System of Particles : Energy and Momentum

Up to now, we have studied motion of a single particle

Now want to look at a system of particles
$\downarrow$
what can we learn about the motion

$\Rightarrow$ to study how the conservation laws such as

Energy
Momentum
Angular Momentum

applied to a system of particles

2 particle system

$\vec{r}_1(t)$, $\vec{r}_2(t)$ $\Rightarrow$ $\vec{v}_1(t)$, $\vec{v}_2(t)$
$\downarrow$ $\downarrow$
$\vec{p}_1(t)$ $\vec{p}_2(t)$

External force
$\downarrow$
due to external source such as gravitational force

Internal force

$\vec{F}_{12}$ force act on particle 1 due to particle 2

$\vec{F}_{21}$ force act on particle 2 due to particle 1

$\vec{F}_{12} = -\vec{F}_{21}$ Newton's third law.

$$\frac{d\vec P_1}{dt} = \vec F_{12} + \vec F_{1,ext}$$

$$\frac{d\vec P_2}{dt} = \vec F_{21} + \vec F_{2,ext}$$

$$\frac{d}{dt}\underbrace{(\vec P_1 + \vec P_2)}_{\vec P} = \underbrace{\vec F_{1,ext} + \vec F_{2,ext}}_{\vec F_{ext}}$$

$\vec P$ → total momentum of the system

Equation of motion for the system determines overall translational motion

$$\frac{d\vec P}{dt} = \vec F_{ext}$$

If $\vec F_{ext} = 0$, then $\frac{d\vec P}{dt} = 0$ and $\vec P = \text{constant}$.

Note: in some problems external forces may vanish on one or more axes but not on all. Linear momentum is a constant for those components

Define $\vec r_{c.m} = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$

$$= \frac{m_1 \vec r_1 + m_2 \vec r_2}{m}$$

$$\Rightarrow m \vec r_{c.m} = m_1 \vec r_1 + m_2 \vec r_2$$

$$m \frac{d\vec r_{c.m}}{dt} = \vec P_1 + \vec P_2$$

$$m \vec V_{c.m} = \vec P_1 + \vec P_2 \quad \Rightarrow \quad v_{c.m.} = \frac{d\vec r_{c.m.}}{dt}$$

$$\Downarrow$$

$$m \frac{d\vec V_{c.m}}{dt} = \frac{d}{dt}(\vec P_1 + \vec P_2) \qquad \vec P = m \vec V_{c.m}$$

$$= \vec F_{ext}$$

Total momentum of the system is the total
mass multiplied by the velocity of the c.m., i.e.,
total $\vec{P}$ is that of a single particle of mass
m moving with a velocity $\vec{v}_{c.m}$

Differentiate again to get

$$\vec{a}_{c.m} = \frac{d\vec{V}_{c.m}}{dt} = \frac{1}{m} \sum m_i \frac{d\vec{V}_i}{dt} = \frac{1}{m} \sum \underbrace{m_i \vec{a}_i}_{\vec{F}_i}$$

$$m\,\vec{a}_{c.m.} = \sum \vec{F}_i$$

$\vec{F}_i \equiv$ force on particle i

Forces on any particle have two sources
Forces on any particle have two sources

. External (from outside the system)

. Internal (from within the system)

From Newton's 3rd law the sum over "internal"
forces cancells in pairs and the overall sum
vanishes.

The c.m. moves like an imaginary particle of
mass m under the influence of the resultant
external force on the system.

If $\sum \vec{F}_{ext} = 0$, then

$$\frac{d\vec{P}}{dt} = m\vec{a}_c = 0$$

Total linear momentum of a system is conserved
if there are no external forces acting on it

For an isolated system of particles, both the
total momentum and velocity of the c.m.
are constant in time.

Linear Momentum

Conservation

$\vec{p} = m\vec{v}$ definition

vector along the direction of velocity

$$\vec{F} = \frac{d}{dt}\vec{p} = \frac{d}{dt} m\vec{v}$$

$$= m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

$m\vec{a}$ useful in solving the rocket problem

[In special relativity]

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

One particle problem.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$\vec{F} = 0 \Rightarrow \vec{p} = \text{constant}$

momentum conservation

Two Particles System

Two particles

m_1 m_2

$\vec{r}_1(t)$ $\vec{r}_2(t)$

$\vec{F}_1$ $\vec{F}_2$

$\parallel$ $\parallel$

$F_{ext,2} + F_{21}$ $\vec{F}_{ext,1} + \vec{F}_{12}$

internal force

Problem

Given $\vec{F}_1, \vec{F}_2$ and initial condition

$\downarrow$

Find $\vec{r}_1(t), \vec{r}_2(t)$

Key observation

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{(Newton's third law}$$

$$\frac{d\vec{P}_1}{dt} = \vec{F}_{12} + F_{ext,1} \qquad (A)$$

$$\frac{d\vec{P}_2}{dt} = \vec{F}_{21} + F_{ext,2} \qquad (B)$$

(A), (B) $\Rightarrow$ Newton's second law

(A) + (B)

$$\frac{d}{dt}(\underbrace{\vec{P}_1 + \vec{P}_2}_{\vec{P} \to \text{total momentum}}) = \underbrace{\vec{F}_{ext,1} + \vec{F}_{ext,2}}_{\vec{F}_{ext}} = \vec{F}_{ext} \qquad (C)$$

If $\vec{F}_{ext} = 0$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{P}_1 + \vec{P}_2) = 0 \quad \Rightarrow \quad \vec{P}_1 + \vec{P}_2 = \text{constant}$$

$$\vec{P}_{1,i} + \vec{P}_{2,i} = \vec{P}_{1,f} + \vec{P}_{2,f}$$

$\downarrow$

momentum conservation

Another way to write the equation

$$\frac{d}{dt}(m_1\vec{V}_1 + m_2\vec{V}_2) = m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2}$$

$$= \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

Define $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

$$= \underbrace{(m_1 + m_2)}_{M}\frac{d^2\vec{R}}{dt^2}$$

(C) can be written as

$$\frac{d}{dt}\vec{P} = \vec{F}_{ext} = M\frac{d^2}{dt^2}\vec{R} \qquad (D)$$

(A), (B)

$$\frac{d^2\vec{r}_1}{dt^2} = \frac{1}{m_1}\vec{F}_{12}$$

$-$

$$\frac{d^2\vec{r}_2}{dt^2} = \frac{1}{m_2}F_{21} = -\frac{1}{m_2}\vec{F}_{12}$$

$$\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)F_{12} \qquad (E)$$

define $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\frac{1}{\mu} = \frac{1}{m_1}+\frac{1}{m_2}$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

reduced mass

(D), (E) $\Longleftrightarrow$ (A), (B) + Newton's third law

In many problems

$\vec{F}_{ext} = 0$ (D) $\Rightarrow \frac{d}{dt}\vec{P} = 0$ $\qquad \vec{R} = \vec{R}_0 + \vec{V}_0 t$

$\Downarrow$

$\vec{P}$ = constant

$\Downarrow$

momentum conservation.

$\vec{R}_0$, $\vec{V}_0$ determined by initial condition

(F)

F_{12} has the form $f(r)\hat{r}$

example gravitational force

$$\mu\frac{d^2\vec{r}}{dt} = f(r)\hat{r} \rightarrow \text{central force problem}$$

(G)

(G) + and initial conditions

$$\Rightarrow \quad \vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$$

$$\vec{R}(t) = \frac{m_1 \vec{r}_1(t) + m_2 \vec{r}_2(t)}{m_1 + m_2}$$

With $\vec{r}(t)$, $\vec{R}(t) \Rightarrow \vec{r}_1(t)$, $\vec{r}_2(t)$

↓

the problem is completely solved

In some problem external force may vanish

↓

momentum conservation

↓

allow us to make general statements about the physics without doing detailed calculations

Example. Collisions

Classification Figure 8-7

Before after

① $M\vec{v}' = m_1\vec{v}_1 + m_2\vec{v}_2$

② $m_1\vec{v}_1' + m_2\vec{v}_2' = m_1\vec{v}_1 + m_2\vec{v}_2$

③ $1 + 2 \rightarrow 3 + 4$

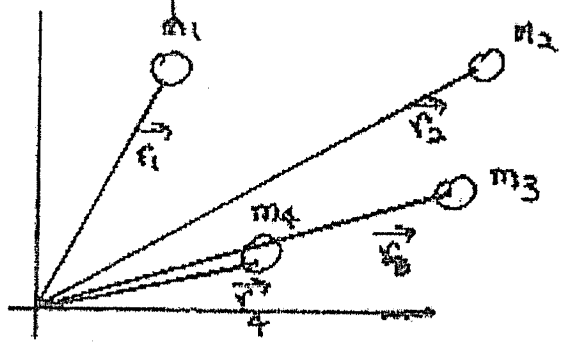
$$m_3\vec{v}_3' + m_4\vec{v}_4' = m_1\vec{v}_1 + m_2\vec{v}_2 \qquad m_4 + m_3 = m_1 + m_2$$

④ $1 + 2 \rightarrow 3 + 4 + 5 + \cdots$

Center-of-Mass

- up to now we have ignored the size of objects
- We will now show that for an object of finite size, the center-of-mass mimics particle motion.

- The position of the center-of-mass is the average position of the mass of the system.



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots m_n\vec{r}_n}{m_1 + m_2 + \cdots m_n}$$

$$\vec{r}_{cm} = (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots m_n\vec{r}_n)/M = \frac{\sum m_i\vec{r}_i}{M}$$

$M \equiv$ Total Mass

In terms of vector components.:

$$x_{cm} = \frac{1}{M}[m_1x_1 + m_2x_2 + \cdots m_nx_n] = \frac{1}{M}\sum m_i x_i$$

$$y_{cm} = \frac{1}{M}[m_1y_1 + m_2y_2 + \cdots m_ny_n] = \frac{1}{M}\sum m_i y_i$$

$$z_{cm} = \frac{1}{M}[m_1z_1 + m_2z_2 + \cdots m_nz_n] = \frac{1}{M}\sum m_i z_i$$

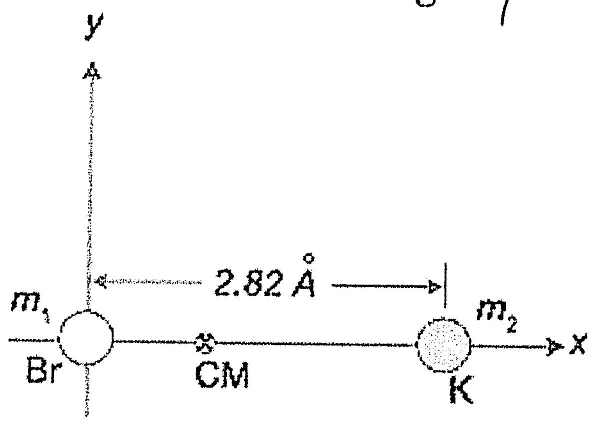
$$\vec{r}_{cm} = x_{cm}\hat{x} + y_{cm}\hat{y} + z_{cm}\hat{z}$$

Example: CM of Two Particles.

[Potassium Bromide KBr]

$y_{cm} = 0$

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$



Atoms of bromine (Br) and potassium (K), regarded as particles.

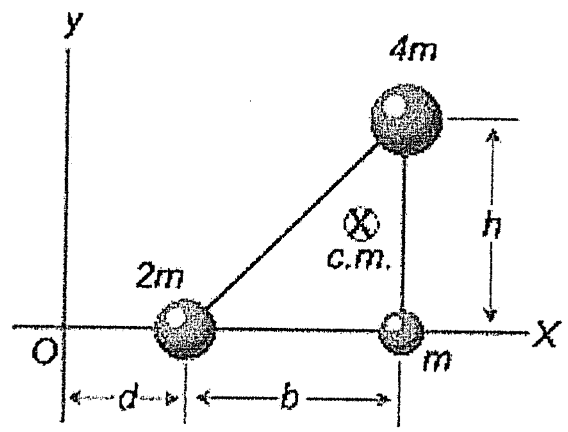
$x_1 = 0 \qquad x_2 = 2.82\,A^\circ$

$m_1 = 79.9\,u \qquad m_2 = 39.1\,u$

$x_{cm} = \dfrac{39.1 \times 2.82\,A^\circ}{79.9 + 39.1} = 0.93\,A^\circ$

[Near heavier atom]

Example: CM of Three Particles



$x_{cm} = \dfrac{\sum m_i x_i}{M} = \dfrac{2md + m(d+b) + 4m(d+b)}{7m}$

$= d + \dfrac{5}{7} b$

$y_{cm} = \dfrac{\sum m_i y_i}{M} = \dfrac{2m(0) + m(0) + 4mh}{7m}$

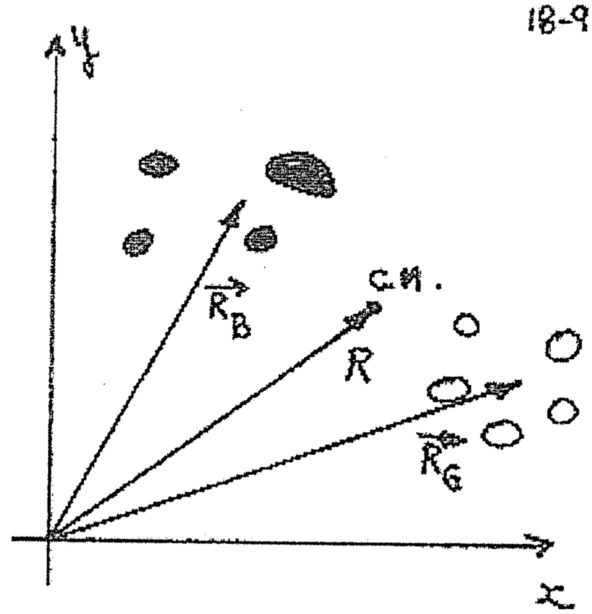
$= \dfrac{4}{7} h$

The position vector of the cm

$r_{cm} = x_{cm}\,\hat{x} + y_{cm}\,\hat{y} = \left(d + \dfrac{5}{7} b\right)\hat{x} + \dfrac{4}{7} h\,\hat{y}$

Groups of Particles

- Divide system of particles into groups.
- Find cm of each group.
- Treat each group as a particle at its cm and find cm of the combined groups.



$$MR = \sum_{i=1}^{l} m_i \vec{r}_i + \sum_{j=l+1}^{n} m_j \vec{r}_j$$

Let $\vec{R}_B = \frac{1}{M_B} \sum_{i=1}^{l} m_i \vec{r}_i$ $\qquad M_B = \sum_{i=1}^{l} m_i$

$$\vec{R}_G = \frac{1}{M_G} \sum_{j=l+1}^{n} m_j \vec{r}_j \qquad M_G = \sum_{j=l+1}^{n} m_j$$

$$M\vec{R} = M_B \vec{R}_B + M_G \vec{R}_G$$

$$M = M_B + M_G$$

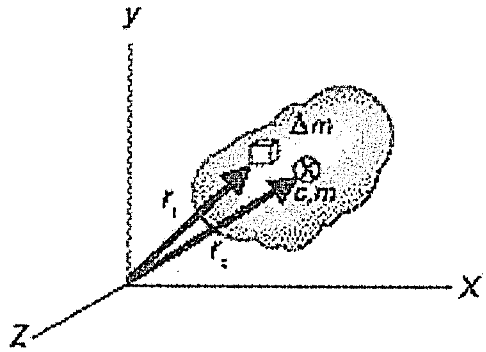
$$\vec{R} = \frac{1}{M} \left[M_B \vec{R}_B + M_G \vec{R}_G \right]$$

CM of Solid Bodies

- Consider objects with continuous distributions of mass.
- Divide body up into elements of mass Δm_i with coordinates $x_i\ y_i\ z_i$.

The x-coordinate of the CM becomes

$$x_c \approx \frac{\sum x_i \Delta m_i}{M}$$



A rigid body can be considered a distribution of small elements of mass Δm_i. The center of mass is located at the vector position r_c, which has coordinates x_c, y_c, and z_c.

Let the number of elements approach infinity.
Then

$$x_c = \lim_{\Delta m_i \to 0} \frac{\sum x_i \Delta m_i}{M} = \frac{1}{M}\int x\, dm$$

Also $$y_c = \frac{1}{M}\int y\, dm$$

$$z_c = \frac{1}{M}\int z\, dm$$ [First moments of mass distr.]

For the position vector of the cm

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$$

Note: From above it follows that the CM of homogeneous, symmetric bodies must lie on an axis of symmetry.

It is often convenient to express the mass distribution in terms of the <u>local density</u> and an element of volume:

$$dm = \rho\, dv \qquad \rho = \rho(x, y, z).$$

Then $$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,\rho\, dv$$

$$x_c = \frac{1}{M}\int x\rho\, dv$$

$$y_c = \frac{1}{M}\int y\rho\, dv$$

$$z_c = \frac{1}{M}\int z\rho\, dv$$

This is a general result even if $\rho(xyz)$ varies throughout the volume. If ρ is a constant then cm is often easily obtained by the symmetry of the object volume.

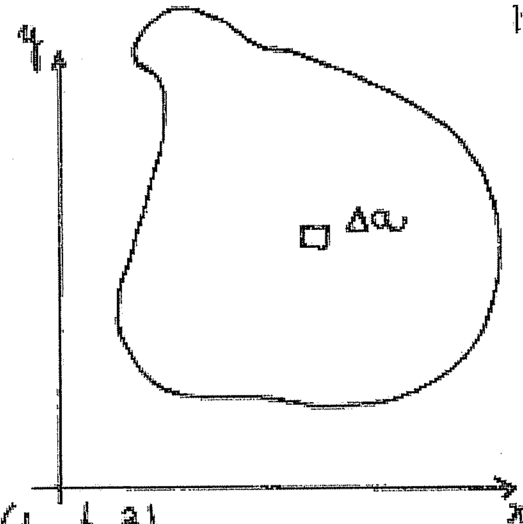
⇒ First moments of the volume distribution if ρ = constant.
Integrals are evaluated over the entire volume.

$dv = dx\,dy\,dz$

18-12

Areas

If object is in the form of a plane sheet of constant thickness t



$$\Delta m_i = \sigma \, \Delta a_i$$

σ — areal mass density (kg/m^2)

$$x_{cm} = \frac{1}{M} \int \sigma x \, da$$

$$y_{cm} = \frac{1}{M} \int \sigma y \, da$$

$\Rightarrow$ First moments of the area

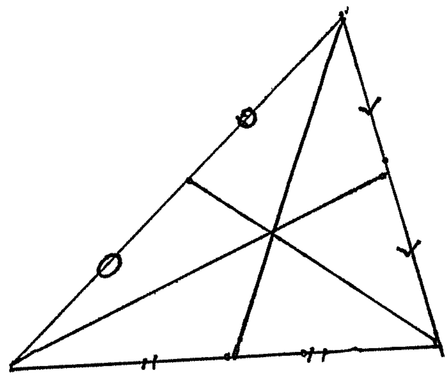
$$da = dx \, dy$$

If $\sigma(x,y) = \sigma_0$, a constant

$$x_{cm} = \frac{1}{A} \int x \, da$$

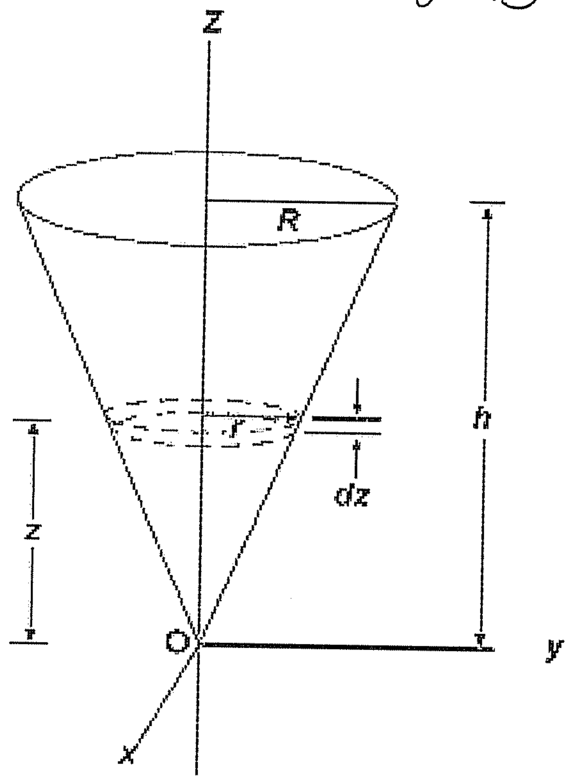
$$y_{cm} = \frac{1}{A} \int y \, da$$

If an object has a point, line or plane of symmetry, the center of must lie on that point, on that line or that plane.



Triangle's center of mass

Example: Right Circular Cone



By symmetry the cm must lie on the axis of the cone.

$x_{cm} = 0 \qquad y_{cm} = 0$

$$z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_0^h \rho z \pi r^2 dz$$

Cone is divided into many cylinders of radius r and thickness dz. Assume ρ = constant

Mass of each cylinder

$dm = \rho\, dV = \rho \pi r^2 dz$

The radius of each cylinder is related to its location z by

$$\frac{r}{z} = \frac{R}{h} \qquad \text{or} \qquad r = Rz/h$$

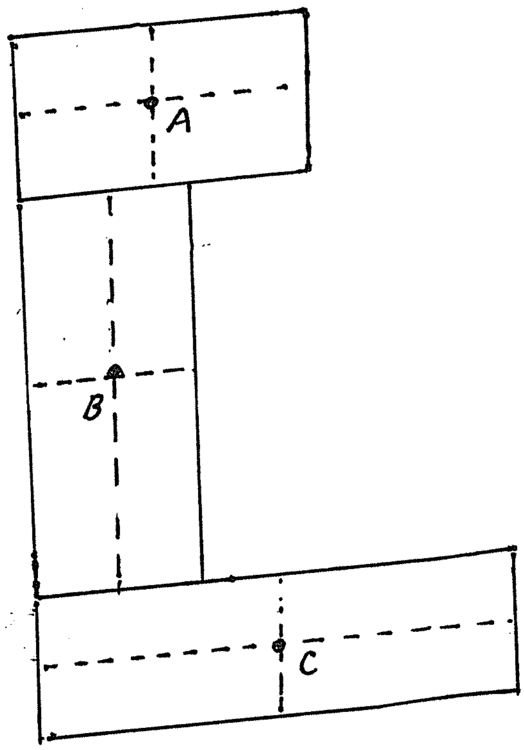
$$\therefore \quad z_{cm} = \frac{1}{M}\int_0^h \frac{\rho\pi R^2}{h^2} z^3 dz = \frac{\rho\pi R^2}{Mh^2}\frac{z^4}{4}\Bigg|_0^h = \frac{\rho\pi R^2 h^2}{4M}$$

The total mass of the cone is equal to its density times the volume.

$M = \rho\pi R^2 h/3$

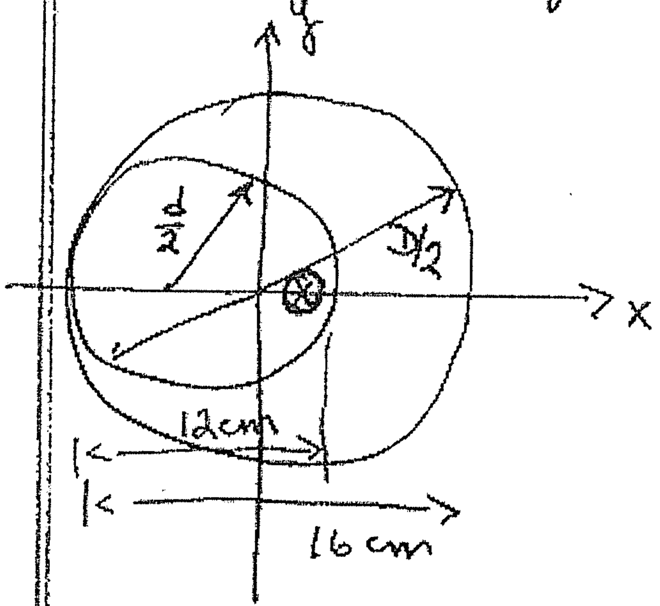
$$\therefore \quad z_{cm} = \frac{3}{4}h \quad \text{or} \quad \frac{1}{4}h \text{ from base}$$

8-16



Example : CM

A homogeneous disk of diameter $D = 16$ cm, contains a circular hole of diameter $d = 12$ cm which is tangent to the circumference as shown. Assume the surface density of the disk is σ g/cm^2.
Locate the center-of-mass.



If body had no cavity its CM would be at $x_D = 0$.
A body with diameter d (negative mass) has a CM at $x_d = -2$ cm.

Treat problem as if it were two particles. Find common CM.

$$y_c = 0$$

$$x_c = \frac{\left(\frac{\pi D^2}{4}\sigma\right)\times 0 - \left(\frac{\pi d^2}{4}\sigma\right)(-2\text{cm})}{\frac{\pi D^2}{4}\sigma - \frac{\pi d^2}{4}\sigma} = \underline{\underline{2.6\text{ cm}}}$$

Motion of the Center-of-Mass

Suppose we take the time derivative of the position vector of the cm. Assuming M is constant (no particles enter or leave system) then we get for the velocity of the cm:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left[m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \cdots \frac{d\vec{r}_n}{dt} \right]$$

$$= \frac{1}{M}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots m_n \vec{v}_n \right)$$

$$= \frac{\vec{P}}{M}$$

or $\vec{P} = M\vec{v}_{cm}$

Total momentum of the system is its total mass multiplied by the velocity of the cm. i.e. total $\vec{P}$ is that of a <u>single particle</u> of <u>mass M</u> moving with a <u>velocity $\vec{v}_{cm}$</u>.

Differentiate again to get the acceleration of the cm:

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum m_i \vec{a}_i$$

$$M\vec{a}_{cm} = \sum \vec{F}_i$$

$\vec{F}_i \equiv$ Force on particle i.

Forces on any particle have two sources:

- External (from outside the system)
- Internal (from within the system)

From Newton's 3rd Law the sum over 'Internal' forces cancells in pairs and the overall sum vanishes.

The net force on the system is due only to external forces.

$$\therefore \sum \vec{F}_{ext} = M\vec{a}_c = \frac{d\vec{P}}{dt} .$$

"The cm moves like an imaginary particle of mass M under the influence of the resultant external force on the system".

If $\sum \vec{F}_{ext} = 0$

$$\frac{d\vec{P}}{dt} = M\vec{a}_c = 0$$

and

$$\boxed{\vec{P} = M\vec{v}_c = \text{constant.}} \quad (\text{when } \sum \vec{F}_{ext} = 0)$$

Total linear momentum of a system is conserved if there are no external forces acting on it.

For an isolated system of particles, both the total momentum and velocity of the cm are constant in time.

Various Comments

Impulse

Exploding Projectile

Rocket , Conveyer

Motion of the center of mass (it is not everything)

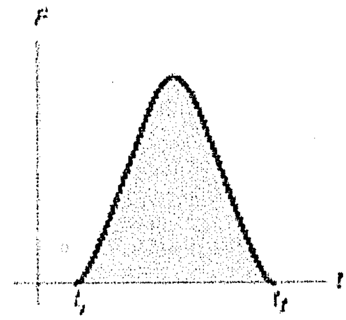
Collisions/Impulse

When two objects collide, the forces they exert on each other usually act only for a short time. Such forces are called impulsive forces.

During the collision the impulsive force produces a large change in the motion of the object while any other forces present produce only small changes usually neglected.

From Newton's 2ND Law

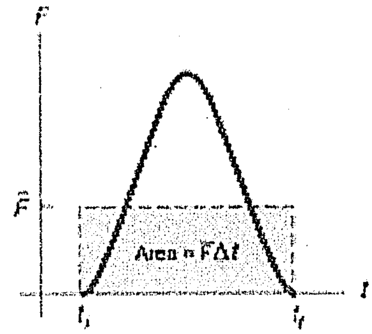
$$\frac{d\vec{p}}{dt} = \vec{F}$$



(A) A force acting on a particle may vary in time. The impulse is the area under the force versus time curve.

During a time interval dt, the momentum changes by

$$d\vec{p} = \vec{F}\,dt$$



(B) The average force (horizontal line) would give the same impulse to the particle in the time Δt as the real time-varying force described in *(A)*.

Integrating over the time of collision

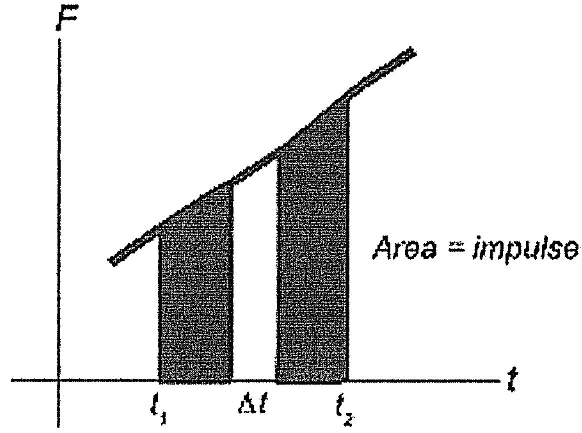
$$\vec{p}_f - \vec{p}_i = \int_{p_i}^{p_f} d\vec{p} = \underbrace{\int_{t_i}^{t_f} \vec{F}\,dt}_{}$$

Area under curve represents Impulse.

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}(t)\, dt$$

[Impulse]

$$\Delta\vec{p} = \vec{p_f} - \vec{p_i} = \vec{I}$$



The area of the small rectangle is approximately equal to the total change in momentum in the interval Δt.

The change in momentum of an object is equal to the impulse acting on it.

Sometimes useful to talk about an average force $\bar{F}$ which acting over the same time interval

$$\Delta t = t_f - t_i$$

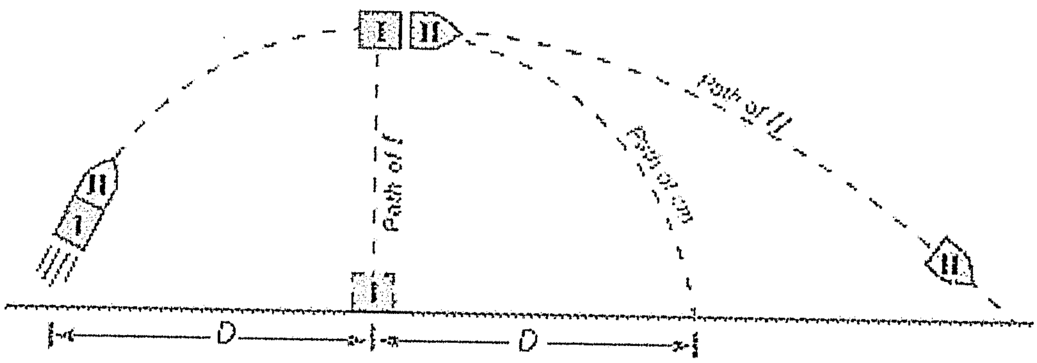
produces the same impulse and consequently the same momentum change.

$$\bar{F} = \frac{1}{\Delta t} \int_{t_i}^{t_f} F(t)\, dt$$

$$\bar{F} \Delta t = (\vec{p_f} - \vec{p_i}) = \vec{I}$$

Example: Exploding Projectile

Rocket fired into the air. At its highest point a distance D from the origin it separates into two parts of equal mass.
Part I falls vertically to the earth.
Where does Part II land? Assume g = constant.



$m_I = m_{II}$

cm follows ballistic trajectory of particle with mass $M = M_I + M_{II}$ and intercepts ground a distance $2D$ from the origin.

Since masses m_I and M_{II} are equal, Part II hits ground a distance D beyond cm or a total distance $3D$ from the origin.

Note: External forces vanish for horizontal motion only. External forces are present for vertical motion.

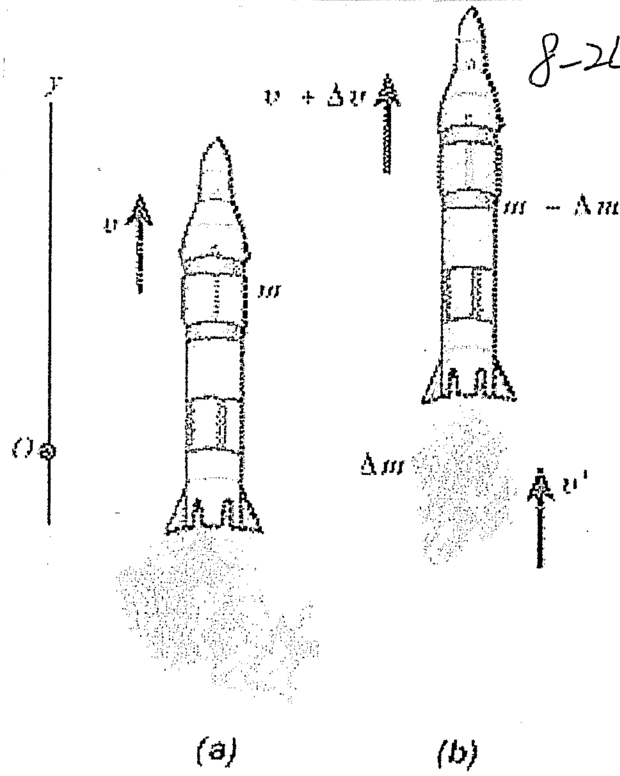
Variable Mass / Rocket Equation

$$F_{ext} = \frac{d\vec{p}}{dt}$$

Rockets

Propelled forward by the ejection of gases.
Force eserted by the gases on the rocket accelerates it
Mass of the rocket decreases

$$\Rightarrow \frac{dm}{dt} < 0$$



(a) Rocket at time t after takeoff, with mass m and upward velocity v. Its momentum is mv.

(b) At time $t + \Delta t$, the mass of the rocket (and unburned fuel) is $m - \Delta m$, its velocity is $v + \Delta v$, and its momentum is $(m - \Delta m)(v + \Delta v)$. The ejected gas has momentum $\Delta m(v - v_r)$.

At time t

mass m and upward velocity v
its momentum is mv

$$P_i = mv$$

At time $t + \Delta t$

$$P_f = (m - \Delta m)(v + \Delta v) + (\Delta m)u$$

$$\Delta p = P_f - P_i$$

$$= \cancel{mv} - \Delta m v + m \Delta v - \Delta m \Delta v + (\Delta m)u - \cancel{mv}$$

double difference
ignore

$$= m\Delta v + \Delta m(u - v)$$

Take the limit $\Delta t \rightarrow dt$ $\quad dm = -\Delta m$

$$F_{ext} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} - (u - v)\frac{dm}{dt}$$

relative velocity of dm with respect to m

$V_r = u - v$ assumed to be zero

$$m \frac{d\vec{V}}{dt} = \vec{F}_{ext} - \vec{v}_r \frac{dm}{dt}$$

$$d\vec{V} = \frac{F_{ext}}{m} dt + \frac{v_r}{m} dm$$

$$\vec{F} = -mg$$

$$v = v_0 - gt - v_r \ln \frac{m}{m_0}$$

$$\Downarrow$$

$$v_r < 0$$

v_0 = initial velocity

Note

$$\frac{dm}{dt} = \lim_{\Delta t \to 0} \frac{m(t+dt) - m(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(m - \Delta m) - m}{\Delta t}$$

$$= - \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t}$$

Note the negative sign

Example: Rocket

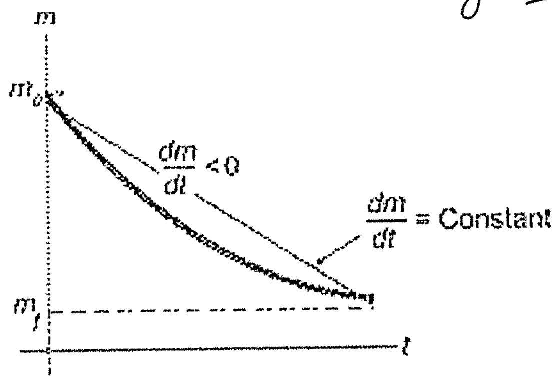
$M_0 = 21{,}000$ kg

$M_f = 6{,}000$ kg (After burnout)

$\frac{dM}{dt} = -190$ kg/s (Rate of fuel exh.)

$v_r = 2800$ m/s

$g = 9.81$ m/s^2



A rocket whose initial total mass is m_0 loses mass as the engine burns fuel. The detailed shape of the curve depends on the burning rate versus time.

a) Thrust: $v_r \frac{dM}{dt} = (2800)(190) = 5.3 \times 10^5$ N

b) $F_{ext} = Mg = 2.1\times10^4$ kg $(9.81) = 2.1\times10^5$ N (initially)

$= (0.6\times10^4)(9.81) = 5.9\times10^4$ N (at burn-out)

Net Force on M:

start: $(5.3\times10^5 - 2.1\times10^5) = 3.2\times10^5$ N $\quad a = 1.52$ gees.

Just Prior to Burn-Out: $(5.3\times10^5 - 5.9\times10^4) = 4.7\times10^5$ N $\quad a = 8.0$ gees

Just After: $= -gM \quad a = -1$ gee

Burn-Out Time: $t = \frac{1.5\times10^4 \text{ kg}}{190 \text{ kg/s}} = 79$ s

If $v_0 = 0$

$v = -9.81(79) + (-2800 \text{ m/s}) \ln \frac{6000}{21000}$

$= 2830$ m/s

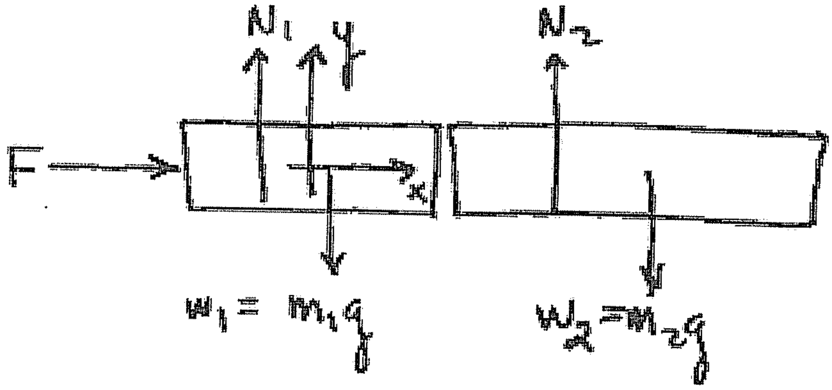
opposite to v.

Example : cm motion

- uniform blocks

$m_1 = 2\,kg \quad L_1 = 20\,cm$

$m_2 = 4\,kg \quad L_2 = 40\,cm.$



a) $F = 12\,\hat{i}$ (N)

$M = m_1 + m_2 = 6$

$$\sum F_{ext} = M a_{cm}$$

$$\vec{F} + \vec{N_1} + \vec{N_2} + \vec{w_1} + \vec{w_2} = M\vec{a}_{cm}$$

$$12\hat{i} = 6\,\vec{a}_{cm}$$

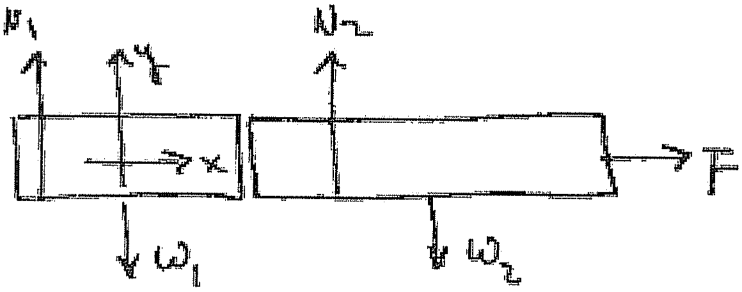
$$\vec{a}_{cm} = 2\hat{i} \quad (m/s^2)$$

$$x_{cm} = \frac{2 \times 0 + 4 \times 30}{2+4} = 20\,cm.$$

$a_{m_1} = 2\hat{i}$

$a_{m_2} = 2\hat{i}$

(b)



$\vec{a}_1 = 0$

$\vec{a}_2 = \frac{12}{4}\hat{i} = 3\hat{i}$

$\vec{a}_{cm} = ?$ [Same as before ??]

$\vec{a}_1 = 0 \qquad \vec{a}_2 = 3\hat{i}$

$\vec{v}_1 = 0 \qquad \vec{v}_2 = 3t\,\hat{i}$

$\vec{r}_1 = 0 \qquad \vec{r}_2 = (0.3\,m + 1.5t^2)\,\hat{i}$

$M\vec{r}_{cm} = \sum m_i \vec{r}_i$

$M\vec{v}_{cm} = \sum m_i \vec{v}_i$

$M\vec{a}_{cm} = \sum m_i \vec{a}_i$

$\therefore \; \vec{a}_{cm} = \frac{1}{6}\left[0 + 4 \times 3\hat{i}\right] = 2\hat{i} \quad (m/s^2)$

$\vec{v}_{cm} = \frac{1}{6}\left[0 + 4 \times 3t\hat{i}\right] = 2t\,\hat{i} \quad (m/s)$

$\vec{r}_{cm} = \frac{1}{6}\left[0 + 4\left[0.3 + 1.5t^2\right]\hat{i}\right]$

$= (0.2 + 1.0t^2)\,\hat{i}$

cm in this case does not remain fixed relative to centers of the two blocks.

Energy of a System of Particles

Total Momentum of a system $\vec{P} = \sum_i \vec{P}_i = m\vec{V}_{c.m.}$

~~K.E. $= \frac{1}{2} m V_{c.m}^2$~~ No !!

The total K.E. is the sum of the K.E. of the individual particle.

For the two-particle system

$$K.E. = \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2}$$

We like to write this in terms of $\vec{v}_{c.m}$

Let us study the motion in the c.m. frame

$$\vec{r}_1{}' = \vec{r}_1 - \vec{r}_{cm}$$

$$\vec{r}_2{}' = \vec{r}_2 - \vec{r}_{c.m}$$

$$\Rightarrow \quad \vec{u}_1 = \vec{v}_1 - \vec{v}_{c.m}$$

$$\vec{u}_2 = v_2 - \vec{v}_{c.m}$$

K.E. of the system

$$= \frac{1}{2} m_1 (\vec{u}_1 + \vec{v}_{c.m})^2 + \frac{1}{2} m_2 (\vec{u}_2 + \vec{v}_{cm})^2$$

$$= \underbrace{\left(\frac{1}{2} m_1 \vec{u}_1^{\,2} + \frac{1}{2} m_2 u_2^2 \right)}_{I} + \underbrace{[m_1 \vec{u}_1 + m_2 \vec{u}_2]}_{II} \cdot \vec{v}_{c.m}$$

$$+ \underbrace{\frac{1}{2} (m_1 + m_2) \vec{v}_{c.m}^{\,2}}_{(III)}$$

$$I = (K.E.)_{int} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$II = [m_1 (\vec{v}_1 - \vec{v}_{c.m}) + m_2 (\vec{v}_2 - \vec{v}_{c.m})] \cdot \vec{v}_{c.m.}$$

$$= [\underbrace{(m_1 \vec{v}_1 + m_2 \vec{v}_2)}_{= (m_1 + m_2) \vec{v}_{c.m}} - (m_1 + m_2) \vec{v}_{c.m}] \cdot \vec{v}_{c.m}$$

$$= 0$$

$$III = \frac{1}{2} m V_{cm}^2$$

↳ translation of c.m.

$$K.E. = \underbrace{(K.E.)_{int}}_{\text{internal kinetic energy}} + \underbrace{\frac{1}{2} m V_{cm}^2}_{\text{translation of c.m.}}$$

$(K.E)_{int}$: rotations, heat

Potential Energy

If internal and external conservative forces act on body, the system will also have potential energy $U = E_p$

$\downarrow$

function of position of all the particles

Total E (mechanical energy) $=$ Total K. E. $+$ Total U.

Gravitational potential energy (near the earth surface)

For a system of (2 particles)

$$U = (m_1 z_1 + m_2 z_2)\, g$$

$$= m z_{c.m}\, g$$

behaves as if entire mass is located at the c. m.

Collision

Definition

Classification

Sticking Collision

One Dimension → general case

Two Dimension → general case

Example

Elastic Collision

One dimension

Two dimension

Ballistic Pendulum

↓

with some estimates

Look at the problem in c. m. system

COLLISIONS

1. Conditions:

An event is a collision if $\Delta t \ll \Delta T$
- time can be separated into before, during + after
- Δt collision time
- ΔT observation time

An event is a collision if $|J_{ext}| \ll |I_{coll}|$ - the impulse of external forces can be neglected and momentum is conserved.

2. Collision Classifications:

Elastic - KE is conserved
Inelastic - KE is not conserved
Completely inelastic - particles stick together after

3. Notation

m_1, m_2 - masses of the two particles
v_{1i}, v_{2i} - initial (before collision) velocities of parts 1, 2.
v_{1f}, v_{2f} - final (after collision) velocities of particles 1, 2.

4. Equations

Conservation of momentum - valid for all collisions:

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

Conservation of kinetic energy - valid only for elastic collisions.

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$

Collisions

We want to study the collision of objects – how they move (velocities) after colliding. In special cases momentum conservation is sufficient. In general it is not enough. Can categorize collisions in terms of two types:

Elastic Collisions

- Interaction forces are all conservative
- Total kinetic energy is the same before and after
- Momentum is conserved

$$K_i = K_f \quad ; \quad \vec{P}_i = \vec{P}_f$$

Inelastic Collisions

- Momentum is conserved
- Total kinetic energy after collision is less than before.

$$\vec{P}_i = \vec{P}_f$$

$$K_i \neq K_f$$

A. Inelastic Collisions

Collisions in which KE is not conserved are called inelastic collisions. Some of the energy is absorbed and converted to other forms.

If the amount of KE absorbed is a maximum that is allowed by momentum conservation, the collision is said to be perfectly inelastic

⇒ No internal KE left only KE of cm.

$$K_{INT} \equiv 0$$

For collisions involving isolated objects, momentum is always conserved.

In practise almost all collisions are inelastic to some degree. To solve inelastic problems is not easy. Need to know how much energy is lost.

The easy problem involves a perfectly inelastic collision (- sticking collision -). Then conservation of momentum gives the answer.

i) Sticking Collision - One Dimension

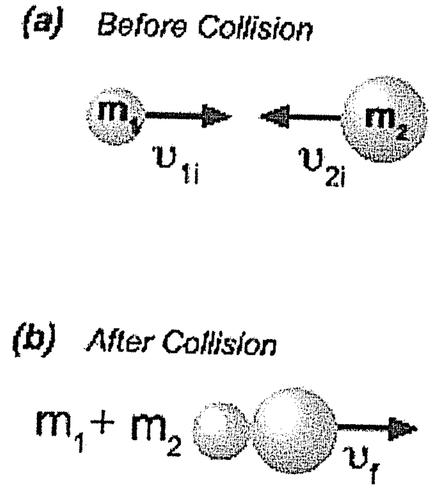
Two particles masses m_1 and m_2 move with velocities v_{1i} and v_{2i} along a st. line. They collide and stick, moving as a unit with velocity v_f after the collision.

Total momentum is conserved.

$$\vec{P}_i = \vec{P}_f$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2}$$



Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before the collision and (b) after the collision.

<u>Example:</u>

Body-2 is at rest, $\vec{v}_{2i} = 0$.

Then $\vec{v}_f = \dfrac{m_1 \vec{v}_{1i}}{m_1 + m_2}$

The corresponding Kinetic Energies are:

$$K_i = \frac{1}{2} m_1 v_{1i}^2$$

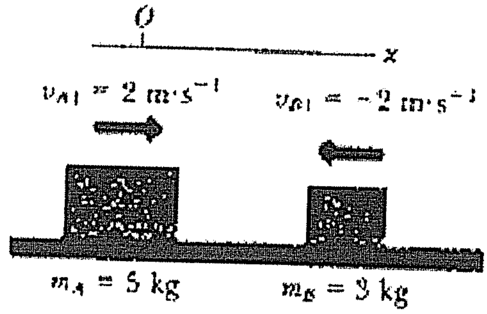
$$K_f = \frac{1}{2}(m_1 + m_2) \cdot \frac{m_1^2 v_{1i}^2}{(m_1 + m_2)^2}$$

$$\frac{K_f}{K_i} = \frac{m_1}{m_1 + m_2} < 1$$

Final KE is always less than initial KE in such collisions.

Example

Two objects collide and stick.



$$\vec{P_i} = \vec{P_f}$$

$$(5\,kg)(2\,m/s) - (3\,kg)(2\,m/s) = (5+3)\,v_f$$

$$v_f = \frac{10-6}{8} = 0.5\,m/s$$ [Positive moving to right].

$$K_i = \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2$$

$$= \frac{1}{2} \times 5 \times 2^2 + \frac{1}{2} \times 3 \times 2^2 = 16\ J$$

$$K_f = \frac{1}{2}(m_A + m_B)\, v_f^2$$

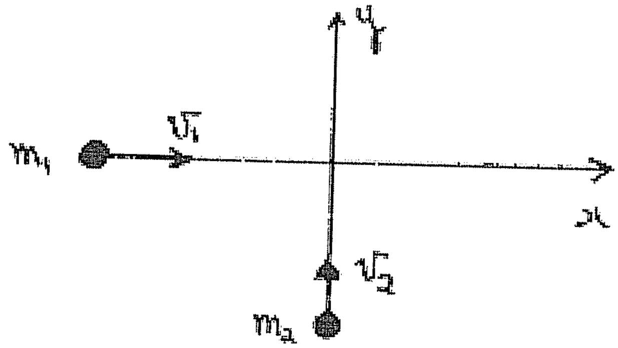
$$= \frac{1}{2} \times 8 \times \left(\frac{1}{2}\right)^2 = 1\ J$$

$$\frac{K_f}{K_i} = \left(\frac{1}{16}\right)$$ Most of the KE was lost!!

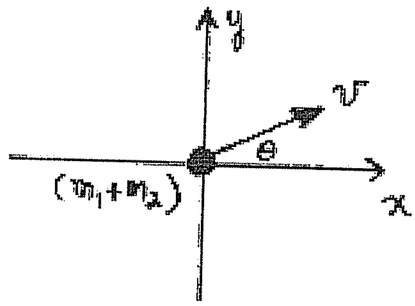
ii) Sticking Collision - Two Dimensions

- Linear momentum conserved.
- Two equations, one for each component.
- Simplest problem is a sticking collision.

Example



Before



After

- Assume no external forces acting

$$\vec{P}_i = \vec{P}_f$$

Conservation of Momentum : x-component

$$m_1 v_1 = V(m_1+m_2)\cos\theta \qquad (1)$$

Conservation of Momentum y-component.

$$m_2 v_2 = V(m_1+m_2)\sin\theta \qquad (2)$$

(2)/(1) $$\tan\theta = \frac{m_2 v_2}{m_1 v_1}$$

from (2) $$v = \frac{m_2}{m_1+m_2}\,\frac{v_2}{\sin\theta}$$

20-6

$m_1 = 70\,kg \qquad v_1 = 2\,m/s$
$m_2 = 50\,kg \qquad v_2 = 3\,m/s$

$$\tan\theta = \frac{50}{70} \times \frac{3}{2}$$

$$\theta = 47^\circ$$

$$v = \frac{50}{50+70} \times \frac{3}{\sin 47^\circ} = 1.71\,m/s$$

Initial KE :

$$K_i = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} \times 70 \times 2^2 + \frac{1}{2} \times 50 \times 3^2$$

$$= 365\,J$$

Final KE :

$$K_f = \frac{1}{2}(m_1+m_2)V^2$$

$$= \frac{1}{2}(50+70)\,1.71^2$$

$$= 175.5\,J$$

KE is lost in this inelastic collision.

Example : 18-5

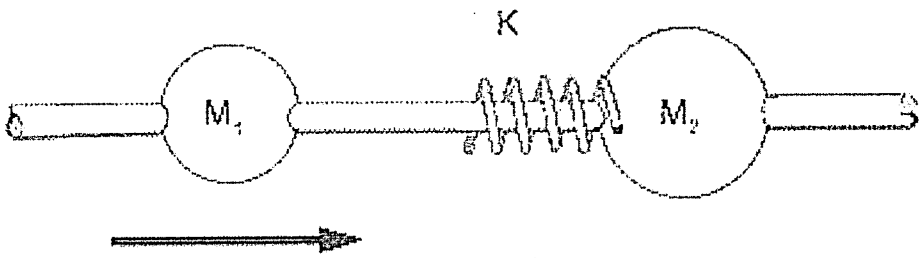
Two particles with mass $m_1 = 2.0\,kg$ and $m_2 = 5.0\,kg$ can slide on a frictionless rod. A spring with $K = 1000\,N/m$ is attached to m_2.

$v_1 = 14\,m/s$

$v_2 = 0.$

a) What is the maximum compression of the spring when the particles collide?

b) What are the final velocities of the particles?



$v_1 = 14.0$ m/s $\quad v_2 = 0.0$ m/s

- When the spring is under maximum compression, the relative velocity of the particles is zero.
- The system then has a velocity v_o.

Conservation of Linear Momentum : [At max. comp.]

$$m_1 v_1 + m_2 \times 0 = (m_1 + m_2)\, v_o$$

$$v_o = \frac{m_1}{m_1 + m_2} v_1 = \frac{2 \times 14}{2+5} = 4\,m/s.$$

Initial KE, before collision

$$K_o = \frac{1}{2} m_1 v_1^2 + \cancel{\frac{1}{2} m_2 \times 0} = \frac{1}{2} \times 2.0 \times (14)^2 = 196\,J$$

At maximum spring compression the remaining KE is

$$K = \frac{1}{2}(m_1 + m_2)v_0^2 = \frac{1}{2}(2+5)\,4^2 = 56\text{ J}$$

Energy difference is stored as PE in the spring.

$$\frac{1}{2}kx^2 = K_0 - K$$

$$x = \sqrt{\frac{2(K_0 - K)}{k}} = \sqrt{\frac{2\times(196-56)}{1000}} = 0.53\text{m}$$

b) When particles separate, the energy stored in the sprin is returned to the particles and total energy and momentum is conserved.

① $m_1 v_1' + m_2 v_2' = m_1 v_1 + \cancel{m_2} \times 0$

② $\frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 = \frac{1}{2}m_1 v_1^2 + \cancel{\frac{1}{2}m_2} \times 0^2$

③ $2v_1' + 5v_2' = 28$ kg.m/s [Sub. in Eq ①]

④ $2v_1'^2 + 5v_2'^2 = 392$ J² [sub in eq ②]

From ⑤ $v_2' = (28 - 2v_1')/5$

$$2v_1'^2 + 5\left[\frac{28 - 2v_1'}{5}\right]^2 = 392$$

	v_1'	v_2'
(NG)!!	14	0
	−6	8

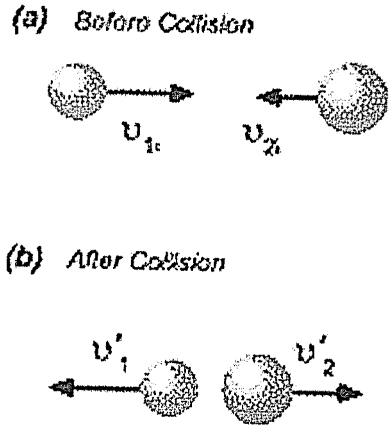
3. Elastic Collisions

- Total energy is conserved.
- Total linear momentum is conserved.

i) Elastic Collision – One Dimension.

- Assume particles moving with velocities $\vec{v}_1$ and $\vec{v}_2$ before the collision.
- Particles move with velocities $\vec{v}_1{}'$ and $\vec{v}_2{}'$ after the collision.

$v > 0$ if particle moves to the right.
$v < 0$ if particle moves to the left



Schematic representation of an elastic head-on collision between two particles: (a) before the collision and (b) after the collision.

Cons. of Momentum: $\vec{P}_i = \vec{P}_f$

① $m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'$

Cons. of Energy: $K_i = K_f$

② $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

We have two equations, can solve for two unknowns.
Given the masses:

$$(v_1, v_2) \Longrightarrow (v_1', v_2')$$

Rewrite Eq. ①: $$m_1(\vec{v}_1 - \vec{v}_1{}') = m_2(\vec{v}_2{}' - \vec{v}_2) \quad ③$$

Rewrite Eq ②: $$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2) \quad ④$$

④/③ $$v_1 + v_1' = v_2' + v_2$$

or $$\boxed{\vec{v}_1 - \vec{v}_2 = \vec{v}_2{}' - \vec{v}_1{}'}$$

The relative velocities of the two particles after the collision is the negative of the relative velocities before the collision for any elastic head-on collision, no matter what the masses.
This can be combined with eq. for cons. of momentum to solve all problems.

<u>Special Cases</u>

- m_1, m_2, v_1 and v_2 are known.
- v_1' and v_2' to be determined

a) Equal Masses : $m_1 = m_2$

Momentum : $\vec{v_1} + \vec{v_2} = \vec{v_1}' + \vec{v_2}'$ ①

Rel Velocities : $\vec{v_1} - \vec{v_2} = \vec{v_2}' - \vec{v_1}'$ ②

① + ② $\quad \vec{v_2}' = \vec{v_1}$

① - ② $\quad \vec{v_1}' = \vec{v_2}$

- Particles exchange velocities during collision
  - Particle-2 acquires velocity of particle-1
  - Particle-1 acquires velocity of particle-2

If $\vec{v_2} = 0$ (Particle-2 initially at rest)

$v_2' = v_1$

$v_1' = 0$

- Particle-1 stops
- Particle-2 moves forward with velocity of particle-1

b) Particle-2 at rest initially, $v_2 = 0$

Momentum: $m_1 v_1 = m_1 v_1' + m_2 v_2'$

Rel. Velocities: $v_1 = -v_1' + v_2'$

$$v_2' = v_1 \left(\frac{2m_1}{m_1+m_2} \right) \quad (5)$$

$$v_1' = v_1 \left(\frac{m_1-m_2}{m_1+m_2} \right) \quad (6)$$

i) $v_2 = 0$, $m_1 = m_2$

$$\left.\begin{aligned} v_2' &= v_1 \\ v_1' &= 0 \end{aligned}\right\} \text{ Same result as Case (a)}$$

ii) $v_2 = 0$, $m_1 \gg m_2$

– A heavy object strikes a light object at rest.

From (5) and (6) $v_2' \simeq 2v_1$

$v_1' \simeq v_1$

- Velocity of incoming particle unchanged
- light particle moves forward with twice the velocity of the heavy object.

iii) $v_2 = 0$, $m_1 << m_2$

- A moving light particle strikes a stationary heavy object.

$$v_2' \sim 0$$
$$v_1' \sim -v_1$$

- Heavy object essentially remains at rest
- Light particle reverses direction and moves of with insident speed.

c) General Solution

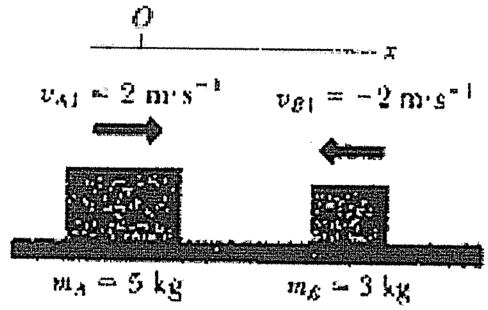
- Can solve original equations for velocities of particles after the collision.
- Usually it is best to start from momentum and energy conservation laws instead of memorizing formulas.

$$v_2' = v_1 \left(\frac{2m_1}{m_1+m_2} \right) + v_2 \left(\frac{m_2 - m_1}{m_1+m_2} \right)$$

$$v_1' = v_1 \left(\frac{m_1 - m_2}{m_1+m_2} \right) + v_2 \left(\frac{2m_2}{m_1+m_2} \right)$$

Example: Elastic 1-D

Conservation of Momentum:



$$(5\,kg)(2\,m/s) + (3\,kg)(-2\,m/s) = 5\,v_{A2} + 3\,v_{B2}$$

$$5v_{A2} + 3v_{B2} = 4 \quad (m/s)$$

Since collision is completely elastic:

$$v_{B2} - v_{A2} = -(v_{B1} - v_{A1})$$

$$= -(-2-2) = 4\ m/s.$$

Solving: $v_{A2} = -1\,m/s$

$v_{B2} = 3\,m/s$

Bodies reverse their directions of motion. A moves to left at 1m/s and B to right at 3m/s.

$$KE_i = \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 = \frac{1}{2}\times 5\times 2^2 + \frac{1}{2}\times 3\times 2^2$$

$$= 16\,J$$

$$KE_f = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2 = \frac{1}{2}\times 5\times 1^2 + \frac{1}{2}\times 3\times 3^2$$

$$= 16\,J.$$

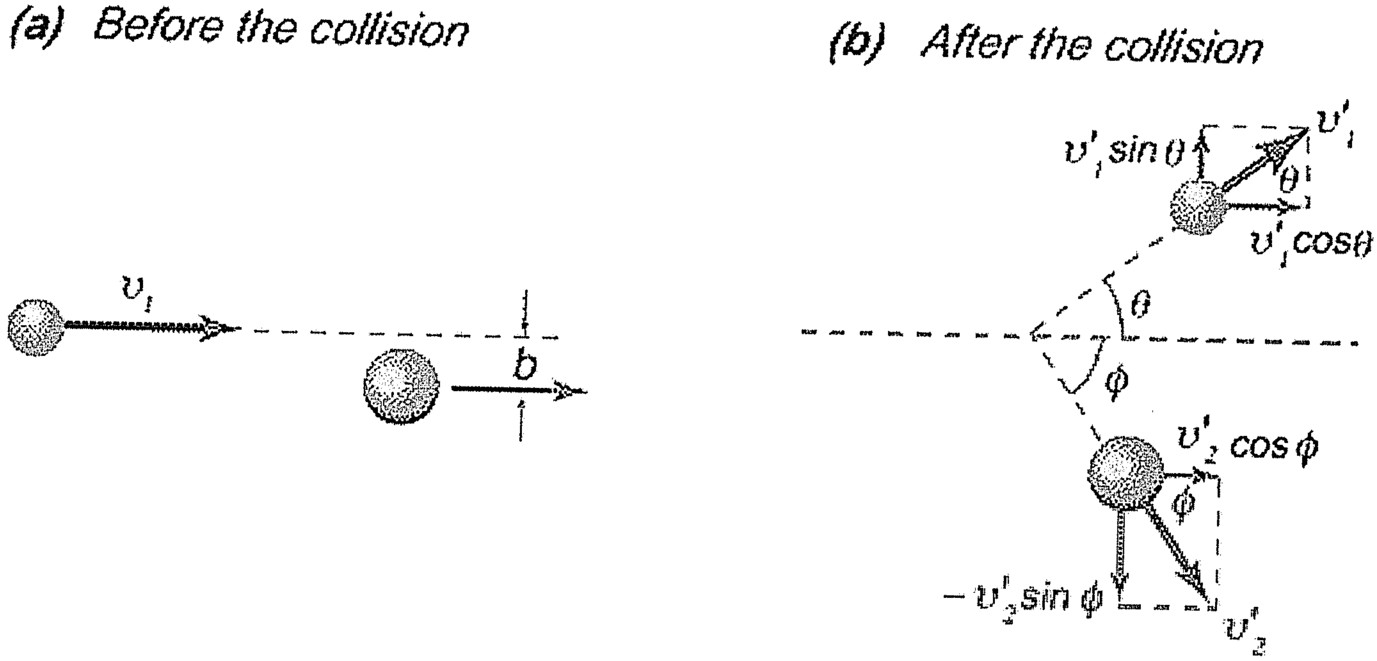
$$KE_i = KE_f$$

ii) Elastic Collisions - Two Dimensions

- Assume special case where target particle is initially at rest, $\vec{v}_2 = 0$.
- Particle collide by interacting. For many forces, gravity, electromag., etc. forces act along line joining the particles.
- The initial particle and scattered particles define a common plane — 2-dimensional problem.

Given initial velocities, we have four unknowns following collision:

$$\left.\begin{matrix} v_{1x}', & v_{1y}' \\ v_{2x}', & v_{2y}' \end{matrix}\right\} \text{ x and y velocities of particles -1 and -2.}$$



Schematic representation of an elastic glancing collision between two particles: (a) before the collision and (b) after the collision. Note that the impact parameter, b, must be greater than zero for a glancing collision.

Cons. of Momentum: 2 equations (x, y)
Cons. of Energy: 1 equation

$\therefore$ Can only get restrictions on the final motion or at least one other quantity must be known.

Cons of Momentum:

$$m_1 \vec{v}_1 = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'$$

$$m_1 v_1 = m_1 v_1' \cos\theta + m_2 v_2' \cos\phi$$

$$0 = m_1 v_1' \sin\theta - m_2 v_2' \sin\phi$$

Cons. of Energy:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

<u>Special Case:</u> $m_1 = m_2$

$$v_1^2 = v_1'^2 + v_2'^2 \qquad (1)$$

$$v_2' \cos\phi = v_1 - v_1' \cos\theta \qquad (2)$$

$$v_2' \sin\phi = v_1' \sin\theta \qquad (3)$$

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$(2)^2 + (3)^2 \quad v_2'^2 = v_1^2 - 2v_1 v_1' \cos\theta + v_1'^2$

Use Eq. ① to eliminate v_2'

$$v_1' = v_1 \cos\theta$$

$$\frac{1}{2} m v_1'^2 = \frac{1}{2} m v_1^2 \cos^2\theta$$

KE of deflected incident projectile after scattering.

We had cons. of momentum :

$$\vec{v_1} = \vec{v_1}' + \vec{v_2}'$$

$$v_1^2 = v_1'^2 + v_2'^2 + 2v_1' v_2' \cos(\theta + \phi) \quad \text{[square]}$$

Comparing with Eq. ①, we must have that

$$\theta + \phi = \pi/2$$

When particles of equal mass collide, the sum of their scattering angles is 90°.

If $\quad m_1 > m_2 \quad \theta \leq \pi/2$

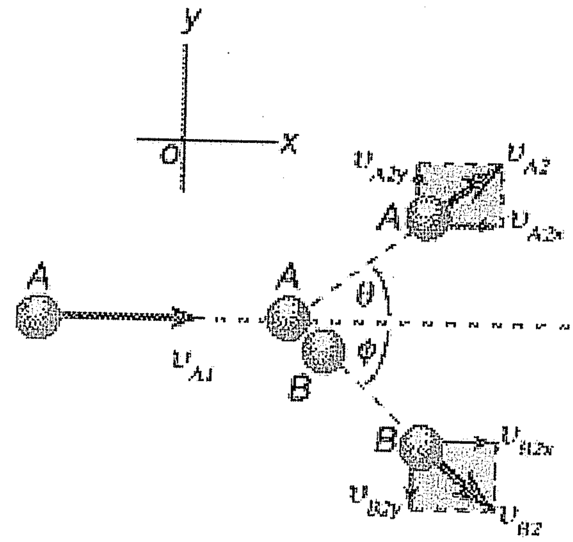
$\quad m_1 < m_2 \quad 0 \leq \theta \leq \pi$

Example: 2-D Elastic Collision

$m_A = 5\,kg \quad v_{A1} = 4\,m/s$

$m_B = 3\,kg \quad v_{B1} = 0$ [stationary]

Assume after collision, $v_{A2} = 2\,m/s$
Find v_{B2}, θ and ϕ



Since collision is elastic
$KE_f = KE_i$

$$\frac{1}{2}(5kg)(4m/s)^2 = \frac{1}{2}(5kg)(2m/s)^2 + \frac{1}{2}(3kg)v_{B2}^2$$

Solving,
$v_{B2} = 4.47\,m/s$

Conservation of x- and y-components of momentum gives,

① $(5kg)(4m/s) = (5kg)(2m/s)\cos\theta + (3kg)(4.47m/s)\cos\phi$

② $0 = (5kg)(2m/s)\sin\theta - (3kg)(4.47m/s)\sin\phi$

Solve Eq.① for $\cos\phi$
Solve Eq.② for $\sin\phi$ } Square and add
$\sin^2\phi + \cos^2\phi = 1$

Then solve for θ and finally ϕ:
$\theta = 36.9^\circ$
$\phi = 26.6^\circ$

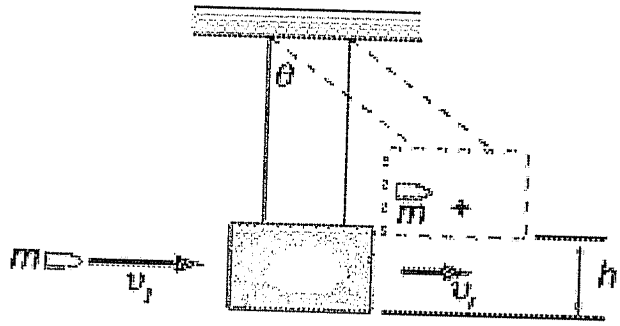


Diagram of a ballistic pendulum. Note that v_f is the velocity of the system right after the perfectly inelastic collision.

A method used to measure the speed of a projectile such as a bullet.

Bullet: mass = m
speed = v_1 (initially)

Block: mass = $M \gg m$
speed = 0 (initially)

After collision, mass $(m+M)$ moves up a height h.

Collision in two parts:

i) Collision and bullet stopped in block.
ii) Motion of block and bullet to maximum height h.
— Recoil —

i) Collision:

- collision time is short
- block does not move during collision
- no net external force during collision which is perfectly inelastic and momentum is conserved.
- The velocity right after the collision is given by:

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$$mv_1 = (m+M)v_f \quad ①$$

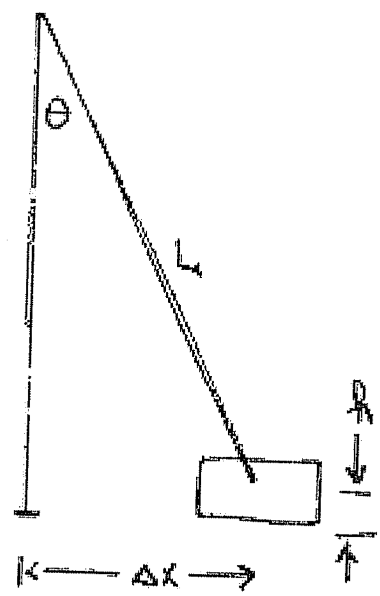
ii) Recoil:

- after collision the system $(m+M)$ has a Kinetic Energy.
- Energy is conserved.
- KE at the bottom is transformed to PE in the block and the bullet at the height h.

$$\frac{1}{2}(m+M)v_f^2 = (m+M)gh$$

$$\therefore v_f = \sqrt{2gh} \quad ②$$

From ① $\quad v_1 = \dfrac{(m+M)}{m}\sqrt{2gh}$



For $\theta \ll 1$

$\Delta x = L\sin\theta \simeq L\theta \Rightarrow \theta \sim \dfrac{\Delta x}{L}$

$h = L(1-\cos\theta)$

$\sim \dfrac{L\theta^2}{2} \sim \dfrac{(\Delta x)^2}{2L}$

$$v_1 = \left(\frac{m+M}{m}\right)\sqrt{2g\left(\frac{\Delta x^2}{2L}\right)}$$

$$= \left(\frac{m+M}{m}\right)\sqrt{\frac{g}{L}}\,(\Delta x)$$

$m = 2.7\,gm \quad L = 1.14\,m$

$M = 3840\,gm \quad \Delta x = 6.5\,cm$

$v_1 = 293\,m/s$

<u>Ballistic Pendulum / Kinetic Energies</u>

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$$K_i = \frac{1}{2} m v_1^2$$

$$K_f = \frac{1}{2}(m+M) v_f^2 = \frac{1}{2}(m+M)\left(\frac{m}{m+M}\right)^2 v_1^2 = \frac{m^2}{2(m+M)} v_1^2$$

$$\frac{K_f}{K_i} = \frac{m}{m+M} \lll 1$$

Most of the initial KE is lost.

<u>Collision Time</u>

Assume bullet deccelerates uniformly in a distance of 0.10 m.

$$v_f^2 - v_0^2 = 2aS \qquad v_f \cong 0.0$$

$$a = -\frac{v_0^2}{2S} \simeq -\frac{(300)^2}{2 \times 0.10} \ m/s^2$$

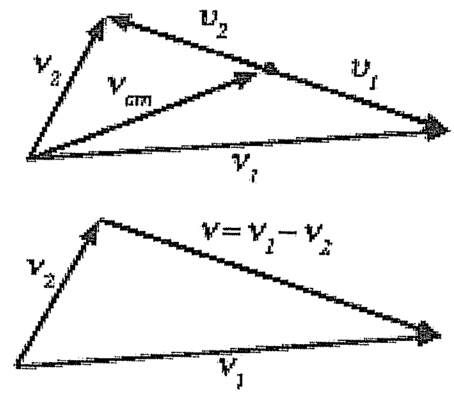
Also $v_f = v_0 + at$

$$t \sim \frac{-v_0}{a} = \frac{-300}{\frac{-(300)^2}{2 \times 0.10}} = .00067\ s$$

Period of pendulum: $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.14}{9.81}} = 2.14\ s$

$t \lll T$ [$\therefore$ good approx.]

Center-of-Mass Frame / Collisions / KE



- 2 Particle System

$\vec{u}_1, \vec{u}_2$: cm velocities

$\vec{v}_1, \vec{v}_2$: Lab velocities

v_{cm} : Velocity of cm.

We had for the Kinetic Energy

$$K = \frac{1}{2} M v_{cm}^2 + K_{INT}$$

$$= \frac{1}{2} M v_{cm}^2 + \underbrace{\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2}_{\text{Internal Energy}}$$

u_1, u_2 : velocities of particles relative to cm frame

$\frac{1}{2} M v_{cm}^2 \rightarrow$ Translational motion of cm. When no external forces act must be conserved.

$$\vec{u}_1 = \vec{v}_1 - \vec{v}_{cm} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_2 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\vec{u}_2 = \vec{v}_2 - \vec{v}_{cm} = v_2 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = -\frac{m_1 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\therefore K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2)^2$$

Relative velocity.

2nd Term: Internal Energy Represents maximum energy available for a totally inelastic collision.

let $\vec{v}_{rel} = \vec{u}_1 - \vec{u}_2$

$= \vec{v}_1 - \vec{v}_2$ } relative velocity of two particles

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \mu v_{rel}^2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$ $\Rightarrow$ reduced mass for a 2-particle system.

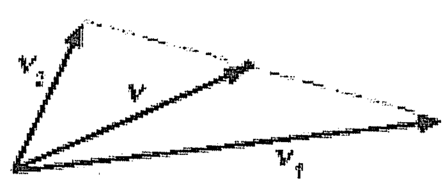
Collisions

In any collision, whether elastic or inelastic, when external forces can be neglected, total momentum is conserved.

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \qquad [\text{constant}]$$

Value of $\vec{P}$ depends on coordinate system, but conservation is true in all frames.

2-Particles



$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{V}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$ (on line joining, $\vec{v}_1, \vec{v}_2$)

Momenta in C-System:

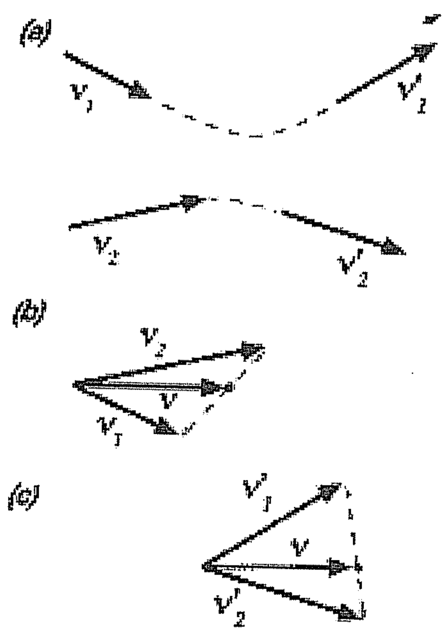
$$\vec{P}_{1c} = m_1 \vec{u}_1 = \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = \mu \, v_{rel}$$

$$\vec{P}_{2c} = m_2 \vec{u}_2 = \frac{-m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = -\mu \, v_{rel}$$

Total Momentum:

$$\vec{P}_c = \vec{P}_{1c} + \vec{P}_{2c} \equiv 0. \qquad [\text{cm-Frame}]$$

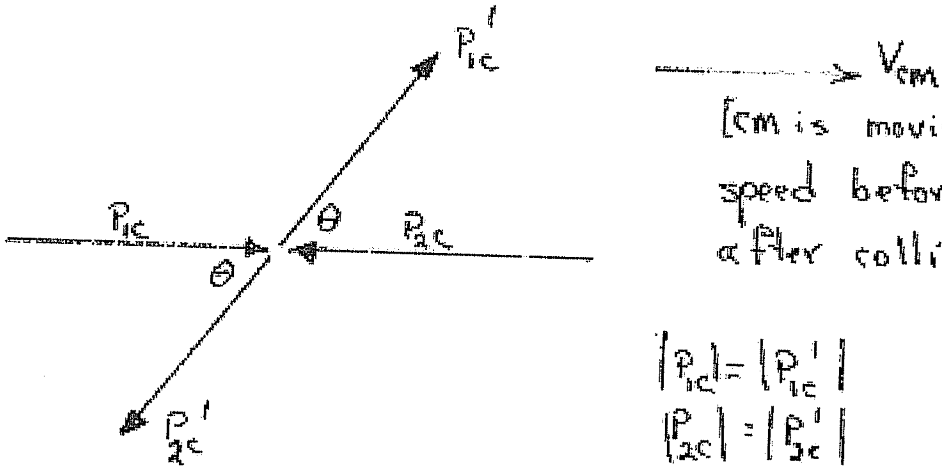
$$\vec{P}_L = \vec{P}_1 + \vec{P}_2 = (m_1 + m_2) V_{cm}$$



What does collision look like in cm frame?

- Initial and final velocities determine the scattering plane.
- Each particle is scattered through the same angle θ.

Elastic:

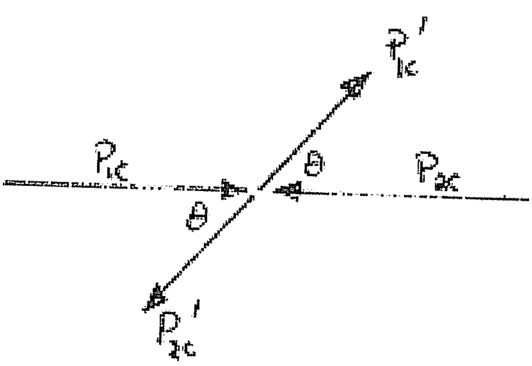


[cm is moving at constant speed before, during and after collision]

$|P_{1c}| = |P_{1c}'|$
$|P_{2c}| = |P_{2c}'|$

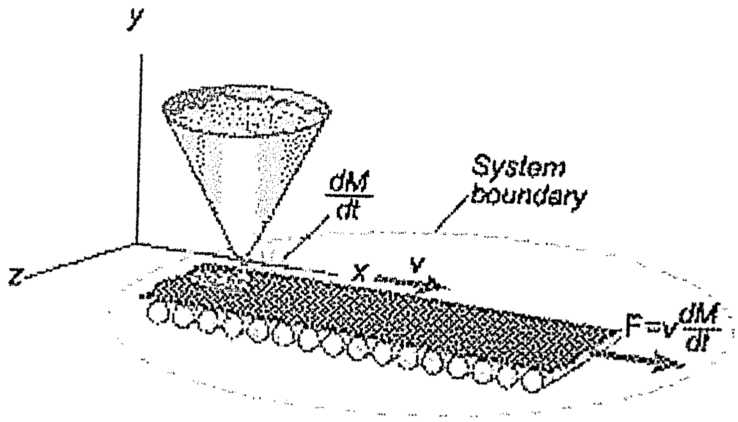
lengths of momentum vectors before and after collision are equal. Energy conserved. Collisions are back-to-back. $\vec{P} = 0$.

Inelastic



Back-to-Back
Momentum magnitudes are reduced following collision.
Loss in KE
Inelastic Collision.

Example: Conveyor Belt



Gravel dropped on belt at rate of 75.0 kg/s.
Belt speed is $v = 2.2$ m/s
What force required to keep belt moving?

Hopper is at rest so $u = 0$

$$\frac{dM}{dt} = 75.0 \text{ kg/s}$$

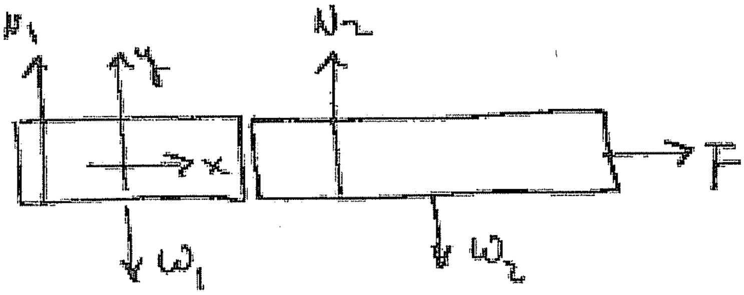
Belt at constant speed

$$\therefore \frac{dv}{dt} = 0$$

$$F_{ext} = \underbrace{M\frac{dv}{dt}}_{=0} - (\underbrace{u}_{u=0} - v)\frac{dM}{dt}$$

$$F_{ext} = v\frac{dM}{dt} = 2.2 \times 75 = 165 \text{ N}$$

(b)



$\vec{a}_1 = 0$

$\vec{a}_2 = \frac{12}{4}\hat{i} = 3\hat{i}$

$\vec{a}_{cm} = ?$ [Same as before ??]

$\vec{a}_1 = 0 \qquad \vec{a}_2 = 3\hat{i}$

$\vec{v}_1 = 0 \qquad \vec{v}_2 = 3t\,\hat{i}$

$\vec{r}_1 = 0 \qquad \vec{r}_2 = (0.3\,m + 1.5t^2)\,\hat{i}$

$M\vec{r}_{cm} = \sum m_i \vec{r}_i$

$M\vec{v}_{cm} = \sum m_i \vec{v}_i$

$M\vec{a}_{cm} = \sum m_i \vec{a}_i$

$\therefore \quad \vec{a}_{cm} = \frac{1}{6}\left[0 + 4 \times 3\hat{i}\right] = 2\hat{i} \quad (m/s^2)$

$\vec{v}_{cm} = \frac{1}{6}\left[0 + 4 \times 3t\hat{i}\right] = 2t\,\hat{i} \quad (m/s)$

$\vec{r}_{cm} = \frac{1}{6}\left[0 + 4[0.3 + 1.5t^2]\hat{i}\right]$

$= (0.2 + 1.0t^2)\,\hat{i}$

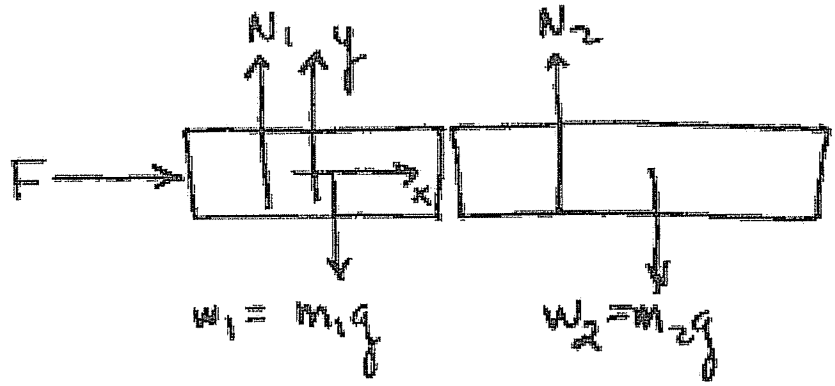
cm in this case does not remain fixed relative to centers of the two blocks.

Example: cm motion

- Uniform blocks

$m_1 = 2kg \quad L_1 = 20cm$

$m_2 = 4kg \quad L_2 = 40cm.$



a) $F = 12\hat{i}$ (N)

$M = m_1 + m_2 = 6$

$$\sum F_{ext} = Ma_{cm}$$

$$\vec{F} + \vec{N_1} + \vec{N_2} + \vec{w_1} + \vec{w_2} = M\vec{a}_{cm}$$

$$12\hat{i} = 6\vec{a}_{cm}$$

$$\vec{a}_{cm} = 2\hat{i} \quad (m/s^2)$$

$$x_{cm} = \frac{2 \times 0 + 4 \times 30}{2+4} = 20 \text{ cm.}$$

$a_{m_1} = 2\hat{i}$

$a_{m_2} = 2\hat{i}$

Forces on any particle have two sources:

- External (from outside the system)
- Internal (from within the system)

From Newton's 3'rd Law the sum over 'Internal' forces cancells in pairs and the overall sum vanishes.

The <u>net</u> force on the system is due only to <u>external</u> forces.

$$\therefore \sum \vec{F}_{ext} = M\vec{a}_c = \frac{d\vec{P}}{dt} .$$

" The cm <u>moves</u> like an <u>imaginary particle of mass M</u> under the influence of the <u>resultant external force</u> on the system".

If $\sum \vec{F}_{ext} = 0$

$$\frac{d\vec{P}}{dt} = M\vec{a}_c = 0$$

and

$$\boxed{\vec{P} = M\vec{v}_c = \text{constant.}} \quad (\text{when } \sum \vec{F}_{ext} = 0)$$

Total linear momentum of a system is conserved if there are no external forces acting on it.

For an isolated system of particles, both the total momentum and velocity of the cm are constant in time.

Motion of the Center-of-Mass

Suppose we take the time derivative of the position vector of the cm. Assuming M is constant (no particles enter or leave system) then we get for the velocity of the cm:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left[m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \cdots\cdots \frac{d\vec{r}_n}{dt} \right]$$

$$= \frac{1}{M}(m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots\cdots m_n \vec{v}_n)$$

$$= \frac{\vec{P}}{M}$$

or $\vec{P} = M\vec{v}_{cm}$

Total momentum of the system is its total mass multiplied by the velocity of the cm. i.e. total $\vec{P}$ is that of a <u>single particle</u> of <u>mass M</u> moving with a <u>velocity $\vec{v}_{cm}$</u>.

Differentiate again to get the acceleration of the cm:

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum m_i \vec{a}_i$$

$$M\vec{a}_{cm} = \sum \vec{F}_i$$

$\vec{F}_i$ ≡ Force on particle i.