Chapter 6 and Chapter 7

Chapter 6

Work and Kinetic Energy

With initial conditions given, Newton's laws of motion relating acceleration to force enable us to predict future of the position and velocity of a particle

With the introduction of work and kinetic energy

$$dW = \vec{F} \cdot d\vec{r}$$

$$K.E. = \frac{1}{2} m\vec{V}^2$$

Work - Kinetic Energy Principle

$$(K.E)_f - (K.E)_i = \int_i^f \vec{F} \cdot d\vec{r}$$

$\uparrow$ $\uparrow$

input

$\vec{F} = m\frac{d\vec{V}}{dt}$ vector equation

involve scalar equation

Work

1 dimension

F_x = constant

Force acting on a particle moving along x.

Under F_x, the particle makes a displacement Δx

$\Delta W = F_x \Delta x$ ↳ displacement of particle

↑ work done by the force F_x, work is a scalar quantity

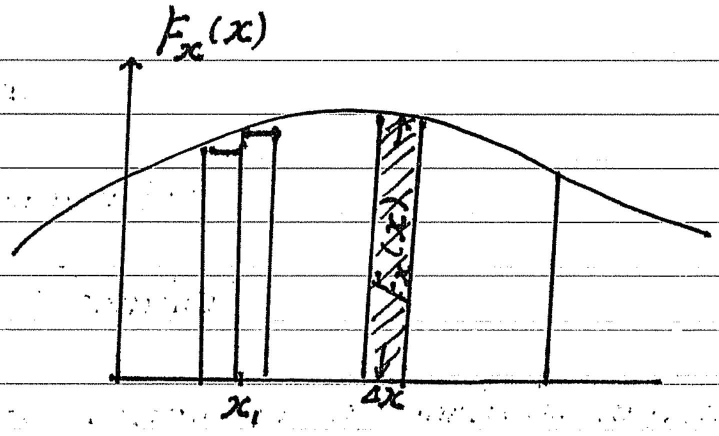
$\Delta W > 0$: force and displacement are in the same direction

$\Delta W < 0$: force and displacement are opposite.

$F_x(x)$ force is a function of position, e.g., spring.

What is $W(a \to b)$ in moving the particle from $x = a$ to $x = b$?

Divide the total displacement into a large number of small intervals Δx



For each interval

$$\Delta W_i = F_x(x_i)\,\Delta x$$

= area of a rectange of height $F_x(x_i)$ and width Δx

Total work from $x = a$ to $x = b$ is the sum of all such intervals

$$W_{a\to b} = \sum_{i=0}^{i=n-1} \Delta W_i = \sum_{i=0}^{i=n-1} F_x(x_i)\,\Delta x$$

In the limiting case

$$\Delta x \to 0,\quad n \to \infty$$

$$W_{a\to b} = \lim_{\Delta x\to 0} \sum_i \underbrace{F_x(x_i)}_{\rightarrow\ \text{integrand}}\,\Delta x$$

$$= \underbrace{\int_a^b}_{\text{definite integral}} F_x(x)\,dx$$

One dimensional
Case

$\underset{a\to b}{\text{Work}}$ ≡ area bounded by the curve $F_x(x)$

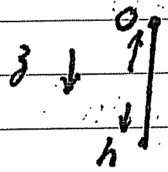
and the lines $x \doteq a$ and $x = b$

and the x-axis

Example $\underset{\substack{\downarrow\\ \text{constant}}}{F} = ma$

a is constant

$$v_f^2 - v_i^2 = 2a(x_f - x_s)$$



free fall

$$v_f^2 - v_i^2 \underset{\substack{\downarrow\\ \text{well known result}}}{=} 2gs$$

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \frac{1}{2} 2 \underbrace{mg}_{F} s$$

$$= F \int_i^f dz$$

$$= mg(z_f - z_i)$$

work - energy principle

Example

$F = -kx$

x is distance from equilibrium

↓

choose $x=0$ at equilibrium.

One dimensional case

(1) Free fall

$$F_x = \text{constant} = -mg = ma$$

$$dW = F\,dx = -mg\,dx$$

$$\frac{1}{2} m v_h^2 - \frac{1}{2} m v_0^2 = \int_0^h F\,dx = -mgh$$

$$\Rightarrow \frac{1}{2} m v_h^2 - \frac{1}{2} m v_0^2 = -mgh$$

$$\Rightarrow \frac{1}{2} m v_h^2 + mgh = \frac{1}{2} m v_0^2$$

$\downarrow$

conservation of mechanical energy

$$E_{p,x} = mgh\Big|_x = mgx$$

$$F_x = -\frac{\partial E_p}{\partial x}$$

$F(x)$ is a function of x only

$$W_{a\to b} = \int_a^b F\,dx$$
$$= \int_a^b m\frac{d^2x}{dt^2}\,dx$$
$$= \int_a^b m\frac{dV}{dt}\,V\,dt$$
$$= \int_a^b m\frac{1}{2}\frac{d(V^2)}{dt}\,dt$$
$$= \underbrace{\tfrac{1}{2}mV_b^2 - \tfrac{1}{2}mV_a^2}_{\text{Work energy relation}}$$

$$\int_a^b \underbrace{F(x)\,dx}_{\text{independent of path in one dimension}}$$

$$\equiv \underbrace{E_p(a) - E_p(b)}_{\text{introduce a minus sign}}$$

$$F = -\frac{\partial E_p}{\partial x}$$

$$\Rightarrow \quad \tfrac{1}{2}mV_b^2 + E_{p,b} = \tfrac{1}{2}mV_a^2 + E_{p,a}$$

mechanical energy conservation.

(2) Simple Harmonic Oscillator

$$F = -kx$$

$$\Rightarrow \quad E_p = \frac{1}{2}kx^2$$

Energy (Mechanical) Conservation

$$\frac{1}{2}mv^2 + \underset{\substack{\downarrow \\ \text{mechanical energy conservation}}}{\frac{1}{2}kx^2} = \underset{\hookrightarrow \text{constant}}{E}$$

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E$$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)}$$

$$\Rightarrow \quad \frac{dx}{\sqrt{\frac{2}{m}E - \frac{k}{m}x^2}} = dt$$

$$\frac{dx}{\underset{\substack{\shortparallel \\ \omega}}{\sqrt{\frac{k}{m}}}\sqrt{\underset{\substack{\shortparallel \\ \alpha^2}}{\frac{2E}{k}} - x^2}}$$

$$\Rightarrow \quad \frac{dx}{\underset{\substack{\downarrow \\ \text{the problem can be solved by integration.}}}{\sqrt{\alpha^2 - x^2}}} = \omega\, dt$$

$$\int \frac{dx}{\sqrt{\alpha^2 - x^2}}$$

$$x = \alpha \sin\theta$$

$$dx = \alpha \cos\theta\, d\theta \qquad\qquad \theta = \sin^{-1}\frac{x}{\alpha}$$

$$\alpha^2 - x^2 = \alpha^2(1 - \sin^2\theta)$$

$$\Rightarrow \quad \int \frac{\alpha\cos\theta\, d\theta}{\alpha\cos\theta} = \theta + c = \omega t$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{\alpha} + c = \omega t$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{\alpha} = \omega t + c'$$

$$\Rightarrow \quad \frac{x}{\alpha} = \sin(\omega t + c')$$

Near $x = x_0$

$$f(x) = f(x_0) + \underset{\substack{\downarrow \\ 0}}{\frac{df}{dx}\Big|_{x=x_0}}(x-x_0) + \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_{x=0}(x-x_0)^2 \underset{\downarrow}{+} \cdots$$

at equilibrium point $x = x_0$ Taylor's series expansion

Near an equilibrium point $x = x_0$, where the higher order terms can be neglected

$$\Rightarrow \quad \text{identify} \quad E_p(x) \underset{\substack{\downarrow \\ \text{simple harmonic oscillation}}}{=} E_p(0) + \frac{1}{2}k(x-x_0)^2$$

Comments

- $$\underset{\substack{\shortparallel \\ P \\ \shortparallel \\ \text{power}}}{\frac{dW}{dt}} = F\frac{dx}{dt} = FV$$

- If $F = \underset{\substack{\downarrow \\ \text{conservative}}}{F} + F_{\text{friction}}$, then there is energy loss

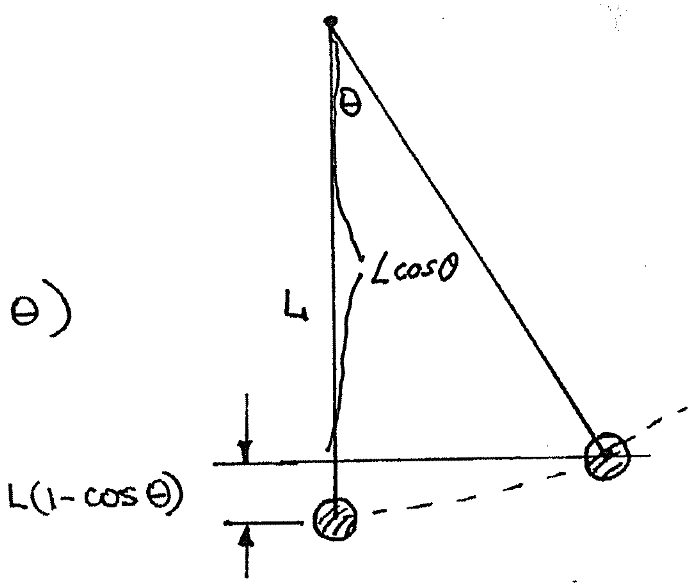
Example $F = -kx - \underset{\substack{\downarrow \\ \text{friction}}}{\lambda v}$

$$\underset{E_p(a) - E_p(b)}{\int_a^b (-kx)\,dx} - \int_a^b \lambda v\,dx = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

$$\Rightarrow \quad \underset{\substack{\shortparallel \\ E_b}}{\left(E_p(b) + \frac{1}{2}mv_b^2\right)} \underset{\substack{\\ - \\ \downarrow \\ \text{energy loss}}}{-} \underset{\substack{\shortparallel \\ E_a}}{\left(E_p(a) + \frac{1}{2}mv_a^2\right)} = -\int_a^b \lambda v\,dx$$

Example - Simple Pendulum

$$\overset{E_p}{\mathcal{U}}(\theta) = mgL(1-\cos\theta)$$



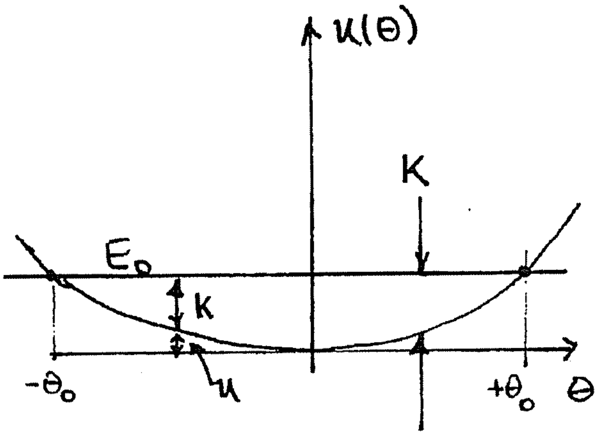
$$E = K + \mathcal{U}$$

$$= \frac{1}{2}mv^2 + mgL(1-\cos\theta)$$

For max. angle

$\theta = \theta_0 \quad v = 0.$

$$\therefore\ E_0 = mgL(1-\cos\theta_0)$$



$$\therefore\ \cancel{m}gL(1-\cos\theta_0) = \frac{1}{2}\cancel{m}v^2 + \cancel{m}gL(1-\cos\theta)$$

$$v^2 = 2gL(\cos\theta - \cos\theta_0)$$

At the bottom, $\theta = 0$

$$v_B = \sqrt{2gL(1-\cos\theta_0)}$$

Example

14-14

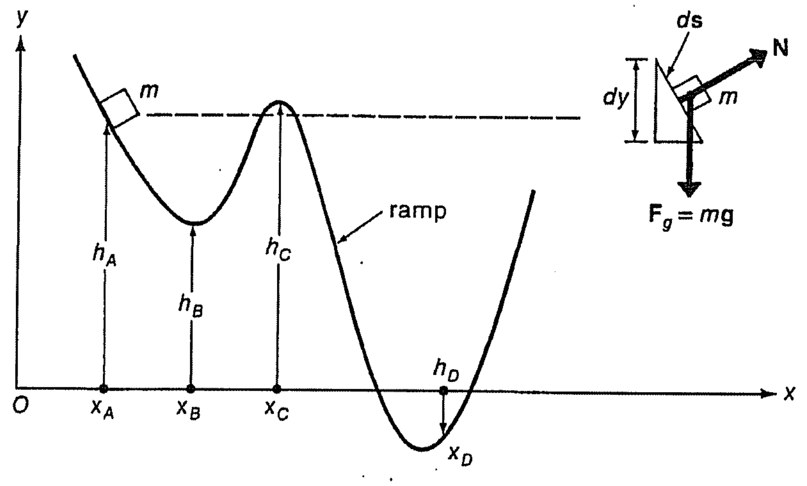
$h_A = 7m$

$h_B = 4m$

$h_C = 7.2m$

$h_D = -1m$

$v_A = 3m/s$ downward and tangent to ramp.



What is speed of particle at $x = x_B, x_C, x_D$?

$\vec{N} \cdot \vec{ds} \equiv 0$, so work is done only by gravitational force.

$$W(A \rightarrow B) = \int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (-mg) y$$

$$W = mg(h_A - h_B)$$

$$\text{Also} \quad W = K_2 - K_1 = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = v_A^2 + 2g(h_A - h_B)$$

$$= 3^2 + 2 \times 9.8 (7 - 4)$$

$$= 67.8 \ (m/s)^2$$

$$v_B = 8.23 \ m/s.$$

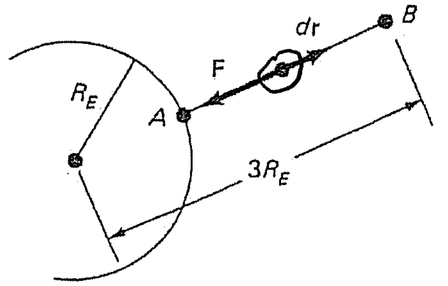
How far up the ramp will particle go after passing $x = x_D$?

h_m will be reached at x_m where $v_m = 0$.

$$0 = v_A^2 + 2g(h_A - h_m) \quad \Rightarrow h_m = h_A + v_A^2/2g \quad h_m = 7.46m$$

Example: Work Done on an Astronaut

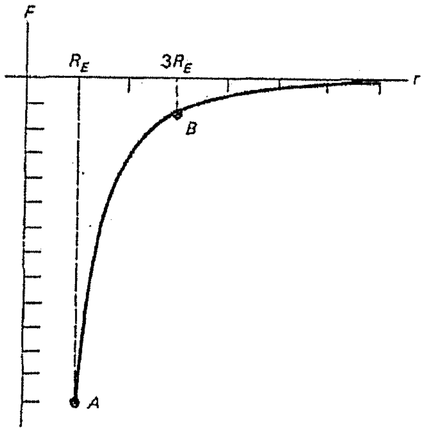
What is the work done by the force of gravity on an 80-kg astronaut in a displacement from point A at the earth's surface to point B whose altitude is $2R_E$. (earth radii).



$$F = -\frac{G M_E m}{r^2}\hat{r}$$

$$W = -\int_{R_E}^{3R_E} \frac{G M_E m}{r^2} dr$$

$$\int \frac{1}{r^2} dr = -\frac{1}{r}$$



Figure

The force acting on an object that is moving away from the earth yields the negative shaded area.

$$W = +\frac{G M_E m}{r}\Bigg|_{R_E}^{3R_E} = G M_E m\left(\frac{1}{3R_E} - \frac{1}{R_E}\right)$$

$$= -\frac{2}{3}\frac{G M_E m}{R_E} = -\frac{2}{3} m g R_E$$

Negative work: Force directed towards earth, displacement is away from earth.

$$W = -3.34 \times 10^9 \text{ J.}$$

Non-Conservative Forces

16-13

If non-conservative forces act on an object, then the change in the KE + PE of the conservative force will be equal to the work done by the friction force.

$$\Delta K + \Delta U = W_{Friction}.$$

$\Delta K \equiv$ change in KE
$\Delta U \equiv$ change in PE
$W_f \equiv$ Work done by frictional force.

$$E_1 = K_1 + U_1$$
$$E_2 = K_2 + U_2$$

$$(E_2 - E_1) = W_{Friction}$$

$$\frac{1}{2}mv_2^2 + U(x_2) - \left[\frac{1}{2}mv_1^2 + U(x_1)\right] = \int_{x_1}^{x_2} f\,dx$$

Example

16-14

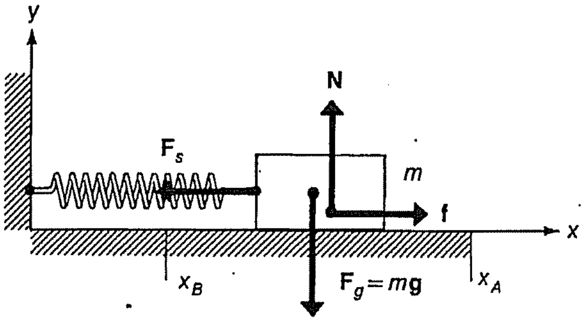
$m = 0.50\,kg$

$k = 50 N/m.$

$\mu_k = 0.20$

$x_A = 0.30m$

$x_B = 0.05m$



Mass released at $x = x_A$ with $v_A = 0$ [Spring is stretched]
what is velocity, v_B when $x = x_B$?

Force of friction $f = \mu_k N = \mu_k mg$

Work done by friction

$$W_f = \int_{x_A}^{x_B} \vec{f} \cdot \vec{dx} = \overbrace{+\mu_k mg\,(x_B - x_A)}^{-0.245\,J}$$

Energy Conservation:

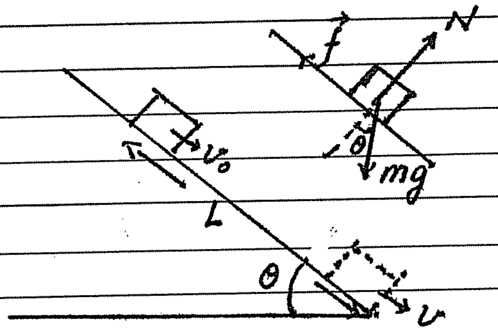
$$\left[\frac{1}{2}mv_B^2 + \frac{1}{2}kx_B^2\right] - \left[\cancel{\frac{1}{2}mv_A^2} + \frac{1}{2}kx_A^2\right] = \mu_k mg\,(x_B - x_A)$$

$$\frac{1}{2}mv_B^2 = \frac{1}{2}k(x_A^2 - x_B^2) + \mu_k mg\,(x_B - x_A)$$

$$= \tfrac{1}{2} \times 50\left(.3^2 - .05^2\right) - 0.245$$

$\underline{v_B} = 2.788\,m/s$

Find the coefficient of friction



$L = 5.0\, m$
$\theta = 36.87°$
$v_0 = 4.0\ m/s$
$v = 6.0\ m/s$
$g = 9.8\ m/sec$

$$W_{if}(\text{cons.}) + W_{if}(\text{non-cons}) = K_f - K_i$$

$$\underset{-fL}{W_{if}}(\text{n. cons.}) = K_f - K_i + (-\underset{\parallel}{W_{if}}(\text{con}))$$

$$E_p(f) - E_p(i)$$

$$= E_f - E_i$$

$$= \tfrac{1}{2}mv^2 + 0 - \tfrac{1}{2}mv_0^2 - mgh$$

$$\Rightarrow \quad fL = mgh + \tfrac{1}{2}mv_0^2 - \tfrac{1}{2}mv^2$$

$$N = mg\cos\theta \qquad h = L\sin\theta$$

$$f = \mu N$$

$$\mu = \frac{f}{N} = \frac{mg\sin\theta - m(v^2 - v_0^2)/2L}{mg\cos\theta}$$

$$= \underset{\frac{3}{4}}{\underset{\updownarrow}{\tan\theta}} - \frac{(v^2 - v_0^2)}{2Lg\underset{\frac{4}{5}}{\underset{\downarrow}{\cos\theta}}} = 0.49$$

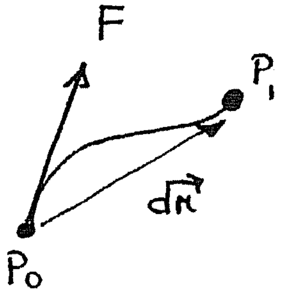
Force ⟷ Potential Energy

We have seen how to calculate the PE given a conservative force:

$$\mathcal{U}(P_1) - \mathcal{U}(P_0) = -\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$$

Can we calculate the force given the PE?

Yes!!!



Assume P_0 and P_1 are separated by the infinitesimal displacement $d\vec{r}$, then differentiating expression for $\mathcal{U}(P_1)$:

$$d\mathcal{U} = \mathcal{U}(P_1) - \mathcal{U}(P_0) = -\vec{F} \cdot d\vec{r} \quad [\text{Inverse of P.E.}]$$

$$= -F_x dx - F_y dy - F_z dz$$

Assume displacement is only along x, $dy = 0$, $dz = 0$.

$$d\mathcal{U} = -F_x dx$$

or $F_x = -\frac{d\mathcal{U}}{dx}$ } differentiate keeping y, z constant.

16-16

Define a special derivative called a partial derivative for one variable at a time,

$$\left.\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ F_y &= -\frac{\partial U}{\partial y} \\ F_z &= -\frac{\partial U}{\partial z} \end{aligned}\right\} \text{ Combining}$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)$$

$$= -\vec{\nabla} U$$

Example: Spring Force

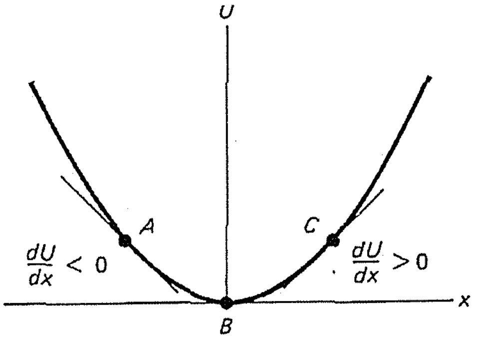
$$U(x) = \frac{1}{2}kx^2$$ PE of an elastic spring.

$$F_x = -\frac{\partial U}{\partial x} = -kx$$

$$F_y = -\frac{\partial U}{\partial y} = 0$$

$$F_z = -\frac{\partial U}{\partial z} = 0$$

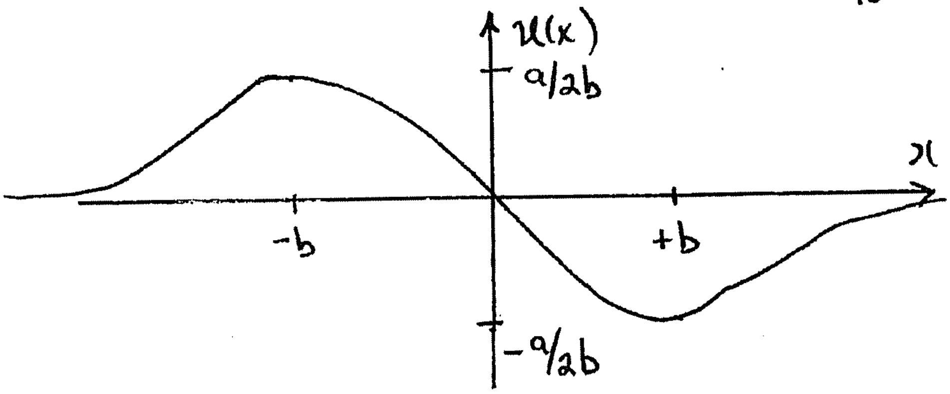
The potential energy $U = {}^1/_2\, kx^2$ is shown for a spring. When the spring is compressed, x<0, the slope is negative, and the force is positive. When the spring is stretched, x>0, the slope is positive, and the force is negative.



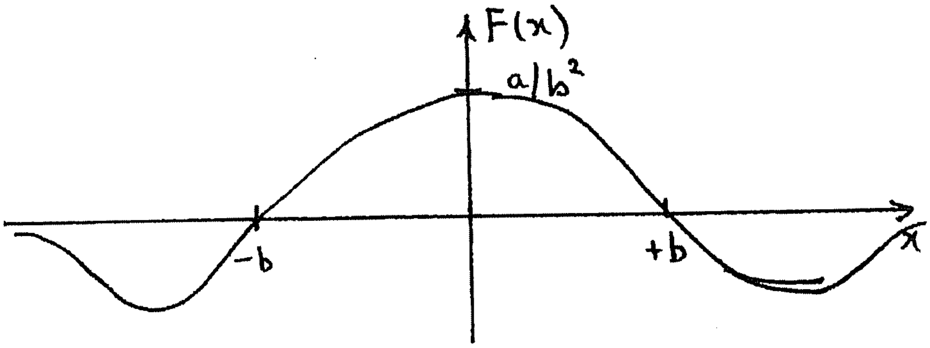
$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$ [Vector Operator]

16-17

Example



$$U(x) = \frac{-ax}{b^2+x^2}$$



$$F(x) = -\frac{\partial U(x)}{\partial x}$$

$$= \frac{a(b^2-x^2)}{(b^2+x^2)^2}.$$

Example

$$U(x,y) = Ax^2y^2$$

$$F_x = \frac{-\partial U}{\partial x} = -2Axy^2$$

$$F_y = \frac{-\partial U}{\partial y} = -2Ax^2y$$

Example:

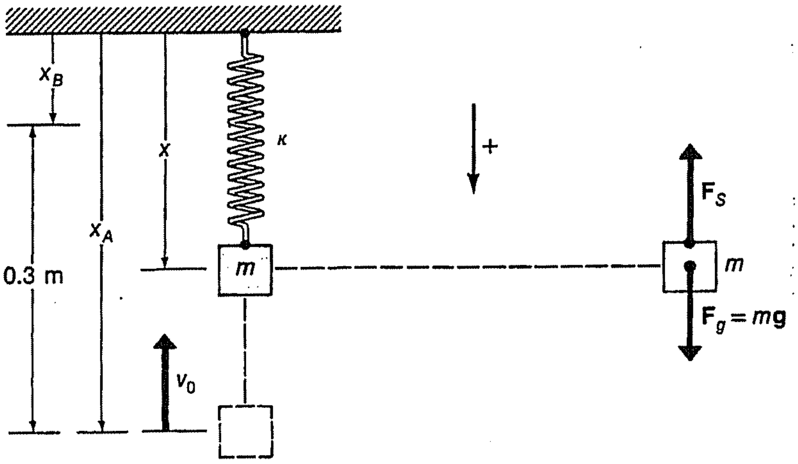
$m = 0.50$ kg.

$k = 50 N/m$

$x_A = 0.50 m$

$x_B = 0.20 m$

$v_0 = 2.0 m/s.$



Mass polled down to x_A, released by imparting an upward velocity v_0. What is velocity of block at x_B?

Net force on block:

$$F = F_g - F_s = mg - kx.$$

Work done, integrate F dx:

$$W(x_A \rightarrow x_B) = \int_{x_A}^{x_B} F dx = \int_{x_A}^{x_B} (mg - kx) dx = \left(mgx - \frac{1}{2} kx^2\right)\Big|_{x_A}^{x_B}$$

$$W = mg(x_B - x_A) - \frac{1}{2} k (x_B^2 - x_A^2)$$

$$= 0.5 \times 9.8 (0.2 - 0.5) - \frac{1}{2} \times 50 (0.2^2 - 0.5^2)$$

$$= -1.47 + 5.25 = 3.78 J$$

Also $W = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

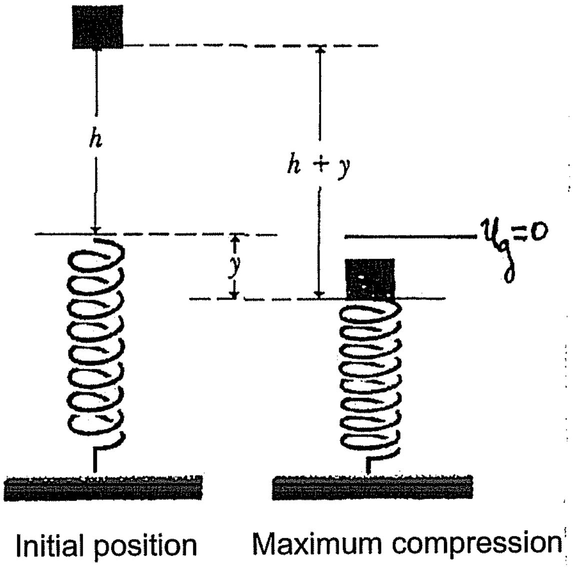
$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = \left[(-2.0)^2 + \frac{2 \times 3.78}{0.5}\right] \Rightarrow v_B = \pm 4.37 \text{ m/s}$$

Example

Dropping a brick onto a spring-mounted platform: using elastic and gravitational potential energies together.

$m = 2.6 kg$
$k = 72 N/m$
$h = 0.55 m$

Mass m dropped from rest on spring. What is maximum compression of spring?



The total fall of the block is h + y.

All conservative forces, PE represents all work done.

$E = K + \mathcal{U} = \text{conserved}$

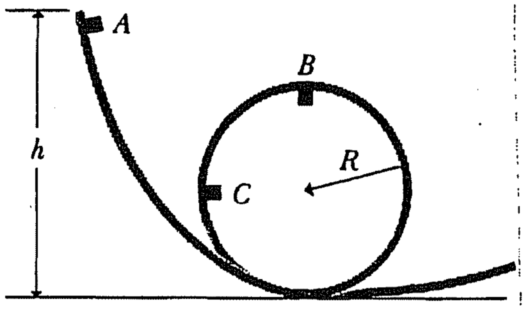
At release, brick is at rest, $v_1 = 0$ and $K_1 = 0$.
At maximum compression, brick is also momentarily at rest, $K_2 = 0$

$$K_1 + \mathcal{U}_1 = K_2 + \mathcal{U}_2$$
$$0 + mgh = 0 - mg(\ y) + \frac{1}{2}ky^2$$
$$y^2 - \frac{2mg}{k}y - \frac{2mg}{k}h = 0$$
$$y = \frac{1}{2}\left[\frac{2mg}{k} \pm \sqrt{\left(\frac{2mg}{k}\right)^2 + \frac{8mgh}{k}}\right]$$

+ → Answer.
− → [Mass + spring together str spring other side]

Example: Loop-the-Loop

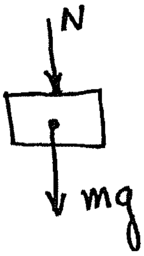
What is the minimum height h (in terms of R) such that an object moves around the loop without falling off the top (B)?



Top of Hoop:

velocity: v_T

$$F_c = mg + N = \frac{m v_T^2}{R}$$



Minimum $v_T \Rightarrow N \equiv 0.$ [contact force vanishes]

$$\therefore v_T = \sqrt{gR}$$

Represents minimum velocity to make the loop.

Conservation of Energy

$$mgH + 0 = mg(2R) + \frac{1}{2} m v_T^2$$

$$v_T^2 = 2g[H-2R]$$

$$v_T = \sqrt{2g(H-2R)}$$

[Energy Considerations]

$$\therefore gR = 2g(H-2R) \Longrightarrow H = \frac{5}{2}R$$

Note: Energy considerations alone not sufficient
Need to apply the second law

Energy Curves

Assume we know the PE curve for a particle moving in one-dimension. What is the description of the particle motion as a function of time?

Given : $\mathcal{U}(x)$
Want : $x(t)$

Consider conservative forces only. Then the total mechanical energy is a constant of the motion.

$$E = K + \mathcal{U} = \text{Constant}$$

$$= \frac{1}{2} m v^2 + \mathcal{U}(x)$$

$$= \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \mathcal{U}(x)$$

Solve for v_x :

$$v_x = \frac{dx}{dt} = \sqrt{\frac{2}{m}\left[E - \mathcal{U}(x)\right]}$$

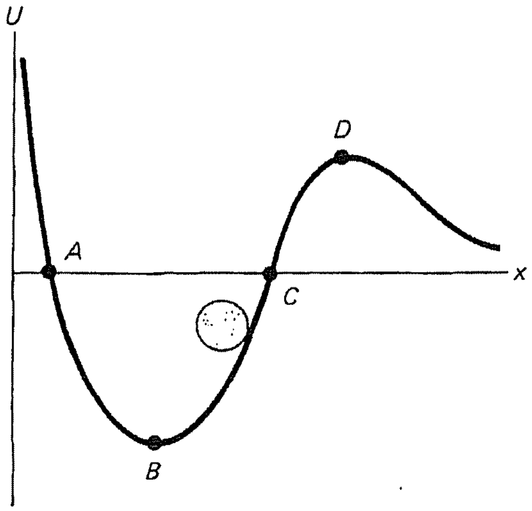
$$\int_{x'=x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\left[E - \mathcal{U}(x')\right]}} = \int_{t'=0}^{t} dt' ,$$

where $x' = x_0$ at $t' = 0$.

In general such an integral is not straightforward to evaluate analytically.

If we look at a PE diagram can we make some general statements about the particle motion?

Example



One dimensional potential for an alpha particle near a gold nucleus.

$r_0 \sim 10^{-13}$ cm

$$F = -\frac{\partial \mathcal{U}(x)}{\partial x}$$

i) At $x = x_B$ and $x = x_D$, $\frac{\partial \mathcal{U}}{\partial x} = 0$ and $F(x) = 0$.

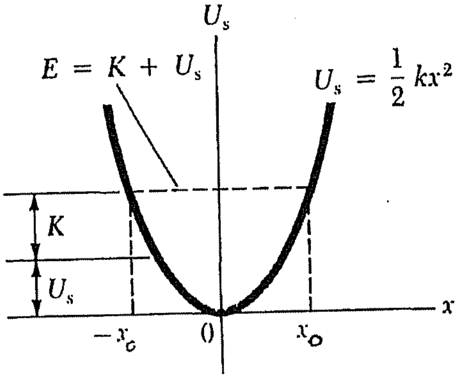
ii) If $\partial \mathcal{U}/\partial x \to 0$ as $x \to \infty$, then $F(x) \to 0$ as $x \to \infty$.

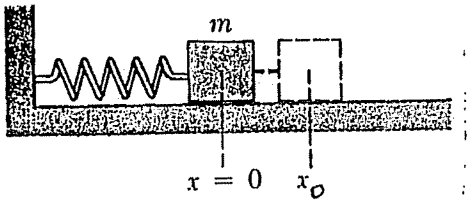
iii) $F(x) > 0$ for $0 < x < x_B$ since $\frac{\partial \mathcal{U}}{\partial x} < 0$. Also true for $x > x_D$. The force is therefore repulsive (away from origin)

iv) $F(x) < 0$ for $x_B < x < x_D$ since slope of $\mathcal{U}(x)$ is positive. Force is attractive (towards the origin).

Strategy: To remember force-potential energy relationship consider a ball rolling on a hilly surface under influence of gravity. The $\mathcal{U}$-axis is "up". The ball is pushed in the direction of F.

Example : Mass + Spring





Mass is pulled to the right $x = x_0$ and released with zero velocity.

$E = \frac{1}{2} k x_0^2$: Initial total energy, remains constant.

$$\therefore \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2 .$$

i) $x > 0$

$$\frac{\partial \mathcal{U}(x)}{\partial x} = kx > 0$$

$F < 0$, attractive ; accelerates mass towards equilibrium position.

ii) $x < 0$

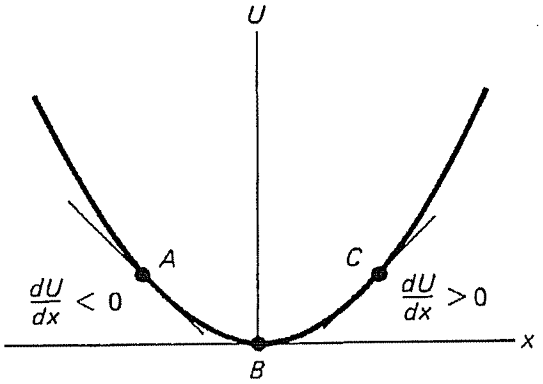
$$\frac{\partial \mathcal{U}(x)}{\partial x} = kx < 0$$

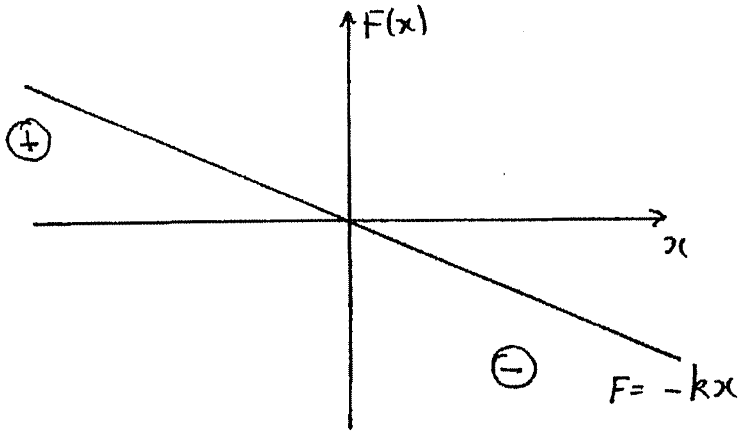
$F > 0$, attractive ; accelerates mass towards equilibrium position.

iii) $x = 0$

$$\frac{\partial u(x)}{\partial x} = 0$$

$\therefore \; F = 0$





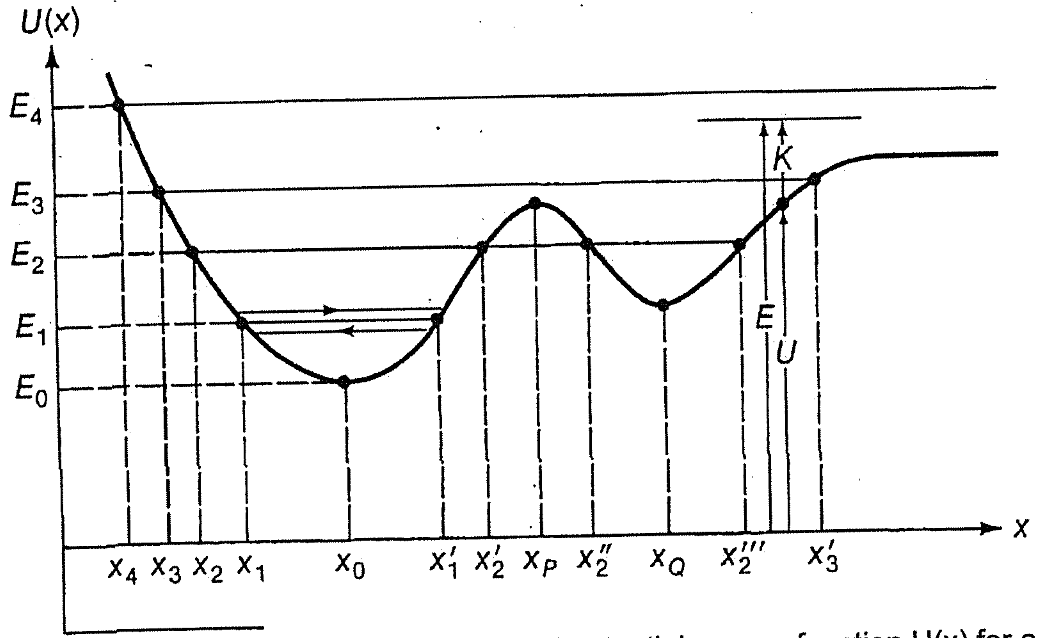
Harmonic Oscillator

Particle moves between turning points, whose limits are defined by the total energy E. As E increases, turning points move further apart. As E increases amplitude of oscillation increases.

If E = 0, $x = 0$ always.

Motion is always completely bounded.

Example



The graphical representation of the one-dimensional constrained potential energy function U(x) for an object. Also shown are various values of the total mechanical energy.

Consider the motion for particles with different total energies $E = E_0, E_1, E_2$ and E_4. The value of E depends upon the initial conditions.

$$v_x = \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

In all cases $U(x) < E$, otherwise v_x is imaginary.

i) $E = E_0$

Particle stays fixed at $x = x_0$. It has PE but can have no KE. Point of <u>stable</u> equilibrium. It is at bottom of lowest valley in the system.

ii) $E = E_1$

Particle oscillates back and forth between the points x_1 and x_1'. Its KE at any point is given by the difference between E_1 and $U(x)$. Starting at x_1

it moves to the right $[F(x) > 0]$ and increases speed up to x_0. Continues to move to right with decreasing speed to x_1', where it stops, turns around and starts back.

$\left.\begin{matrix} x_1 \\ x_1' \end{matrix}\right\}$ turning points / bounded motion [Particle is trapped in a potential well]

iii) $E = E_2$

Four turning points. Particle can move in <u>one</u> of <u>two</u> valleys, depending on where it is initially.
Quantum Mechanics: Particle can jump from one valley to the other → tunnelling. Classically not allowed!!

iv) $E = E_4$

One turning point only.
$U(x) < E_4$ for all $x > x_4$.

Particle moving to left varies in speed as it passes over the valleys. It reverses direction at $x = x_4$ and moves with $x > x_4$ indefinitely → it never returns. <u>Unbounded motion.</u>

<u>Equilibrium</u>

$x = x_0$: Force on either side of x_0 acts to return particle to x_0.
"Stable Equilibrium"

$x = x_p$: Force acts to move particle away from $x = x_p$
"Unstable Equilibrium"

Equilibrium/Stability

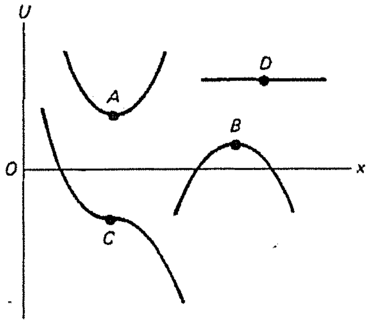
How do we tell if an equilibrium point is stable or unstable?

Mathematically: Examine sign of $\frac{d^2U}{dx^2}$!!

$\frac{d^2U}{dx^2} > 0$

Potential Minimum

stable.

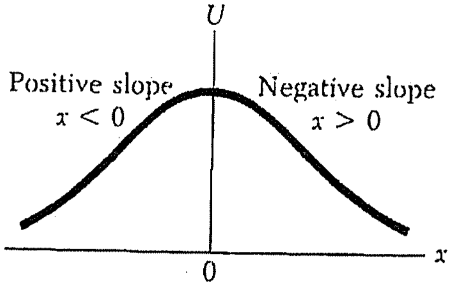


Points (A, B, C, D) on the potential-energy versus position graph with zero slope are equilibrium positions. The equilibrium is characterized as stable (A), unstable (B), or neutral (D), according to the graph in the neighborhood of the equilibrium point. For a curve like that near point C the equilibrium cannot be characterized simply as stable, unstable, or neutral.

$\frac{d^2U}{dx^2} < 0$

Potential Maximum

Unstable.



A plot of U versus x for a system that has a position of unstable equilibrium, located at x=0. In this case, the force of the system for finite displacements is directed away from x=0.

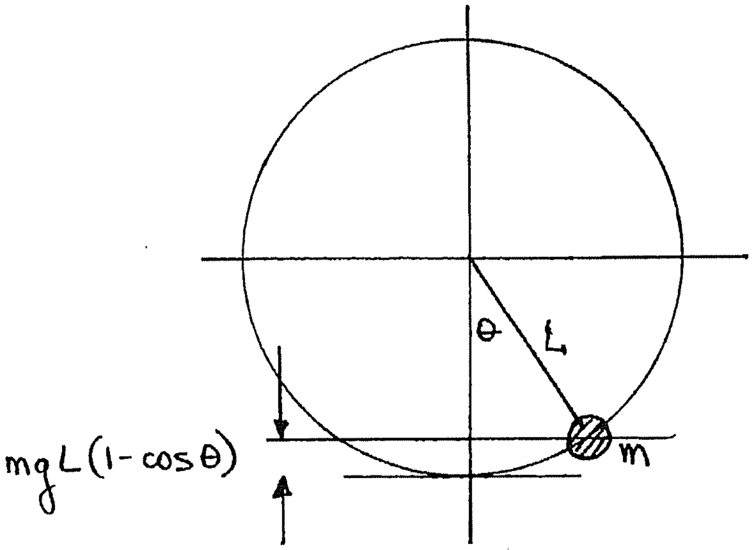
$\frac{d^2U}{dx^2} = 0$

U is constant over a region.

$F = 0$

Neutral equilibrium.

Pendulum

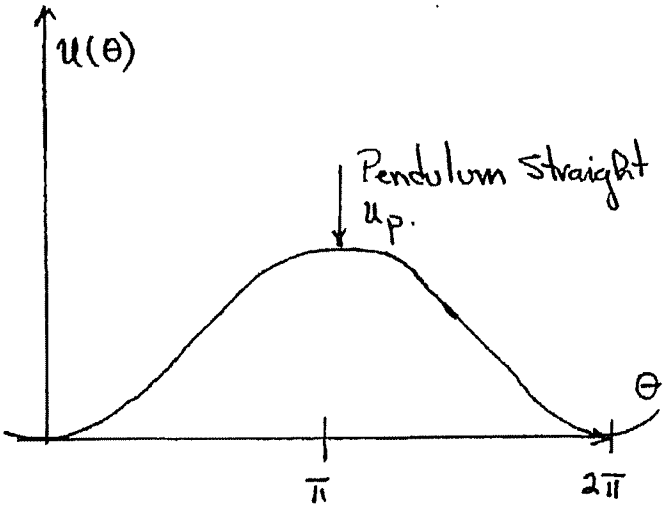


$$\frac{d^2 U}{d\theta^2} > 0 \quad \text{at} \quad \theta = 0$$

(stable)

$$\frac{d^2 U}{d\theta^2} < 0 \quad \text{at} \quad \theta = \pi$$

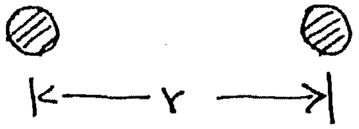
(unstable)



Example: Diatomic Molecule

Force between atoms in a diatomic molecule has its origin in the interactions between the electrons and nuclei in each atom. For simple atoms the potential energy is, to a good approx., represented by the Lennard-Jones potential (6, 12).

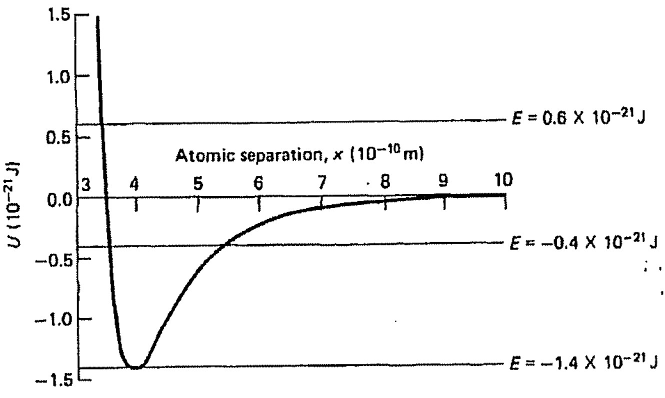
$$U(r) = U_0\left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right]$$



$r \equiv$ distance between atoms

$U_0, a \equiv$ constants.

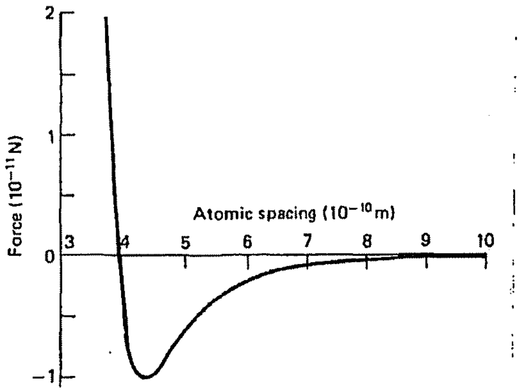
$$\left.\begin{array}{l} U_0 = 5.6 \times 10^{-21}\ \text{J} \\ a = 3.5 \times 10^{-10}\ \text{m} \end{array}\right\} O_2$$



$$F = -\frac{dU}{dr}$$

$$= -U_0\left[-12\frac{a^{12}}{r^{13}} + 6\frac{a^{6}}{r^{7}}\right]$$

$$= \frac{6U_0}{a}\left[2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right]$$



At equilibrium $F = 0$

$$\therefore\ 2\left(\frac{a}{r_0}\right)^{13} - \left(\frac{a}{r_0}\right)^{7} = 0$$

$$\left(\frac{a}{r_0}\right)^{6} = \frac{1}{2}$$

$$r_0 = 2^{1/6}a = 2^{1/6}\left(3.5 \times 10^{-10}\ \text{m}\right)$$

$$= 3.9 \times 10^{-10}\ \text{m} \quad (O_2\ \text{molecule})$$

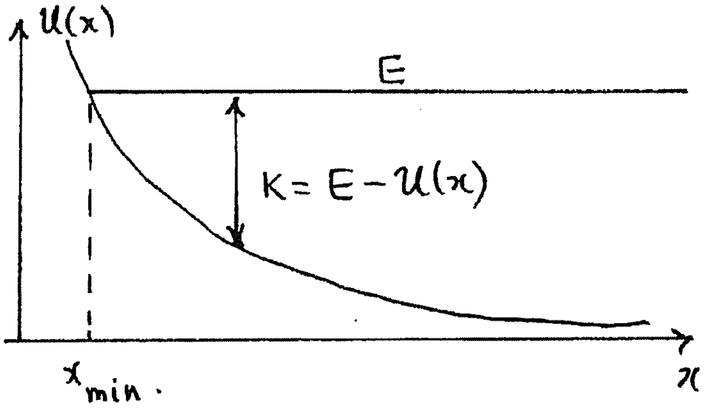
$$\boxed{2^{1/6} = 1.122}$$

Example

Repulsive square law force. [eg. like point charges]

$F(x) = \frac{A}{x^2}$ $\qquad A > 0$

$U = \frac{A}{x}$



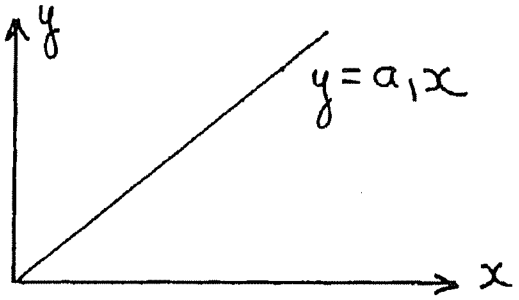
Assume particles have total energy E.
Particle approaching the origin comes as close as x_{min}, reverses direction and then moves out forever.

$x_{min} = \frac{A}{E}$ $\qquad$ [i.e. $U(x_{min}) = E$]

Least Time Trajectories

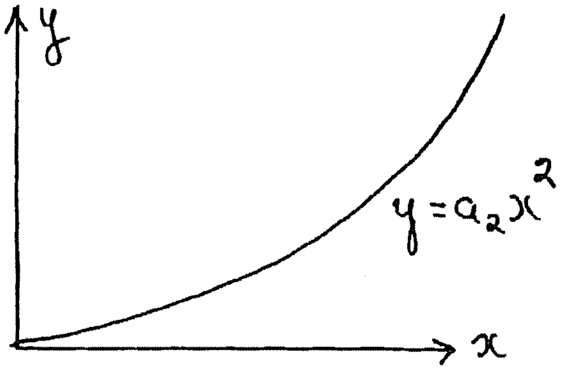
1. Straight Line:

$y = a_1 x$



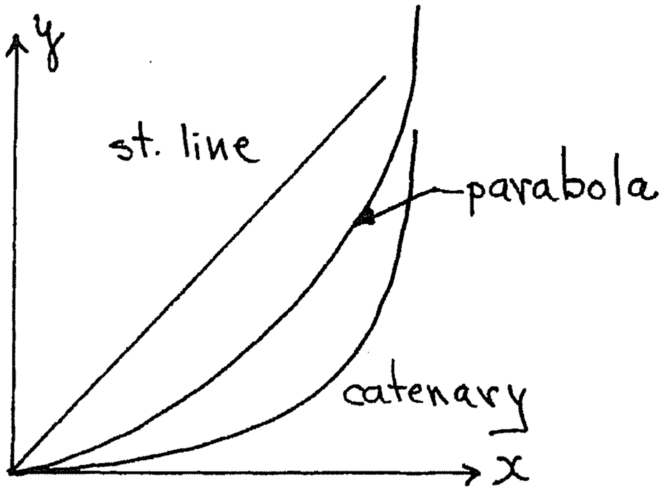
2. Parabola:

$y = a_2 x^2$



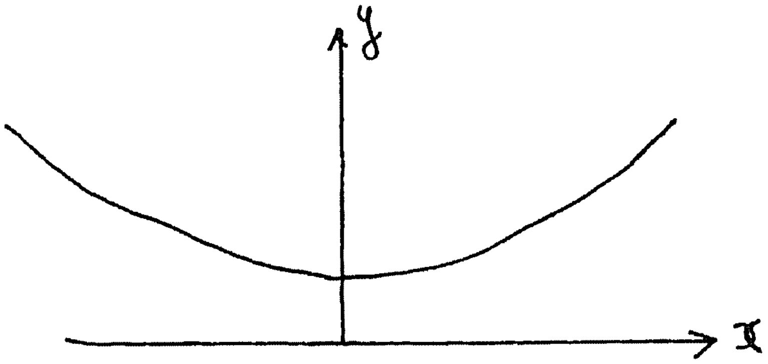
3. Catenary:

$y = \frac{1}{a} \cosh ax$



Catenary

- Suspend a cable between two posts and let it hang under own weight.
- The shape the cable takes is called a catenary



$$y = \frac{1}{2a}\left(e^{ax} + e^{-ax}\right)$$

$$= \frac{1}{a}\cosh ax$$

$$y(x=0) = 1/a$$

- catenary has lowest average center-of-gravity
- catenary has smallest potential energy
- nature shapes curve to minimize PE

Power

- Power is defined as the time rate of doing work.
- If an external force applied to an object does an amount of work Δw in the time interval Δt, the average power is

$$\overline{P} = \frac{\Delta w}{\Delta t}.$$

- The instantaneous power, P, is the limiting value of the average power as Δt approaches zero.

$$P \equiv \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

$$[P] = J/s$$

$$= \underline{W}\text{atts}$$

British: $1\,hp = 550\ ft \cdot lbs/s$

$$= 745.7\,W$$

Power ⟷ Force

We can express the power in terms of the force acting and the velocity of the object. For a small displacement $d\vec{r}$ the work done is

$$dW = \vec{F} \cdot d\vec{r}$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

Note: Velocity depends on the frame of reference.
Power also depends upon reference frame.

Energy ⟷ Power

$$E = \int_{t_1}^{t_2} P\,dt$$

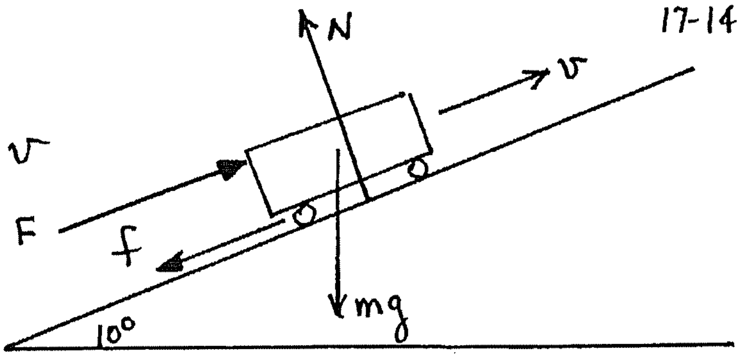
$$= Pt \qquad [\text{Constant } P]$$

$$1\,\text{kWh} = 3.6 \times 10^6\,\text{J} \qquad [1000\text{W rate for } 1\text{h}]$$

$$\left.\begin{array}{ll} 1\,\text{kilocalorie} & = 4.187 \times 10^3\,\text{J} \\ 1\,\text{Btu} & = 1.055 \times 10^3\,\text{J} \end{array}\right\} \text{Fuels}$$

Example

Car moves at constant v
up a plane



$m = 1400\,kg$
$v = 80\,km/h = 22\,m/s$
$f = 700\,N$

Net force exerted by engine to move car up at constant v:

$$F = f + mg\sin\theta$$

$$= 700 + 1400 \times 9.81 \times \sin 10^{\circ}$$

$$= 3100\,N$$

$$P = \vec{F}\cdot\vec{v} \qquad [\vec{F} \text{ is parallel to } \vec{v}]$$

$$= 3100 \times 22 = 6.8 \times 10^{4}\,W$$

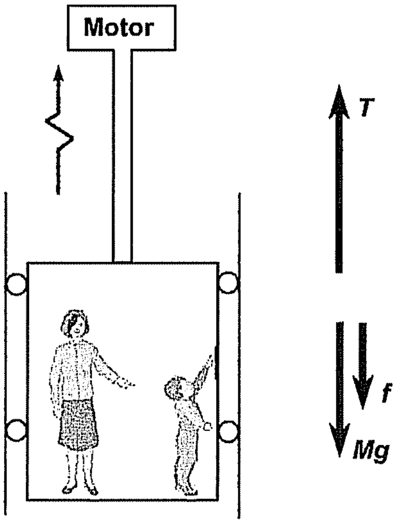
$$= 91\,hp.$$

$1\,hp = 746\,W$

Example

m (elevator+load) = 1800 kg.

f = 4000 N



a) What is horsepower required to move elevator up at constant speed of 3 m/s?

Motor supplies required tension.

$$T - f - Mg = 0$$

$$T = f + Mg$$
$$= 4\times10^3 + 1.8\times10^3 \times 9.81$$
$$= 2.16\times10^4 \text{ N}$$

$$P = \vec{T}\cdot\vec{v} = Tv$$
$$= 2.16\times10^4 \times 3$$
$$= 6.48\times10^4 \text{ W}$$
$$= 64.8 \text{ kW}$$
$$= 86.9 \text{ hp.}$$

b) If elevator accelerates upward at 1.0 m/s^2?

$$T - f - Mg = Ma$$
$$T = M(a+g) + f$$
$$= 2.34\times10^4 \text{ N}$$

v ≡ instantaneous speed.

$P = Tv = (2.34\times10^4)v$ Watts   Power increases with inc. speed

Forms of Energy

- electrical
- chemical
- heat
- nuclear

Atomic/Nuclear:

1eV = electron-volt

= 1.602×10^{-19} J

Power Plants:

1-kilowatt hour = 3.6×10^{6} J

Fuels:

1 kilocalorie = 4.187×10^{3} J

1 Btu = 1.055×10^{3} J

Relativity/Einstein

$$\text{Mass} \longleftrightarrow \text{Energy}$$

$$E = mc^2$$

$c = 3 \times 10^{8}$ m/s

mass has energy
energy has mass } nuclear power
nuclear weapons
particle reactions

Two or Three Dimensions

$dW = \vec{F} \cdot d\vec{r}$

$W_{a\to b} = \int_a^b \vec{F} \cdot d\vec{r}$

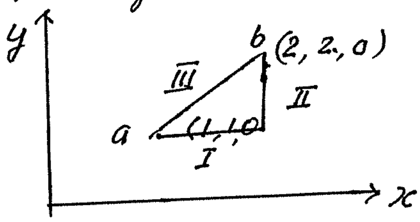
$\downarrow$

Line integral

The line integral usually are path dependent.

Example:

$\vec{V} = y^2 \hat{i} + 2x(y+1)\hat{j}$



$a \to b$ $\quad I \to II$

(I) $\quad d\vec{r} = dx\,\hat{i}$

$I \qquad y = 1$

$\vec{V} = \hat{i} + 2x(2)\hat{j}$

$I = \int_{(1,1,0)}^{(2,1,0)} \vec{V} \cdot dr = \int_1^2 dx = 1$

(II) $\quad d\vec{r} = dy\,\hat{j} \qquad x = 2$

$\int \vec{V} \cdot d\vec{r} = \int_1^2 4(y+1)\,dy = \frac{3}{2} + 4$

$a \to b$ along III

$x = y \qquad dx = dy$

$\int \vec{v} \cdot d\vec{r} = \int_1^2 x^2\,dx + \int_1^2 2x(x+1)\,dx$

Clearly $\int_{I+II} \neq \int_{III}$

If $\int_a^b \vec{F} \cdot d\vec{r}$ is independent of path, then $\vec{F}$ is conservative

The following statements are equivalent

- $\vec{F}$ is conservative
- $\int \vec{F} \cdot d\vec{r}$ is independent of path
- $\oint \vec{F} \cdot d\vec{r} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

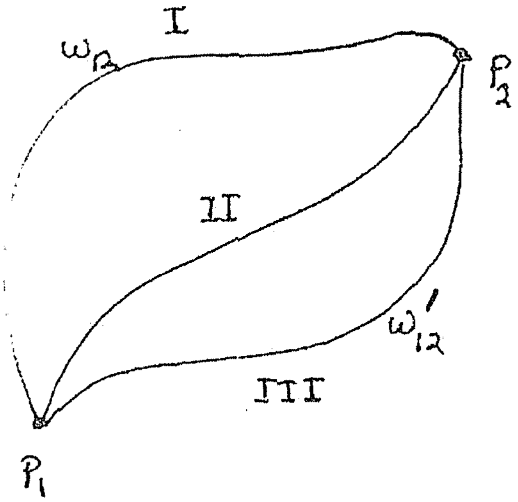
$$\vec{F} = -\nabla E_p$$

Conservative Forces

For the gravitational force we obtained Conservation of Energy by summing the kinetic and potential energies:

$$E = K + \mathcal{U}(z) \qquad \text{[Constant - Conserved]}$$

What are the essential requirements for other such conservative forces, i.e. that a potential energy is to exist?



Consider a particle which moves from P_1 to P_2 with a force $\vec{F}$ acting on it. Assume $\vec{F}$ is a function only of position but does not depend explicitly on time. Since particle moves, force is a function of time but we assume that at any given location the force is the same no matter when the particle arrived there.

The work done by the force $\vec{F}$ on the particle

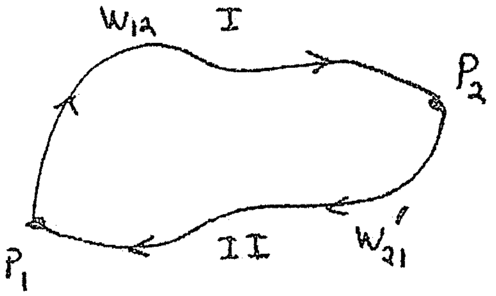
$$W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$ in moving from $1 \rightarrow 2$ along path - I

$$W_{12}' = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$ work done in moving from 1→2 along primed path - II.

<u>Definition</u>: $\vec{F}$ is a <u>conservative force</u> if the work depends only on the position of the points P_1 and P_2 but not on the shape of the path between P_1 and P_2.

i.e. $W_{12} = W_{12}'$ for any two paths.

<u>Closed Path - Round Trip</u>



Particle moves from $P_1 \to P_2$ and then back to P_1. If the force is conservative the total work is exactly zero for a round trip along a closed path.

$$W_{12} + W_{21}' = W_{12} - W_{12}' = 0$$

$$\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} + \int_{P_2}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \equiv 0.$$

$$\oint \vec{F} \cdot d\vec{r} = 0$$

↑ Line integral around a closed loop.

- Gravity is a conservative force.
- Spring is a conservative force.
- Friction is not a conservative force since it always opposes motion.

$$\oint \vec{F} \cdot d\vec{r} \neq 0 \qquad \text{[Friction]}$$

Work done against a non-conservative force is not recoverable ; it goes into heat or is dissipated in some other way. Work also depends on path length and not simply on the location of end points. [Macroscopic Viewpoint]

Potential Energy of Conservative Forces

PE can only be defined for conservative forces. Take a reference point P_0 to which is assigned a potential energy which has some value $\mathcal{U}(P_0)$ — any arbitrary number. Often convenient to take $\mathcal{U}(P_0) \equiv 0$.

Then any other point P has the potential energy which is given by:

$$\boxed{\mathcal{U}(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0)}$$

↑ Evaluated along any path between P_0 and P since for conservative forces it is path independent.

Does this function for PE have the correct properties? Calculate the change in PE between the points P_1 and P_2.

$$\mathcal{U}(P_2) - \mathcal{U}(P_1) = -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0) - \left[-\int_{P_0}^{P_1} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0)\right]$$

$$= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} + \int_{P_0}^{P_1} \vec{F}\cdot d\vec{r}$$

$$= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} - \int_{P_1}^{P_0} \vec{F}\cdot d\vec{r}$$

$$\mathcal{U}(P_2) - \mathcal{U}(P_1) = -\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

The change in PE equals the negative of the work done by the force between the two points.

$\mathcal{U}(P_0)$: Drops out in final result. Choice of P_0 does not matter. In all applications only differences in potential are significant and we often use $\mathcal{U}(P_0) \equiv 0$.

Mechanical Energy

We had previously that the change in kinetic energy of the particle equals the work ;

$$K_2 - K_1 = W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

∴ For any conservative force we must have

$$K_2 - K_1 = \mathcal{U}(P_1) - \mathcal{U}(P_2)$$
$$K_2 + \mathcal{U}(P_2) = K_1 + \mathcal{U}(P_1)$$

$$E = K + \mathcal{U} = [\text{constant}]$$

$E \equiv$ Total Mechanical Energy

[Law of Conservation of Mechanical Energy]

PE: Gravity (Near Earth)

$$\vec{F} = -mg\,\hat{z}$$

$$u(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r}' + u(P_0)$$

Assume $u(P_0) = 0$ for $P_0 \Rightarrow x=0,\ y=0,\ z=0$

$$u(P) = -\cancel{\int_0^x F_x\,dx'} - \cancel{\int_0^y F_y\,dy'} - \int_0^z F_z\,dz'$$

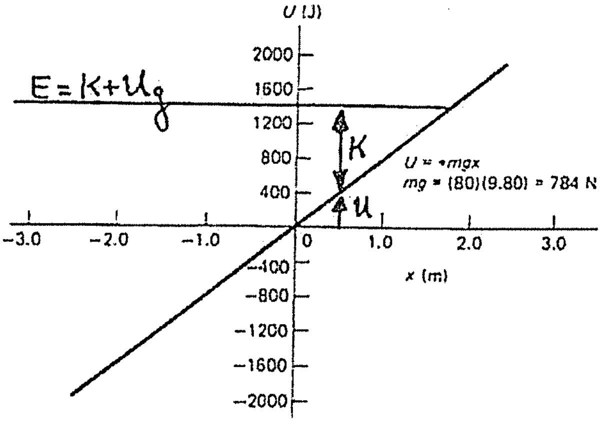
$$= -\int_0^z F_z\,dz' = -\int_0^z (-mg)\,dz'$$

$$u(z) = mgz$$

Total mechanical energy is

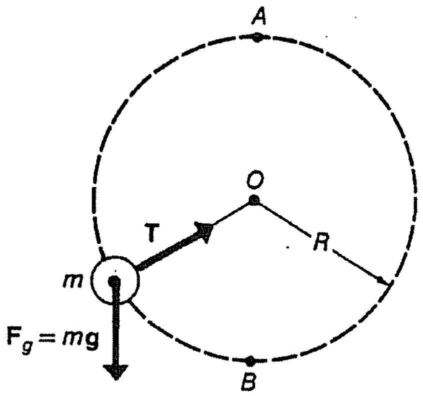
$$E = K + u = \frac{1}{2}mv^2 + mgz.$$

The potential energy U = mgx of an 80-kg mass is shown versus height x near the surface of the earth.



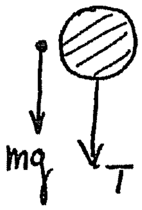
Example

A rock of mass m is tied to the end of a string and whirled in a circular path in the vertical plane.
What is the minimum speed v_0 that the rock has at the bottom (B) if the rock is to pass the top (A) with the string remaining taut?



- Energy considerations alone not sufficient.
- Need to apply Newton's 2ND Law.

At the top the minimum speed occurs as tension vanishes, $T \rightarrow 0$.
Net downward force is



$$F_g = mg = m\frac{v_T^2}{R}$$

[centripetal acceleration in circular path]

$$v_T^2 = gR$$

$$W(A \rightarrow B) = mgh = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_T^2$$

$$v_0^2 = v_T^2 + 4gR = gR + 4gR = 5gR$$

$$v_0 = \sqrt{5gR}$$

Summary of Last Two Lecture

One Dimensional Case

$F_x(x)$

$$dW = F_x(x)\,dx$$

$\downarrow$ work done by the particle due to the fore

$$\int_a^b F_x\,dx = W_{ab}$$

$\Downarrow$ Newton's second law

work - energy relationship

$$\underset{\substack{\downarrow \\ K_b}}{\tfrac{1}{2}mV_b^2} - \underset{\substack{\downarrow \\ K_a}}{\tfrac{1}{2}mV_a^2}$$

$$\int_a^b F_x\,dx = E_p(a) - E_p(b)$$

$$E(a) = -\int_0^a F(x)\,dx + E_p(0)$$

$$\int_{x_0}^{x} F(x)\,dx = -E_p(x) + E_p(x_0)$$

$\downarrow$ reference point can be choosen at x_0

$$F_x = -\frac{dE_p}{dx}$$

Example

$$F = -kx$$

$$\Rightarrow \quad E_p(x) = \frac{1}{2}kX^2 + \overset{\substack{\text{constant} \\ \uparrow}}{E_p(0)}$$

$E_p(0) = C = 0$. (reference point)

Note: the result is indepent of choice

$$\Rightarrow \quad \frac{1}{2} m V_b^2 + E_p(b) = \frac{1}{2} m V_a^2 + E_p(a)$$

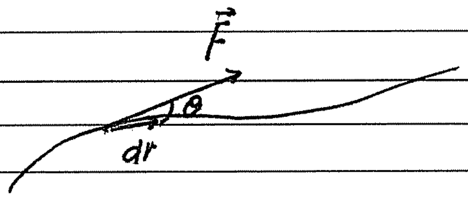
$\downarrow$

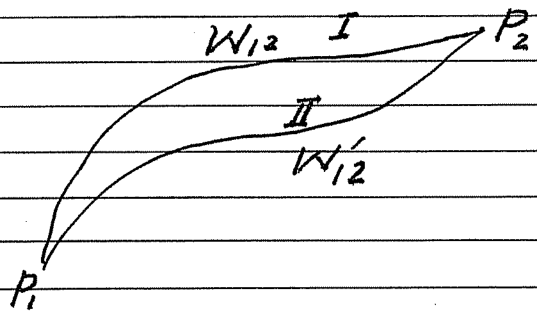
conservation of $\overset{\text{mechanical}}{\text{energy}}$.

Three dimensional case

$$\vec{F} = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k}$$

$$dW = \vec{F} \cdot d\vec{r} \quad = |\vec{F}|\, dr \cos\theta$$





W_{12} = work done by the force $\vec{F}$ on the particle in moving from 1 → 2 along path I

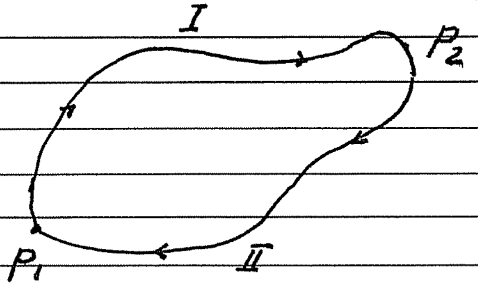
$$= \int_{P_1 \, (I)}^{P_2} \vec{F} \cdot d\vec{r}$$

W_{12}' = work done by the force $\vec{F}$ on the particle in moving from 1 → 2 along path II

Definition $\vec{F}$ is a conservative if the work depends only on the position of the point P_1 and P_2 but not the path between P_1 and P_2

$W_{12} = W_{12}'$ for any two paths

Closed path - Round trip



Particle moves from $P_1 \rightarrow P_2$ and then back to P_1

If the force is conservative the total work is exactly zero for a round trip along a closed path

$$\underset{I}{W_{12}} + \underset{II}{W'_{21}} = \underset{I}{W_{12}} - \underset{II}{W'_{12}} \underset{\substack{\uparrow \\ \text{if the force is} \\ \text{conservative.}}}{=} 0 \quad \Rightarrow \quad \oint \vec{F} \cdot d\vec{r} = 0$$

Through Stoke's theorem in calculus, we can show that the necessary and sufficient condition for

$$\oint \vec{F} \cdot d\vec{r} = 0$$

is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial F_z(x,y,z)}{\partial y} - \frac{\partial F_y(x,y,z)}{\partial z} = 0$$

$$-\frac{\partial F_z(x,y,z)}{\partial y} + \frac{\partial F_x(x,y,z)}{\partial z} = 0$$

$$\frac{\partial F_y(x,y,z)}{\partial x} - \frac{\partial F_x(x,y,z)}{\partial y} = 0$$

Partial derivative

$$\frac{\partial F_z(x, y, z)}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{F_z(x, y+\Delta y, z) - F_z(x, y, z)}{\Delta y}$$

Example

$$F_z \ (x, y, z) = axyz^2$$

$$\frac{\partial F_z(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{a(x+\Delta x)yz^2 - axyz^2}{\Delta x}$$

$$= ayz^2$$

$$\frac{\partial F_z(x, y, z)}{\partial z} = \lim_{\Delta z \to 0} \frac{axy(z+\Delta z)^2 - axyz^2}{\Delta z}$$

$$= axy(2z)$$

When take partial derivative with respect to x; y, z are fixed

Theorem If $\vec{F}(x, y, z) = f(r)\,\vec{r}$ (i.e. it is a central force) ⇒ it is a conservative force.

We shall show the first of these relations

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\vec{F} = f(r)\,\vec{r} = f(r)\,x\,\hat{i} + f(r)\,y\,\hat{j} + f(r)\,z\,\hat{k}$$

$$\frac{\partial F_z}{\partial y} = z\frac{\partial}{\partial y} f(r)$$

$$= z \frac{df(r)}{dr} \frac{\partial r}{\partial y}$$

$\vec{F} = \frac{-\alpha \vec{r}}{r^3}$ Gravitation force

$r = (x^2+y^2+z^2)^{1/2}$

$F_x = \frac{x}{r^3}$

$F_y = \frac{y}{r^3}$

$F_z = \frac{z}{r^3}$

Condition for $\vec{F}$ to be conservative (related to the Stoke's theorem)

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0, \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0, \quad \text{and} \quad \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0$$

$$\frac{\partial}{\partial y} \frac{z}{(\sqrt{x^2+y^2+z^2})^3} = z \frac{\partial}{\partial y} \frac{1}{u^{3/2}}$$

$$\boxed{u = x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial y} = 2y$$

$$= z \frac{\partial}{\partial y} u^{-3/2}$$

$$= z\left(-\frac{3}{2}\right) u^{-5/2} \underbrace{\frac{\partial u}{\partial y}}_{2y} = \frac{-3}{2} 2u^{-5/2} yz \; (-\alpha)$$

$$\frac{\partial}{\partial z} \frac{y}{\sqrt{x^2+y^2+z^2}} = y \frac{\partial}{\partial z} \frac{1}{u^{3/2}}$$

↓ using the same procedure

$$= -\frac{3}{2} 2u^{-5/2} yz \cdot (-\alpha)$$

⇒ check

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \quad \frac{\partial F_z}{\partial y} = z\frac{df(r)}{dr}y \; = y\,z\frac{df}{dr}$$

$$\frac{\partial F_y}{\partial z} = y \frac{df}{dr}\frac{\partial r}{\partial z} = yz\frac{df}{dr}$$

↓

check

We can prove the other two components

↓

thus the proof.

We can show that gravity is a conservative force.

spring is a conservative force.

Friction is not a conservative force since it always opposes motion

$$\oint \vec{F}\cdot d\vec{r} \neq 0 \qquad \text{(friction)}$$

Work done against a non-conservative force is not recoverable; it goes into heat or is dissipated in some other way

Potential Energy of Conservative Forces

P. E. can only be defined for conservative

Take a reference point P_0 to which is assigned a potential energy $E_p(P_0)$

Then any other point P has the potential energy

$$E_p(P) \equiv -\int_{\underset{\uparrow}{P_0}}^{P} \vec{F}\cdot d\vec{r} + E_p(P_0)$$

evaluated along any path between P_0 and P since for conservative force it is path independent.

$$\underset{\overset{\parallel}{U(p)}}{E_p(p)} = -\int_{P_0}^{p} \vec{F}\cdot d\vec{r} + \underset{\overset{\parallel}{U(P_0)}}{E_p(P_0)}$$

$$\begin{aligned} E_p(P_2) - E_p(P_1) &= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} + E_p(P_0) \\ &\quad - \left[-\int_{P_0}^{P_1} \vec{F}\cdot d\vec{r} + E_p(P_0)\right] \\ &= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} - \int_{P_1}^{P_0} \vec{F}\cdot d\vec{r} \\ &= -\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} \end{aligned}$$

The change in PE equals the negative work done by the force between the two points

$E_p(P_0)$: Drop out in final result

Choice of P_0 does not matter

In all applications, only difference in potential are significant and we often use $E_p(P_0)=0$

Mechanical Energy

Work - Energy Theorem

$$(K.E.)_2 - (K.E.)_1 = W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

For any conservative force

$$(K.E.)_2 - (K.E.)_1 = E_p(P_1) - E_p(P_2)$$

$$(K.E.)_2 + E_p(2) = (K.E.)_1 + E_p(P_1)$$

$$E = (K.E.) + E_p = \text{constant}$$

total mechanical energy

Law of conservation of mechanical energy

Example :

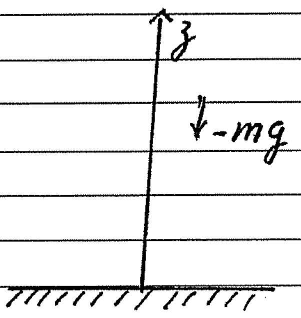
1. Gravity (Near Earth)

$\vec{F} = -mg\hat{z}$

$E_p(p) = -\int_{P_0}^{p} \vec{F} \cdot d\vec{r} + E_p(P_0)$

Take $E_p(P_0) = 0$

for P_0 $(x=0, y=0, z=0)$



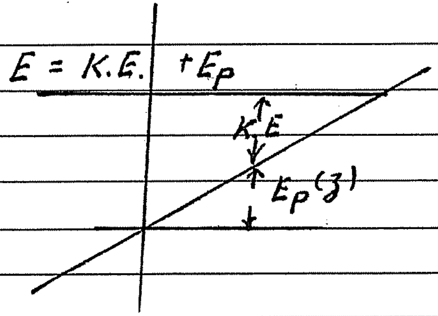
$$E_p(p) = -\int_0^x F_x\, dx' - \int_0^y F_y\, dy' - \int_0^z F_z\, dz'$$

$$= -\int_0^z F_z\, dz' = -\int_0^z (-mg)\, dz'$$

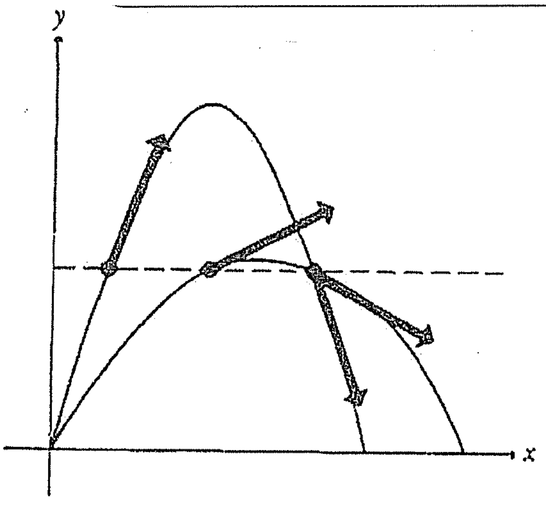
$\Rightarrow E_p(z) = mgz$

Total mechanical energy is

$$E = K.E. + E_p = \frac{1}{2}mv^2 + mgz$$



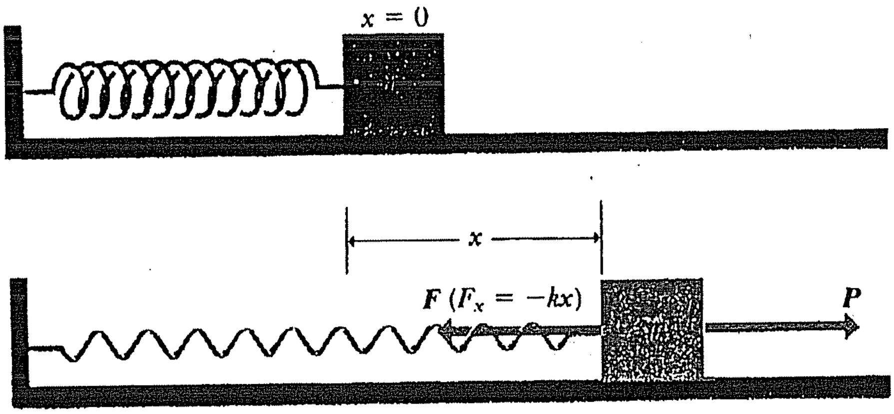
Ballistic trajectories –
only force is weight, neglect air resistance.
Two trajectories, same initial speed (i.e. same total energy) different launch angles.
At points with the same elevation, PE is the same, ∴ KE is the same and the speed is the same.



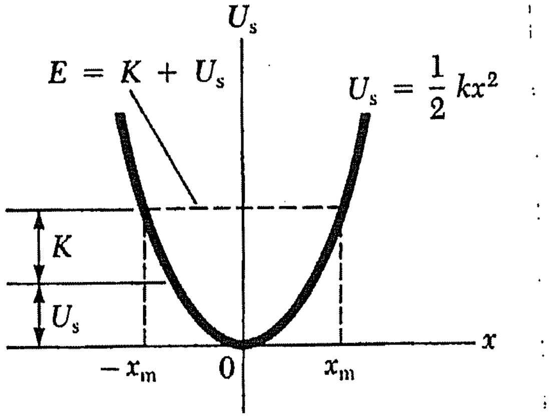
For the same initial speed, the speed is the same at all points at the same elevation.

PE : Spring

16-7



For reference point we choose point P_0 at $x=0$, spring is in equilibrium.



$$U(P_0) \equiv 0.$$

$$\underset{\underset{E_p(x)}{\downarrow}}{U(x)} = -\int_0^x F_x(x')\,dx'$$

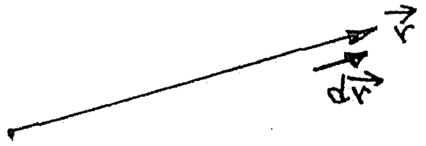
$$= -\int_0^x (-kx')\,dx'$$

$$U(x) = \frac{1}{2}kx^2$$

The total mechanical energy for the mass-spring system is :

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

PE : Gravitational (General)



$$F_g = -\frac{GmM_E}{r^2}\hat{r}$$

$$U(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r} + U(P_0)$$

$$\overset{\shortparallel}{E_p(P)}$$

$$U(r) = \int_{\infty}^{r} \frac{GmM_E}{r'^2}\,dr' + U(P_0)$$

$$U(r) = -\frac{GmM_E}{r'}\Bigg|_{\infty}^{r} + U(P_0)$$

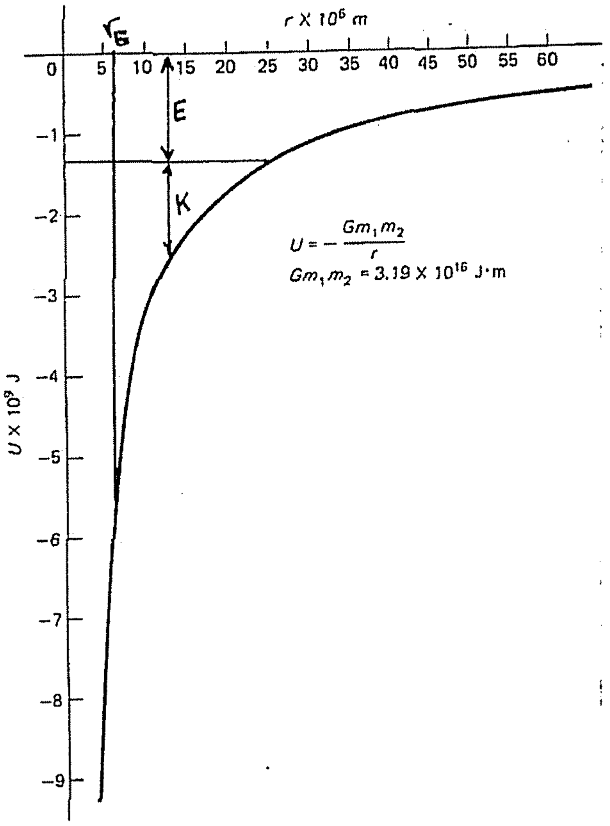
$$U(P_0) = U(\infty) = 0$$

$$\boxed{U(r) = -\frac{GmM_E}{r}}$$

Total Mechanical Energy

$$E = K + U = \frac{1}{2}m\vec{v}^2 - \frac{GmM_E}{r}$$

The mutual gravitational potential energy U = - Gm1m2/r between an 80-kg object and the earth is shown versus their separation r.



Potential Energy → Force

$$E_p(\vec{r}+d\vec{r}) - E_p(\vec{r}) = -\vec{F}\cdot d\vec{r}$$

$$= -F_x\,dx - F_y\,dy - F_z\,dz$$

Take displacement along dx ; i.e. $dy=0$, $dz=0$

$$E_p(x+\Delta x, y, z) - E_p(x,y,z) = -F_x$$

$$\Rightarrow \quad F_x(x,y,z) \underset{\substack{\text{definition of}\\ \text{partial derivative}}}{=} -\frac{\partial}{\partial x}E_p(x,y,z)$$

Same argument

$$\Rightarrow \quad F_y = -\frac{\partial E_p}{\partial y}$$

$$F_z = -\frac{\partial E_p}{\partial z}$$

Combine these results

$$\Rightarrow \quad \vec{F} = -\left(\frac{\partial E_p}{\partial x}\hat{i} + \frac{\partial E_p}{\partial y}\hat{j} + \frac{\partial E_p}{\partial z}\hat{k}\right)$$

$$= -\underset{\substack{\downarrow\\ \text{gradient of } E_p}}{\vec{\nabla}} E_p$$

Example

(1) $E_p(x,y,z) = \frac{1}{2}kx^2$

$F_x = -kx$

(2) $E_p(x) = \dfrac{-ax}{b^2+x^2}$

$$F_x = -\frac{dE_p}{dx} = \frac{a(b^2-x^2)}{(b^2+x^2)^2}$$

(3) $E_p(x,y) = Ax^2y^2$

$$F_x = -\frac{\partial E_p}{\partial x} = -2Axy^2$$

$$F_y = -\frac{\partial E_p}{\partial y} = -2Ax^2y^2$$

(4) $E_p = -\frac{\alpha}{r} = -\frac{\alpha}{\sqrt{x^2+y^2+z^2}}$

$$F_x = +\alpha\left(-\frac{1}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{\alpha x}{(x^2+y^2+z^2)^{3/2}}$$

Same method gives

$$F_y = -\frac{\alpha y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_z = -\frac{\alpha z}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow \vec{F} = -\frac{\alpha}{(x^2+y^2+z^2)^{3/2}}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= -\frac{\alpha\vec{r}}{r^3}$$

Forces $\longleftrightarrow$ Potential Energy

Force	$\vec{F}(x)$	$\underset{E_p(x)}{\underset{\shortparallel}{\mathcal{U}(x)}}$	x_0	$F(x_0)$
gravity (near)	$-mg\,\hat{j}$	mgy	$y=0$	$-mg\,\hat{j}$
spring	$-kx\,\hat{i}$	$\frac{1}{2}kx^2$	$x=0$	0
gravity (far)	$-\frac{GmM_E}{r^2}\,\hat{r}$	$-\frac{GmM_E}{r}$	$r=\infty$	0

often convenient to choose reference point where $\vec{F}(x_0)=0$ and $\mathcal{U}(x_0)\equiv 0$. ✓

$$W_{if}(\text{Cons.}) + W_{if}(\text{N-Cons}) = \underset{(K.E.)_f-(K.E.)_i}{\underset{\shortparallel}{K_f - K_i}} \qquad [\text{WE Theorem}]$$

$$W_{if}(\text{NCons}) = \underset{(K.E.)_f-(K.E.)_i}{\underset{\shortparallel}{K_f - K_i}} + \left(-W_{if}(\cancel{N}\text{-Cons})\right)$$

$$= K_f - K_i + \mathcal{U}_f - \mathcal{U}_i$$

$$W_{if}(\text{N.Cons}) = E_f - E_i \qquad [\text{Modified W-E}]$$

$$E = \underset{(K.E)+E_p}{\underset{\downarrow}{K+\mathcal{U}}} \qquad \text{Total M.E.}$$

$$E_f = K_f + \mathcal{U}_f = E_i = K_i + \mathcal{U}_i \quad \underline{\underline{\text{if}}} \quad [W_{if}(\text{N.Cons}) \equiv 0]$$

superposition

16-9

Several conservative forces acting on an object : $\vec{F}_A$, $\vec{F}_B$, $\vec{F}_C$

$$W_{Total} = \int \vec{F}_A \cdot d\vec{r} + \int \vec{F}_B \cdot d\vec{r} + \int \vec{F}_c \cdot d\vec{r}$$

$$W_{Total} = \int \vec{F}_R \cdot d\vec{r}$$

$\vec{F}_R \equiv$ Resoltant Force.

$$E_p = E_{P,A} + E_{P,B} + E_{P,C}$$
$$\mathcal{U} = \mathcal{U}_A + \mathcal{U}_B + \mathcal{U}_C$$

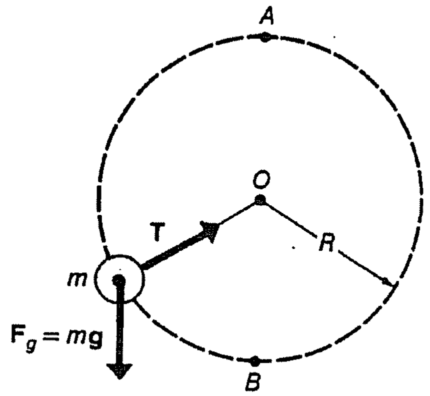
Net Potential Energy → sum of all the individual potential energies for each force.

Total mechanical energy is conserved.

$$\boxed{K_i + \Sigma \mathcal{U}_i = K_f + \Sigma \mathcal{U}_f = E}$$

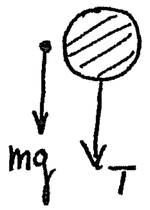
Example

A rock of mass m is tied to the end of a string and whirled in a circular path in the vertical plane. What is the minimum speed v_0 that the rock has at the bottom (B) if the rock is to pass the top (A) with the string remaining taut?



- Energy considerations alone not sufficient.
- Need to apply Newton's 2ND Law.

At the top the minimum speed occurs as tension vanishes, $T \rightarrow 0$.
Net downward force is



$$F_g = mg = m\frac{v_T^2}{R}$$ [centripetal acceleration in circular path]

$$v_T^2 = gR$$

$$W(A \rightarrow B) = mgh = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_T^2$$

$$v_0^2 = v_T^2 + 4gR = gR + 4gR = 5gR$$

$$v_0 = \sqrt{5gR}$$

Example

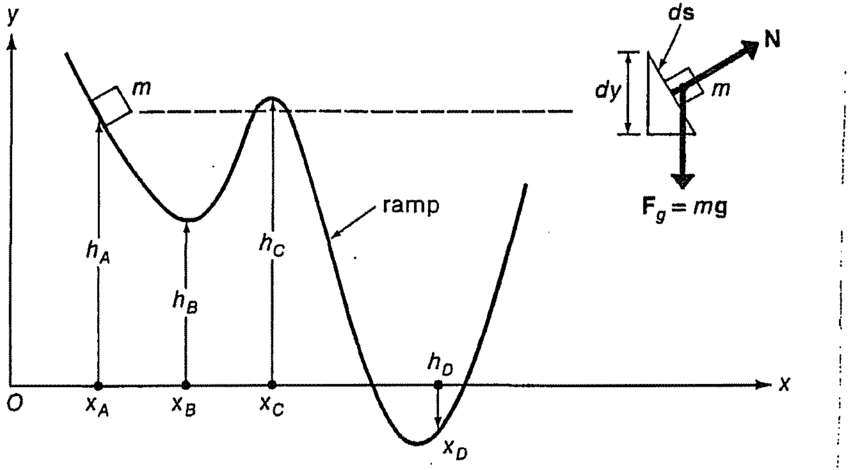
$h_A = 7m$

$h_B = 4m$

$h_C = 7.2m$

$h_D = -1m$

$v_A = 3m/s$ downward and tangent to ramp.



What is speed of particle at $x = x_B, x_C, x_D$?

$\vec{N} \cdot \vec{ds} \equiv 0$, so work is done only by gravitational force.

$$W(A \rightarrow B) = \int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (-mg) y$$

$$W = mg(h_A - h_B)$$

Also $W = K_2 - K_1 = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = v_A^2 + 2g(h_A - h_B)$$

$$= 3^2 + 2 \times 9.8 (7-4)$$

$$= 67.8 \, (m/s)^2$$

$$v_B = 8.23 \, m/s.$$

How far up the ramp will particle go after passing $x = x_D$?

h_m will be reached at x_m where $v_m = 0$.

$0 = v_A^2 + 2g(h_A - h_m) \Rightarrow h_m = h_A + v_A^2/2g \quad h_m = 7.46m$

第七章 功能

第一節 功和動能

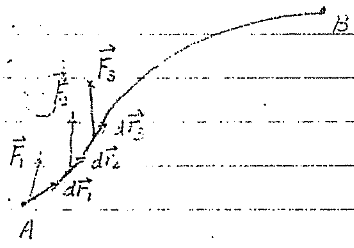
簡介 引用功和動能的觀點給出了一種獲得解決動力學的另一方法在此方法中所遭遇的(1)不是向量而是純量（在牛頓定律中之公式則是向量公式）除了提供一種解題方法之外更重要的是引進了能量守恆定律的觀念（當所有的能量均被考慮進去時，能量守恆定律的正確性範圍是極廣的，也就是說，即是當牛頓定律並不成立時，它仍能成立，這節中我們也將討論功率的觀念及功和動能間的關係

基本觀念

功的定義　當一粒子作了一無窮小的位移 $d\vec{r}$ (1) 時其所受的力為 $\vec{F}$ 則在這段位移中，此力對它所做的功(2)

$$dW = \vec{F}\cdot d\vec{r} \qquad (1)$$

當一粒子由 A 點行經一途徑 AB 抵達 B 點則外力對它所做的功仍是



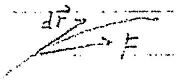
$$W_{A\to B} = \vec{F}_1\cdot d\vec{r}_1 + \vec{F}_2\cdot d\vec{r}_2 + \vec{F}_3\cdot d\vec{r}_3 + \cdots \qquad (2)$$

取極限則得(3)

$$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{r}, \qquad (3)$$

$$dW = \vec{F}\cdot d\vec{r}$$

令 $ds = |d\vec{r}|$，則



$$dW = |\vec{F}|\ ds\ (\cos\theta)$$

$$= F_t\ ds \qquad (4)$$

F_t 是力在沿切線方向之分量

$d\vec{r}$ 是沿軌道切線方向之無窮小向量，將它寫成直角坐標形式

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \tag{5}$$

通常 $\vec{F}$ 是坐標 x, y, z 的函數，可寫成

$$\vec{F} = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k} \tag{6}$$

則 $\vec{F}\cdot d\vec{r} = F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,z)\,dz$ (7)

因此 $W_{A\to B} = \int F_x(x,y,z)\,dx + \int F_y(x,y,z)\,dy + \int F_z(x,y,z)\,dz$ (8)

瞬時功率之定義 $P(t) = \frac{dW}{dt} = \vec{F}\cdot\vec{v}^{(4)}$ (9)

平均功率之定義 $\bar{P}\big|_{\Delta t} = \frac{\text{在時間 } \Delta t \text{ 中,力對粒子所作之功}^{(5)}}{\Delta t}$ (10)

以下，我們將討論功與動能間之關係

$$W_{A\to B} = \int_A^B F_t\,dS$$

由第3章第 3 節中得知對沿任意軌跡一粒子其所受之切線加速度為

$$a_t = \frac{dv}{dt}^{(6)}, \quad v \equiv |\vec{v}| \tag{11}$$

由牛頓定律得知該粒子所受沿切線方向的力為

$$F_t = m\frac{dv}{dt}$$

故$^{(7)}$ $W_{A\to B} = \int_A^B m\frac{dv}{dt}\,ds$

$$= \int_A^B mv\,dv = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2{}^{(8)} \tag{12}$$

動能的定義，一質量為 m 之粒子以速度 $\vec{v}$ 進行，則其動能$^{(9)}$

$$K \equiv \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} \tag{13}$$

(12) 式仍是功與動能定理. 它謂外力對該粒子所作之功等於該粒子動能的增加. 由此式得知，功與能間有着密切的關係.

當一粒子受到幾個力 $\vec{F}_1, \vec{F}_2, \cdots$ 其合力為 $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdot\vec{F}_N$ 則

$$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{r} = W_{1,A\to B} + W_{2,A\to B} + \cdots W_{N,A\to B} \tag{14}$$

此處 $W_{i,A\to B} = \int_A^B \vec{F}_i \cdot d\vec{r}$ $(i=1,2,\ldots N)$ (5)

討論

(1) $d\vec{r}$ 是沿切線方向之一無窮小向量. 在此段位移中, 粒子所受的力可視為一定向量

(2) 注意此為外力對該質點所作的. 可正, 可負也可為零, 功是一純量

(3) 通常此一線積分與由 A 至 B 所經的路程有關

(4) 一般工程上常用的公式

(5) 注意平均功率與瞬時功率的不同.

(6) 注意此一公式的普遍性

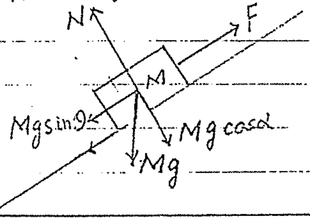
(7) $W_{A\to B}$ 與所行之途徑有關 而 v_B 也與所行之途徑有關

(8) 此式之正確性極為普遍與所受的力是否保守力無關

(9) 以此種方法引入動能之方向較自然. 同時注意一粒子之動能恆為正值.

用.

當作用於一粒子上之力及運動途徑為已知時, 則可求該力對該粒子. 由此可以利用功和動能定理求得該粒子動能的變化. 當必須注意當利用功和動能定理時, 功是所有外力對該粒子所作功的和. 我們將舉例來說明利用這些觀念。



假設有一平行於斜面之力 F 沿着斜面將一質量為 M 之木塊拉上 S 之距離 若其初速為 v_i 求末速
(木塊與斜面間之動摩擦係數為 μ)

$$\int \vec{F}_R \cdot d\vec{r} = \frac{Mv_f^2}{2} - \frac{Mv_i^2}{2}$$

此處 $\vec{F}_R$ 是所有作用於木塊力之和. $d\vec{r}$ 是沿着斜面的方向, 因此 N 及 $Mg\cos\alpha$ 均不作功因為它們與位移之方向相垂直.

$(F - Mg\sin\alpha - \mu Mg\cos\alpha)S = \frac{1}{2}Mv_f^2 - \frac{1}{2}Mv_i^2$

由上式，v_f 馬上即可求出。此題也可以用牛頓定律求出。取沿斜面向上的方向為 x 軸。木塊沿 x 方向的加速度為

$a = (F - Mg\sin\alpha - \mu Mg\cos\alpha)/M = $ 常數

因為木塊之運動為等加速度運動

$v_f^2 - v_i^2 = 2aS$

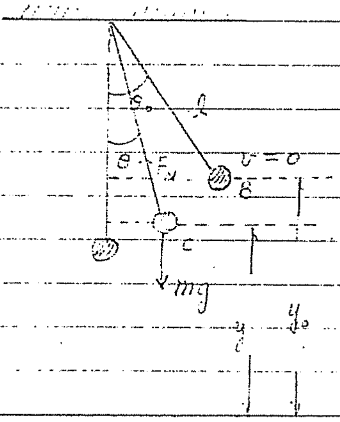
將 a 代入上式，可求得 v_f。其結果與以上利用功與動能定理所得之結果相同。

在計算 $W = \int_A^B \vec{F}\cdot d\vec{r}$ 若 $\vec{F}$ 之大小及方向在粒子由 A 運行至 B 途中保持不變 則 $W = \vec{F}\cdot(\vec{r}_B - \vec{r}_A)$

證: $\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B F_x\,dx + \int_A^B F_y\,dy + \int_A^B F_z\,dz$

$= F_x(x_B - x_A) + F_y(y_B - y_A) + F_z(z_B - z_A)$

$= \vec{F}\cdot(\vec{r}_B - \vec{r}_A)$



一單擺之擺長為 l 拉至與垂直方向之夾角為 θ_0 後釋手，試求當擺至 θ 時之速度。由 B, C 之過程中 繩子對其拉力均與位移之方向相垂直 因此不對質量 m 作功 重力之方向及大小均不改變 其對質點 m 所作的功是 $W = -mg(y - y_0) = mg(y_0 - y) = mgl(\cos\theta - \cos\theta_0)$

利用功與動能定理 $\frac{1}{2}mv^2 = mgl(\cos\theta - \cos\theta_0)$

$\Rightarrow v = \sqrt{2gl(\cos\theta - \cos\theta_0)}$ 與質量 m 無關

題

) 有一重量為 20 lbf 之物体置於一平面上 物体與平面間之摩擦力為 15 lbf.

若你用一水平方向之力使此物体以 [illegible] ft/sec 之速度走了 30 ft

(a) 你需加多大的力？ [K]

(b) 你需作之功為何？ [T]

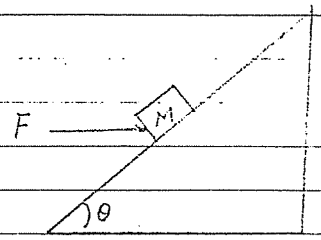
(c) 所有外力之和對此物体所作之功為何？ [D]

(d) 利用功—能定律, 此物体之動能變化為何？ [I]

(e) 此物体之動能為何？ [A]

(f) 假若你停止推它, 此一物体能走多遠？ [Q]

2)



若以一固定的水平力將一質量為 M 之物体推上一斜面。此一斜面与地面之交角為 θ.

$$W = \int_{\text{starting point}}^{\text{end point}} \vec{F} \cdot d\vec{r}$$

當此物体由某一起點至某一終點求外力對此物体所作的功時

(a) $d\vec{r}$ 為何？ [G]

(b) $\vec{F} \cdot d\vec{r}$ 為何？ [S],

(c) 利用 $\vec{F}$ 的一特性來簡化這個積分後求此積分

除了外力外, 此物体尚受地球之重力, 求重力對此物体所作之功時

(d) $d\vec{r}$ 為何？[B] $\vec{F} \cdot d\vec{r}$ 為何？ [H]

表面施於此物体一接觸力。求接觸力對此物体所作之功時

中

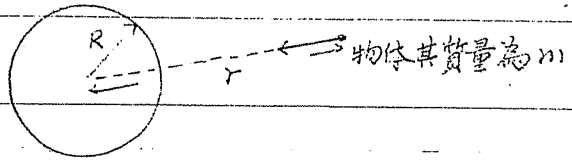
分類：
編號： 6
總號：

(a") $d\vec{r}$ 為何？ [C] , (b") $\vec{F} \cdot d\vec{r}$ 為何？ [C]

(d) 物体由斜面之中点以 V_0 之速度向上運行，利用功與能的關係，計算當此物体沿斜面走了 D 之距離後（到A点時），該物体之速度。

(e) 物体開始時之情況与在(d)項時之情況相同，只是它以 V_0 之速度向下，利用能与功之關係，求當此物体到達 A点時之速度。 [E]

(f) 解釋 (d) 与 (e) 項之結果，兩者之整個運動有何不同？ [R]



一物体 m 在離地心為之距離為 r 時，它所受地球之吸力為 $\frac{Km}{r^2}$，此處 $K = 3.99 \cdot 10^{14}\ m^3/sec^2$

(a) 我們已知在地球表面上，此物体所受地球之吸力為何？ [M]

(b) 由以上的結果，計算地球之半徑 R [J]

(c) 將此物体沿軸方向由地球表面移至離地心 r (r > R) 處，需作多少功？ [L]

(d) 將此物体沿軸方向由地球表面移至無窮遠處，需作多少功？ [L]

一個物体受一外力運動，此外力給此物体之功率為一常數，此一外力可否是一固定力？請略予說明 [U]

1馬力等於 550 ft-lbf/sec 也即是 746 瓦特。假如若想要將一重為 3000lbf 之車由靜止加速至每小時 60 英哩，一了 200 馬力之引擎需連需操作多久？ [P]

這個車子的運動可否為等加速度運動？ [W]

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1280 ft-lbf [B] 与答案 G 相同，[C] 0 ， [D] 0 ，[E] 与答案

同 [F] 4.06×10^9 焦耳 [G] 一個極小之位移向量，其大小是一途徑之

方向是沿著斜面向上的方法。 [H] $-Mg\sin\theta\, ds$ (I) 不變

.38 $\times 10^6$ m [K] 15 lbf [L] $Km\left[\frac{1}{R} - \frac{1}{r}\right]$ [M] 物体之重量 mg

$\left[\frac{2D}{M}(F\cos\theta - Mg\sin\theta) + V_0^2\right]^{1/2}$ [O] 与答案[G]相同 [P] 3.52秒

85.3 呎 [R] 在第二種情形下，此物体先向下運動，停止後向上運動

通過出發點時以 V_0 之速度向上運動。以後之情形与第一種情形完

一樣 [S] $[F\cos\theta]\, ds$ ，此處 ds 是沿斜面之途徑長度 [T] 450 lbf

] 不可能 $P = \vec{F}\cdot\vec{v}$ 若功率為定值時 當 $\vec{v}$ 增加時 $\vec{F}$ 必須相對的

因此不可能是等加速度運動。 [V] $\int \vec{F}\cdot d\vec{r} = \int F\cos\theta\, ds = F\cos\theta \int ds$

$F\cos\theta$（起，訖點之距離）。 [W] 不可能。見答案 [U]

第二節 位能 保守力 能量守恆定律及直線碰撞

一粒子由 A 點行至 B 點，其合力對之所作的功是一線積分 $\int_A^B \vec{F}_R\cdot d\vec{r}$ 是所有作用於此粒子之力的和。通常此一這個線積分不單與其端點（兩點間之位置）有關，同時與其所行之途徑也有關，唯有對於某類稱為保守力的力，其這個線積分（也即是外力對該粒子所作之功）與其所行之途徑無關而只與其端點有關。對這些力，此一線積分可以位能差來取求，在地面上物體所受之重力即是一保守力

基本觀念

在上節所討論之功－能關係(1)

$$\int_A^B \vec{F}_R\cdot d\vec{r} = \frac{1}{2} mv_B^2 - \frac{1}{2} mv_A^2 \qquad (1)$$

中 $\vec{F}_R$ 是所有加之於質量為 m 粒子之力的和，此式左邊之積分通常相當複雜，它不但與其起點 A 及終點 B 有關，同時也與其行經的路徑有關，但對有某種力則此積分與所經之途徑無關而只與其端點的位置有關(2),(3),(4)

$$\int_A^B \vec{F}_R\cdot d\vec{r} = E_p(A) - E_p(B) \qquad (2)$$

此處 $E_p(A)$ 是一純量，為端點 A 之位置的函數

$E_p(B)$ 是一純量，為端點 B 之位置的函數

$E_p(A)$，$E_p(B)$ 分別稱為粒子在 A 點及 B 點之位能。

而滿足(2)式之力則稱為保守力。將(1)，(2)兩式合併則可得

$$\frac{1}{2} mv_B^2 - \frac{1}{2} mv_A^2 = E_p(A) - E_p(B) \qquad (3)$$

也即是

$$E_p(A) + \frac{1}{2} mv_A^2 = E_p(B) + \frac{1}{2} mv_B^2 \qquad (4)$$

在上式中，左手邊只與 A 點的性質有關，它代表在 A 點之總能量（位能加

動能)。很顯然的，其右手邊則代表在 B 點之總能量。上式是一守恆公式，謂在 A 點之能量與在 B 點(或在任何點)之能量相同。因此 (4) 式被稱為能量守恆公式。

在直線踫撞中，我們定義一 Q 值

$$Q = (k.E.)_i - (k.E)_f \qquad (5)$$

$(k.E.)_i$, $(k.E)_f$ 分別為踫撞前與踫撞後該系統之動能。

若 $Q=0$，此一踫撞稱為彈性踫撞。$Q>0$ 則稱為非彈性踫撞，$Q<0$ 則稱為爆炸性踫撞。

討論

(1) 此式取之上節，在此強調者是此處之 $\vec{F}$ 是所有作用於此粒子之力的和。

(2) 由 (2) 式可得 $\oint \vec{F}\cdot d\vec{r} = 0$。$\oint$ 表示由 A 點走任一途徑後回到 A 點的積分。

(3) 此處定義是位能差。此為重要的物理量。位能的原點取於何處並無特殊物理意義。

(4) 並非所有力均滿足此條件。滿足此條件之力稱為保守力

題

(1) 一個 [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

(2) 一 [illegible] $\vec{F}$ [illegible]

分類：
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總號：

力之影响，此物体由地面升至離地面高度為 H 之處，之開始時之速度為 0．

H 時之速度為 V（向上）

a) 將在上昇期間作用於此物体所有力之自由力圖繪出。[R]

b) 寫出這個系統中功與能（尚未引入位能）之關係，並將各力所做之功分別寫出。[C]

c) 將能計算之積分算出．[I]

d) 重新寫出功與能之關係，但現在將保守力所做之功寫成位能 [M]

e) 比較 (b)，(d) 之結果 [A]

f) 將此物体提升至高度 H 中，$\vec{F}$ 所作之功是多少？[O]

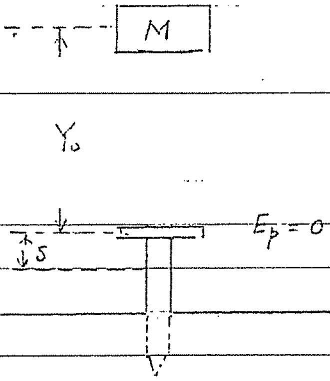
個釘子有一部分已釘進了木板。

現在利用一質量為 M 下落將其再

釘深一點。此物体之質量為 M 由

高度為 Y_0 處墮下而將釘子多釘深

S 之距離



a) 令釘子之頂端之位能為 0，當 M 放開時其位能為何？[D]

b) 當時其動能為何？[N]

c) 當質量將釘子打下 S 距離後其位能為何？[J]

d) 此時其動能為何？[B]

e) 將功與能關係（將保守力所作之功寫成位能差）寫出 [F]

f) 質量 M 對釘子所作之功是多少？[Q]

g) 釘子對質量所作之功是多少？[L]

h) 若在釘子在使質量 M 減慢的過程中對 M 所施之力為固定時，F_N 之大小為何？[E]

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(將此題與第五章第二節第二題比較兩種不同的解法)。

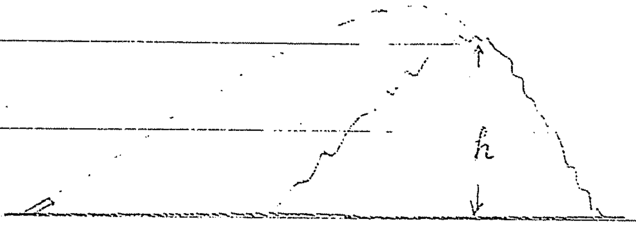
7) 一由地平面上之砲所發射之砲彈離開砲口時之速率是 200 ft/sec

此砲彈擊中位於 100 ft 山頂之

目標

(a) 利用能量守恆，計算砲彈擊中

目標時之速率。 [H]



(b) 若此砲彈之重量為 16 lbf，而

其擊中目標時之速率為 150 ft/sec

則砲彈在飛行所損失之能量（大部分是由於空氣之阻力 [S]

答案

[A] 它們相同，[B] 0，[C] $\int_{\text{Ground}}^{\text{pt. }H} \vec{F}\cdot d\vec{r} + \int_{\text{Ground}}^{\text{Pt. }H} (-Mg)\,ds = \frac{1}{2}MV^2$

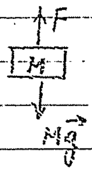
[D] MgY_0，[E] $Mg(1+\frac{Y_0}{S})$ [F] $\int_{\text{Start}}^{\text{End}} \vec{F}_N\cdot d\vec{r} + MgY_0 + MgS = \frac{MV_2^2}{2} - \frac{MV_1^2}{2}$

[G] $-0.9\,E_k$ [H] 183.3 ft/sec [I] 第二個積分是 $-MgH$

[J] $-MgS$ [K] $0.1\,V_1$ [L] $-Mg(Y_0+S)$ [M] $\int_{\text{Ground}}^{\text{Pt. }H} \vec{F}\cdot d\vec{r}$

$+E_p$(地面) $-E_p$(在 H 點) $= \frac{1}{2}MV^2 -$ [E_k(在地面時) $= 0$]

[N] 0 [O] $\frac{1}{2}MV^2 + MgH$ [P] 與 V_1 相同 [Q] $+Mg(Y_0+S)$

[R] 

[S] 2775 ft-lbf

第三節 位能之運動, 位能圖, 轉向點

簡介: 若一粒子所受之力為保守力 同時在位能在空間的函數及總能量已知, 則置於此空間中粒子之運動情況(如所受之力, 它的加速度及速度等等) 均可決定. 在此節中我們將討論如何由位能來決定作用力, 同時我們也將討論如何利用位能圖來研討運動的性質

基本觀念

若一粒子所受之力為保守力 則

$$\int_A^B \vec{F}\cdot d\vec{r} = E_p(A) - E_p(B) \qquad (1)$$

在一度空間中

$$E_p(x_B) - E_p(x_A) = -\int_A^B F(x)\,dx \qquad (2)$$

因此, 很明顯的

$$F(x) = -\frac{dE_p}{dx} \qquad (3)$$

推廣至三度空間, 則 $E_p(x,y,z)$ 是三度空間座標之函數

在 (1) 式中取 A 点之坐標為 x, y, z

B 点之坐標為 $x+\Delta x, y, z$

若 Δx 很小, 則第 (1) 式之左邊為 $\vec{F}\cdot\Delta x\,\hat{i} = (F_x\hat{i}+F_y\hat{j}+F_z\hat{k})\Delta x\cdot\hat{i}$

$$= F_x\,\Delta x \qquad (4)$$

第一式之右邊為 $E_p(x,y,z) - E_p(x+\Delta x,y,z)$ (5)

因此,

$$F_x = -\left[\frac{E_p(x+\Delta x,y,z) - E_p(x,y,z)}{\Delta x}\right] \qquad (6)$$

取 $\Delta x \to 0$ 則得

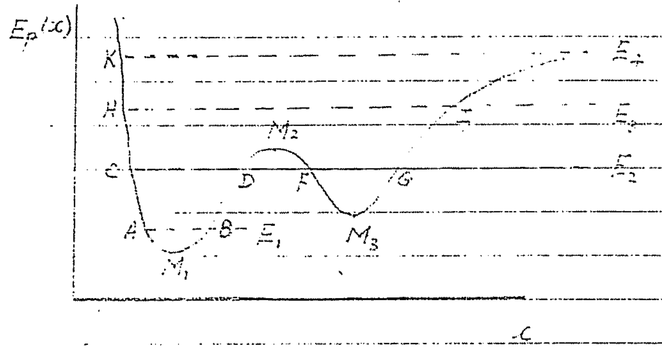
$$F_x = -\frac{\partial E_p(x,y,z)}{\partial x} \qquad \partial \to 偏微分 \qquad (7a)$$

同理，$F_y = -\frac{\partial E_p(x,y,z)}{\partial y}$ (7b)

$F_z = -\frac{\partial E_p(x,y,z)}{\partial z}$ (7c)

因此，$\vec{F} = -\frac{\partial E_p}{\partial x}\hat{i} - \frac{\partial E_p}{\partial y}\hat{j} - \frac{\partial E_p}{\partial z}\hat{k}$ (8)

以下，我們將利用以上的結果及位能圖來討論運動的性質



在 0 至 M_1 $-\frac{dE_p(x)}{dx} > 0$ 力是向右

M_1 至 M_2 $-\frac{dE_p(x)}{dx} < 0$ 力是向左

M_2 至 M_3 $-\frac{dE_p(x)}{dx} > 0$ 力是向右

M_3 至 ∞ $-\frac{dE_p(x)}{dx} < 0$ 力是向左

M_1, M_2, M_3 是位能之最大或最小值之點，因此，

$$\frac{dE_p}{dx} = 0$$

因此，在這些點，粒子所受之力為 0。因此這些點是平衡點(2)

在 M_1, M_3：E_p 是一個局部極小值，在這兩點若作一小位移則所受的力使其趨向走回原來之位置，因此稱為穩定平衡。

在 M_2：E_p 是一個局部極大值，在這點若作一小位移則所受的力使其走離原來之位置，因此稱為不穩定平衡

$E - E_p = E_k$ 必須 >0，此運動情況方為可能

當 $E = E_1$ 時，E_k 在 $A \rightarrow B$ 區域中為正值，因此是准許區，在 A, B 點，$v = 0$，而力的方向也是使其走回准許區，因此被稱為轉向點(3)，其他是禁區

當 $E = E_2$ 時在 C→D 及 F→G 間是准許區 其他是禁區

當 $E = E_3$ 時在 H→I 間是准許區 其他是禁區

當 $E = E_4$ 時由 [illegible] 範圍為准許區 這是一個非有限的運動 也即是說

此時質點之運動並不局限於一有限的區域內.

討論

(1) $U(x,y,z)$ 是一純量 它是一個 x,y,z 的函數 即是對應於每一點有一純量 這叫做純量場。由此純量場可得 $\nabla U = \frac{\partial U(x,y,z)}{\partial x}\hat{i} + \frac{\partial U(x,y,z)}{\partial y}\hat{j} + \frac{\partial U(x,y,z)}{\partial z}\hat{k}$ 叫做 U 之梯度。對應於每一點有一向量，因此是有一向量場。

(2) 在平衡點質點所受之力為 0 也即是其加速度為 0 但通常，在此處質點之速度不為 0.

(3) 在轉向點，質點之速度為 0。但通常在此處質點之加速度不為 0.

習題 (在此小節中，我們暫時不考慮單位).

(1) 假設一質點之質量為 $M = 2$ 其位能函數為 $E_p(x) = 3x^2 - x^3$ (將 E_p 與 x 之關係繪出，以便答覆以下之問題). 已知在某時，此質點位於 $x = +1$ 處其能量為 3 向右運動.

(a) 在此處之動能為何？ [K]

(b) 它的速度為何？ [N]

(c) 此時，該質點所受之力為何？ [E]

(d) 此時，該質點所受之加速度 (大小及方向) 為何？ [I]

(e) 在此點，質點之速率是在增加或是減少？ [P] 試說明之 [S]

分類：
編號： 1
總號：

充資料

用能量守恒定律解一度空間的問題

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + E_p(x) = \frac{1}{2}mv^2 + E_p(x) \quad (1)$$

若在 $t=0$ 時，$x = x_0$, $V = V_0$ 則可利用上式求出 E $t=0$ 時之能量，由於能量是守恒的，因此在任何時間其值不變，所以上式中 E 是一已知常數

$$\frac{dx}{dt} = \left[\frac{2(E - E_p(x))}{m}\right]^{\frac{1}{2}}$$

$$\int_{x_0}^{x} \frac{dx}{\left[\frac{2}{m}(E - E_p(x))\right]^{\frac{1}{2}}} = \int_0^t dt = t \quad (2)$$

我們可以利用上式找出 x 和 t 之關係

例： 若 $F = -kx$，則 $E_p(0) - E_p(x) = \int_0^x (-kx)\,dx = -\frac{1}{2}kx^2 \quad (3)$

若是我們定 $x=0$ 處之位能為 0，則 $E_p(x) = \frac{1}{2}kx^2 \quad (4)$

將 (4) 代入 (2) 式則得

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = t$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{k}{m}}\sqrt{\frac{2E}{k} - x^2}} = t$$

$$= \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \underbrace{\sqrt{\frac{k}{m}}}_{\omega}\, t \qquad \omega^2 \equiv \frac{k}{m} \quad (5)$$

積分公式 $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c \quad (6)$

證明 令 $x = a\sin u$, $u = \sin^{-1}\frac{x}{a}$

$dx = a\cos u\, du$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos u\, du}{\sqrt{a^2 - a^2\sin^2 u}} = \int du = u = \sin^{-1}\frac{x}{a}$$

利用 (6) 式，(5) 式可寫成

$$\sin^{-1}\frac{x}{\sqrt{\frac{2E}{k}}} - \sin^{-1}\frac{x_0}{\sqrt{\frac{2E}{k}}} = \omega t$$

$$\Rightarrow x = \sqrt{\frac{2E}{k}}\sin\left[\omega t + \sin^{-1}\frac{x_0}{\sqrt{\frac{2E}{k}}}\right] \quad (7)$$

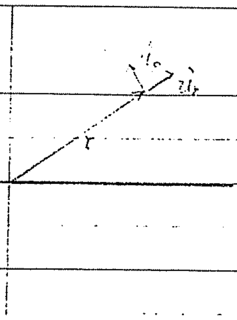
分類：
編號：　2
總號：

利用能量及角動量守恆定律解中心力的問題。中心力時質點的運動是在一平面上，同時角動量守恆，我們將用極角座標

$$\hat{u}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{u}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\vec{r} = r\hat{u}_r$$

$$d\vec{r} = d(r\hat{u}_r)$$
$$= r\,d\hat{u}_r + dr\,\hat{u}_r$$



$$d\hat{u}_r = -\sin\theta\,d\theta\,\hat{i} + \cos\theta\,d\theta\,\hat{j} = d\theta\,\hat{u}_\theta$$

$$\vec{F} = F_r\,\hat{u}_r + F_\theta\,\hat{u}_\theta$$

$$E_p(r,\theta) - E_p(r+dr,\theta) = F_r\,dr$$

$$\Rightarrow \quad F_r = -\frac{\partial E_p}{\partial r} \qquad (8)$$

$$E_p(r,\theta) - E_p(r,\theta+d\theta) = F_\theta\,r\,d\theta$$

$$\Rightarrow \quad F_\theta = -\frac{1}{r}\frac{\partial E_p}{\partial \theta} \qquad (9)$$

此質點對原點之力距之大小（方向是垂直於平面）

$$|\vec{\tau}| = |r\hat{u}_r \times (F_r\hat{u}_r + F_\theta\hat{u}_\theta)|$$

$$= rF_\theta = -\frac{\partial E_p}{\partial \theta} \qquad (10)$$

$$\vec{v} = \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta \qquad (11)$$

$$|\vec{L}| = m|r\hat{u}_r \times (\frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta)|$$

↳ 對原點之角動量

$$= mr^2\frac{d\theta}{dt}$$

中心力時，E_p 只是 r 之函數

此時能量及角動量守恆公式可寫成

$$E = \frac{1}{2}m[(\frac{dr}{dt})^2 + r(\frac{d\theta}{dt})^2] + E_p(r) \qquad (12)$$

$$L = mr^2\frac{d\theta}{dt} \qquad (13)$$

分類：
編號： 3
總號：

此處 E 及 L 均為常數，可由在 t=0 時之位置及速度決定

由 (12), (13) 式可得

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{L^2}{2mr^2} + E_p(r)}_{E_{p,eff}(r)} \qquad (14)$$

$$= \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + E_{p,eff}(r) \qquad (15)$$

$$E_{p,eff}(r) = \frac{L^2}{2mr^2} + E_p(r) \qquad (16)$$

比較 (1) 及 (15) 式，顯然他之間的數學結構相同，因此解法也相同

$$\int_{r_0}^{r} \frac{dr}{\{\frac{2}{m}[E - E_{p,eff}(r)]\}^{\frac{1}{2}}} = t \qquad (17)$$

由(17) 式可得 $r(t)$

由 (13) 式

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \qquad (18)$$

$$\theta - \theta_0 = \int_0^t \frac{L}{m[r(t)]^2} \, dt \qquad (19)$$

由 (14) 式 可得

$$\frac{dr}{dt} = \left(\frac{2}{m} (E - E_{eff}(r)\right)^{\frac{1}{2}} \qquad (20)$$

與第 (18) 式合併可得

$$\frac{dr}{d\theta} = \frac{\left(\frac{2}{m} (E - E_{eff}(r)\right)^{\frac{1}{2}}}{(L/mr^2)} \qquad (21)$$

因此

$$\int_{r_0}^{r} \frac{dr}{(\frac{m}{L}) r^2 \{\frac{2}{m}(E - E_{eff}(r))\}^{\frac{1}{2}}} = \theta - \theta_0 \qquad (22)$$

中心力的運動軌跡於是可由上式解出。

分類：
編號： 4
總號：

(f) 當此質點向右進行時由 E_p 在 $x=1$ 附近的形狀來決定其動能是在增加或是減少 [C] 這與 (e) 項之結果相符嗎？ [I]

(g) 在何處，此質點（向右進行）之速率為 0？（只需在 E_p 與 x 關係之圖上表出即可） [A]

(h) 在此 $v=0$ 處，決定此質點加速度之方向。由此決定在 $v=0$ 以後之瞬間，該質點的速度之方向為何？ [O]

(i) 若 $E=3$，在 E_p 與 x 之關係圖中將禁區標出？ [G]

(j) 若 $E=3$，此質點運動之轉向點位於何處？ [Y]

(k) 若質量為 $M=2$ 在 $t=0$ 時，位於 $x=-2$，速度 $=0$.

(i) 它的動能為何？ [EE]

(ii) 它的位能為何？ [T]

(iii) 它的總能量為何？ [X]

(iv) 它所受之力（大小及方向）為何？ [AA]

(v) 它的加速度之大小為何？ [V]

(vi) 它的加速度之方向為何？ [Z]

(vii) 當該質點抵達 $x=0$ 時，其速度為何？ [DD]

(viii) 當該質點到達 $x=2$ 時，其速度為何？ [U]

(ix) 當該質點到達 $x=2$ 時，其加速度為何？ [BB]

(x) 它在回轉以前，能走多遠？ [W]

(xi) 試解釋以上之結果。 [CC]

一位能函數為 $E_p=-\frac{1}{2}kr^2=-\frac{1}{2}k(x^2+y^2+z^2)$，此處 k 為一常數。

分類：
編號： 5
總號：

(a) 求在 r 處，此質點所受力之大小為何？[B]

(b) 此力之方向為何？[M]

(c) 求 F_x, F_y, F_z [Q]

(d) 利用向量加法，求以上三分量所組成之力。將此結果與 (a), (b) 之結果相比 [F]

(3) 若 $E_p = \frac{k}{r}$，此處 $r = \sqrt{x^2+y^2+z^2}$，$k = 10$ Newton-m^2。

(a) 求在 $x = 0$, $y=1$, $z=2$ (以 m 為單位) 之 F_x, F_y, 及 F_z [H]

(b) 合力之大小為何？[D]

(c) 此合力之方向為何？[R]

答案

[A] $x^3-3x^2+3=0$ 之三個解之一。令三解為 $x_1, x_2,$ 及 x_3 ($x_1<x_2<x_3$, x_1 為負)，x_2 是此題之解 [B] kr [C] 減少 [D] 2 Newtons [E] $F_x=-3$ 向左 [F] $|\vec{F}| = \sqrt{k^2x^2+k^2y^2+k^2z^2} = kr$ [G] $x<x_1$, $x_2<x<x_3$ 為禁區 [H] $F_x = 0$, $F_y = \frac{2}{\sqrt{5}} = 0.89$ Newtons, $F_z = \frac{4}{\sqrt{5}} = 1.79$ Newtons [I] $a_x = -\frac{3}{2}$ 向左 [J] 向左 [K] $E_k = 1$ [L] 是的，動能及速率均減少 [M] 沿 $\vec{r}$ (由原點至我們考慮之點) 之方向 [N] $V_x = +1$ [O] a_x 向左，因此 $V_x = 0$ 以後之瞬間 V_x 向左 [P] 減少 [Q] $F_x = kx$, $F_y = ky$, $F_z = kz$ [R] 在 XY 平面上，與 x 軸之交角為 $\theta = \tan^{-1}2$，由原點向外。[S] $V_x>0$, $a_x<0$ 因此 V_x 減少 [T] 20 [U] 4 [V] 12 [W] 到無窮遠處 [X] 20 [Y] X_1 及 x_2 [Z] 向右 [AA] 24 向右 [BB] 0 [CC] 位能一直減少，總能量為定值，動能一直增加，永

不為零，所以 v 永遠不改變方向 [DD] $\sqrt{20}$ [EE] 0