec{\omega}\times\vec{v}')$

與在物體在 S' 坐標系統之速度有關

若一河流在北半球向東流則在 S' 坐標系統中 $\vec{v}' = -v'\hat{j}'$

則　$-2(\vec{\omega}\times\vec{v}') = -v'\omega\sin\lambda\,\hat{i}' + \omega v'\cos\lambda\,\hat{k}'$

因此克勞雷加速度有一向南之分向量

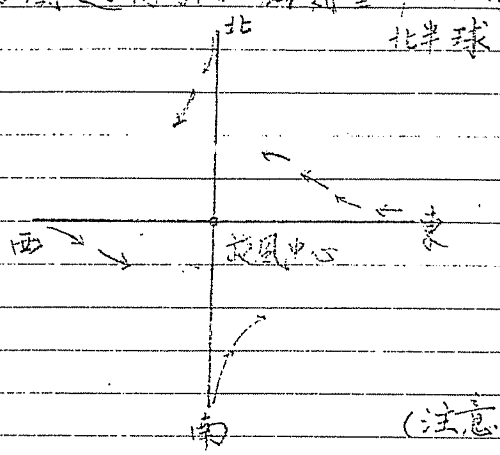


所以在北半球向東流之河流其右岸所受之侵蝕會比其左岸所受之侵蝕為烈.

在北半球中若一物体之方向為向東(西)進行則其克勞雷加速度有沿南(北)之分向量

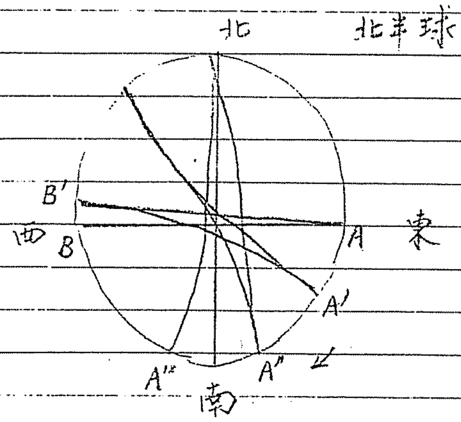
〃〃〃〃〃〃〃〃〃〃〃〃〃〃〃北(南)〃〃〃〃〃〃〃〃〃〃有沿東(西)之分向量

所以當風趨向於一低氣壓中心時會反時鐘旋轉如下圖所示



(注意 在每一点均有一向旋風中心之加速度 此一加速度恒大於克勞雷加速度)

佛肖單擺



由A点向西出發 由於克勞雷加速度有沿北之分量 擺在完成一次擺動將不會到達B点 反而到達B'点 而由B'回来時則到達A'点, 以此類推, 在1851年佛肖用67米長的單擺證實了此現象 他發現A点沿反時鐘方向每小時轉11°15'

同時此一實驗也證實了地球的自轉

4. 習題

1. 一粒子在北緯45°以对地球500米/秒之速度向南進行

(a) 求其離心加速度 (C)

(b) 求其克勞雷加速度 (B)

2. 一物体在北緯41度由200米高處自由下落 求其落點與其向地心方面

分類：
編號： 8
總號：

之距離爲何？ (A)

答案

(A) 東方 3.27 厘米處　(B) $5.15 \times 10^{-2}\ m/sec^2$ 向西

(C) $-1.67 \times 10^{-2}\ m/sec^2\ \hat{i}' + 1.67 \times 10^{-2}\ m/sec\ \hat{k}'$

6. 例題

在北半球緯度 $\lambda°$ 處以 v_0 之速度向上拋出一物体，求當該物体落回地面時其位置向西移動了多少？

若向上拋時之速度爲 V' 則其克勞雷加速度爲

$$-2\vec{\omega} \times \vec{V}'$$

$$= -2 \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ \omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & V' \end{vmatrix}$$

$$= 2\omega V'\cos\lambda\ \hat{j}'$$ 向西之加速度

$$\frac{d^2y'}{dt^2} = 2\omega V'\cos\lambda$$

V' 是沿 $\hat{j}'$ 軸之速度，它滿足之公式爲

$$V' = v_0 - gt$$

因此 $\frac{d^2y'}{dt^2} = 2\omega(v_0 - gt)\cos\lambda$

積分一次得 $\frac{dy'}{dt} = 2\omega(v_0 t - \frac{1}{2}gt^2)\cos\lambda + C$

在 $t=0$ 時只有沿 $\hat{j}'$ 軸之速度，因此 $t=0$ 時 $\frac{dy'}{dt} = 0$

將上兩式合併可得 $C = 0$

$$\frac{dy'}{dt} = 2\omega(v_0 t - \frac{1}{2}gt^2)\cos\lambda$$

將此式再積分一次，同時記住當 $t=0$ 時 $y=0$，可得

分類：
編號： 9
總號：

$$y' = 2\omega\left(\frac{1}{2}v_0 t^2 - \frac{1}{6}gt^3\right)\cos\lambda$$

當該物体到達最高點時其時間為 $t_0 = \frac{v_0}{g}$ (由 $v'=0$ 解出)

代入 y' 及 $\frac{dy'}{dt}$ 之公式，可得當該物体抵達最高點時

$$y' = \frac{2}{3}\frac{v_0^3}{g^2}\omega\cos\lambda \quad ✓$$

$$\frac{dy'}{dt} = 2\omega\left(v_0\frac{v_0}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2\right)\cos\lambda = \omega\frac{v_0^2}{g}\cos\lambda \quad ✓$$

現在我們討論該物体之下落部分

$$\frac{d^2y'}{dt^2} = 2\omega V'\cos\lambda$$

$$V' = -g(t-t_0)$$

$$\frac{d^2y'}{dt^2} = -2\omega\left[g(t-t_0)\right]\cos\lambda$$

積分一次得

$$\frac{dy'}{dt} = -2\omega\left[\frac{1}{2}gt^2 - gt_0t\right]\cos\lambda + C$$

當 $t = t_0 = \frac{v_0}{g}$ 時 $\frac{dy'}{dt} = \omega\frac{v_0^2}{g}\cos\lambda$

因此 $\omega\frac{v_0^2}{g}\cos\lambda = +\omega g\frac{v_0^2}{g^2}\cos\lambda + C$

所以 $C = 0$

將 $\frac{dy'}{dt} = -2\omega\left[\frac{1}{2}gt^2 - gt_0t\right]\cos\lambda$ 再積分一次

$$y' = -2\omega\left[\frac{1}{6}gt^3 - \frac{1}{2}gt_0t^2\right]\cos\lambda + C'$$

當 $t = t_0 = \frac{v_0}{g}$ 時 $y' = \frac{2}{3}\frac{v_0^3}{g^2}\omega\cos\lambda$

所以 $\frac{2}{3}\frac{v_0^3}{g^2}\omega\cos\lambda = -2\omega\left[\frac{1}{6}g\frac{v_0^3}{g^3} - \frac{1}{2}g\frac{v_0^3}{g^3}\right]\cos\lambda + C'$

由上可解得 $C' = 0$

$$y' = -2\omega\left[\frac{1}{6}gt^3 - \frac{1}{2}gt_0t^2\right]\cos\lambda$$

落地時 $t = \frac{2v_0}{g}$

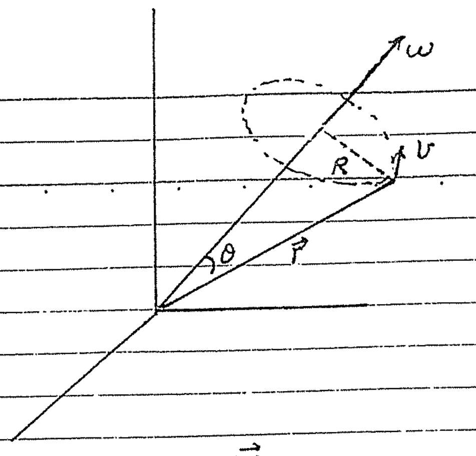
分類：
編號：10
總號：

所以 $y' = -2w\left[\frac{1}{6}g\cdot\frac{8v_0^3}{g^3} - \frac{1}{2}g\,\frac{v_0}{g}\,\frac{4v_0^2}{g^2}\right]\cos\lambda$

$$= -2w\cos\lambda\cdot\frac{v_0^3}{g^2}\Big[\underbrace{\frac{4}{3}-2}_{\frac{4-6}{3}}\Big] = \frac{4}{3}\,\omega\cos\lambda\,\frac{v_0^3}{g^2}$$

因此當該物体落回地面位於其拋出處之西方 $\frac{4}{3}\omega\cos\lambda\,\frac{v_0^3}{g^2}$.

分類：	
編號：	
總號：	



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$v \perp \vec{\omega}$

$v \perp \vec{r}$ $\quad \frac{d}{dt}(\vec{r}\cdot\vec{r}) = 0 \Rightarrow \vec{r}\cdot\vec{v} = 0$

$|\vec{r}|, |\vec{v}|$ 均不為 0

$\vec{r} \perp \vec{v}$

$\Rightarrow \quad \vec{v} \parallel \vec{\omega}\times\vec{r}$

由右手定則

$$R = |\vec{r}|\sin\theta$$

$$|v| = R\omega = \omega|\vec{r}|\sin\theta = |\vec{\omega}\times\vec{r}|$$

因此得證

$S \qquad \hat{i}, \hat{j}, \hat{k}, t$

$S' \qquad \hat{i}'(t), \hat{j}'(t), \hat{k}'(t), t$

$$\frac{d\vec{r}_{AO'}}{dt} = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}', \quad \frac{d\hat{j}'}{dt} = \vec{\omega}\times\hat{j}', \quad \frac{d\hat{k}'}{dt} = \vec{\omega}\times\hat{k}'$$

$$\frac{d\vec{r}_{AO'}}{dt} = \vec{V}' + \vec{\omega}\times\vec{r}_{AO'}$$

$$\frac{d\vec{r}_{AO}}{dt} = \vec{V} \qquad \vec{r}_{AO} \qquad \vec{r}_{OO'} = 0$$

$$\frac{d\vec{V}}{dt} = \vec{V}' + \vec{\omega}\times\vec{r}_{AO} = \vec{V}' + \vec{\omega}\times\vec{r}$$

$$\vec{a} = \frac{d\vec{V}'}{dt} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

$$\vec{V}' = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}'$$

$$\frac{d\vec{V}'}{dt} = \frac{d^2x'}{dt^2}\hat{i}' + \frac{d^2y'}{dt^2}\hat{j}' + \frac{d^2z'}{dt^2}\hat{k}' + \frac{dx'}{dt}\frac{d\hat{i}'}{dt} + \frac{dy'}{dt}\frac{d\hat{j}'}{dt} + \frac{dz'}{dt}\frac{dz'}{dt}$$

$$= \vec{a}' + \vec{\omega} \times \vec{V}'$$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{V}'}{dt} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \vec{a}' + \vec{\omega} \times \vec{V}' + \vec{\omega} \times \underbrace{\frac{d\vec{r}}{dt}}_{\vec{V}}$$

$$= \vec{a}' + \vec{\omega} \times \vec{V}' + \vec{\omega} \times (\vec{V}' + \vec{\omega} \times \vec{r})$$

$$= \vec{a} = \vec{a}' + 2\vec{\omega} \times \vec{V}' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a}' = \vec{a} - \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{離心加速度}} - \underbrace{2\vec{\omega} \times \vec{V}'}_{\text{克勞雷加速度}}$$

應用:

地球表面座標 S' $\longleftrightarrow$ S'' $\qquad$ S

只差一固定位移向量

S'': 輔助座標系統

S: 以地心為原點之固定座標系統

S': 隨地球轉動地面之座標系統

分類：
編號：
總號：

地球 as a rotating system

Need Fig. 6.4 and 9.7

Note, we want to work in the moving system of the earth

Near the surface of the earth

$$\vec{a}' = \vec{a} - 2\vec{\omega}\times\vec{V}' - \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

Use the coordinate as shown in the next page

We shall notice

$$\omega = \frac{2\pi}{86400\,\text{sec}} \sim \underbrace{7\times10^{-5}}_{\text{small}}\ \text{rad/sec}$$

$\vec{a} = -g\hat{k}$

$\hat{i} = \hat{e}_x$ pointing south

$\hat{j} = \hat{e}_y$ pointing east

$\hat{k} = \hat{e}_k$ outward from the surface of the earth

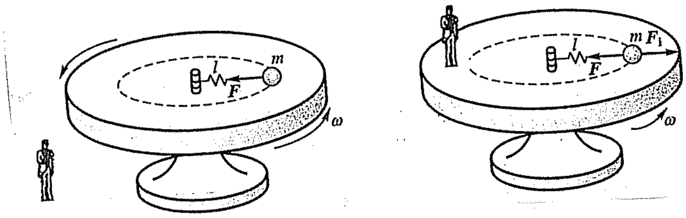
$\omega_x = -\omega\cos\lambda$

$\omega_y = 0$

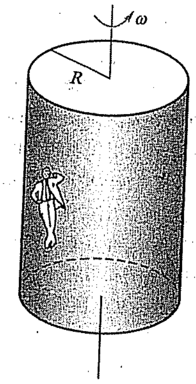
$\omega_z = \omega\sin\lambda$

We have omitted the "prime" in our notation

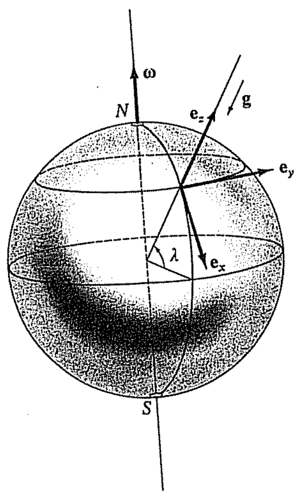
it is important to
remember
that we
are working
in the
rotating frame.


m
l
F
ω
m F₁

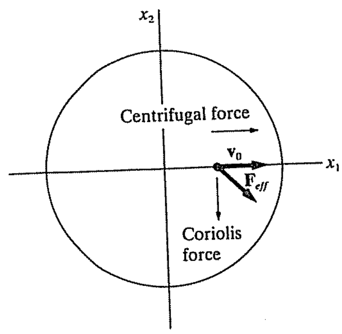
1


ω
R

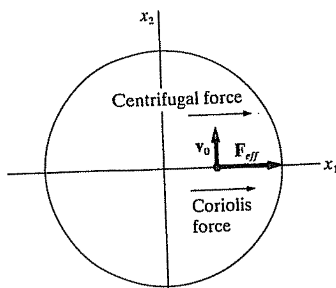
2


ω
N
e_z
g
e_y
λ
e_x
S

3


x_2
Centrifugal force
v_0
x_1
F_{eff}
Coriolis
force

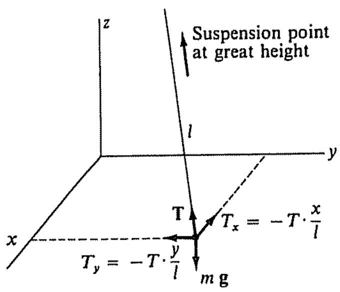
(a)

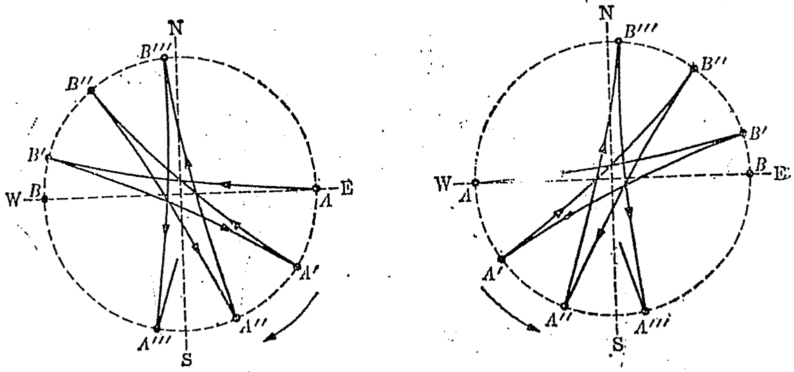

x_2
Centrifugal force
v_0
F_{eff}
x_1
Coriolis
force

4

(b)

5


z
Suspension point
at great height
l
y
T
$T_x = -T \cdot \frac{x}{l}$
x
$T_y = -T \cdot \frac{y}{l}$
m g


N
B'''
B''
B'
W
B
E
A
A'
A''
A'''
S

分類：	
編號：	
總號：	

See the figure

$\hat{i}'$ 向南

$\hat{j}'$ 向東　　　見圖

$\hat{k}'$ 向上

$\vec{\omega} = (-\omega\cos\lambda,\ 0,\ \omega\sin\lambda)$

$-2(\vec{\omega}\times\vec{V}')$

$$= -2\begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ V_x' & V_y' & V_z' \end{vmatrix}$$

向東　$V_y' > 0$　$V_x' = V_z' = 0$

$$\cancel{=-2\,|} \quad = -2\begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & V_y' & 0 \end{vmatrix}$$

$$= +2\hat{i}'\omega\sin\lambda V_y'$$

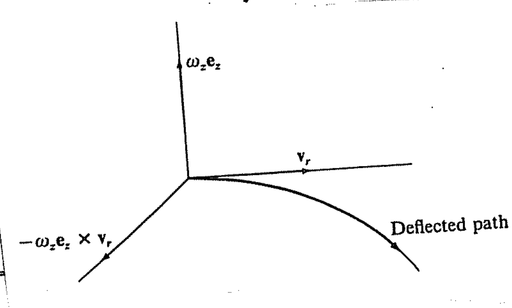
向南

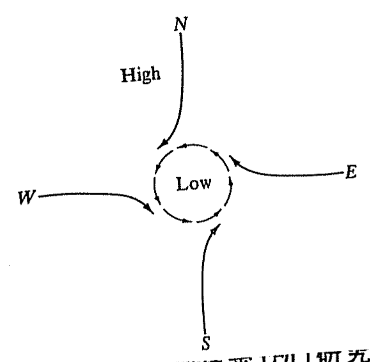
向北　$-\hat{i}'$

$$-2\begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ -V_x' & 0 & 0 \end{vmatrix}$$

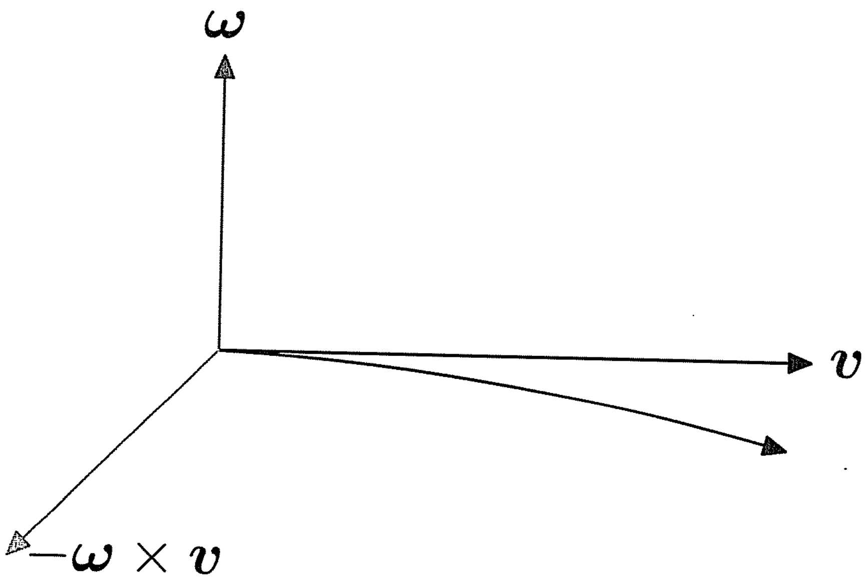
$$= \overset{①}{-}2\,\omega\sin\lambda V_x' \overset{②}{\hat{j}'}$$

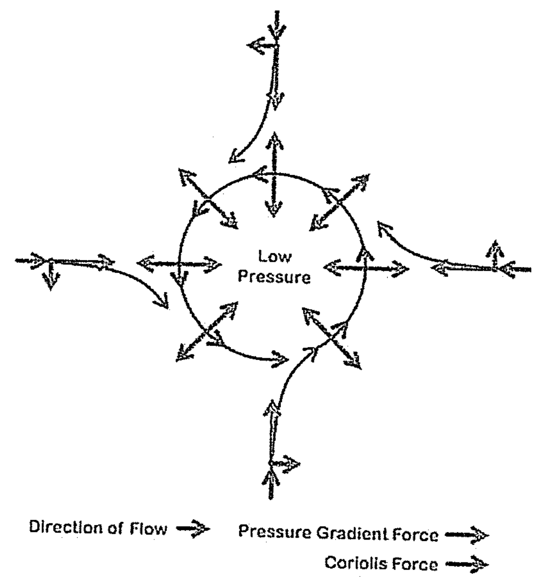
$= +$　向東





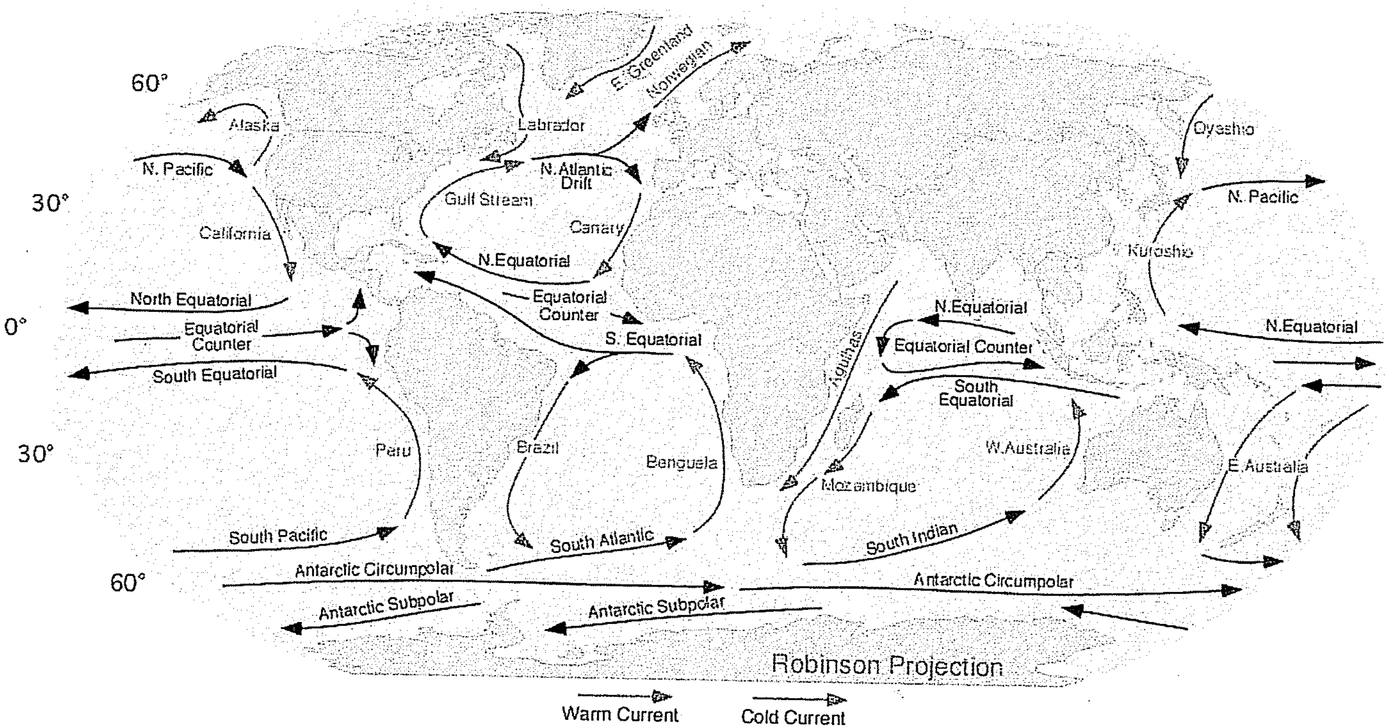
The Coriolis force is proportional to the velocity and the mass. Consider objects moving on the earth's surface, and in the northern hemisphere the upward component of the angular velocity will deflect moving object to the right.





In the regions of low pressure, the wind will blow inwards, deflected to move in a counter-clockwise direction. Cyclones, typhoons, etc. in the northern hemisphere will move likewise.

Ocean currents are also affected by the Coriolis force.



Now we shall study the free fall body.

Since ω is small, we shall neglect the centrifugal acceleration.

$$\frac{d\vec{v}}{dt} = \vec{g} - 2\vec{\omega}\times\vec{v}$$

Write $\underbrace{\vec{v} = \vec{v}_1}_{\text{dominant part}} + \underbrace{\vec{v}_2}_{\text{correction}}$

$$\frac{d\vec{v}_1}{dt} + \frac{d\vec{v}_2}{dt} = \vec{g} - 2\vec{\omega}\times(\vec{v}_1 + \vec{v}_2)$$

$$\frac{d\vec{v}_1}{dt} = \vec{g} \quad \Rightarrow \quad \vec{v}_1 = \vec{g}t$$ (Note: the initial condition)

$$\frac{d\vec{v}_2}{dt} = 2\vec{v}_1\times\vec{\omega}$$
$$= 2\vec{g}t\times\vec{\omega}$$

$$\vec{g}\times\vec{\omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -g \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \end{vmatrix}$$

$$= \omega g\cos\lambda\, \hat{j}$$

pointing toward east

$$\vec{r}_1 = \vec{r}_0 + \frac{1}{2}\vec{g}t^2$$

$$\vec{r}_2 = \frac{1}{3}\vec{g}t^3\times\vec{\omega}$$

$$t = \sqrt{\frac{2h}{g}} \quad \Rightarrow \quad y = \frac{1}{3}t^3\omega g\cos\lambda$$
$$= \frac{1}{3}\sqrt{\frac{2h}{g}}\,\frac{2h}{g}\,\omega g\cos\lambda$$
$$= \frac{2h}{3}\sqrt{\frac{2h}{g}}\,\omega\cos\lambda$$

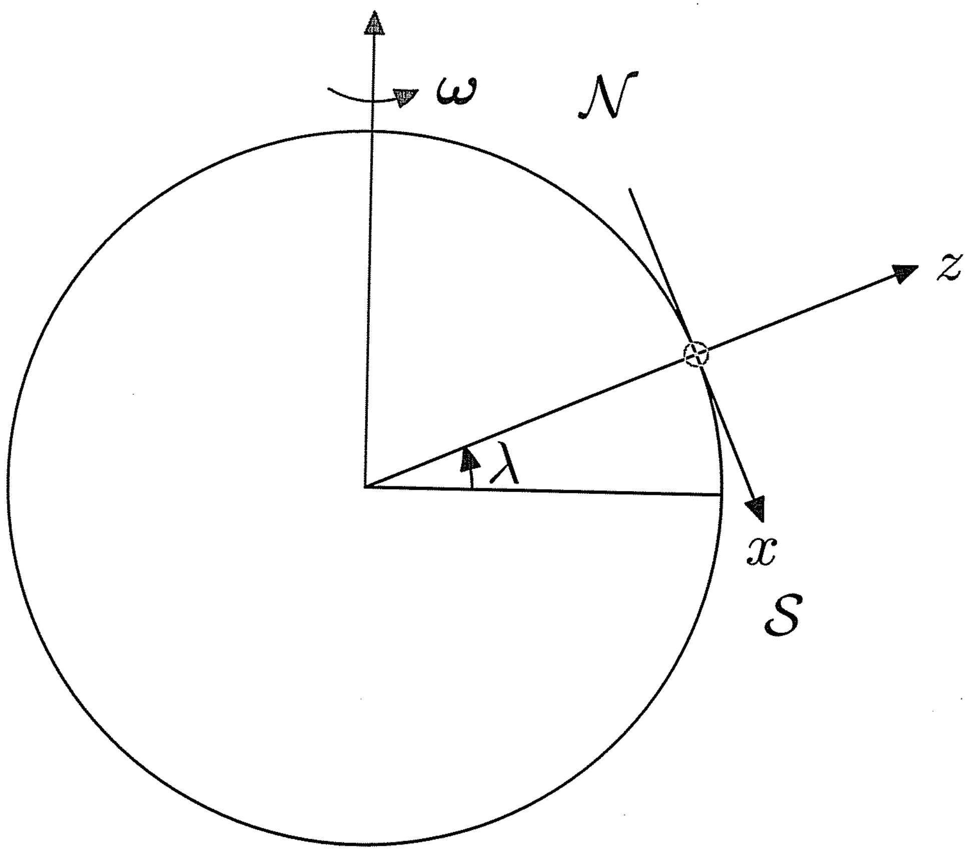
分類：
編號：
總號：

If $\lambda = 23°$, $h = 100\,m$, $\omega = \frac{2\pi}{1\,day} \sim 7\cdot 10^{-5}$

$\Rightarrow \quad y \simeq 2\ cm.$

Worked Examples:

1. Let us calculate the deflection of a free falling body from the vertical caused by the earth's rotation.



Let the vertical be defined as the direction determined by $\boldsymbol{g}$. The centrifugal force is to be neglected. Hence

$$\frac{d\boldsymbol{v}}{dt} = 2\boldsymbol{v} \times \boldsymbol{\omega} + \boldsymbol{g}\,.$$

The deflection is very small. So we can write $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, a dominant part plus a correction.

$$\frac{d\boldsymbol{v}_1}{dt} + \frac{d\boldsymbol{v}_2}{dt} = 2(\boldsymbol{v}_1 + \boldsymbol{v}_2) \times \boldsymbol{\omega} + \boldsymbol{g}\,.$$

The equation for $\boldsymbol{v}_1$ describes the free fall without rotation.

$$\frac{d\boldsymbol{v}_1}{dt} = \boldsymbol{g} \quad \Rightarrow \quad \boldsymbol{v}_1 = \boldsymbol{g}t\,,$$

Neglecting the second order term, the differential equation for $\boldsymbol{v}_2$ is

$$\frac{d\boldsymbol{v}_2}{dt} = 2\boldsymbol{g}t \times \boldsymbol{\omega}\,.$$

Hence we get

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 + \frac{1}{2}\boldsymbol{g}t^2\,,$$
$$\boldsymbol{r}_2 = \frac{1}{3}\boldsymbol{g}t^3 \times \boldsymbol{\omega}\,.$$

We now choose the z-axis to be the vertically upward direction, the x-axis pointing south, and the y-axis the eastward direction. Then

$$\boldsymbol{g} \times \boldsymbol{\omega} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & -g \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \end{vmatrix} = \omega g \cos\lambda \boldsymbol{j}\,.$$

Approximating the time of fall to the ground as $t = \sqrt{2h/g}$, where h is the height the particle begins free falling, the deflection is then eastward with displacement

$$y = \frac{2h}{3}\sqrt{\frac{2h}{g}}\omega\cos\lambda\,.$$

If $\lambda = 23°$, $h = 100$ m, then using $\omega = 2\pi/1$ day $\sim 7 \times 10^{-5}$, we can compute y to be

$$y \approx 2 \text{ cm}\,.$$

$$\frac{d^2\vec{r}'}{dt^2} = \vec{g} - 2(\vec{\omega}\times\vec{v}')$$
$$= -g\hat{k}' + 2\omega\cos\lambda\,\dot{y}\,\hat{i}' - 2(\omega\cos\lambda\,\dot{x}' + \omega\sin\lambda\,\dot{z}')\hat{j}' + 2\omega\sin\lambda\,\dot{y}\,\hat{k}'$$

$$\begin{cases} \ddot{x}' = 2\omega\cos\lambda\,\dot{y}' \\ \ddot{y}' = -2(\omega\cos\lambda\,\dot{x}' + \omega\sin\lambda\,\dot{z}') \\ \ddot{z}' = -g + 2\omega\sin\lambda\,\dot{y}' \end{cases}$$

equation of motion in the moving (rotating) frame

Initial conditions

At $t=0$ $\quad \dot{x}' = \dot{y}' = \dot{z}' = 0$

$x' = y' = 0, \quad z' = h$

$\dot{x}' = 2\omega\cos\lambda\, y' + C_1$

$\dot{y}' = -2(\omega\cos\lambda\, x' + \omega\sin\lambda\, z') + C_2$

Put in the initial condition $\quad C_1 = 0, \quad C_2 = 2\omega\sin\lambda\, h$

$\dot{x}' = 2\omega\cos\lambda\, y \quad , \quad \dot{y}' = -2(\omega\cos\lambda\, x' + \omega\sin\lambda\, z') + 2\omega\sin\lambda\, h$

$\ddot{z}' = -g + 2\omega\sin\lambda\,\dot{y}' = -g - 4\omega^2\sin\lambda\, x'[\cos\lambda\, x' + \sin\lambda(z'-h)]$

$\cong -g$

$\dot{z}' = -gt + C_3$ put in the initial condition $\downarrow$

$C_3 = 0$

$\dot{z}' = -gt$

$\ddot{y}' = [-2\omega\cos\lambda\,\dot{x}' + (-2\omega\sin\lambda)(-gt)$

$= \underbrace{-4\omega^2\cos\lambda\, y}_{\text{neglect}} + 2\omega\sin\lambda\, gt$

$\dot{y}' = \omega g\sin\lambda\, t^2 + \underbrace{C_4}_{0}$ from initial condition

$y' = \frac{1}{3}\omega g\sin\lambda\, t^3 + \underbrace{C_5}_{=0}$

$y' = \frac{1}{3}\omega g\sin\lambda\, t^3$

$h \simeq \frac{1}{2}gt^2 \qquad t = \sqrt{\frac{2h}{g}} \qquad y' = \frac{1}{3}\omega g\sin\lambda\left(\sqrt{\frac{2h}{g}}\right)^3$

Foucault Pendulum

$$\vec{a}' = \vec{a} - 2\vec{\omega} \times \dot{\vec{r}}'$$

$\downarrow$

Coriolis acceleration

S system

z axis is along the local vertical

Interest in the rotation of the plane of oscillation

$\downarrow$

consider the motion of
the pendulum blob
in the x, y
plane

Limit the motion to oscillations of small amplitude

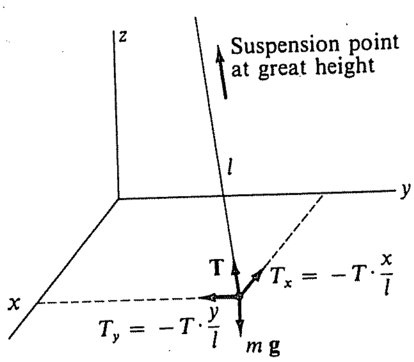
with the horizontal excursions small compared

to the length of the pendulum

$\downarrow$

$\dot{z}$ is small compared to $\dot{x}, \dot{y}$
and
can be neglected

In the S system



分類:
編號:
總號:

The Foucault Pendulum

S system
(inertial coordinate)

$$\vec{T} = (\vec{T}\cdot\hat{i})\hat{i} + (\vec{T}\cdot\hat{j})\hat{j} + (\vec{T}\cdot\hat{k})\hat{k}$$

$\downarrow$
tension

Proof: $\vec{T} = T_x\hat{i} + T_y\hat{j} + T_z\hat{k}$

$$(\vec{T}\cdot\hat{i}) = T_x$$

$$(\vec{T}\cdot\hat{j}) = T_y$$

$$(T\cdot\hat{k}) = T_z$$

$$\Rightarrow \vec{T} = T\cos\alpha\,\hat{i} + T\cos\beta\,\hat{j} + T\cos\gamma\,\hat{k}$$

$$= -T\left(\frac{x}{l}\right)\hat{i} - T\left(\frac{y}{l}\right)\hat{j} + T\left(\frac{l-z}{l}\right)\hat{k}$$

$$= -T/l\ (\underbrace{x\hat{i} + y\hat{j} + z\hat{k}}_{\vec{r}}) + T\hat{k}$$

$$\vec{F} = \vec{T} + m\vec{g} = m\vec{a}$$

$$\vec{g} = -g\hat{k}$$

In S' system

$$\vec{a}' = \vec{a} - 2\,(\vec{\omega}\times\vec{V}') - \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

small, shall be neglected

Write out the equation of motion

$$\ddot{x}' = -\frac{T}{ml}x' + 2m\omega\dot{y}'\cos\lambda$$

$$\ddot{y}' = -\frac{T}{ml}y - 2\ \omega(\dot{x}'\cos\lambda + \dot{z}'\sin\lambda)$$

$$\ddot{z}' = \frac{T}{m} - \frac{T}{ml}z - g + 2\omega\dot{y}'\sin\lambda$$

Assuming that the bob of the pendulum undergoes small oscillation about the equilibrium position so that its motion can be assumed to take place in a horizontal plane $\Rightarrow \dot{z}' = 0, \ddot{z}' = 0$

$$0 = T - mg + 2m\omega\dot{y}\sin\lambda$$

$$\frac{l-z}{l} \simeq 1$$

$$\Rightarrow \quad T = mg - 2m\omega\dot{y}'\sin\lambda$$

$$\ddot{x}' = -\frac{gx'}{l} + \frac{2\omega x'\dot{y}'\sin\lambda}{l} + 2\omega\dot{y}'\cos\lambda$$

$$\ddot{y} = -\frac{gy'}{l} + \frac{2\omega y'\dot{y}\sin\lambda}{l} - 2\omega\dot{x}'\cos\lambda$$

ω, $\underbrace{x', \ y'}_{\ll l}$ are small

the second term can be neglected

$$\Rightarrow \quad \ddot{x}' = -\frac{gx'}{l} + 2\omega\dot{y}\cos\lambda$$

$$\ddot{y}' = -\frac{gy'}{l} - 2\omega\dot{x}\cos\lambda$$

Let $K^2 = g/l$, $\alpha = \omega\cos\lambda$

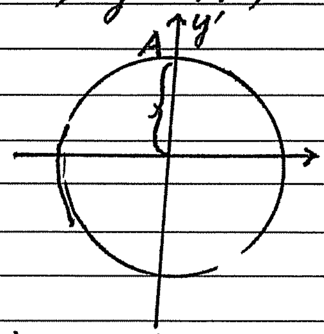
$$\Rightarrow \quad \ddot{x}' = -K^2x' + 2\alpha\dot{y}' \qquad (A)$$

$$\ddot{y}' = -K^2y' - 2\alpha\dot{x} \qquad (B)$$

This is the equation of motion.

Choose the initial condition

$x'=0$, $\dot{x}'=0$, $y'=A$, $\dot{y}'=0$ at time $t=0$



$\Rightarrow$ mathematical problem

$\downarrow$

useful to use complex number

(A) + i (B)

$$\ddot{x}'+i\ddot{y}' = -K^2(x'+iy') + 2\alpha\underset{\parallel}{(\dot{y}'-i\dot{x}')}$$

$$2i\alpha(\dot{x}'+i\dot{y}')$$

Let $u = x'+iy'$ (complex number), then

$$\ddot{u} = -K^2u - 2i\alpha\dot{u} \qquad i^2=-1 \quad i=\sqrt{-1}$$

$$\Rightarrow \quad \ddot{u} + \underline{2i\alpha}\dot{u} + \underline{K^2}u = 0$$

complex equation constants

decoupled.

Ansatz $u = ce^{\gamma t}$ c, γ are constant

$$\Rightarrow \quad \gamma^2 + 2i\alpha\gamma + K^2 = 0$$

$$\gamma_\pm = (-2i\alpha \pm \sqrt{-4\alpha^2-4K^2})/2 = -i\alpha \pm i\sqrt{\alpha^2+K^2}$$

$$\alpha^2 = \omega^2\cos^2\lambda \ll \frac{g}{l} = K^2 \qquad \downarrow K^2$$

$$\Rightarrow \quad \gamma = -i\alpha \pm iK.$$

General solution

$$u = (C_1 + iC_2)e^{-i(\alpha-K)t} + (C_3 + iC_4)e^{-i(\alpha+K)}$$

C_1, C_2, C_3, C_4 are real.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$x' + iy' = (C_1 + iC_2)(\cos(\alpha-K)t - i\sin(\alpha-K)t)$$
$$+ (C_3 + iC_4)(\cos(\alpha+K)t - i\sin(\alpha+K)t\}$$

Equating the real and imaginary part

$$x' = C_1\cos(\alpha-K)t + C_2\sin(\alpha-K)t + C_3\cos(\alpha+K)t + C_4\sin(\alpha+K)t$$

$$y' = -C_1\sin(\alpha-K)t + C_2\cos(\alpha-K)t - C_3\sin(\alpha+K)t + C_4\cos(\alpha+K)t$$

C_1, C_2, C_3, C_4 are to be determined by initial conditions.

$t=0, \quad x'=0 \qquad C_1 + C_3 = 0 \Rightarrow C_1 = -C_3$

$$\dot{x}' = -C_1(\alpha-K)\sin(\alpha-K)t + C_2(\alpha-K)\cos(\alpha-K)t$$
$$- C_3(\alpha+K)\sin(\alpha+K)t + C_4(\alpha+K)\cos(\alpha+K)t$$

$\dot{x}' = 0 \qquad C_2(\alpha-K) + C_4(\alpha+K) = 0$

$$C_4 = C_2\left(\frac{K-\alpha}{K+\alpha}\right) = C_2\left(\frac{\sqrt{g/l} - \omega\cos\lambda}{\sqrt{g/l} + \omega\cos\lambda}\right)$$

$\downarrow$

~ 1

$\Rightarrow C_4 = C_2$

$$\Rightarrow x' = C_1\cos(\alpha-K)t + C_2\sin(\alpha-K)t$$
$$- C_1\cos(\alpha+K)t + C_2 \quad (\alpha+K)t$$

$$y' = -C_1\sin(\alpha-K)t + C_2\cos(\alpha-K)t$$
$$+ C_1\sin(\alpha+K)t + C_2\cos(\alpha+K)t$$

分類:
編號:
總號:

$$\dot{y}' = -C_1(\alpha-K)\cos(\alpha-K)t + C_2(\alpha-K)(-\sin(\alpha-K)t)$$

$$+C_1(\alpha+K)\cos(\alpha+K)t + C_2(\alpha+K)(-\sin(\alpha+K)t\}$$

$$\dot{y}'_{\substack{at\\t=0}}=0 \Rightarrow \quad C_1=0$$

$$y' = C_2\cos(\alpha-K)t + C_2\cos(\alpha+K)t$$

$$y'_{at\ t=0}=A \qquad C_2=\frac{1}{2}A$$

$\Rightarrow$

$$x' = \frac{1}{2}A\sin(\alpha-K)t + \frac{1}{2}A\sin(\alpha+K)t$$

$$= A\cos Kt\sin\alpha t = A\cos\sqrt{\frac{g}{\ell}}\,t\,\sin(\omega\cos\lambda\, t)$$

$$y' = \frac{1}{2}A\cos(\alpha-K)t + \frac{1}{2}A\cos(\alpha+K)t$$

$$= A\cos Kt\cos\alpha t = A\cos\sqrt{g/\ell}\,t\cos(\omega\cos\lambda\, t)$$

This is the final result

$$\vec{r}' = x'\hat{i}' + y'\hat{j}'$$

$$= A\cos\sqrt{\frac{g}{\ell}}\,t\ \hat{n}'$$

$$\hat{n}' = \underbrace{\sin(\omega\cos\lambda)t\ \hat{i}' + \cos(\omega\cos\lambda\, t)\,\hat{j}'}_{\text{a time-dependent}}$$

$\cos\sqrt{\frac{g}{\ell}}\,t$ term has a period $2\pi\sqrt{\frac{\ell}{g}} = T_1$

When it going from $t=0$ to $t=T_1$

$n' = \hat{j}' \rightarrow$

$$\hat{n}' = \sin\left(\omega\cos\lambda\ 2\pi\sqrt{\frac{\ell}{g}}\right)\hat{i}' + \cos\left(\omega\cos\lambda\cdot\frac{2\pi\sqrt{\ell}}{g}\right)\hat{j}'$$

$$\sim \hat{j}'$$

slowly varying

$\therefore \vec{r}'$ is $\sim$ along $\hat{n}'$ the magnitude change from A to $-A$.

rather quickly

$$\hat{n}' = \hat{i}' \sin(\omega\cos\lambda)t + \hat{j}' \cos(\omega\cos\lambda)t$$

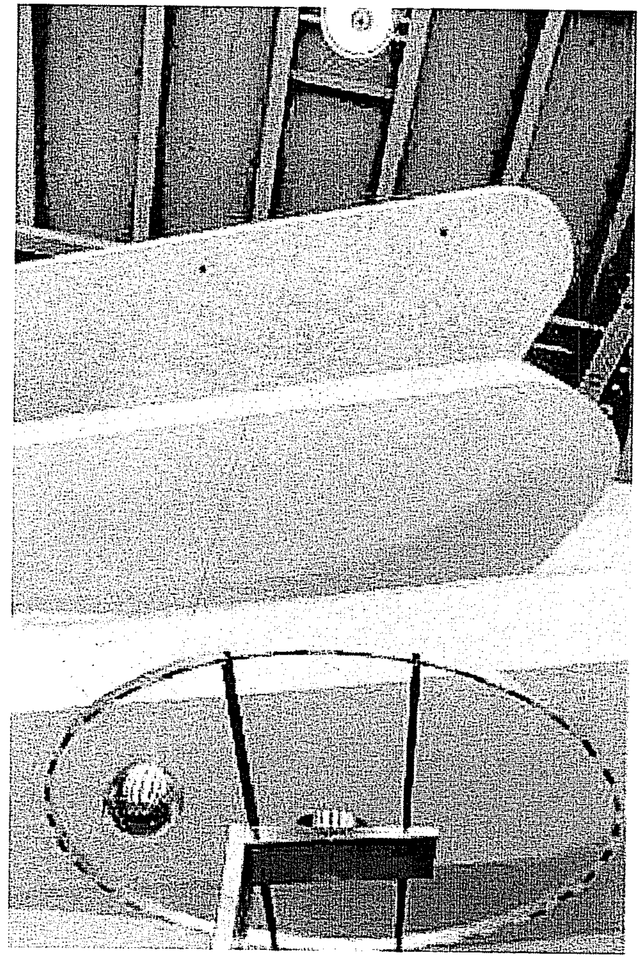
After $t = \frac{2\pi}{4\omega\cos\lambda} \Rightarrow \hat{n}' = \frac{\sqrt{2}}{2}\hat{i}' + \frac{\sqrt{2}}{2}\hat{j}'$

The period of $\hat{n}'$

$$T = \frac{2\pi}{\omega\sin\lambda} = \frac{1}{\sin\lambda} \text{ day}$$

Foucault Pendulum

Foucault in 1851 suspended a 28-kg bob with a 67-metre wire from the dome of the Panthéon in Paris to demonstrate that the earth really rotates.



2. We derive here how the plane of oscillation changes due to the rotation of the earth. We can make the approximation $\dot{z} = 0$, and the Coriolis force is

$$2m \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dot{x} & \dot{y} & 0 \\ \omega_x & 0 & \omega_z \end{vmatrix} = 2m(\dot{y}\omega_z \boldsymbol{i} - \dot{x}\omega_z \boldsymbol{j} - \dot{y}\omega_x \boldsymbol{k}) \,.$$

We consider the horizontal motion.

$$\begin{aligned} \ddot{x} &= -\omega_0^2 x + 2\dot{y}\omega_z \,, \\ \ddot{y} &= -\omega_0^2 y - 2\dot{x}\omega_z \,, \end{aligned}$$

where ω_0 is just the angular frequency of the pendulum. We can set $w = x + iy$, then

$$\ddot{w} = -\omega_0^2 w - 2i\omega_z \dot{w} \,.$$

As usual, we set $w = Ae^{\alpha t}$, then

$$\alpha^2 + 2i\omega_z \alpha + \omega_0^2 = 0\,.$$

$$\therefore \qquad \alpha = -i\omega_z \pm i\sqrt{\omega_z^2 + \omega_0^2} \approx -i\omega_z \pm i\omega_0\,.$$

So

$$w = e^{-i\omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)\,.$$

Without the first term we get the usual harmonic oscillation of a spherical pendulum in a plane. The phase $e^{-i\omega_z t}$ just rotates the plane of oscillation clockwise in the northern hemisphere with a period given by

$$T = \frac{2\pi}{\omega \sin\lambda} = \frac{1}{\sin\lambda}\text{day}\,.$$

The Foucault pendulum at UN takes 36 hours and 45 minutes to complete a cycle.

$$T_x \cong -T\frac{x}{\ell}$$

$$T_y \cong -T\frac{y}{\ell}$$

$$T_z \simeq T = mg$$

$$\vec{a} = -\frac{T}{m\ell}(x\hat{i} + y\hat{j}) = -\frac{T}{m\ell}\vec{r} = -\frac{g}{\ell}\underset{\vec{r}'}{\underset{\parallel}{\vec{r}}}$$

$$\vec{\omega} \times \vec{v}' \qquad \qquad \omega_0^2 \equiv \frac{g}{\ell}$$

$$= \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & 0 \end{vmatrix}$$

$$= \hat{i}'(-\dot{y}\omega\sin\lambda) + \hat{j}'\omega\sin\lambda\,\dot{x}' - \hat{k}'\dot{y}\,\omega\cos\lambda$$

The equation for x', y' component

$$\frac{d^2x'}{dt^2} = -\omega_0^2 x' + \dot{y}'\omega\sin\lambda$$

$$\frac{d^2y'}{dt^2} = -\omega_0^2 y' - \dot{x}'\omega\sin\lambda$$

This is the equation describing the motion in the x', y' plane

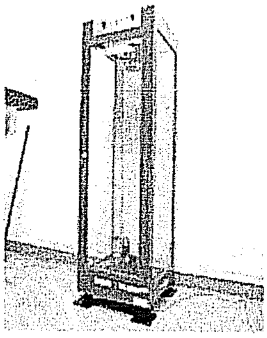
With the initial conditions, we can solve for the problem

↓

plane of oscillation
rotates
clockwise
in the northern
hemisphere
with a period
given by

$$T = \frac{2\pi}{\omega\sin\lambda} = \frac{1}{\sin\lambda} \text{ day}$$

佛科擺（Foucault Pendulum）



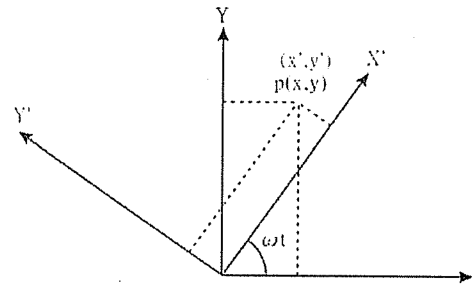
要描述一物體的運動必須選擇一參考座標系，然後才能對這一座標系敘述其位置和速度。討論質點的運動時，我們的參考系可以固定在任何物體上；我們甚至還可以選擇轉動座標，或者是加速座標。不過，通常所選擇的座標以能讓我們所要描述的運動問題愈簡單爲原則。

（一）慣性座標

若牛頓第二定律 $F = ma$ 能夠在參考座標系 A 中成立，則稱 A 爲一「慣性座標系」。也就是說，如果在 A 座標系內一不受任何外力作用的物體，永遠沿一直線做等速度運動，或永遠靜止時，則這一座標系便稱爲慣性座標系。而我們通常將固定在遙遠的 fixed star 上參考座標視爲慣性座標。

（二）非慣性座標

若我們將一參考座標系固定在地球表面上，從一些在地面上所做的實驗結果，我們曉得這一參考座標系大致是一慣性座標系。例如，不受外力作用的氣墊車會在一光滑水平面上沿一直線做等速運動；或者一物體斜向拋射時的運動路徑可用 $F = ma$ 完美的解釋。但是如果物體的運動時間很長，或移動的距離很大時，我們就不能再將固定在地球表面上的參考座標 視爲慣性座標，而要視爲非慣性座標。



（圖 1）轉動座標

座標（ $X'-Y'$ ）以 ω 角速度沿 z 軸對座標（ $X-Y$ ）旋轉

位置分量間的關係

$$\Rightarrow \begin{cases} x = x'\cos\omega t - y'\sin\omega t \\ y = x'\sin\omega t + y'\cos\omega t \\ z = z' \end{cases}$$

ps： $\vec{\omega} = \omega\vec{k} = \omega\vec{k}'$

速度分量間的關係

$$\Rightarrow \begin{cases} v_x = v_{x'}\cos\omega t - v_{y'}\sin\omega t - \omega x'\sin\omega t - \omega y'\cos\omega t \\ v_y = v_{x'}\sin\omega t + v_{y'}\cos\omega t + \omega x'\cos\omega t - \omega y'\sin\omega t \\ v_z = v_{z'} \end{cases}$$

加速度分量間關係

$$\Rightarrow \begin{cases} a_x = a_{x'}\cos\omega t - a_{y'}\sin\omega t - 2\omega v_{x'}\sin\omega t - 2\omega v_{y'}\cos\omega t - \omega^2 x'\cos\omega t + \omega^2 y'\sin\omega t \\ a_y = a_{x'}\sin\omega t + a_{y'}\cos\omega t + 2\omega v_{x'}\cos\omega t - 2\omega v_{y'}\sin\omega t - \omega^2 x'\sin\omega t - \omega^2 y'\cos\omega t \\ a_z = a_{z'} \end{cases}$$

所以 $a = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$

$$\begin{aligned} &= a_{x'}(\cos\omega t\ \vec{i} + \sin\omega t\ \vec{j}) + a_{y'}(\cos\omega t\ \vec{j} - \sin\omega t\ \vec{i}) + a_{z'}\vec{k} \\ &\quad +2\omega v_{x'}(\cos\omega t\ \vec{j} - \sin\omega t\ \vec{i}) - 2\omega v_{y'}(\cos\omega t\ \vec{i} + \sin\omega t\ \vec{j}) \\ &\quad -\omega^2 x'(\cos\omega t\ \vec{i} + \sin\omega t\ \vec{j}) + \omega^2 y'(\sin\omega t\ \vec{i} - \cos\omega t\ \vec{j}) \\ &= a_{x'}\vec{i}' + a_{y'}\vec{j}' + a_{z'}\vec{k}' - 2\omega v_{y'}\vec{i}' + 2\omega v_{x'}\vec{j}' - \omega^2 x'\vec{i}' - \omega^2 y'\vec{j}' \\ &= a' + 2\omega v_{x'}(\vec{k}'\times\vec{i}') + 2\omega v_{y'}(\vec{k}'\times\vec{j}') + \omega^2 x'\vec{k}'\times(\vec{k}'\times\vec{i}') + \omega^2 y'\vec{k}'\times(\vec{k}'\times\vec{j}') \\ &= a' + 2\omega\vec{k}'\times(v_{x'}\vec{i}' + v_{y'}\vec{j}') + \omega\vec{k}'\times(\omega\vec{k}'\times x'\vec{i}') + \omega\vec{k}'\times(\omega\vec{k}'\times y'\vec{j}') \\ &= a' + 2\omega\vec{k}'(v_{x'}\vec{i}' + v_{y'}\vec{j}' + v_{z'}\vec{k}') + \omega\vec{k}'\times[\omega\vec{k}'\times(x'\vec{i}' + y'\vec{j}' + z'\vec{k}')] \end{aligned}$$

（利用 $\vec{k}'\times\vec{k}' = 0$）

$$\begin{aligned} &= \vec{a}' + 2\omega\vec{k}'\times\vec{v}' + \omega\vec{k}'\times(\omega\vec{k}'\times\vec{r}') \\ &= \vec{a}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}') \end{aligned}$$

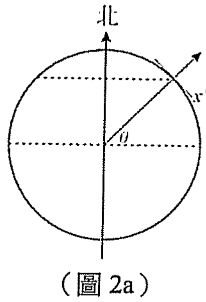
所以得 $\vec{a} = \vec{a}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$

在慣性座標系中：

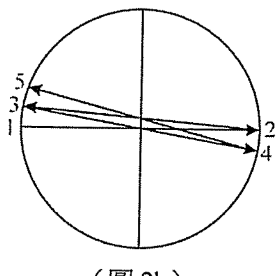
$$F = ma = m[\vec{a}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')]$$

$$m\vec{a}' = F - \underbrace{2m\vec{\omega}\times\vec{v}'}_{\text{柯氏力}} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r}')}_{\text{離心力}}$$

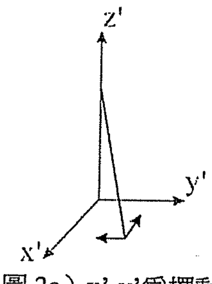
在緯度 ϑ 的北半球：



（圖 2a）



（圖 2b）



（圖 2c）x'-y'為擺動面

在非慣性座標系中，單擺擺動會因地球自轉而有角度變化。如圖 2b

圖 2b 中，擺錘從位置 1 移動至位置 5 時：$y' = A\cos\omega_0 t$ ，$\omega_0 = (\frac{g}{L})^{\frac{1}{2}}$ ，A 為擺錘的振福，即圓內的半徑；L 為擺長。

$$v_{y'} = \frac{dy'}{dt} = -A\omega_0\sin\omega_0 t$$

圖 2c 中，在 x' 方向，擺錘受到柯氏力和張力的水平分力影響

柯氏力$=2m\omega\sin\vartheta v_{y'}=-2m\omega A\omega_0\sin\vartheta\sin\omega_0 t$

張力的（x' 方向）水平分力$=-T(\dfrac{x}{L})$，又$T\sim mg$

所以　$a'_x=-2\omega A\omega_0\sin\vartheta\sin\omega_0 t-\dfrac{gx'}{L}$

$$a'_x=-2\omega A\omega_0\sin\vartheta\sin\omega_0 t-\omega_0^2 x'$$

By initial condition：$t=0$、$x'=0$、$v'_x=0$

$$x'=A\omega\sin\vartheta\ [t\cos\omega_0 t-\frac{\sin\omega_0 t}{\omega_0}]$$

擺錘由位置 2 到位置 3 所需的時間　$t=\dfrac{\pi}{\omega_0}$

此時位置 3 的位置爲　$x'=-\dfrac{A\omega\pi\sin\vartheta}{\omega_0}$

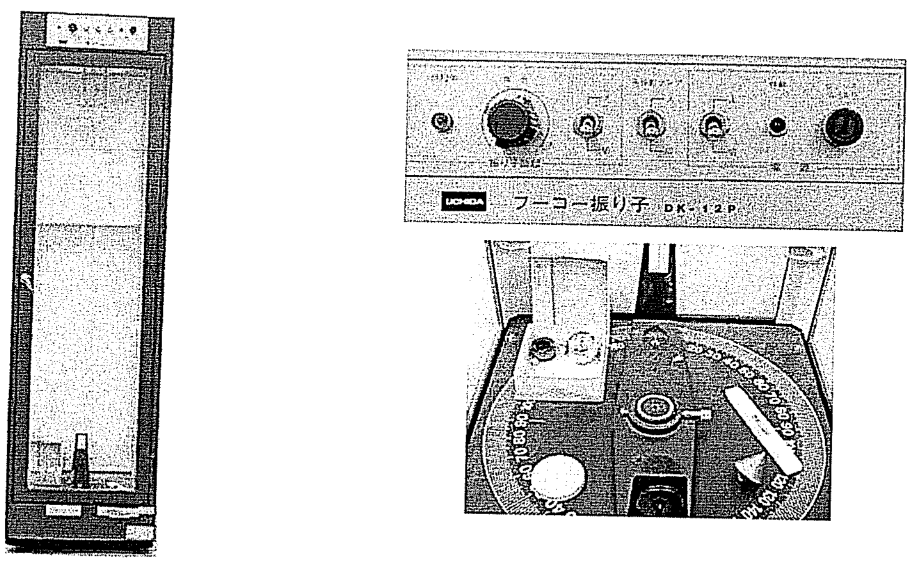
$$\overset{\frown}{35}\sim\frac{A\omega\pi\sin\vartheta}{\omega_0}$$

在　$t=\dfrac{\pi}{\omega_0}$　內　振動面旋轉的角度$=\dfrac{\omega\pi\sin\vartheta}{\omega_0}$

所以，佛科擺進動的角速度$=\dfrac{\dfrac{\omega\pi\sin\vartheta}{\omega_0}}{\dfrac{\pi}{\omega_0}}=\omega\sin\vartheta$

因此，佛科擺進動一圈的週期$T=\dfrac{2\pi}{\omega\sin\vartheta}=\dfrac{1}{\sin\vartheta}$天（$\dfrac{2\pi}{\omega}=$一天）。

整體實驗裝置圖：



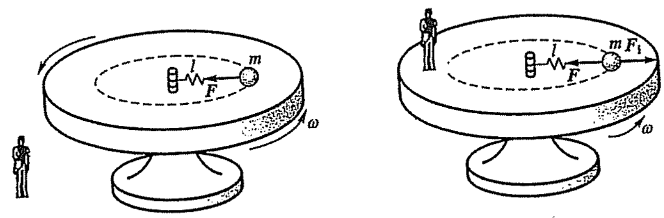
（圖 3）

分類:
編號:
總號:

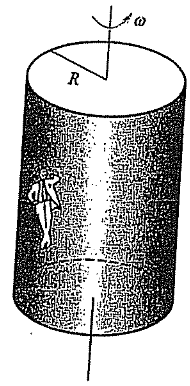
Foucault Pendulum   Plane of oscillation changes due to the rotation of the earth.

Horizonal motion

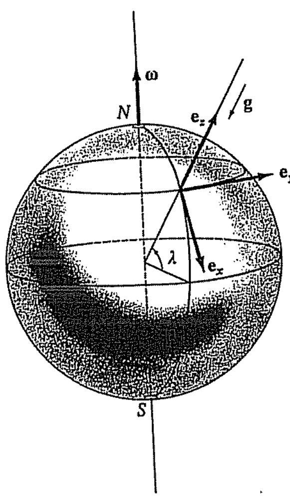
$$\vec{a} = \vec{g} + \frac{\vec{T}}{m} - \underbrace{2\vec{\omega}\times\vec{v}'}_{\text{Coriolis acceleration}}$$


m
l
F
ω
m F_1
l
F
ω

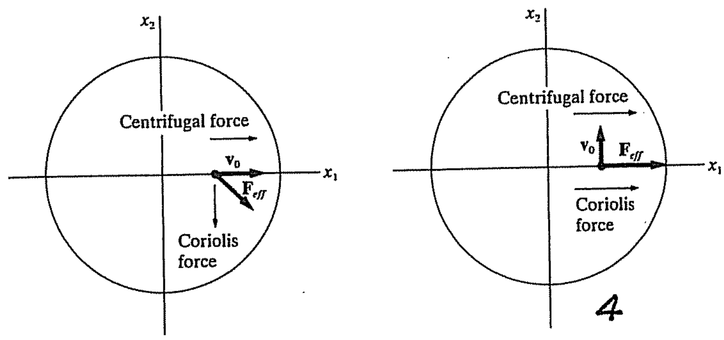
1


ω
R

2


ω
N
e_z
g
e_y
λ
e_x
S

3

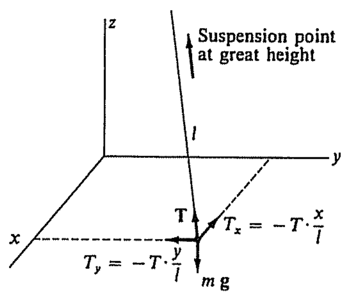

x_2
Centrifugal force
v_0
F_{eff}
x_1
Coriolis
force
x_2
Centrifugal force
v_0
F_{eff}
x_1
Coriolis
force

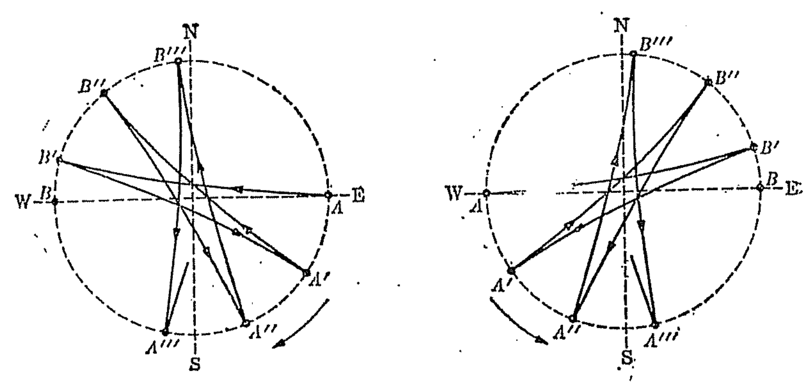
(a)

(b)

4

5


z
Suspension point
at great height
l
y
T
$T_x = -T \cdot \frac{x}{l}$
x
$T_y = -T \cdot \frac{y}{l}$
m g


N
B'''
B''
B'
W
B
A
E
A'
A''
A'''
S
N
B'''
B''
B'
B
E
W
A
A'
A''
A'''
S

Marion and Thornton
"Classical Dynamics
of
Particles and Systems"
Third Edition
Chapter 9

CHAPTER 9

Motion in a Noninertial Reference Frame

9.1 Introduction

The advantage of choosing an inertial reference frame to describe dynamic processes was made evident in the discussions in Chapters 2 and 6. It is, of course, always possible to express the equations of motion for a system in an inertial frame. But there are types of problems for which these equations would be extremely complex, and it becomes easier to treat the motion of the system in a noninertial frame of reference.

To describe, for example, the motion of a particle on or near the surface of the earth, it is tempting to do so by choosing a coordinate system fixed with respect to the earth. We know, however, that the earth undergoes a complicated motion, compounded of many different rotations (and hence accelerations) with respect to an inertial reference frame identified with the "fixed" stars. The earth coordinate system is, therefore, a *noninertial* frame of reference; and, although the solutions to many problems can be obtained to the desired degree of accuracy by ignoring this distinction, many important effects result from the noninertial nature of the earth coordinate system.

In analyzing the motion of rigid bodies in the following chapter, we also find it convenient to use noninertial reference frames and therefore make use of much of the development presented here.

9.2 Rotating Coordinate Systems

Let us consider two sets of coordinate axes. Let one set be the "fixed" or inertial axes, and let the other be an arbitrary set that may be in motion with respect to the inertial system. We designate these axes as the "fixed" and "rotating" axes, respectively. We use x_i' as coordinates in the fixed system and x_i as coordinates in

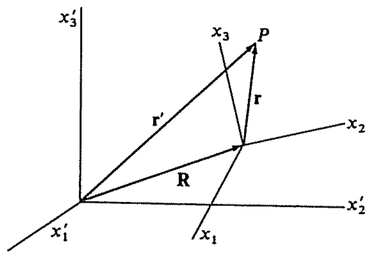


Figure 9-1

the rotating system. If we choose some point P, as in Figure 9-1, we have

$$\mathbf{r}' = \mathbf{R} + \mathbf{r} \tag{9.1}$$

where $\mathbf{r}'$ is the radius vector of P in the fixed system and $\mathbf{r}$ is the radius vector of P in the rotating system. The vector $\mathbf{R}$ locates the origin of the rotating system in the fixed system.

We may always represent an arbitrary infinitesimal displacement by a pure rotation about some axis called the **instantaneous axis of rotation**. For example, the instantaneous motion of a disk rolling down an inclined plane can be described as a rotation about the point of contact between the disk and the plane. Therefore, if the x_i system undergoes an infinitesimal rotation $\delta\theta$, corresponding to some arbitrary infinitesimal displacement, the motion of P (which, for the moment, we consider to be at rest in the x_i system) can be described in terms of Equation 1.106 as

$$(d\mathbf{r})_{\text{fixed}} = d\boldsymbol{\theta} \times \mathbf{r} \tag{9.2}$$

where the designation "fixed" is explicitly included to indicate that the quantity $d\mathbf{r}$ is measured in the x_i', or *fixed*, coordinate system. Dividing this equation by dt, the time interval during which the infinitesimal rotation takes place, we obtain the time of rate change of $\mathbf{r}$ as measured in the fixed coordinate system:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \frac{d\boldsymbol{\theta}}{dt} \times \mathbf{r} \tag{9.3}$$

or, since the angular velocity of the rotation is

$$\boldsymbol{\omega} \equiv \frac{d\boldsymbol{\theta}}{dt} \tag{9.4}$$

we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \mathbf{r} \qquad \text{(for } P \text{ fixed in } x_i \text{ system)} \tag{9.5}$$

This same result was determined in Section 1.15.

If we allow the point P to have a velocity $(d\mathbf{r}/dt)_{\text{rotating}}$ with respect to the x_i

system, this velocity must be added to $\boldsymbol{\omega} \times \mathbf{r}$ to obtain the time rate of change of $\mathbf{r}$ in the fixed system:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r} \tag{9.6}$$

EXAMPLE 9.1

Consider a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ in the rotating system. Let the fixed and rotating systems have the same origin. Find $\dot{\mathbf{r}}'$ in the fixed system by direct differentiation if the angular velocity of the rotating system is $\boldsymbol{\omega}$ in the fixed system.

Solution: We begin by taking the time derivative directly

$$\begin{aligned}\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \frac{d}{dt}\left(\sum_i x_i\mathbf{e}_i\right)\\ &= \sum_i (\dot{x}_i\mathbf{e}_i + x_i\dot{\mathbf{e}}_i)\end{aligned} \tag{9.7}$$

The first term is simply $\dot{\mathbf{r}}_r$ in the rotating system, but what are the $\dot{\mathbf{e}}_i$?

$$\dot{\mathbf{r}}_r = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \dot{\mathbf{r}}_r + \sum_i x_i\dot{\mathbf{e}}_i \tag{9.8}$$

Look at Figure 9-2 and examine which components of ω_i tend to rotate $\mathbf{e}_1$. We see that ω_2 tends to rotate $\mathbf{e}_1$ toward the $-\mathbf{e}_3$ direction and that ω_3 tends to rotate $\mathbf{e}_1$ toward the $+\mathbf{e}_2$ direction. We therefore have

$$\frac{d\mathbf{e}_1}{dt} = \omega_3\mathbf{e}_2 - \omega_2\mathbf{e}_3 \tag{9.9a}$$

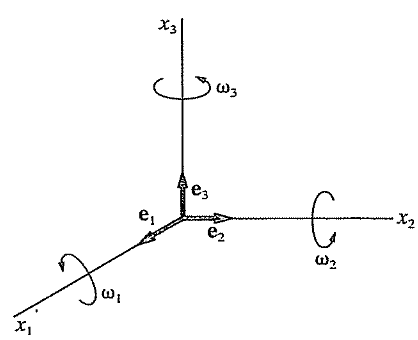


Figure 9-2

Similarly, we have

$$\frac{d\mathbf{e}_2}{dt} = -\omega_3\mathbf{e}_1 + \omega_1\mathbf{e}_3 \tag{9.9b}$$

$$\frac{d\mathbf{e}_3}{dt} = \omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2 \tag{9.9c}$$

In each case the direction of the time derivative of the unit vector must be perpendicular to the unit vector in order not to change its magnitude.

Equations 9.9 can be written

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \tag{9.10}$$

and Equation 9.8 becomes

$$\begin{aligned}\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \dot{\mathbf{r}}_r + \sum_i \boldsymbol{\omega} \times x_i\mathbf{e}_i \\ &= \dot{\mathbf{r}}_r + \boldsymbol{\omega} \times \mathbf{r}\end{aligned} \tag{9.11}$$

which is the same result as Equation 9.6.

Although we chose the displacement vector $\mathbf{r}$ for the derivation of Equation 9.6, the validity of this expression is not limited to the vector $\mathbf{r}$. In fact, for an arbitrary vector $\mathbf{Q}$, we have

$$\boxed{\left(\frac{d\mathbf{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{Q}} \tag{9.12}$$

Equation 9.12 is an important result.

We note, for example, that the angular acceleration $\dot{\boldsymbol{\omega}}$ is the same in both the fixed and rotating systems:

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \boldsymbol{\omega} \equiv \dot{\boldsymbol{\omega}} \tag{9.13}$$

since $\boldsymbol{\omega} \times \boldsymbol{\omega}$ vanishes and where $\dot{\boldsymbol{\omega}}$ designates the common value in the two systems.

Equation 9.12 may now be used to obtain the expression for the velocity of the point P as measured in the fixed coordinate system. From Equation 9.1 we have

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} \tag{9.14}$$

so that

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r} \tag{9.15}$$

If we define

$$\mathbf{v}_f \equiv \dot{\mathbf{r}}_f \equiv \left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} \tag{9.16a}$$

$$\mathbf{V} \equiv \dot{\mathbf{R}}_f \equiv \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} \tag{9.16b}$$

$$\mathbf{v}_r \equiv \dot{\mathbf{r}}_r \equiv \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} \tag{9.16c}$$

we may write

$$\boxed{\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}} \tag{9.17}$$

where

$\mathbf{v}_f$ = Velocity relative to the fixed axes

$\mathbf{V}$ = Linear velocity of the moving origin

$\mathbf{v}_r$ = Velocity relative to the rotating axes

$\boldsymbol{\omega}$ = Angular velocity of the rotating axes

$\boldsymbol{\omega} \times \mathbf{r}$ = Velocity due to the rotation of the moving axes

9.3 Centrifugal and Coriolis Forces

We have seen that Newton's equation $\mathbf{F} = m\mathbf{a}$ is valid only in an inertial frame of reference. The expression for the force on a particle can therefore be obtained from

$$\mathbf{F} = m\mathbf{a}_f = m\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}} \tag{9.18}$$

where the differentiation must be carried out with respect to the fixed system. Differentiating Equation 9.17 we have

$$\begin{aligned}\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}} &= \left(\frac{d\mathbf{V}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} \\ &\quad + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}}\end{aligned} \tag{9.19}$$

We denote the first term by $\ddot{\mathbf{R}}_f$:

$$\ddot{\mathbf{R}}_f \equiv \left(\frac{d\mathbf{V}}{dt}\right)_{\text{fixed}} \tag{9.20}$$

The second term can be evaluated by substituting $\mathbf{v}_r$ for $\mathbf{Q}$ in Equation 9.12:

$$\left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{v}_r$$

$$= \mathbf{a}_r + \boldsymbol{\omega} \times \mathbf{v}_r \tag{9.21}$$

where $\mathbf{a}_r$ is the acceleration in the rotating coordinate system. The last term in Equation 9.19 can be obtained directly from Equation 9.6:

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{9.22}$$

Combining Equations 9.18–9.22, we obtain

$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r \tag{9.23}$$

To an observer in the rotating coordinate system, however, the effective force on a particle is given by*

$$\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r \tag{9.24}$$

$$= \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \tag{9.25}$$

The first term, $\mathbf{F}$, is the sum of the forces acting on the particle as measured in the fixed inertial system. The second $(-m\ddot{\mathbf{R}}_f)$ and third $(-m\dot{\boldsymbol{\omega}} \times \mathbf{r})$ terms result because of the translational and angular acceleration, respectively, of the moving coordinate system relative to the fixed system.

The quantity $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the usual *centrifugal force* term and reduces to $-m\omega^2 r$ for the case in which $\boldsymbol{\omega}$ is normal to the radius vector. Note that the minus sign implies that the centrifugal force is directed *outward* from the center of rotation (see Figure 9-3).

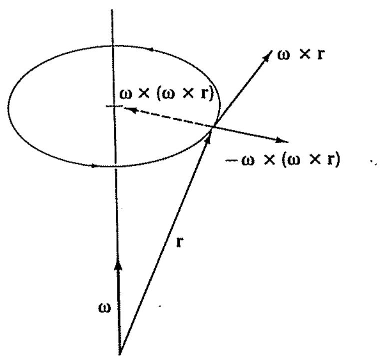


Figure 9-3

*This result was published by G. G. Coriolis in 1835. The theory of the composition of accelerations was an outgrowth of Coriolis's study of water wheels.

The last term in Equation 9.25 is a totally new quantity that arises from the motion of the particle in the rotating coordinate system. This term is called the **Coriolis force**. Note that the Coriolis force does indeed arise from the *motion* of the particle, since the force is proportional to v_r and hence vanishes if there is no motion.

Since we have used (on several occasions) the term "centrifugal force" and have now introduced the Coriolis force, we must now inquire about the physical meaning of these quantities. It is important to realize that the centrifugal and Coriolis forces are not forces in the usual sense of the word; they have been introduced in an artificial manner as a result of our arbitrary requirement that we be able to write an equation resembling Newton's equation which is at the same time valid in a noninertial reference frame; that is, the equation

$$\mathbf{F} = m\mathbf{a}_f$$

is valid only in an inertial frame. If, in a rotating reference frame, we wish to write (let $\ddot{\mathbf{R}}_f$ and $\dot{\boldsymbol{\omega}}$ be zero for simplicity)

$$\mathbf{F}_{\mathrm{eff}} = m\mathbf{a}_r$$

then we can express such an equation in terms of the real force $m\mathbf{a}_f$ as

$$\mathbf{F}_{\mathrm{eff}} = m\mathbf{a}_f + \text{(noninertial terms)}$$

where the "noninertial terms" are identified as the centrifugal and Coriolis "forces." Thus, for example, if a body rotates about a fixed force center, the only real force on the body is the force of attraction toward the force center (and gives rise to the *centripetal* acceleration). An observer moving with the rotating body, however, measures this central force and, in addition, notes that the body does not fall toward the force center. To reconcile this result with the requirement that the net force on the body vanish, the observer must postulate an additional force—the centrifugal force. But the "requirement" is artificial; it arises solely from an attempt to extend the form of Newton's equation to a noninertial system, and this can be done only by introducing a fictitious "correction force." The same comments apply for the Coriolis force; this "force" arises when an attempt is made to describe motion relative to the rotating body.

In spite of their artificiality, the concepts of centrifugal and Coriolis forces are useful. To describe the motion of a particle relative to a body rotating with respect to an inertial reference frame is a complicated matter. But the problem can be made relatively easy by the simple expedient of introducing the "noninertial forces," which then allows the use of an equation of motion resembling Newton's equation.

EXAMPLE 9.2

A student is performing measurements with a hockey puck on a large merry-go-round with a smooth horizontal surface. The merry-go-round has a constant angular velocity ω. (a) Find the effective force on the hockey puck after it is given a radial push. (b) Find the effective force on the hockey puck if it is given a push perpendicular to the radial direction.

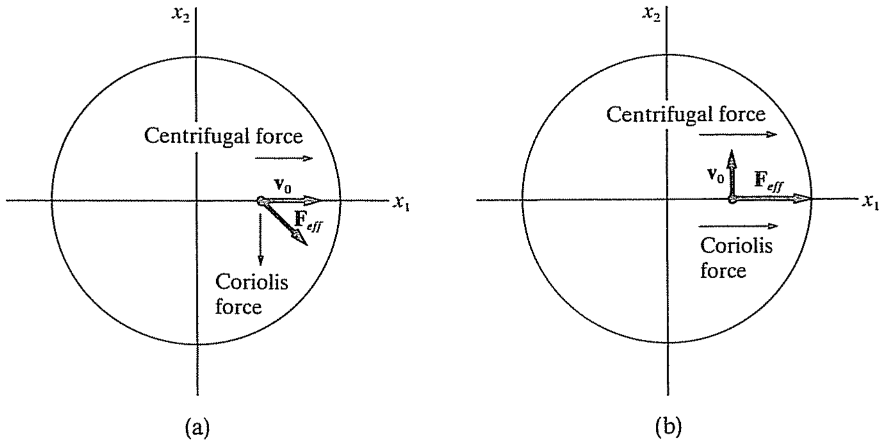


Figure 9-4

Solution: Refer to Figure 9-4, where the fixed and moving coordinate systems are fixed at the center. From Equation 9.25 the effective force is

$$\mathbf{F}_{\mathbf{eff}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r. \tag{9.26}$$

where $\mathbf{F}$ and the two system acceleration terms are zero. The friction force is also assumed to be zero. In Figure 9-4a let $\mathbf{v}_r = v_0\mathbf{e}_1$, $\boldsymbol{\omega} = \omega\mathbf{e}_3$, $\mathbf{r} = x_0\mathbf{e}_1$.

$$\mathbf{F}_{\mathbf{eff}} = m\omega^2 x_0\mathbf{e}_1 - 2m\omega v_0\mathbf{e}_2 \tag{9.27}$$

The directions of the centrifugal, Coriolis, and effective forces are shown in Figure 9-4a.

If the initial velocity of the hockey puck is $\mathbf{v}_r = v_0\mathbf{e}_2$ at the same place on the merry-go-round, the effective force becomes

$$\begin{aligned}\mathbf{F}_{\mathbf{eff}} &= m\omega^2 x_0\mathbf{e}_1 + 2m\omega v_0\mathbf{e}_1 \\ &= (m\omega^2 x_0 + 2m\omega v_0)\,\mathbf{e}_1\end{aligned} \tag{9.28}$$

The centrifugal term is the same, but the Coriolis term forces the puck to the right with respect to the direction of its velocity, as shown in Figure 9-4b.

The reader should attempt such experiments to compare experimental results in the fixed and moving coordinate systems. Computer simulations of the two systems are also enlightening.

9.4 Motion Relative to the Earth

The motion of the earth with respect to an inertial reference frame is dominated by the earth's rotation about its axis, the effects of the other motions (revolution about the sun, motion of the solar system with respect to the local galaxy, etc.) being small by comparison. Therefore, to a good approximation (see Problem 9-1) we can consider a coordinate system fixed relative to the earth to be

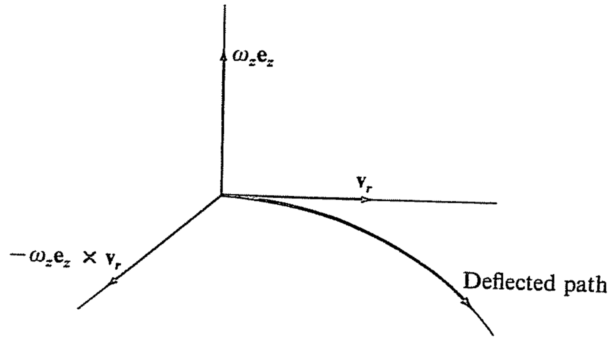


Figure 9-5

in pure rotation with respect to an inertial frame of reference, and we can therefore apply Equation 9.25 to the problems of motion on or near the surface of the earth.

The angular velocity vector ω, which represents the earth's rotation about its axis, is directed in a northerly direction. Therefore, in the Northern Hemisphere, ω has a component ω_z directed *outward* along the local vertical. If a particle is projected in a horizontal plane (in the local coordinate system at the surface of the earth) with a velocity $\mathbf{v}_r$, then the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}_r$ has a component in the plane of magnitude $2m\omega_z v_r$ directed toward the *right* of the particle's motion (see Figure 9-5), and a deflection from the original direction of motion results.*

Since the magnitude of the horizontal component of the Coriolis force is proportional to the vertical component of ω, the portion of the Coriolis force producing deflections depends on the latitude, being a maximum at the North Pole and zero at the Equator. In the Southern Hemisphere, the component ω_z is directed *inward* along the local vertical, and hence all deflections are in the opposite sense from those in the Northern Hemisphere.†

It is also interesting to note that the radial flow of air masses from high pressure regions into low pressure regions, since they are always deflected to the right by the Coriolis force (in the Northern Hemisphere), produce cyclonic motion (Figure 9-6). The actual motion of air masses is much more complicated, but the qualitative features of cyclonic motion are correctly given by considering the effects of the Coriolis force. The motion of water in whirlpools is (at least in principle) a similar situation, but in actuality other factors (various perturbations

---

*Poisson discussed the deviation of projectile motion in 1837.

†During the naval engagement near the Falkland Islands early in World War I, the British gunners were surprised to see their accurately aimed salvos falling 100 yards to the left of the German ships. The designers of the sighting mechanisms were well aware of the Coriolis deflection and had carefully taken this into account, but they apparently were under the impression that all sea battles took place near 50° N latitude and never near 50° S latitude. The British shots, therefore, fell at a distance from the targets equal to *twice* the Coriolis deflection.

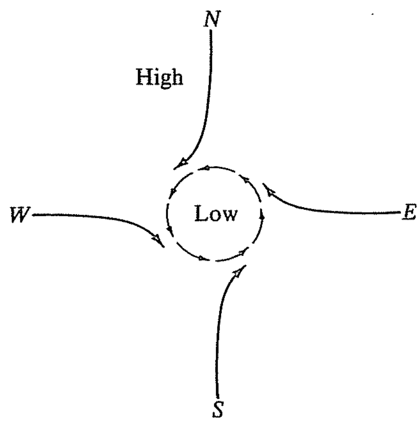


Figure 9-6

and residual angular momentum) dominate the Coriolis force, and whirlpools are found with both directions of flow. (Even under laboratory conditions it is extremely difficult to isolate the Coriolis effect.)

We note that if we are considering motion in the earth's gravitational field, then the quantity that we call the acceleration due to gravity (i.e., g or the vector $\mathbf{g}$) is actually a combination of the gravitational acceleration proper (as defined from the universal gravitation law) and the apparent outward acceleration (i.e., the centrifugal acceleration) resulting from the fact that our coordinate system is fixed with respect to the rotating earth; that is, the effective $\mathbf{g}$ is defined only in terms of measurements that we make: the magnitude is determined by the period of a pendulum and the direction by the direction that a plumb bob assumes at equilibrium. Thus $\mathbf{g}$ already includes the term $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. Because of this fact, the direction of $\mathbf{g}$ at a given point is in general slightly different from the true vertical (defined as the direction of the line connecting the point with the center of the earth; see Problem 9-9).

EXAMPLE 9.3

Find the horizontal deflection resulting from the Coriolis force of a particle falling freely in the earth's gravitational field.

Solution: The value of ω that occurs in the force equation (Equation 9.25) is that of the earth's rotation:

$$\omega = \frac{2\pi \text{ rad/day}}{86{,}400 \text{ s/day}} \cong 7.29 \times 10^{-5} \text{ rad/s}$$

The acceleration of the particle is given by

$$\mathbf{a}_r = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{v}_r$$

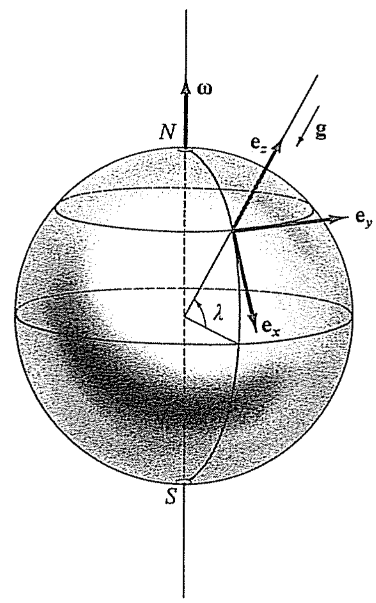


Figure 9-7

where **g** is the acceleration due to gravity (i.e., the effective **g** that includes the centrifugal acceleration). We choose a z-axis directed vertically outward from the surface of the earth. With this definition of $\mathbf{e}_z$, we complete the construction of a right-hand coordinate system by specifying that $\mathbf{e}_x$ be in a southerly and $\mathbf{e}_y$ in an easterly direction, as in Figure 9-7. We make the approximation that the distance of fall is sufficiently small that g remains constant during the process.

Since we have chosen the origin O of the rotating coordinate system to lie in the Northern Hemisphere,* we have

$$\omega_x = -\omega\cos\lambda$$
$$\omega_y = 0$$
$$\omega_z = \omega\sin\lambda$$

Although the Coriolis force produces small velocity components in the $\mathbf{e}_y$ and $\mathbf{e}_x$ directions, we can certainly neglect $\dot{x}$ and $\dot{y}$ compared to $\dot{z}$, the vertical velocity. Then, approximately,

$$\dot{x} \cong 0$$
$$\dot{y} \cong 0$$
$$\dot{z} \cong -gt$$

*The point O does not move in a precisely uniform manner with respect to an inertial reference frame. For motion near the surface of the earth, however, the terms proportional to $\ddot{\mathbf{R}}_f$ and $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ contribute little and can be neglected.

where we obtain $\dot{z}$ by considering a fall from rest. Therefore we have

$$\boldsymbol{\omega} \times \mathbf{v}_r \cong \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ 0 & 0 & -gt \end{vmatrix}$$

$$\cong -(\omega g t \cos \lambda)\mathbf{e}_y$$

The components of $\mathbf{g}$ are

$$g_x = 0$$
$$g_y = 0$$
$$g_z = -g$$

so the equations for the components of $\mathbf{a}_r$ (neglecting terms in ω^2; see Problem 9-10) become

$$(\mathbf{a}_r)_x = \ddot{x} \cong 0$$
$$(\mathbf{a}_r)_y = \ddot{y} \cong 2\omega g t \cos \lambda$$
$$(\mathbf{a}_r)_z = \ddot{z} \cong -g$$

Thus the effect of the Coriolis force is to produce an acceleration in the $\mathbf{e}_y$, or easterly, direction. Integrating $\ddot{y}$ twice, we have

$$y(t) \cong \tfrac{1}{3}\omega g t^3 \cos \lambda$$

where $y = 0$ and $\dot{y} = 0$ at $t = 0$. The integration of $\ddot{z}$ yields the familiar result for the distance of fall,

$$z(t) \cong z(0) - \tfrac{1}{2}gt^2$$

and the time of fall from a height $h = z(0)$ is given by

$$t \cong \sqrt{2h/g}$$

Hence the result for the eastward deflection d of a particle dropped from rest at a height h and at a northern latitude λ is*

$$d \cong \tfrac{1}{3}\omega \cos \lambda \sqrt{8h^3/g} \tag{9.29}$$

An object dropped from a height of 100 m at latitude 45° is deflected approximately 1.55 cm (neglecting the effects of air resistance).

*The eastward deflection was predicted by Newton (1679), and several experiments (notably those of Robert Hooke) appeared to confirm the results. The most careful measurements were probably those of F. Reich (1831; published 1833), who dropped pellets down a mine shaft 188 m deep and observed a mean deflection of 28 mm. This is smaller than the value calculated from Equation 9.29, the decrease being due to air resistance effects. In all the experiments, a small southerly component of the deflection was observed—and remained unaccounted for until Coriolis's theorem was appreciated (see Problem 9-10).

EXAMPLE 9.4

To demonstrate the power of the Coriolis method for obtaining the equations of motion in a noninertial reference frame, rework the last example but use only the formalism previously developed—the theory of central-force motion.

Solution: If we release a particle of small mass from a height h above the earth's surface, the path the particle describes is a conic section—an ellipse with $\varepsilon \cong 1$ and with one focus very close to the earth's center. If r_0 is the earth's radius and λ the (northern) latitude, then at the moment of release, the particle has a horizontal velocity in the eastward direction:

$$v_{\text{hor}} = r\omega \cos \lambda = (r_0 + h)\omega \cos \lambda$$

and the angular momentum about the polar axis is

$$l = mrv_{\text{hor}} = m(r_0 + h)^2 \omega \cos \lambda \tag{9.30}$$

The equation of the path is*

$$\frac{\alpha}{r} = 1 - \varepsilon \cos \theta \tag{9.31}$$

if we measure θ from the initial position of the particle (see Figure 9-8). At $t = 0$

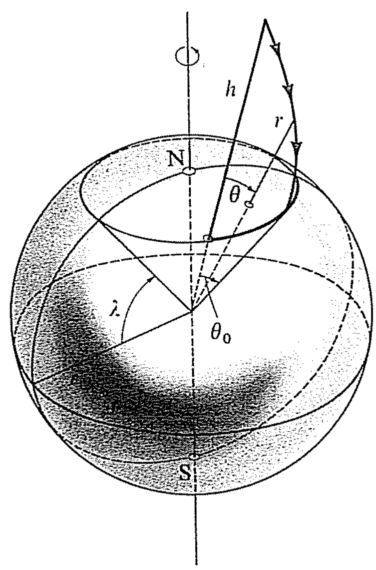


Figure 9-8

*Notice that there is a change of sign between Equation 9.31 and Equation 7.41 due to the different origins for θ in the two cases.

we have

$$\frac{\alpha}{r_0 + h} = 1 - \varepsilon$$

so Equation 9.31 can be written as

$$r = \frac{(1 - \varepsilon)(r_0 + h)}{1 - \varepsilon \cos \theta} \tag{9.32}$$

From Equation 7.12 for the areal velocity we can write

$$\frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{l}{2m}$$

Thus the time t required to describe an angle θ is

$$t = \frac{m}{l} \int_0^\theta r^2 \, d\theta$$

Substituting into this expression the value of l from Equation 9.30 and r from Equation 9.32, we find

$$t = \frac{1}{\omega \cos \lambda} \int_0^\theta \left(\frac{1 - \varepsilon}{1 - \varepsilon \cos \theta} \right)^2 d\theta \tag{9.33}$$

If we let $\theta = \theta_0$ when the particle has reached the earth's surface ($r = r_0$), then Equation 9.32 becomes

$$\frac{r_0}{r_0 + h} = \frac{1 - \varepsilon}{1 - \varepsilon \cos \theta_0}$$

or, inverting,

$$\begin{aligned} 1 + \frac{h}{r_0} &= \frac{1 - \varepsilon \cos \theta_0}{1 - \varepsilon} \\ &= \frac{1 - \varepsilon[1 - 2\sin^2(\theta_0/2)]}{1 - \varepsilon} \\ &= 1 + \frac{2\varepsilon}{1 - \varepsilon} \sin^2 \frac{\theta_0}{2} \end{aligned} \tag{9.34}$$

from which we have

$$\frac{h}{r_0} = \frac{2\varepsilon}{1 - \varepsilon} \sin^2 \frac{\theta_0}{2}$$

Since the path described by the particle is almost vertical, little change occurs in the angle θ between the position of release and the point at which the particle reaches the surface of the earth; θ_0 is therefore small and $\sin(\theta_0/2)$ can be approximated by its argument:

$$\frac{h}{r_0} \cong \frac{\varepsilon \theta_0^2}{2(1 - \varepsilon)} \tag{9.35}$$

If we expand the integrand in Equation 9.33 by the same method used to obtain Equation 9.34, we find

$$t = \frac{1}{\omega \cos \lambda} \int_0^\theta \frac{d\theta}{\{1 + [2\varepsilon/(1-\varepsilon)]\sin^2(\theta/2)\}^2}$$

and since θ is small, we have

$$t \cong \frac{1}{\omega \cos \lambda} \int_0^\theta \frac{d\theta}{[1 + \varepsilon\theta^2/2(1-\varepsilon)]^2}$$

Substituting for $\varepsilon/2(1-\varepsilon)$ from Equation 9.35 and writing $t(\theta = \theta_0) = T$ for the total time of fall, we obtain

$$\begin{aligned} T &\cong \frac{1}{\omega \cos \lambda} \int_0^{\theta_0} \frac{d\theta}{[1 + (h\theta^2/r_0\theta_0^2)]^2} \\ &\cong \frac{1}{\omega \cos \lambda} \int_0^{\theta_0} \left(1 - \frac{2h}{r_0\theta_0^2}\theta^2\right) d\theta \\ &= \frac{1}{\omega \cos \lambda}\left(1 - \frac{2h}{3r_0}\right)\theta_0 \end{aligned}$$

Solving for θ_0, we find

$$\theta_0 \cong \frac{\omega T \cos \lambda}{1 - 2h/3r_0} \cong \omega T \cos \lambda \left(1 + \frac{2h}{3r_0}\right)$$

During the time of fall T, the earth turns through an angle ωT, so the point on the earth directly beneath the initial position of the particle moves toward the east by an amount $r_0 \omega T \cos \lambda$. During the same time, the particle is deflected toward the east by an amount $r_0\theta_0$. Thus the net easterly deviation d is

$$\begin{aligned} d &= r_0\theta_0 - r_0\omega T \cos \lambda \\ &= \tfrac{2}{3} h\omega T \cos \lambda \end{aligned}$$

and using $T \cong \sqrt{2h/g}$ as in the preceding example, we have, finally,

$$d \cong \frac{1}{3}\omega \cos \lambda \sqrt{\frac{8h^3}{g}}$$

which is identical with the result obtained previously (Equation 9.29).

EXAMPLE 9.5

The effect of the Coriolis force on the motion of a pendulum produces a **precession**, or rotation with time of the plane of oscillation. Describe the motion of this system, called a **Foucault pendulum.***

*Devised in 1851 by the French physicist Jean Lèon Foucault (1819–1868).

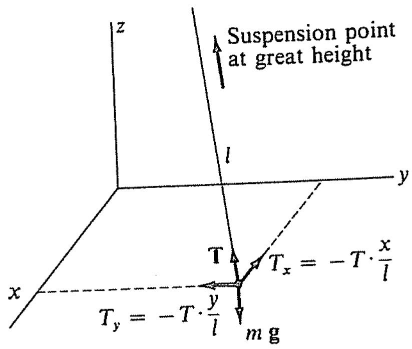


Figure 9-9

Solution: To describe this effect, let us select a set of coordinate axes with origin at the equilibrium point of the pendulum and z-axis along the local vertical. We are interested only in the rotation of the plane of oscillation—that is, we wish to consider the motion of the pendulum bob in the $x-y$ plane (the horizontal plane). We therefore limit the motion to oscillations of small amplitude, with the horizontal excursions small compared to the length of the pendulum. Under this condition, $\dot{z}$ is small compared to $\dot{x}$ and $\dot{y}$ and can be neglected.

The equation of motion is

$$\mathbf{a}_r = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\boldsymbol{\omega} \times \mathbf{v}_r \tag{9.36}$$

where $\mathbf{T}/m$ is the acceleration produced by the force of tension $\mathbf{T}$ in the pendulum suspension (Figure 9-9). We therefore have, approximately,

$$\left.\begin{aligned} T_x &= -T\cdot\frac{x}{l} \\ T_y &= -T\cdot\frac{y}{l} \\ T_z &\cong T \end{aligned}\right\} \tag{9.37}$$

As before,

$$g_x = 0$$
$$g_y = 0$$
$$g_z = -g$$

and

$$\omega_x = -\omega\cos\lambda$$
$$\omega_y = 0$$
$$\omega_z = \omega\sin\lambda$$

with

$$(\mathbf{v}_r)_x = \dot{x}$$
$$(\mathbf{v}_r)_y = \dot{y}$$
$$(\mathbf{v}_r)_z = \dot{z} \cong 0$$

Therefore

$$\boldsymbol{\omega} \times \mathbf{v}_r \cong \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

so that

$$\left.\begin{aligned} (\boldsymbol{\omega} \times \mathbf{v}_r)_x &\cong -\dot{y}\omega\sin\lambda \\ (\boldsymbol{\omega} \times \mathbf{v}_r)_y &\cong \dot{x}\omega\sin\lambda \\ (\boldsymbol{\omega} \times \mathbf{v}_r)_z &\cong -\dot{y}\omega\cos\lambda \end{aligned}\right\} \tag{9.38}$$

Thus the equations of interest are

$$\left.\begin{aligned} (\mathbf{a}_r)_x = \ddot{x} &\cong -\frac{T}{m}\cdot\frac{x}{l} + 2\dot{y}\omega\sin\lambda \\ (\mathbf{a}_r)_y \cong \ddot{y} &\cong -\frac{T}{m}\cdot\frac{y}{l} - 2\dot{x}\omega\sin\lambda \end{aligned}\right\} \tag{9.39}$$

For small displacements, $T \cong mg$. Defining $\alpha^2 \equiv T/ml \cong g/l$, and writing $\omega_z = \omega\sin\lambda$, we have

$$\left.\begin{aligned} \ddot{x} + \alpha^2 x &\cong 2\omega_z\dot{y} \\ \ddot{y} + \alpha^2 y &\cong -2\omega_z\dot{x} \end{aligned}\right\} \tag{9.40}$$

We note that the equation for $\ddot{x}$ contains a term in $\dot{y}$, and that the equation for $\ddot{y}$ contains a term in $\dot{x}$. Such equations are called **coupled equations**. A solution for this pair of coupled equations can be effected by adding the first of the above equations to i times the second:

$$(\ddot{x} + i\ddot{y}) + \alpha^2(x + iy) \cong -2\omega_z(i\dot{x} - \dot{y}) = -2i\omega_z(\dot{x} + i\dot{y})$$

If we write

$$q \equiv x + iy$$

we then have

$$\ddot{q} + 2i\omega_z\dot{q} + \alpha^2 q \cong 0$$

This equation is identical with the equation that describes damped oscillations (Equation 3.35), except that here the term corresponding to the damping factor is purely imaginary. The solution (see Equation 3.37) is

$$q(t) \cong \exp[-i\omega_z t][A\exp(\sqrt{-\omega_z^2 - \alpha^2}\,t) + B\exp(-\sqrt{-\omega_z^2 - \alpha^2}\,t)] \tag{9.41}$$

If the earth were not rotating, so that $\omega_z = 0$, then the equation for q would become

$$\ddot{q}' + \alpha^2 q' \cong 0, \qquad \omega_z = 0$$

from which it is seen that α corresponds to the oscillation frequency of the pendulum. This frequency is clearly much greater than the angular frequency of the earth's rotation. Therefore $\alpha \gg \omega_z$, and the equation for $q(t)$ becomes

$$q(t) \cong e^{-i\omega_z t}(Ae^{i\alpha t} + Be^{-i\alpha t}) \tag{9.42}$$

We can interpret this equation more easily if we note that the equation for q' has the solution

$$q'(t) = x'(t) + iy'(t) = Ae^{i\alpha t} + Be^{-i\alpha t}$$

Thus

$$q(t) = q'(t) \cdot e^{-i\omega_z t}$$

or

$$\begin{aligned} x(t) + iy(t) &= [x'(t) + iy'(t)] \cdot e^{-i\omega_z t} \\ &= [x' + iy'][\cos \omega_z t - i \sin \omega_z t] \\ &= [x' \cos \omega_z t + y' \sin \omega_z t] \\ &\quad + i[-x' \sin \omega_z t + y' \cos \omega_z t] \end{aligned}$$

Equating real and imaginary parts,

$$\left.\begin{aligned} x(t) &= x' \cos \omega_z t + y' \sin \omega_z t \\ y(t) &= -x' \sin \omega_z t + y' \cos \omega_z t \end{aligned}\right\}$$

We can write these equations in matrix form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos \omega_z t & \sin \omega_z t \\ -\sin \omega_z t & \cos \omega_z t \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \tag{9.43}$$

from which (x, y) may be obtained from (x', y') by the application of a rotation matrix of the familiar form

$$\lambda = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \tag{9.44}$$

Thus the angle of rotation is $\theta = \omega_z t$, and the plane of oscillation of the pendulum therefore rotates with a frequency $\omega_z = \omega \sin \lambda$. The observation of this rotation gives a clear demonstration of the rotation of the earth.*

*Vincenzo Viviani (1622–1703), a pupil of Galileo, had noticed as early as about 1650 that a pendulum undergoes a slow rotation, but there is no evidence that he correctly interpreted the phenomenon. Foucault's invention of the gyroscope in the year following the demonstration of his pendulum provided an even more striking visual proof of the earth's rotation.

Problems

9-1. Calculate the centrifugal acceleration, due to the earth's rotation, on a particle on the surface of the earth at the Equator. Compare this result with the gravitational acceleration. Compute also the centrifugal acceleration due to the motion of the earth about the sun and justify the remark made in the text that this acceleration may be neglected compared to the acceleration due to axial rotation.

9-2. An automobile drag racer drives a car with acceleration a and instantaneous velocity v. The tires (of radius r_0) are not slipping. Find which point on the tire has the greatest acceleration relative to the ground. What is this acceleration?

9-3. In Example 9.2, assume that the coefficient of static friction between the hockey puck and a horizontal rough surface (on the merry-go-round) is μ_s.

(a) How far away from the center of the merry-go-round can the hockey puck be placed without sliding?

(b) If the coefficient of kinetic friction is μ_k and the hockey puck is moving outward radially with velocity v on the merry-go-round, how far away from the center can the puck move without slipping?

9-4. A bucket of water is set spinning about its symmetry axis. Determine the shape of the water in the bucket.

9-5. Obtain an expression for the angular deviation of a particle projected from the North Pole in a path which lies close to the earth. Is the deviation significant for a missile that makes a 4800-km flight in 10 min? What is the "miss distance" if the missile is aimed directly at the target? Is the miss distance greater for a 19,300-km flight at the same velocity?

9-6. If a particle is projected vertically upward to a height h above a point on the earth's surface at a northern latitude λ, show that it strikes the ground at a point $\frac{4}{3}\omega \cos \lambda \sqrt{8h^3/g}$ to the west. (Neglect air resistance, and consider only small vertical heights.)

9-7. If a projectile is fired due east from a point on the surface of the earth at a northern latitude λ with a velocity of magnitude V_0 and at an angle of inclination to the horizontal of α, show that the lateral deflection when the projectile strikes the earth is

$$d = \frac{4V_0^3}{g^2} \cdot \omega \sin \lambda \cdot \sin^2 \alpha \cos \alpha$$

where ω is the rotation frequency of the earth.

9-8. In the preceding problem, if the range of the projectile is R for the case $\omega = 0$, show that the change of range due to the rotation of the earth is

$$\Delta R = \sqrt{\frac{2R^3}{g}} \cdot \omega \cos \lambda \left(\cot^{\frac{1}{2}} \alpha - \frac{1}{3} \tan^{\frac{3}{2}} \alpha \right)$$

9-9. Show that the angular deviation ε of a plumb line from the true vertical at a point on the earth's surface at a latitude λ is

$$\varepsilon = \frac{r_0 \omega^2 \sin \lambda \cos \lambda}{g - r_0 \omega^2 \cos^2 \lambda}$$

where r_0 is the radius of the earth. What is the value (in seconds of arc) of the maximum deviation?

9-10. Refer to Example 9.3 concerning the deflection of a particle falling in the earth's gravitational field. Perform a calculation in second approximation (i.e., retain terms in ω^2) and show that there is a *southerly* deflection

$$d_s \cong \frac{2}{3} \frac{h^2 \omega^2}{g} \sin \lambda \cos \lambda$$

9-11. Consider the description of a particle's motion in a coordinate system in uniform rotation with respect to an inertial reference frame. Obtain the Lagrangian for the particle. Next calculate the Hamiltonian and attempt to identify this quantity with the total energy. (Are all of the requirements for this identification met?) The expression for the total energy thus obtained is the standard formula $\frac{1}{2}mv^2 + U$ plus an additional term. Show that the extra term is the *centrifugal potential energy*. Finally, show that it is possible to define an *effective potential* for the problem which is exactly that used in the central-force problem (see Equation 7.34).

Chapter 7

ACCELERATED COORDINATE SYSTEMS

The simple form of Newton's second law,

$$\mathbf{F} = m\frac{d^2\mathbf{r}_\mathbf{I}}{dt^2} \tag{7.1}$$

for a particle of mass m holds only in inertial coordinate systems (unaccelerated and not rotating with respect to the distant stars), as denoted by the $\mathbf{I}$ subscript above. On the other hand, physical events are sometimes more simply described with reference to an accelerated or rotating coordinate system. For example, observations of motion on the earth's surface are more simply expressed in terms of a coordinate system fixed on the rotating earth than in terms of an inertial coordinate system. For this reason it is useful to derive the form of the second law which directly applies in accelerated reference frames.

7.1 Transformation to Moving Coordinate Frames

To transform the law of motion to an accelerated reference frame, we first need to relate the time derivatives of vector quantities in moving and fixed coordinate systems. In a moving frame, an arbitrary vector quantity $\mathbf{A}$ can be written

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} \tag{7.2}$$

where the directions of the unit vectors change in time. The time derivative of $\mathbf{A}$ is

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}\hat{\mathbf{x}} + \frac{dA_y}{dt}\hat{\mathbf{y}} + \frac{dA_z}{dt}\hat{\mathbf{z}}\right) + \left(A_x\frac{d\hat{\mathbf{x}}}{dt} + A_y\frac{d\hat{\mathbf{y}}}{dt} + A_z\frac{d\hat{\mathbf{z}}}{dt}\right) \tag{7.3}$$

The first term on the right-hand side of this equation is the time rate of change of $\mathbf{A}$ with reference to the axes of the accelerated frame. We denote this time rate of change of $\mathbf{A}$ in the moving frame by

$$\frac{\delta\mathbf{A}}{\delta t} \equiv \frac{dA_x}{dt}\hat{\mathbf{x}} + \frac{dA_y}{dt}\hat{\mathbf{y}} + \frac{dA_z}{dt}\hat{\mathbf{z}} \tag{7.4}$$

The second term in (7.3) is due to the rotation of the coordinate system, which causes the direction of the unit vectors to change with time. Since the coordinate system is rigid, we can directly apply the result of (6.64),

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}\times\mathbf{r} \tag{7.5}$$

to find the time derivatives of the unit vectors

$$\frac{d\hat{\mathbf{x}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{x}} \qquad \frac{d\hat{\mathbf{y}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{y}} \qquad \frac{d\hat{\mathbf{z}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{z}} \tag{7.6}$$

Here $\boldsymbol{\omega}$ is the angular velocity of rotation of the accelerated frame relative to a fixed frame. Upon substitution of (7.4) and (7.6) into (7.3), we have

$$\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \tag{7.7}$$

or more simply,

$$\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times\mathbf{A} \tag{7.8}$$

Accordingly, the time derivative $d\mathbf{A}/dt$ in a fixed reference frame consists of a part $\delta\mathbf{A}/\delta t$ from the time rate of change of $\mathbf{A}$ relative to the axes of the moving frame and a part $\boldsymbol{\omega}\times\mathbf{A}$ from the rotation of these axes relative to the fixed axes. It follows from (7.8) that the time derivative of $\boldsymbol{\omega}$ is independent of the coordinate frame.

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{\delta\boldsymbol{\omega}}{\delta t} = \dot{\boldsymbol{\omega}} \tag{7.9}$$

We next apply the result in (7.8) to the vector $\mathbf{r}$, which specifies the location of a particle with respect to the moving axes. The first time derivative is

$$\frac{d\mathbf{r}}{dt} = \frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\times\mathbf{r} \tag{7.10}$$

The second time derivative can likewise be evaluated with the aid of (7.8).

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}\left(\frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\times\mathbf{r}\right) = \frac{\delta}{\delta t}\left(\frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\times\mathbf{r}\right) + \boldsymbol{\omega}\times\left(\frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\times\mathbf{r}\right) \\ &= \frac{\delta^2\mathbf{r}}{\delta t^2} + \frac{\delta\boldsymbol{\omega}}{\delta t}\times\mathbf{r} + 2\boldsymbol{\omega}\times\frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\end{aligned} \tag{7.11}$$

To find the law of motion in the accelerated/rotating frame, we relate the location of the particle in the accelerated and inertial frames by the vector $\mathbf{R}$ connecting the origins of the two frames

$$\mathbf{r}_\mathrm{I} = \mathbf{r} + \mathbf{R} \tag{7.12}$$

as illustrated in Fig. 7-1. Then, substituting the result in (7.11) into

$$\frac{d^2\mathbf{r}_\mathrm{I}}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} + \frac{d^2\mathbf{R}}{dt^2} \tag{7.13}$$

we get

$$\frac{d^2\mathbf{r}_\mathrm{I}}{dt^2} = \frac{\delta^2\mathbf{r}}{\delta t^2} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\frac{\delta\mathbf{r}}{\delta t} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \frac{d^2\mathbf{R}}{dt^2} \tag{7.14}$$

The form of Newton's law in the non-inertial frame now follows directly from (7.1) and (7.14)

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F} - m\left[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \frac{d^2\mathbf{R}}{dt^2}\right] \tag{7.15}$$

where $\mathbf{v} = \delta\mathbf{r}/\delta t$ is velocity and $\delta^2\mathbf{r}/\delta t^2$ is the acceleration of a particle as observed in the moving coordinate system.

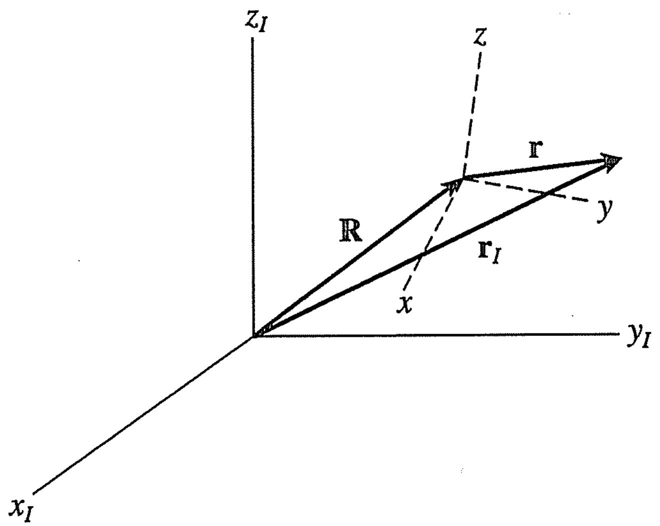


FIGURE 7-1. Inertial and accelerated coordinate frames.

7.2 Fictitious Forces

We can write the equation of motion (7.15) for an accelerated frame in a form similar to (7.1) for an inertial frame.

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F}_\mathrm{eff} \tag{7.16}$$

The acceleration $\delta^2\mathbf{r}/\delta t^2$ observed in the moving frame is generated by the *effective force*

$$\mathbf{F}_\mathrm{eff} = \mathbf{F} - m\left[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \frac{d^2\mathbf{R}}{dt^2}\right] \tag{7.17}$$

The names associated with the so-called *fictitious force* terms on the right-hand side of (7.17) are

Centrifugal force:

$$\mathbf{F}_{cf} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7.18}$$

Coriolis force:

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega}\times\mathbf{v} \tag{7.19}$$

Azimuthal force:

$$\mathbf{F}_{az} = -m\dot{\boldsymbol{\omega}}\times\mathbf{r} \tag{7.20}$$

Translational force:

$$\mathbf{F}_{tr} = -m\frac{d^2\mathbf{R}}{dt^2} \tag{7.21}$$

The centrifugal force of (7.18) is due to the rotational motion of the coordinate system. Since $\boldsymbol{\omega}\cdot\mathbf{F}_{cf} = 0$, the centrifugal force is perpendicular to the rotation axis $\hat{\boldsymbol{\omega}}$. If the angular velocity $\boldsymbol{\omega}$ is chosen to lie along the z axis of the moving frame, as in Fig. 7-2, then

$$\begin{aligned}\mathbf{F}_{cf} &= -m\left[\boldsymbol{\omega}(\boldsymbol{\omega}\cdot\mathbf{r}) - \mathbf{r}\omega^2\right] = m\omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \\ &= m\omega^2\boldsymbol{\rho}\end{aligned} \tag{7.22}$$

where $\boldsymbol{\rho}$ is the cylindrical-radius vector to the particle from the z axis. The centrifugal force is directed radially outward from the axis of rotation. The result in (7.22) is the same as (5.14).

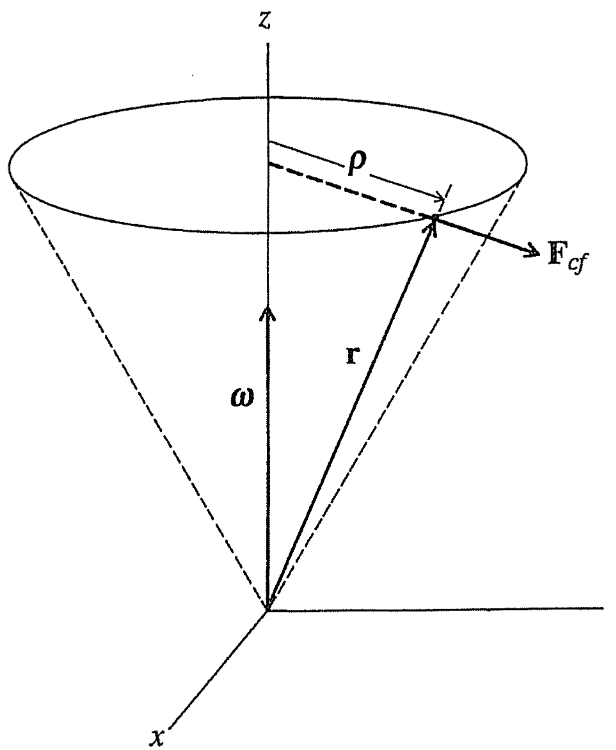


FIGURE 7-2. Centrifugal force $\mathbf{F}_{cf}$ due to rotation with angular velocity $\boldsymbol{\omega}$.

The centrifugal force, along with gravity, accounts for the parabolic shape of the surface of a spinning pail of water. Because of viscous forces the water in a uniformly spinning pail will reach an equilibrium condition where it rotates as if it were a rigid body (*i.e.*, each element has the same angular velocity). Using (7.17), the equation of motion of a small mass of water m on the surface in a frame rotating with the pail is

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F}' + m\mathbf{g} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7.23}$$

where the force $\mathbf{F}'$, due to the pressure gradient, is normal to the surface. If the pressure gradient had a component tangent to the surface the pressure would vary along the surface, but this cannot happen since the surface is at atmospheric pressure. Since in equilibrium $\delta^2\mathbf{r}/\delta t^2 = 0$, the *effective-gravity* term

$$\mathbf{g}_{\text{eff}} \equiv \mathbf{g} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7.24}$$

must also be normal to the surface, by (7.23). In cylindrical coordinates, $\mathbf{g}_{\text{eff}}$ is given by

$$\mathbf{g}_{\text{eff}} = -g\hat{\mathbf{z}} + \omega^2\rho\hat{\boldsymbol{\rho}} \tag{7.25}$$

From the geometry of Fig. 7-3, the normal requirement on $\mathbf{g}_{\text{eff}}$ can be written

$$\tan\theta = \frac{dz}{d\rho} = \frac{\omega^2\rho}{g} \tag{7.26}$$

Integration gives

$$z = \frac{\omega^2}{2g}\rho^2 + \text{constant} \tag{7.27}$$

which is a parabolic shape. The solution in (7.27) can alternatively be found from (7.25) by potential-energy methods. The potential energy due to the force $m\mathbf{g}_{\text{eff}}$ is

$$V(z,\rho) = m(gz - \tfrac{1}{2}\omega^2\rho^2) \tag{7.28}$$

as verified by computing $\mathbf{F} = mg_{\text{eff}} = -\boldsymbol{\nabla}V$. Since there can be no component of force tangential to the surface in equilibrium, the potential energy in (7.28) must be constant on the surface, and the result in (7.27) is thus obtained.

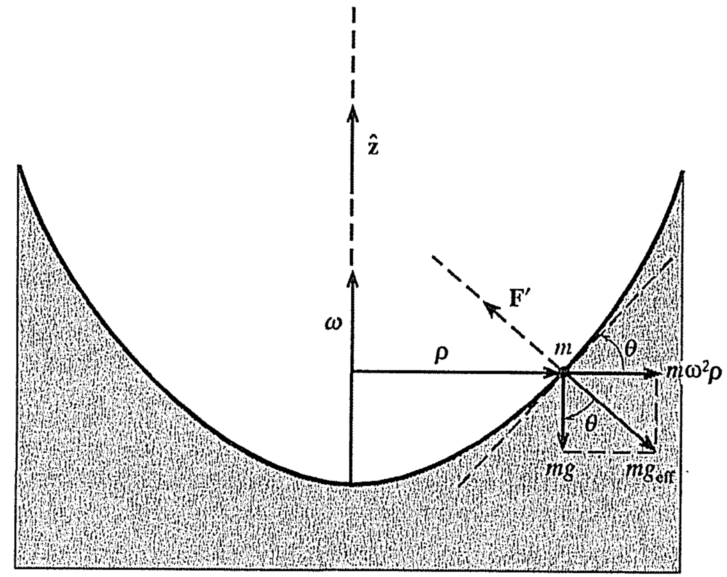


FIGURE 7-3. Parabolic surface of a spinning pail of water.

There are two important aspects of this spinning pail phenomenon which might be mentioned, one of historical and philosophical importance and the other of current practical importance in the construction of telescope mirrors. The *water pail experiment* was of great significance in Isaac Newton's formulation of mechanics. He noted that a spinning water pail achieves a curved surface while one that is not rotating has a flat surface; Newton deduced that a reference frame not rotating with respect to the stars was fundamental.

In the construction of the post-Palomar generation of large land-based optical telescopes short focal lengths must be utilized to minimize the weight and consequent expense. To construct such a mirror by the old technology would require starting with a thick blank disk and then laboriously grinding out the concave shape. An important innovation is to spin a molten pyrex or quartz blank inside a furnace. It obtains precisely the parabolic shape that is needed to focus a distant star into a point image and can be cast with uniform thickness helping to reduce the weight. The surface is then coated with a thin layer of aluminum to make it reflecting. Photos of such a telescope mirror being made by this process are shown in Fig. 7-4.

(a) (b)

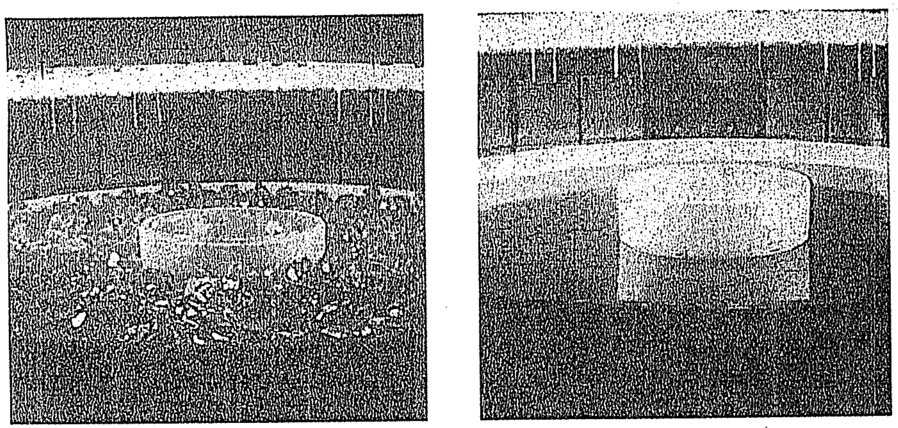


FIGURE 7-4. A parabolic mirror for the WIYN (Wisconsin-Indiana-Yale-NOAO) telescope being made by the spin-casting process. (a) The initial stage of casting and (b) after spin casting. The central plug leaves a hole for light transmission after reflection by a secondary mirror.

The Coriolis force in (7.19) is present when the particle is in motion relative to the rotating coordinate system. Since $\boldsymbol{\omega} \cdot \mathbf{F}_{\text{Cor}} = 0$ and $\mathbf{v} \cdot \mathbf{F}_{\text{Cor}} = 0$, this force is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{v}$. The effects of the Coriolis force are important in such problems as calculations of long-range artillery and ballistic missile trajectories and in the description of large-scale atmospheric weather phenomena.

The azimuthal force in (7.20) occurs only when the angular-velocity vector changes with time. Inasmuch as $\mathbf{r} \cdot \mathbf{F}_{\text{az}} = 0$, this force always points in a direction perpendicular to $\mathbf{r}$. If $\boldsymbol{\omega}$ is changing in magnitude but constant in direction, the azimuthal force acts toward maintaining the rotational velocity of the particle.

The translation force in (7.21) is due to the acceleration of the origin of the moving frame relative to an inertial frame. In the special case when the motion of the accelerated coordinate system is purely translational (that is, $\boldsymbol{\omega} = 0$), the equation of motion in (7.16) and (7.17) reduces to

$$m\frac{\delta^2 \mathbf{r}}{\delta t^2} = \mathbf{F} - m\frac{d^2\mathbf{R}}{dt^2} \tag{7.29}$$

The problem of a pendulum with a moving support provides an interesting application of (7.29). We choose the origin of the moving coordinate system to coincide with the instantaneous location of the support, as illustrated in Fig. 7-5. We shall restrict our discussion to angular motion in the x, y plane defined by $\mathbf{F}$ and $\mathbf{R}$. In terms of the tension $\mathbf{T}$ and the gravitational force $mg\hat{\mathbf{x}}$ acting on the pendulum bob, the equation of motion in the moving frame is

$$m\frac{\delta^2 \mathbf{r}}{\delta t^2} = \mathbf{T} + m(g - A_x)\hat{\mathbf{x}} - mA_y\hat{\mathbf{y}} \tag{7.30}$$

where $\mathbf{A} = d^2\mathbf{R}/dt^2$ is the translational acceleration. Since physical motion occurs along the θ direction, it is advantageous to write (7.30) in polar coordinates using (2.124),

$$\frac{\delta^2 \mathbf{r}}{\delta t^2} = \hat{\mathbf{r}}(\ddot{r} - r\dot{\theta}^2) + \hat{\boldsymbol{\theta}}(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{7.31}$$

and the relations

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \\ \hat{\mathbf{y}} &= \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta \end{aligned} \tag{7.32}$$

obtained from (2.125). Since $T_\theta = 0$, we find

$$\begin{aligned} ml\ddot{\theta} &= -m(g - A_x)\sin\theta - mA_y\cos\theta \\ -ml\dot{\theta}^2 &= T_r + m(g - A_x)\cos\theta - mA_y\sin\theta \end{aligned} \tag{7.33}$$

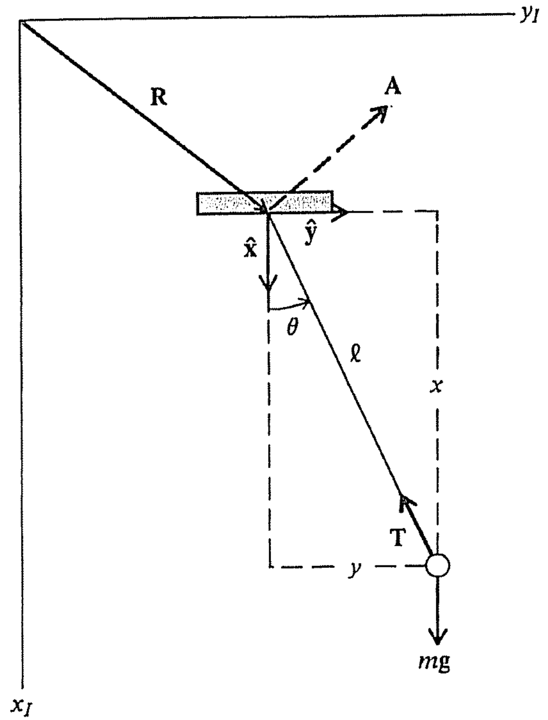


FIGURE 7-5. Pendulum with a support which moves with acceleration $\mathbf{A} \equiv \ddot{\mathbf{R}}$.

The angular motion of the pendulum bob is therefore determined by

$$\ddot{\theta} + \left[\frac{g}{l} - \frac{A_x(t)}{l}\right] \sin\theta = -\frac{A_y(t)}{l} \cos\theta \tag{7.34}$$

This is the same as obtained more laboriously by the Lagrangian method in Chapter 3 (except with the x and y axes interchanged). For uniform vertical acceleration of the support ($A_x =$ constant and $A_y = 0$), the natural frequency of small oscillations for $A_x < g$ is

$$\omega_0 = \sqrt{\frac{g - A_x}{l}} \tag{7.35}$$

When $A_x = g$, the pendulum undergoes free-fall motion and it behaves as if the gravity field has vanished. The fact that gravity can be made to disappear (or appear) locally by a coordinate transformation led Einstein to a theory of gravity, the general theory of relativity, in which gravity is linked closely to geometry.

7.3 Motion on the Earth

For the motion of a particle moving near the surface of the earth, it is convenient to choose a coordinate system that is fixed on the earth's surface. We consider in Fig. 7-6 first a reference frame S' rotating with a constant angular velocity ω relative to S_I (the inertial frame). The frame S' is fixed in the rotating earth and its origin is at the CM of the earth, coinciding with the origin of S_I. The translational vector $\mathbf{R}$ in (7.15) is thus zero. As shown in Fig. 7-6 the location of mass m is given by $\mathbf{r}'$ and its equation of motion from (7.15) is

$$m\frac{\delta^2 \mathbf{r}'}{\delta t^2} = \mathbf{F}' + m\mathbf{g} - m\left[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') - 2\boldsymbol{\omega}\times\mathbf{v}'\right] \tag{7.36}$$

The net force acting on m in the inertial system has been separated into the gravitational force mg and any other forces $\mathbf{F}'$.

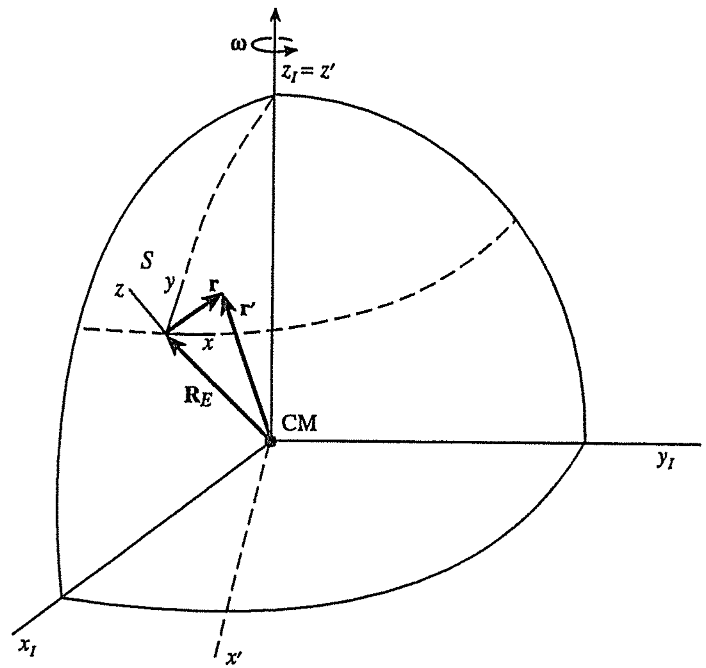


FIGURE 7-6. Inertial reference frame S_I, earth-fixed frame S' whose origin is at the earth CM, and earth-fixed local frame S whose origin is on the earth's surface.

The origin of a local reference frame S fixed on the earth's surface is given by $\mathbf{R}_E$. The location of m in S is specified by the vector $\mathbf{r}$ and hence

$$\mathbf{r}' = \mathbf{R}_E + \mathbf{r} \tag{7.37}$$

Since $\mathbf{R}_E$ is a fixed vector in S', $\delta\mathbf{r}'/\delta t = \delta\mathbf{r}/\delta t$ from (7.37); (7.36) then becomes

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F}' + m\mathbf{g} - m[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{R}_E+\mathbf{r})) - 2\boldsymbol{\omega}\times\mathbf{v}] \tag{7.38}$$

Due to the earth's large size and its relatively slow rotation the fictitious force corrections are small compared to gravity. The centrifugal force is proportional to the square of the small quantity ω and therefore we can neglect $\mathbf{r}$ compared to $\mathbf{R}_E$ in this term. Thus to a very good approximation the centrifugal force is constant. The Coriolis force is linear in ω and depends on the state of motion relative to S. The equation of motion relative to S is then well approximated by

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F}' + m\mathbf{g} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}_E) - 2m\boldsymbol{\omega}\times\mathbf{v} \tag{7.39}$$

If the earth were perfectly spherical and isotropic, $\mathbf{g}$ would be constant in magnitude and directed toward the center of the earth. In fact, local irregularities, distortions from sphericity, and deviations from uniform density cause slight variations in $\mathbf{g}$ at different points on the earth.

The condition for a particle at rest on the earth ($\mathbf{v} = 0$) to be in equilibrium ($\delta^2\mathbf{r}/\delta t^2 = 0$) from (7.39) is

$$\mathbf{F}' = -m[\mathbf{g} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}_E)] \tag{7.40}$$

For example, if m is the bob on a plumb line, the tension $\mathbf{F}'$ in the string is opposite to the direction determined by

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}_E) \tag{7.41}$$

The plumb bob thus points in the direction of $\mathbf{g}_{\text{eff}}$. We conclude that $\mathbf{g}_{\text{eff}}$ is the effective gravitational acceleration on the earth. The magnitude of the correction term to $\mathbf{g}$ is

$$|\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}_E)| = \omega^2 R_E \sin\theta \tag{7.42}$$

where θ is the colatitude angle between $\boldsymbol{\omega}$ and $\mathbf{R}_E$. For the earth's angular velocity of rotation,

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{24\times 3{,}600} = 0.727\times 10^{-4}\,\text{rad/s} \tag{7.43}$$

we find that the correction term is small,

$$\omega^2 R_E\sin\theta = (0.727\times10^{-4})^2(6{,}371\times10^3)\sin\theta \simeq 0.03\sin\theta\ \text{m/s}^2 \tag{7.44}$$

This correction is less than 0.3% of g, but nonetheless measurable. The direction of the correction is radially outward from the rotation axis, as illustrated in Fig. 7-7.

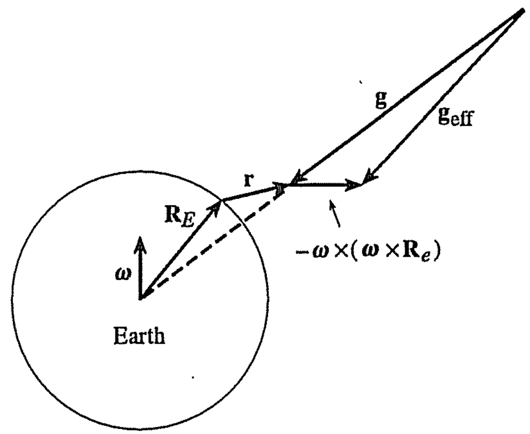


FIGURE 7-7. Effective gravitational acceleration $\mathbf{g}_{\text{eff}}$ (with the relative magnitude of the centrifugal acceleration exaggerated for clarity).

The differential equation (7.39) which describes the motion of a particle on the earth can be expressed in terms of $\mathbf{g}_{\text{eff}}$ in (7.41) as

$$m\frac{\delta^2\mathbf{r}}{\delta t^2} = \mathbf{F}' + m\mathbf{g}_{\text{eff}} - 2m\boldsymbol{\omega}\times\mathbf{v} \tag{7.45}$$

For convenience we choose the z axis of the coordinate system so that

$$\mathbf{g}_{\text{eff}} = -g_{\text{eff}}\hat{\mathbf{z}} \tag{7.46}$$

The y axis is taken to point north and the x axis east, as pictured in Fig. 7-8. The components of $\boldsymbol{\omega}$ along these axes are

$$\boldsymbol{\omega} = 0\hat{\mathbf{x}} + \omega\sin\theta\hat{\mathbf{y}} + \omega\cos\theta\hat{\mathbf{z}} \tag{7.47}$$

where θ is the colatitude angle as measured from the north polar axis.

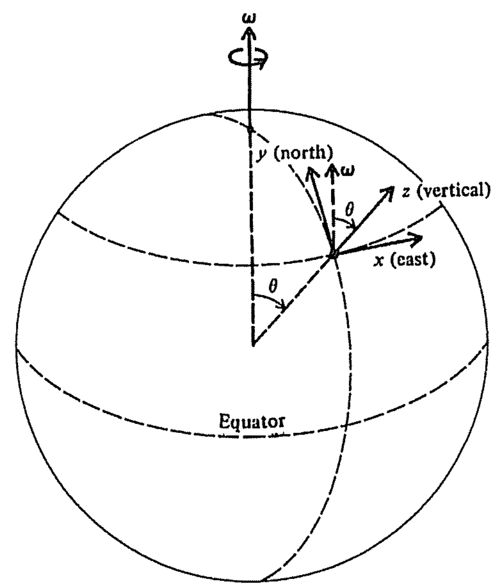


FIGURE 7-8. Coordinate frame fixed on the surface of the earth at colatitude angle θ.

Substituting (7.47) into the Coriolis force term,

$$\mathbf{F}_{\text{Cor}} = -2m\boldsymbol{\omega}\times\mathbf{v} \tag{7.48}$$

of (7.45), we have

$$\mathbf{F}_{\text{Cor}} = 2m\omega[(v_y\cos\theta - v_z\sin\theta)\hat{\mathbf{x}} - v_x\cos\theta\hat{\mathbf{y}} + v_x\sin\theta\hat{\mathbf{z}}] \tag{7.49}$$

The direction of deflection of the particle from its direction of motion due to the Coriolis force follows directly from (7.49). In the Northern Hemisphere, $0 \le \theta \le \pi/2$, we find

Velocity direction	Deflection direction
North ($v_y > 0$)	East
East ($v_x > 0$)	South and up
South ($v_y < 0$)	West
West ($v_x < 0$)	North and down
Up ($v_z > 0$)	West
Down ($v_z < 0$)	East

For motion parallel to the earth's surface ($v_z = 0$), the particle is always deflected to the right in the Northern Hemisphere and to the left in the Southern Hemisphere.

The trade winds and weather circulations of high- and low-pressure areas are striking examples of Coriolis force effects. The equatorial region of the earth generally receives more heat from the sun. The warm air rises and is replaced by a flow of air from the temperate regions. The air moving south from the Northern Hemisphere is deflected westward by the Coriolis effect. This accounts for the steady prevailing winds to the west and south, known as the *trade winds.*

On a smaller scale a low-pressure region in the Northern Hemisphere on the order of 200 km across is associated with a counterclockwise circulation of the air because of the Coriolis force effect on the air flowing in. The pressure gradient is largely balanced by the Coriolis force. Under certain circumstances this cyclonic motion builds up to great intensity and destructive power in the form of a hurricane, cyclone, or typhoon. High-pressure areas force air outward. This airflow deflects to the right and produces clockwise circulation in the Northern Hemisphere. Vortices on a still smaller scale such as tornados, dust devils, water spouts, and the bathtub vortex are not directly influenced by Coriolis effects to any great extent. Nevertheless, some of these vortices often have a counterclockwise motion because of general counterclockwise movements which spawn them.

7.4 Foucault's Pendulum

In 1851 Jean Foucault exhibited a pendulum at the Pantheon in Paris which through Coriolis force dramatically illustrated the rotation of the earth. Today Foucault pendulums are on exhibit in many public buildings and planetariums. One of the most famous hangs in the United Nations Building in New York, as illustrated in Fig. 7-9. The Foucault pendulum is a simple plane pendulum which can oscillate a long time without being appreciably damped by friction. Its oscillation plane is observed to rotate slowly with time, confirming in a dramatic way that a reference frame in which the distant stars appear fixed is more fundamental than one in which the earth is fixed and the stars rotate about the earth.

The motion of the Foucault pendulum can be determined from (7.45). We take $\mathbf{r}$ to represent the distance of the bob of mass m from its equilibrium position. At rest the pendulum hangs along the direction $\mathbf{g}_{\text{eff}}$, and

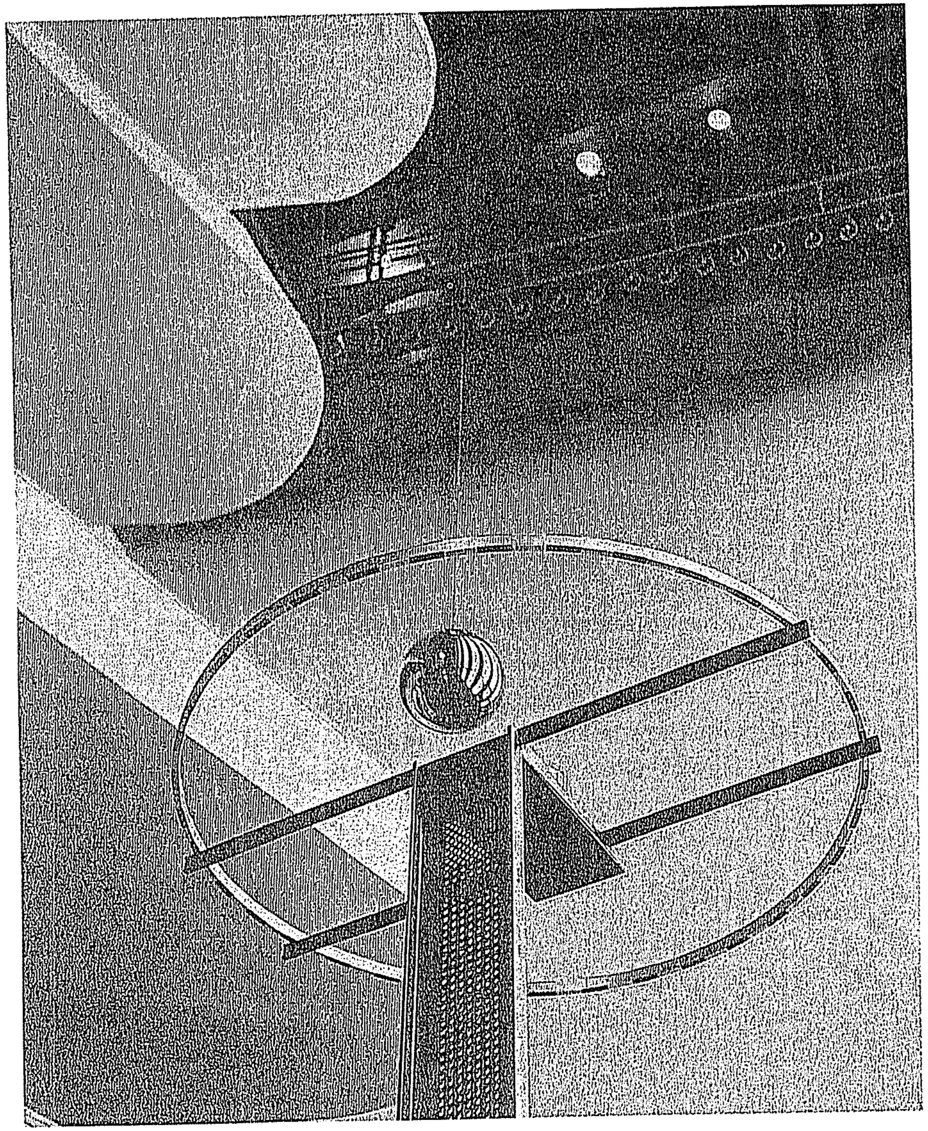


FIGURE 7-9. Foucault pendulum which hangs in the United Nations Building in New York City. (*Photo courtesy of United Nations.*)

the tension in the string is $\mathbf{F}' = -m\mathbf{g}_{\text{eff}}$. If the earth did not rotate, the Coriolis force term in (7.45) would not be present and the motion would occur in a fixed plane. With Coriolis force present the bob will deflect to the right, out of its plane as shown in Fig. 7-10. On the return swing the pendulum bob again deflects to the right and after one period the pendulum plane has rotated clockwise as viewed from above.

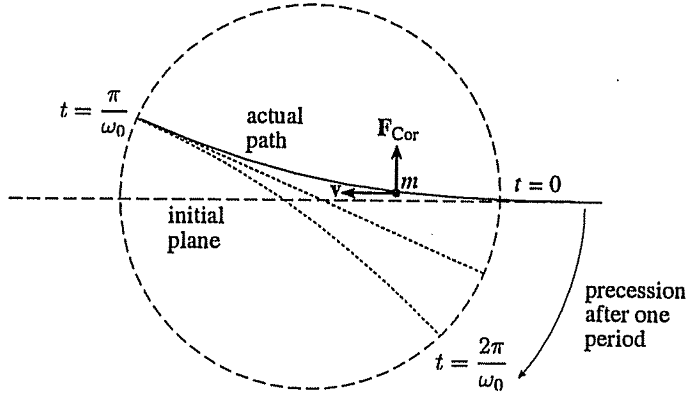


FIGURE 7-10. Deflection of Foucault pendulum bob by Coriolis force as viewed from above. The precession angle is greatly exaggerated in the figure.

A way to obtain the Foucault precession frequency is to view the system from a new frame S_F which rotates with the angular velocity $\boldsymbol{\omega}_F$ of the pendulum relative to our local earth fixed frame S. Starting from the equation of motion (7.45) in S, we transform it to frame S_F using (7.11)

$$m\frac{\bar{\delta}^2\mathbf{r}}{\bar{\delta}t^2} = (\mathbf{F}' + m\mathbf{g}_{\text{eff}} - 2m\boldsymbol{\omega}\times\mathbf{v}) - m\boldsymbol{\omega}_F\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}_F\times\mathbf{v}_F \quad (7.50)$$

Here $\bar{\delta}^2\mathbf{r}/\bar{\delta}t^2$ is the second time derivative of $\mathbf{r}$ relative to the axes of the S_F frame. The particle velocity in the S_F frame is

$$\mathbf{v}_F = \mathbf{v} + \boldsymbol{\omega}_F\times\mathbf{r} \quad (7.51)$$

Substituting (7.51) into (7.50) gives

$$m\frac{\bar{\delta}^2\mathbf{r}}{\bar{\delta}t^2} = \mathbf{F}' + m\mathbf{g}_{\text{eff}} + m\boldsymbol{\omega}_F\times(\boldsymbol{\omega}_F\times\mathbf{r}) - 2m(\boldsymbol{\omega}+\boldsymbol{\omega}_F)\times\mathbf{v} \quad (7.52)$$

To see the advantage of viewing the Foucault pendulum from a frame rotating with angular velocity $\boldsymbol{\omega}_F$ we observe

1. The Foucault pendulum precesses about the vertical axis. Thus we take

$$\boldsymbol{\omega}_F = \omega_F\hat{\mathbf{z}} \quad (7.53)$$

where $\hat{\mathbf{z}}$ is the vertical direction at the earth's surface.

2. For small displacements the pendulum motion is nearly perpendicular to $\hat{\mathbf{z}}$.
3. The $\boldsymbol{\omega}_F \times (\boldsymbol{\omega}_F \times \mathbf{r})$ term is small compared to the already small centrifugal term in $\mathbf{g}_{\text{eff}}$. It can be neglected.
4. For small pendulum displacements v_z is negligible; using (7.47)

$$(\boldsymbol{\omega}+\boldsymbol{\omega}_F)\times\mathbf{v} = -\hat{\mathbf{x}}v_y(\omega\cos\theta+\omega_F)+\hat{\mathbf{y}}v_x(\omega\cos\theta+\omega_F)-\hat{\mathbf{z}}v_x(\omega\sin\theta) \tag{7.54}$$

Thus the pendulum motion will remain in its initial plane in this frame if

$$\omega_F = -\omega\cos\theta \tag{7.55}$$

An earth-fixed observer in S thus sees the pendulum plane precessing slowly clockwise (in the northern hemisphere) with angular frequency $\omega_F = -\omega\cos\theta$. We note that in the southern hemisphere $\cos\theta$ is negative and ω_F automatically adjusts in sign. The time required for the pendulum plane to precess by 2π is

$$\tau_P = \frac{2\pi}{\omega_F} = \frac{(1\text{ day})}{\cos\theta} \tag{7.56}$$

The precession vanishes at the equator and is a maximum at the north pole, where the pendulum precesses clockwise through a complete revolution every 24 h. From the viewpoint of an observer in space, the oscillation plane at the north pole remains fixed, while the earth turns counterclockwise beneath it.

李怡嚴

又因 $\frac{\Delta r}{\Delta t}=v'_r$　　$\frac{\Delta\theta}{\Delta t}=\omega$

故 $$a_\theta=2\omega v'_r,$$

因此, $$F_\theta=2m\omega v'_r,$$

F_θ 爲彈簧施於質點的沿 $\hat{\theta}$ 方向的外力，其方向與 $\hat{\theta}$ 方向相同。對 S 坐標上的觀察者而言，這是質點所受的唯一與半徑方向垂直的外力，這是使質點有加速度分量 a_θ 所需的眞實的外力。但是，對 S' 坐標上的觀察者而言，質點是以等速度沿半徑方向運動，因此，與半徑垂直的方向沒有已知的力，所以這一觀察者將斷定有一等於 "$-2m\omega v_r'$" 的力量施於質點上，這便在 S' 坐標上所看的柯里歐利力。當然，對 S' 而言，在半徑方向仍有離心力 "$m\omega^2\mathbf{r}'$" 存在；爲着使質點對 S' 作等速度運動，必另外有一眞正的外力 "$-m\omega^2\mathbf{r}'$" 施於質點上。對 S 坐標上的觀察者而言，沿半徑方向只有 "$-m\omega^2\mathbf{r}'$" 的外力存在，而沒有"離心力"這項力量存在。

如果我們不限制質點沿徑向等速度運動，而是先將它用一根繩子拉住並使其隨 S' 坐標轉動，則質點對 S' 坐標的速度爲零，而對 S 坐標質點則作等速圓周運動。然後我們將質點放開使其在桌面上運動，若桌面爲完全光滑的，則在 S' 坐標上的觀察者將看到質點的路徑爲如圖(2–13)之形狀。這位觀察者說質點因受到離心力及柯里歐利力的作用而呈現這種運動。當質點被放開時，離心力使其向外加速。但是，一當質點有一速度時，質點即受到 "$-2m\boldsymbol{\omega}\times\mathbf{v}'$" 的柯里歐利力的作用。因爲柯里歐利力與 $\mathbf{v}'$ 垂直（由 $\boldsymbol{\omega}\times\mathbf{v}'$ 之向

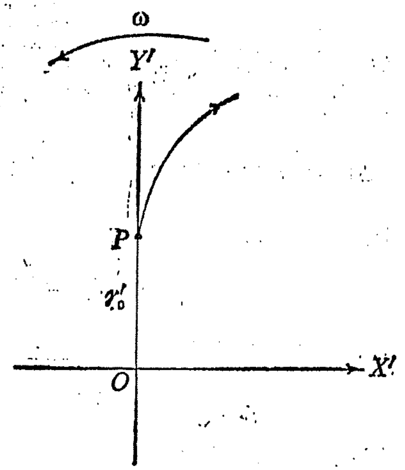


圖 2–13　質點在光滑桌面上運動的路徑

量積可知)，故它只使質點的運動方向改變，而致有如圖(2–13)的路徑。大家在高中時學過，作用於一運動中的電荷的磁力是與其速度垂直的；這一磁力只能改變帶電粒子的運動方向而不能改變其速率。現在，柯里歐利力也與 $\mathbf{v}'$ 垂直，故也只能改變質點的運動方向（對 S' 坐標而言），而不能改變其速率。因爲桌面爲完全光滑，故站在 S 坐標上的觀察者將看到質點沿一直線等速運動，其路徑並不像圖(2–13)那樣複雜。

我們知道地球對固定的恒星 (fixed stars) 以等角速度在轉動，因此，固定在地面上的參考系爲一轉動坐標；所以在地面上觀測物體的運動時，必定有一些現象是由離心力及柯里歐利力所引起的。當然，這些假力所引起的效應不應很大，否則我們便無法由靠近地面所作的實驗或觀測歸納出牛頓運動定律。我們現在先討論一些與離心力有關的現象。我們曉得地球在赤道的部分稍微凸出，而像一個橢圓體；這一現象便是由於地球沿南北軸自轉時，在赤道的離心力比其他部分大而引起的。另一現象是在地面上測得的有效重力加速度(可用單擺測量)隨緯度之增加而稍微變大。這一現象不能單獨用地球在赤道稍微凸出的事實來解釋，它大部分是由離心力引起的。如果地球是球形，其半徑爲 R，則在赤道上所量到的重力加速度(包含離心力)最少應比在北極所量得的小 $\omega^2R\simeq3.4$ 厘米/秒²。實際量度的差別約爲 5.2 厘米/秒²。這一差別可由地球像一橢圓體的事實解釋之。

柯里歐利力所引起的效應與物體對地面之速度有關。現在我們先估計一物體靠近地面垂直自由降落時柯里歐利力所引起的效應。如果地球不動，則物體必沿地球之半徑方向降落。爲簡單起見，我們只討論在赤道自由降落之情形。設一物體自 h 米高處落下，初速爲零。令 S' 坐標固定在地面上，x' 軸向東方，y' 軸向北方，而 z' 軸垂直向上（如圖 2–14）。則由(2·29)式可知質點對 S' 坐標的運動方程式爲

$$m\mathbf{a}'=\mathbf{F}-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})-2m\boldsymbol{\omega}\times\mathbf{v}', \quad (2\cdot30)$$

此處 $\mathbf{F}$ 爲地球予物體的重力，$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$ 爲 S' 坐標之原點 O' 對地心的向心加速度，$\mathbf{R}$ 爲地心至 O' 點的位置向量，$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')$ 爲質點對

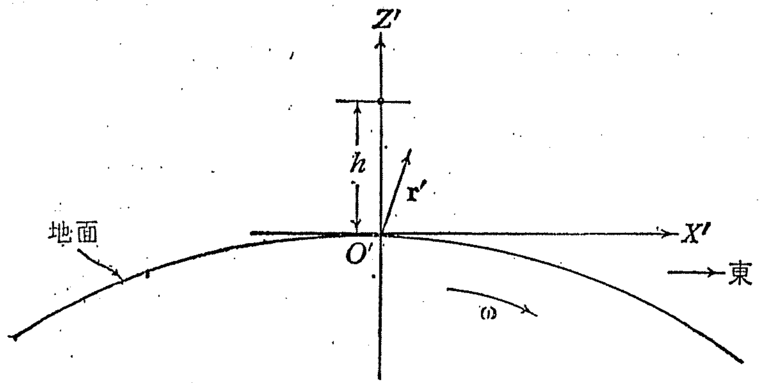


圖 2-14 柯里歐利力對自由落體的影響

O' 點的向心加速度，因此質點對地心的向心加速度爲 $\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r}')]$，而"$-m\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r}')]$"則爲物體所受的離心力。因爲實際測量重力加速度時已包含這一項離心力，故"有效"重力加速度（即實際測得的）與萬有引力 $\mathbf{F}$ 及離心力之關係爲

$$\begin{aligned} m\mathbf{g} &= \mathbf{F}-m\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r}')] \\ &= \mathbf{F}-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')。 \end{aligned} \qquad (2\cdot31)$$

若質點離地面不很高，$\mathbf{g}$ 大約爲常數，其方向垂直向下（即 $-\mathbf{k}'$ 方向），因此，質點 m 對 S' 坐標的運動方程式爲

$$m\mathbf{a}'=-mg\mathbf{k}'-2m\boldsymbol{\omega}\times\mathbf{v}' \qquad (2\cdot32)$$

現在，$\boldsymbol{\omega}=\omega\mathbf{j}'$，$\mathbf{v}'=\dot{x}'\mathbf{i}'+\dot{y}'\mathbf{j}'+\dot{z}'\mathbf{k}'$，$\mathbf{a}'=\ddot{x}'\mathbf{i}'+\ddot{y}'\mathbf{j}'+\ddot{z}'\mathbf{k}'$，

故上式可被寫成

$$\begin{aligned} m(\ddot{x}'\mathbf{i}'+\ddot{y}'\mathbf{j}'+\ddot{z}'\mathbf{k}') &= -mg\mathbf{k}'-2m\omega\mathbf{j}'\times(\dot{x}'\mathbf{i}'+\dot{y}'\mathbf{j}'+\dot{z}'\mathbf{k}') \\ &= -mg\mathbf{k}'+2m\omega\dot{x}'\mathbf{k}'-2m\omega\dot{z}'\mathbf{i}' \end{aligned}$$

或者，

$$\ddot{x}'=-2\omega\dot{z}' \qquad (2\cdot33)$$

$$\ddot{y}'=0$$

$$\ddot{z}'=-g+2\omega\dot{x}'\simeq-g \qquad (2\cdot34)$$

上式中 ω 及 $\dot{x}'$ 皆很小，故與 g 比較時可被忽略不計。因 $t=0$ 時 $z'=h$，$\dot{z}'=0$，故積分(2·34)式可得

$$\dot{z}'=-gt, \qquad z'=h-\frac{g}{2}t^2.$$

因此(2·33)式可被寫成 $\ddot{x}'=-2\omega\dot{z}'=2\omega gt$，

因 $t=0$ 時，$\dot{x}'=0$，$x'=0$，故將上式積分之可得

$$x'=\frac{1}{3}\omega gt^3.$$

當質點落到地面時 $z'=0$，所需之時間爲 $t=\sqrt{2h/g}$。將此值代入上式得，$x'=\frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{\frac{3}{2}}$。從這一結果我們曉得由於柯里歐利力的影響質點自由降落時並不鉛垂落下，而是稍微偏向東方落下。若質點自 100 米高處釋放，則將地球的角速度及重力加速度代入上面之結果，可求出質點落到地面後向東偏差 2.2 厘米。這種微小的偏差在日常的經驗裏當然看不出來。但是，有些場合柯里歐利力是不能忽略的，並且是產生某些現象的主要原因。例如，旋風 (cyclone) 及反旋風 (anticyclone) 的形成，甚至於整個氣象，都是受到柯里歐利力的影響。我們曉得風是空氣從高氣壓處流到低氣壓處所產生的氣流。如果地面有一低氣壓區，則空氣由四周向此低壓區流動時即發生旋風；龍卷風及颱風便是屬於這類風。若在北半球，氣體由西向東流時，氣體所受的柯里歐利力的方向係向南，故氣流向南偏，即向前進方向的右邊偏。同理空氣由南方，東方，或北方流向低壓區時的路徑大致如圖 2-15 所示之情形。因此，空氣便在低壓區周圍依逆時鐘的方向旋轉，即在北半球的旋風的旋轉方向爲逆時鐘的。反之，在南半球產生的旋風係依順時鐘的方向旋轉。反旋風係由一高氣壓區域產生的。當空氣由高壓區向四面八方流開時，由於柯里歐利力的影響，在北半球的流動路徑大約如圖(2-16)的情形，故在北半球所產生的反旋風依順時針的方向旋轉。在南半球反旋風則依反時鐘的方向旋轉。季候風的方向也是由於柯里歐利力的影響而產生