Angular Momentum

1. Single Particle

$\vec{L}_o = \vec{r} \times \vec{P}$ definition

↓ respect to a point

$\frac{d\vec{L}_o}{dt} = \vec{r} \times \vec{F} = \vec{\tau}_o = \vec{N}_o$ Equation of motion

2. A System of Particles

$\vec{L}_{tot} = \sum m_i \vec{r}_i \times \vec{v}_i = \sum_i \vec{r}_i \times \vec{P}_i$ definition

$$\frac{d\vec{L}_{tot}}{dt} = \sum \vec{r}_i \times \vec{F}_i^{ext}$$

3. Seperation of the c.m. part

$$\vec{L}_o = \sum \vec{r}_i \times m_i \vec{V}_i$$

Define $\vec{\rho}_i = \vec{r}_i - \vec{R}_{c.m}$

$$\vec{L}_o = \sum_i \vec{r}_i \times m_i \vec{V}_i$$

$$= \sum_i (\vec{\rho}_i + \vec{R}_{c.m}) \times m_i \vec{V}_i$$

$$= \underbrace{\sum_i \vec{\rho}_i \times m_i \vec{v}_i}_{\vec{L}_{c.m.}} + \underbrace{\sum \vec{R}_{c.m} \times m_i \vec{V}_i}_{= \vec{R}_{c.m} \times \vec{P}}$$

$\vec{L}_{c.m.}$ ⇑ definition

$$\frac{d\vec{L}_o}{dt} = \frac{d}{dt}\vec{L}_{c.m.} + \frac{d}{dt}(\vec{R}_{c.m.} \times \vec{P})$$

$$\frac{d}{dt}(\vec{R}_{c.m.} \times \vec{P}) = \frac{dR_{c.m.}}{dt} \times \vec{P} + \vec{R}_{c.m} \times \frac{d\vec{P}}{dt}$$

$$\frac{dR_{c.m.}}{dt} \times \vec{P} = \frac{1}{M}\vec{P} \times \vec{P} = 0$$

$$\vec{R}_{c.m} \times \frac{d\vec{P}}{dt} = \vec{R}_{c.m} \times F_{tot}^{ext} = \vec{\tau}_{tot}^{ext} - \vec{\tau}_{c.m}^{ext}$$

$$\vec{\tau}_{tot} = \sum \vec{r}_i \times \vec{F}_i^{ext}$$

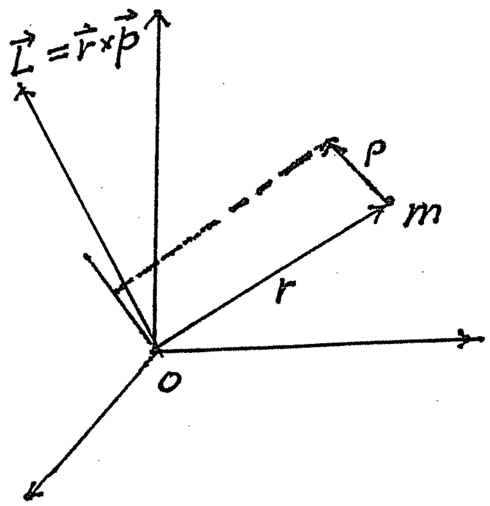
$$= \sum_i (\vec{R}_{c.m} + \rho_i) \times F_i^{ext}$$

$$= (\vec{R}_{cm} \times \vec{F}_{tot} + \sum_i \rho_i \times \vec{F}_i^{ext})$$

$$\frac{d\vec{L}_o}{dt} = \tau_{tot}^{ext} \quad ; \quad \frac{d\vec{L}_{c.m}}{dt} = \tau_{c.m}^{ext}$$

Angular Momentum for a Particle

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$



Since $\vec{L}$ depends on the reference point O

↓ should say

the angular momentum with respect to O

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

↓ torque on the particle with respect to the point O

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{\tau} \,(= \vec{N})$$

Angular momentum theorem

$$\frac{d\vec{L}}{dt} = \vec{N} = \vec{\tau}$$

The rate of change of the angular momentum of a particle around some point O equals the torque on the particle with respect to O.

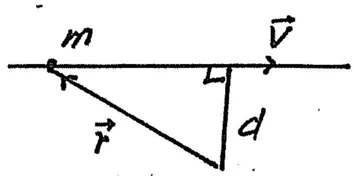
Conservation of Angular Momentum

If no torque acts on a particle, the angular momentum of that particle with respect to O is constant in time.

Example

(i) Central force problem
↓
the orbits of the planets around the Sun are planar
and
Kepler's second law

(ii) Angular momentum is not solely associated with closed orbits.



A free particle of mass m moves along a straight line with velocity $\vec{v}$ in an inertial frame

$$\vec{L}_o = \vec{r} \times m\vec{v}$$

↓
⊥ to the plane pointing into the plane of the paper

$$L_o = |\vec{L}_o| = m v d$$

↑ (d) is the ⊥ distance from the line to O

We have used the result when we discussed the Rutherford Scattering.

Torque and Angular Momentum Around an Axis

Torque around an axis is simply the projection of the torque vector on the line.

$$\vec{\tau} = \vec{r} \times F$$
$$\parallel$$
$$\vec{N}$$

In Cartesian coordinates

$$N_x = yF_z - zF_y$$
$$N_y = zF_x - xF_z$$
$$N_z = xF_y - yF_x$$

$\downarrow$

projection of $\vec{N}$ on z axis

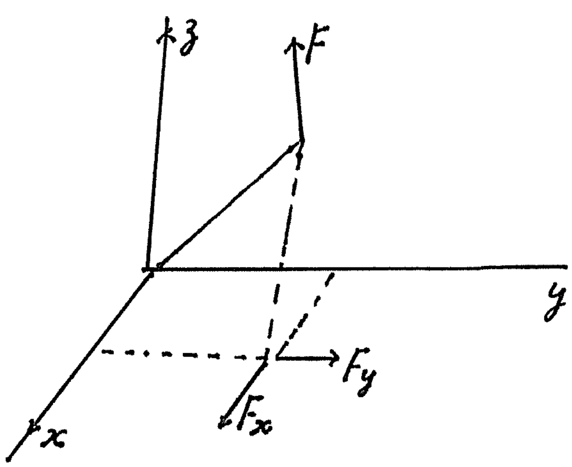
$\Downarrow$

torque around the z axis

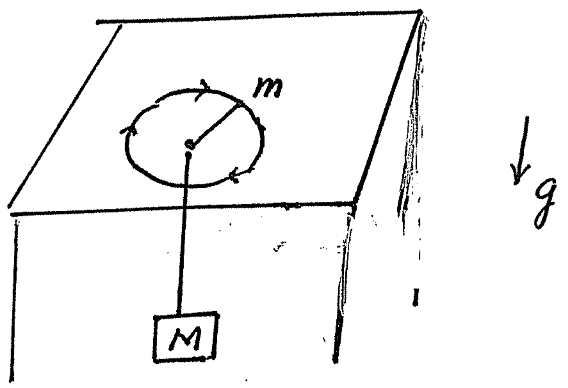
F_y acts over the arm x $\Longrightarrow$ "create" a rotation in the positive direction around the z-axis

F_x acts over the arm y $\Rightarrow$ "create" a rotation in the negative direction around the z-axis

In angular momentum around some axis, this is the projection of $\vec{L}$ onto the axis



Example



Particle of mass in circular motion on a smooth horizontal table

↓

uniform circular motion with velocity v_0 radius R_0

$R_0 \to R_1$

(a) Calculate v_1

(b) Calculate increase in K.E. of the particle

(c) Calculate the work done by the string.

(a) string force has no torque around the center of the circle ⇒ angular momentum is conserved

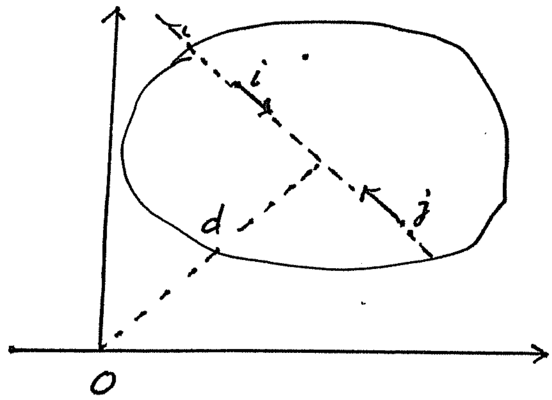
$$m v_0 R_0 = m v_1 R_1 = m v r$$

(b) $\Delta K.E. = \frac{1}{2} m (v_1^2 - v_0^2) = \frac{1}{2} m v_0^2 \left(\frac{R_0^2}{R_1^2} - 1 \right)$

(c) $W = \int \vec{F} \cdot d\vec{r} = -\int_{R_0}^{R_1} m \frac{v^2}{r} dr$

$$= -\int_{R_0}^{R_1} m \frac{v_0^2 R_0^2}{r^2 r} dr = \frac{1}{2} m v_0^2 \left[\frac{R_0^2}{R_1^2} - 1 \right]$$

The Angular Momentum Theorem for a System of Particles



The contribution of the internal force to the torque vanishes

$$\vec{N}_{ij} = \vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}$$

$$= (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \qquad (\vec{F}_{ij} = -\vec{F}_{ij})$$

$$= 0$$

since $(\vec{r}_i - \vec{r}_j)$ and $\vec{F}_{ij}$ are parallel.

$$\Rightarrow \quad \frac{d}{dt} \vec{L}^{tot} = \sum_i N_i^{ext}$$

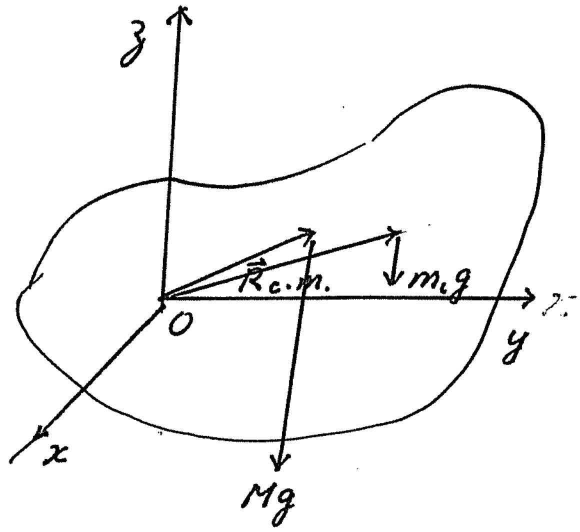
The time derivative of total angular momentum for a particle system relative to a point O fixed in an inertial frame, equals the sum of the torques of the external forces, around O

When the sum of external torque around a point O (fixed in an inertial frame) vanishes, the total angular momentum of the particle system relative to the point O is a constant of motion.

No external forces act on a system of particles $\Rightarrow$ a closed system.

For a closed system the total angular momentum is conserved.

Center of Gravity



$$\vec{N}_o = \sum m_i \vec{r}_i \times \vec{g}$$
$$= M\vec{R}_{c.m.} \times \vec{g}$$
$$= \vec{R}_{c.m.} \times Mg$$

The total torque of gravity on a body can be calculated by assuming the entire mass to be concentrated at the center of mass.

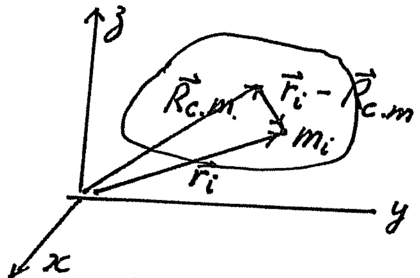
⇒ center of mass ⟷ center of gravity

⇒ torque of gravity about the C.M is zero

↓

a body in a homogeneous gravitational field and supported at the C.M. is in equilibrium.

Angular Momentum Around the Center of Mass



$$\vec{L}_o = \sum \vec{r}_i \times m_i \vec{V}_i$$

$$= \sum \underbrace{(\vec{r}_i - \vec{R}_{c.m})}_{\rho_i} \times m_i \vec{V}_i + \sum \vec{R}_{c.m.} \times m_i \vec{V}_i$$

$$= \vec{L}_{c.m.} + \vec{R}_{cm} \times \vec{P}$$

$\vec{L}_{c.m.}$: independent of the choice of origin.

$\vec{R}_{cm} \times \vec{P}$: depend on choice of the origin O

$$= \vec{R}_{c.m} \times \vec{P} + \sum \rho_i \times \vec{P}_i$$

$$\frac{dL_o}{dt} = \frac{d\vec{L}_{c.m.}}{dt} + \frac{d}{dt}(\vec{R}_{c.m} \times \vec{P})$$

$$\vec{\tau}_{tot} = \sum \vec{r}_i \times \vec{F}_i = \sum_i (\vec{R}_{c.m} + \rho_i) \times \vec{F}_i$$

$$= \sum_i \vec{R}_{c.m} \times \vec{F}_i + \sum (\rho_i \times \vec{F}_i)$$

$$= \vec{R}_{c.m.} \times \vec{F}_{tot} + \underbrace{\sum_i \rho_i \times \vec{F}_i}_{\vec{\tau}_{cm}}$$

$$\frac{d\vec{L}_o}{dt} = \frac{d\vec{L}_{c.m}}{dt} + \frac{d}{dt}(\vec{R}_{c.m.} \times \vec{P})$$

$$\frac{d}{dt}(\vec{R}_{c.m.} \times \vec{P}) = \frac{d\vec{R}_{c.m}}{dt} \times \vec{P} + \vec{R}_{c.m} \times \frac{d\vec{P}}{dt}$$

$$= \frac{1}{M} \vec{P} \times \vec{P} + \vec{R}_{c.m} \times \vec{F}_{tot}$$

$$\Rightarrow \quad \vec{R} \times \vec{F}_{tot} = \frac{d}{dt}(\vec{R} \times \vec{P})$$

and

$$\frac{d\vec{L}_{c.m}}{dt} = \frac{d}{dt}\vec{L}_{c.m} = \vec{\tau}^{\,ext}_{c.m}$$

10-4 CONSERVATION OF ANGULAR MOMENTUM

In Eq. 10-9, we found that the time rate of change of the total angular momentum of a system of particles about a point fixed in an inertial reference frame (or about the center of mass) is equal to the net *external* torque acting on the system; that is

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}. \quad (10\text{-}9)$$

If the net external torque acting on the system is zero, then the angular momentum of the system does not change with time ($d\vec{L}/dt = 0$). Thus

$$\vec{L} = \text{a constant} \quad \text{or} \quad \vec{L}_i = \vec{L}_f. \quad (10\text{-}15)$$

In this case the initial angular momentum is equal to the final angular momentum. Equation 10-15 is the mathematical statement of the principle of *conservation of angular momentum:*

If the net external torque acting on a system is zero, the total vector angular momentum of the system remains constant.

This is the second of the major conservation laws we have discussed. Along with conservation of linear momentum, conservation of angular momentum is a general result that is valid for a wide range of systems. It holds true in both the relativistic limit and in the quantum limit. No exceptions have ever been found.

Like conservation of linear momentum in a system on which the net external *force* is zero, conservation of angular momentum applies to the total angular momentum of a system of particles on which the net external *torque* is zero. The angular momentum of individual particles in a system may change due to *internal* torques (just as the linear momentum of each particle in a collision may change due to *internal* forces), but the total remains constant.

Angular momentum is (like linear momentum) a *vector* quantity so that Eqs. 10-15 is equivalent to three one-dimensional equations, one for each coordinate direction through the reference point. Conservation of angular momentum therefore supplies us with three conditions on the motion of a system to which it applies. *Any component of the angular momentum will be constant if the corresponding component of the torque is zero;* it might be the case that only one of the three components of torque is zero, which means that only one component of the angular momentum will be constant, the other components changing as determined by the corresponding torque components.

For a system consisting of a rigid body rotating with angular speed ω about an axis (the z axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega, \quad (10\text{-}16)$$

where L_z is the component of the angular momentum along the rotation axis and I is the rotational inertia for this same axis. *If no net external torque acts, then L_z must remain constant.* If the rotational inertia I of the body changes (from I_i to I_f)—for example, by a change in the distance of parts of the body from the axis of rotation—there must be a compensating change in ω from ω_i to ω_f. The principle of conservation of angular momentum in this case is expressed as $L_{iz} = L_{fz}$ or

$$I_i\omega_i = I_f\omega_f. \quad (10\text{-}17)$$

Equation 10-17 holds not only for rotation about a fixed axis but also for rotation about an axis through the center of mass of a system that moves so that the axis always remains parallel to itself (see the discussion at the beginning of Section 9-7).

Conservation of angular momentum is a principle that regulates a wide variety of physical processes, from the subatomic world to the motion of acrobats, divers, and ballet dancers, to the contraction of stars that have run out of fuel, and to the condensation of galaxies. The following examples show some of these applications.

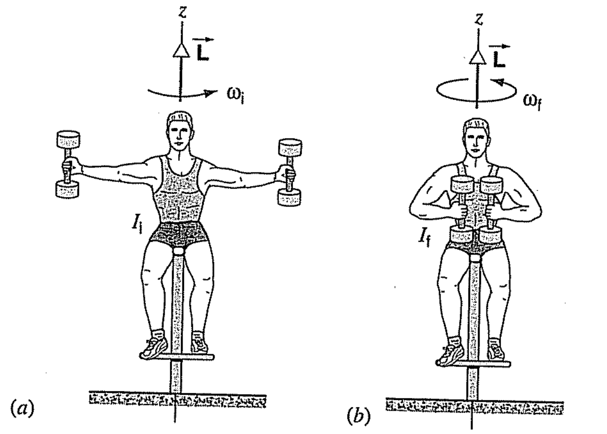


FIGURE 10-12. (*a*) In this configuration, the system (student + weights) has a larger rotational inertia and a smaller angular velocity. (*b*) Here the student has pulled the weights inward, giving a smaller rotational inertia and hence a larger angular velocity. The angular momentum $\vec{L}$ has the same value in both situations.

The Spinning Skater

A spinning ice skater pulls her arms close to her body to spin faster and extends them to spin slower. When she does this, she is applying Eq. 10-17. Another application of this principle is illustrated in Fig. 10-12, which shows a student sitting on a stool that can rotate freely about a vertical axis. Let the student extend his arms holding the weights, and we will set him into rotation at an angular velocity ω_i. His angular momentum vector $\vec{L}$ lies along the vertical axis (z axis) in the figure.

The system consisting of student + stool + weights is an isolated system on which no external vertical torque acts. The vertical component of angular momentum must therefore be conserved.

When the student pulls his arms (and the weights) closer to his body, the rotational inertia of his system is reduced from its initial value I_i to a smaller value I_f, because the weights are now closer to the axis of rotation. His final angular speed, from Eq. 10-17, is $\omega_f = \omega_i(I_i/I_f)$, which is greater than his initial angular velocity (because $I_f < I_i$), and the student rotates faster. To slow down, he need only extend his arms again.

The Springboard Diver*

Figure 10-13*a* shows a diver leaving the springboard. As she jumps, she pushes herself slightly forward so that she acquires a small rotational speed, just enough to carry her head-first into the water as her body rotates through one-half revolution during the arc.

While she is in the air, no external torques act on her to change her angular momentum about her center of mass. (The only external force, gravity, acts *through* her center of mass and thus produces no torque about that point. We neglect air resistance, which could produce a net torque and change her angular momentum.) When she pulls her body into the *tuck position,* she lowers her rotational inertia, and therefore according to Eq. 10-17 her angular velocity must increase. The increased angular velocity enables her to complete $1\frac{1}{2}$ revolutions where she had previously completed only one-half revolution (Fig. 10-13*b*). At the end of the dive, she pulls back out into the *layout position* and slows her angular speed as she enters the water.

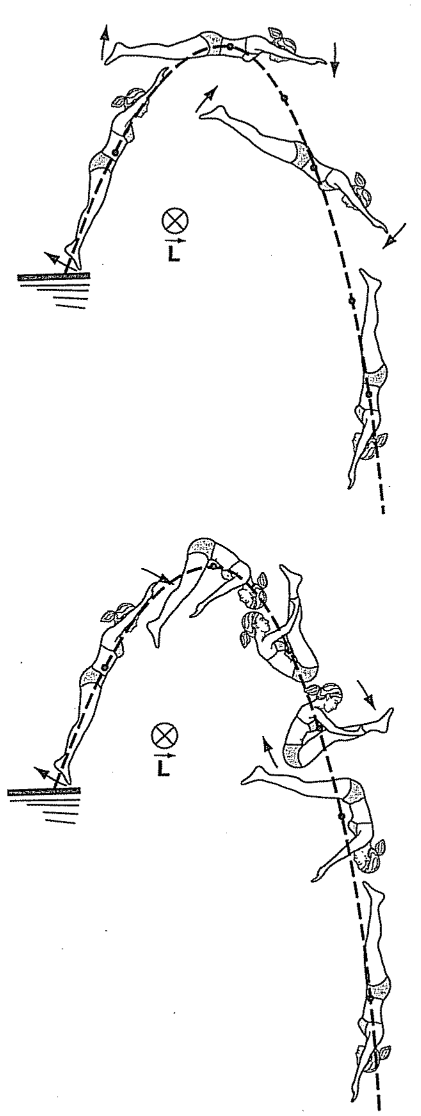


FIGURE 10-13. (*a*) A diver leaves the springboard in such a way that the springboard imparts to her an angular momentum $\vec{L}$. She rotates about her center of mass (indicated by the dot) by one-half revolution as the center of mass follows the parabolic trajectory. (*b*) By entering the tuck position, she reduces her rotational inertia and thus increases her angular velocity, enabling her to make $1\frac{1}{2}$ revolutions. The external forces and torques on her are the same in (*a*) and (*b*), as indicated by the constant value of the angular momentum $\vec{L}$.

* See "The Mechanics of Swimming and Diving," by R. L. Page, *The Physics Teacher,* February 1976, p. 72; "The Physics of Somersaulting and Twisting," by Cliff Frohlich, *Scientific American,* March 1980, p. 155.

The Rotating Bicycle Wheel

Figure 10-14*a* shows a student seated on a stool that is free to rotate about a vertical axis. The student holds a bicycle wheel that has been set spinning. When the student inverts the spinning wheel, the stool begins to rotate (Fig. 10-14*b*).

No net vertical torque acts on the system consisting of student + stool + wheel, and therefore the vertical (z) component of the total angular momentum of the system must remain constant. Initially, the z component of the angular momentum of the rotating wheel is $+L_w$. The total initial angular momentum of the system is then $L_{iz} = +L_w$. When the wheel is turned over (as a result of an *internal* torque in the system), the z component of the total angular momentum must remain constant. The z component of the final total angular momentum is $L_{fz} = L_s + (-L_w)$, where L_s is the angular momentum of student + stool and $-L_w$ is the angular momentum of the inverted wheel. Conservation of angular momentum (in the absence of external torque) requires that $L_{iz} = L_{fz}$, and so the student and stool will rotate with angular momentum $L_s = +2L_w$.

We can also consider this situation from the standpoint of two separate systems, one being the wheel and the other being the student + stool. Neither of these systems is now isolated: the student's hand forms the connection between them. When the student attempts to invert the wheel, she must apply a torque to change the wheel's angular momentum. The force she exerts on the wheel to produce that torque is returned by the wheel as a reaction force on her, by Newton's third law. This external force on the system of student + stool causes that system to rotate. In this view the student exerts an external torque on the wheel to change its angular momentum, while the wheel exerts a torque on the student to change her angular momentum. If we consider the complete system consisting of student + stool + wheel, as we did above, this torque is an internal torque that did not enter into out calculations. Whether we consider the torque as internal or external depends on how we define our system.

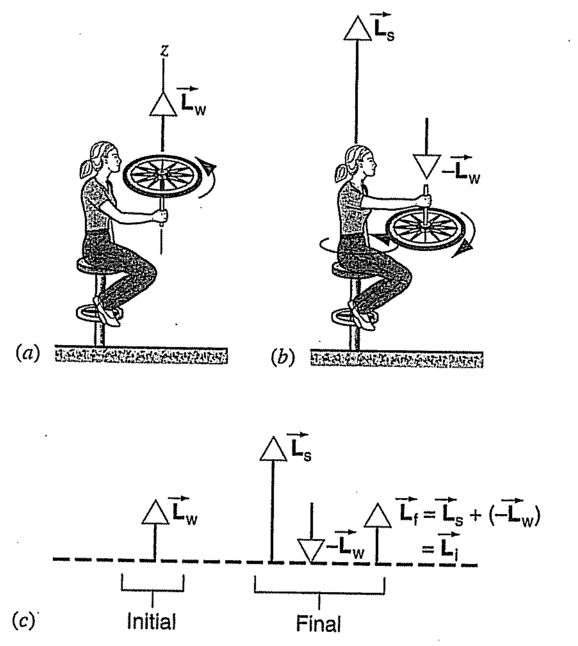


FIGURE 10-14. (*a*) A student holds a rotating bicycle wheel. The total angular momentum of the system is $\vec{L}_w$. (*b*) When the wheel is inverted, the student begins to rotate. (*c*) The total final angular momentum must be equal to the initial angular momentum.

The Stability of Spinning Objects

Consider again Fig. 10-3*b*. An object moving with linear momentum $\vec{p} = M\vec{v}$ has a *directional stability;* a deflecting force provides the impulse, corresponding to a sideways momentum increment $\Delta\vec{p}_\perp$, and as a result the direction of motion is changed by an angle $\theta = \tan^{-1}(\Delta p_\perp/p)$. The larger is the momentum p, the smaller is the angle θ. The same deflecting force is less effective in diverting an object with large linear momentum than it is in diverting an object with small linear momentum.

Angular momentum provides an object with *orientational stability* in much the same way. A rapidly spinning object (as in Fig. 10-4*b*) has a certain angular momentum $\vec{L}$. A torque $\vec{\tau}$ perpendicular to $\vec{L}$ changes the direction of $\vec{L}$, and therefore the direction of the axis of rotation, by an angle $\theta = \tan^{-1}(\Delta L_\perp/L)$. Once again, the larger is the angular momentum L, the less successful a given torque will be in changing the direction of the axis of the spinning object.

When we give an object rotational angular momentum about a symmetry axis, we in effect stabilize its orientation and make it more difficult for external forces to change its orientation. There are many common examples of this effect. A riderless bicycle given a slight push is able to remain upright for a far longer distance than we might expect. In this case it is the angular momentum of the spinning wheels that gives the stability. Minor bumps and curves of the roadway, which might otherwise topple or deflect a nonrotating object balanced on so narrow a base as a bicycle tire, have less effect in this case because of the tendency of the angular momentum of the wheels to fix their orientation.*

A football is thrown for a long forward pass such that it rotates about an axis that is roughly parallel to its translational velocity. This stabilizes the orientation of the football and keeps it from tumbling, which makes it possible to throw more accurately and catch more effectively. It also keeps the smallest profile of the football in the forward direction, thereby minimizing air resistance and increasing the range.

It is important to stabilize the orientation of a satellite, particularly if it is using its thrusters to move to a specific orbital position (Fig. 10-15). The orientation might be changed, for example, by friction from the thin residual atmosphere at orbital altitudes, by the solar wind (a beam of charged particles from the Sun), or by impacts from tiny

* See "The Stability of the Bicycle," by David E. H. Jones, *Physics Today,* April 1970, p. 34.

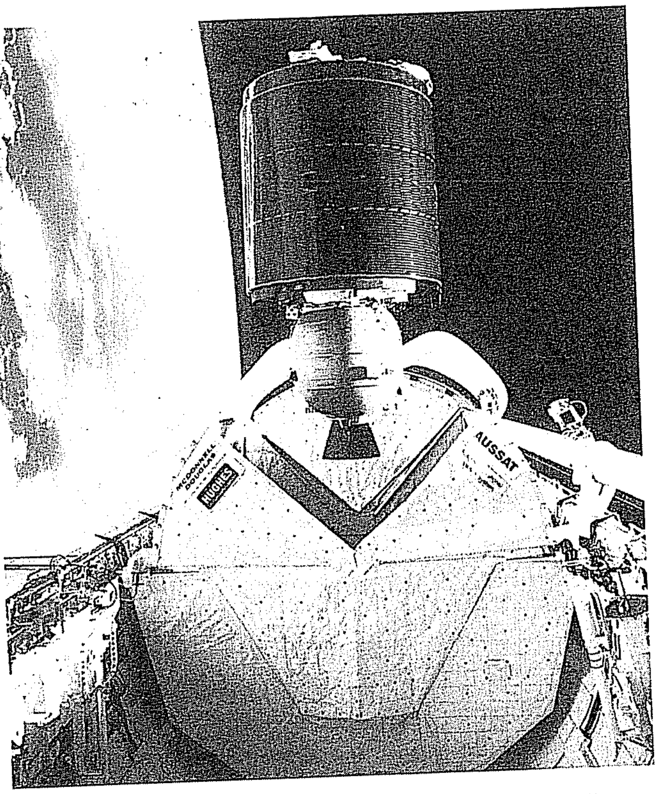


FIGURE 10-15. Deployment of a communications satellite from the bay of the space shuttle. The satellite is made to spin about its central axis (the vertical axis in this photo) to stabilize its orientation in space as it makes its way upward to its geosynchronous orbit.

meteoroids. To reduce the effects of such encounters, the craft is made to spin about an axis thereby stabilizing its orientation.

Collapsing Stars

Most stars rotate, as our Sun does. It turns once on its axis every month or so. (The Sun is a ball of gas and does not rotate quite like a rigid body; the regions near the poles have a rotational period of about 37 days, but the equator rotates once every 26 days.) The Sun is kept from collapsing by *radiation pressure,* in essence the effect of impulsive collisions of the emerging radiation with the atoms of the Sun. When the Sun's nuclear fuel is used up, the radiation pressure will vanish, and the Sun will begin to collapse, its density correspondingly increasing. At some point the density will become so great that the atoms simply cannot be crowded any closer together, and the collapse will be halted. This is the *white dwarf* stage, where the Sun will end its life.

In stars more than about 1.4 times as massive as the Sun, however, the gravitational force is so strong that the atoms cannot prevent further collapse. The atoms are in effect crushed by gravity, and the collapse continues until the nuclei are touching one another. The star has in effect become one giant atomic nucleus; it is called a *neutron star.* The radius of a neutron star of about 1.5 solar masses is about 11 km.

Suppose the star began its collapse like our Sun, rotating once every month. The forces during the collapse are clearly internal forces, which cannot change the angular momentum. The final angular speed is therefore related to the initial angular speed by Eq. 10-17: $\omega_f = \omega_i(I_i/I_f)$. The ratio of the rotational inertias will be the same as the ratio of the squares of the radii: $I_i/I_f = r_i^2/r_f^2$. If the initial radius were about the same as the Sun's (about 7×10^5 km), then

$$\frac{I_i}{I_f} = \frac{r_i^2}{r_f^2} = \frac{(7 \times 10^5 \text{ km})^2}{(11 \text{ km})^2} = 4 \times 10^9.$$

That is, its rotational speed goes from once per month to 4×10^9 per month, or more than 1000 revolutions per second!

Neutron stars can be observed from Earth, because (again like the Sun) they have magnetic fields that trap electrons, and the electrons are accelerated to very high tangential speeds as the star rotates. Such accelerated electrons emit radiation, which we see on Earth somewhat like a searchlight beacon as the star rotates. These sharp pulses of radiation earned rotating neutron stars the name *pulsars.* A sample of the radiation observed from a pulsar is shown in Fig. 10-16.

Conservation of angular momentum applies to a wide variety of astrophysical phenomena. The rotation of our galaxy, for example, is a result of the much slower initial rotation of the gas cloud from which the galaxy condensed; the rotation of the Sun and the orbits of the planets were determined by the original rotation of the material that formed our solar system.

SAMPLE PROBLEM 10-4. A 120-kg astronaut, carrying out a "space walk," is tethered to a spaceship by a fully extended cord 180 m long. An unintended operation of the propellant pack causes the astronaut to acquire a small tangential velocity of 2.5 m/s. To return to the spacecraft, the astronaut begins pulling along the tether at a slow and constant rate. With what force must the astronaut pull at distances of (*a*) 50 m and (*b*) 5 m from the spacecraft? What will be the astronaut's tangential speed at these points?

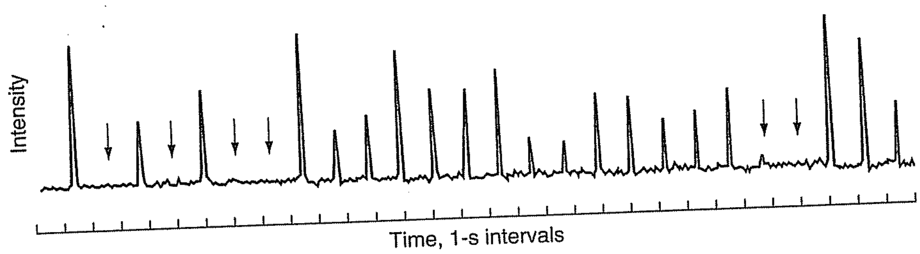


FIGURE 10-16. Electromagnetic pulses received on Earth from a rapidly rotating neutron star. The vertical arrows suggest pulses too weak to be detected. The interval between pulses is remarkably constant, being equal to 1.187,911,164 s.

Solution No external torques act on the astronaut, so that conservation of angular momentum holds. That is, the astronaut's initial angular momentum relative to the spaceship as origin ($Mv_i r_i$) when beginning to pull on the tether must equal the angular momentum (Mvr) at any point in the motion. Thus

$$Mvr = Mv_i r_i$$

or

$$v = \frac{v_i r_i}{r}.$$

The centripetal force at any stage is given by

$$F = \frac{Mv^2}{r} = \frac{Mv_i^2 r_i^2}{r^3}.$$

Initially, the required centripetal force is

$$F = \frac{(120\text{ kg})(2.5\text{ m/s})^2}{180\text{ m}} = 4.2\text{ N (about 1 lb)}.$$

(*a*) When the astronaut is 50 m from the spacecraft, the tangential speed is

$$v = \frac{(2.5\text{ m/s})(180\text{ m})}{50\text{ m}} = 9.0\text{ m/s},$$

and the centripetal force is

$$F = \frac{(120\text{ kg})(2.5\text{ m/s})^2(180\text{ m})^2}{(50\text{ m})^3} = 194\text{ N (about 44 lb)}.$$

(*b*) At 5 m from the ship, the speed goes up by a factor of 10 to 90 m/s, while the force increases by a factor of 10^3 to 1.94×10^5 N, or about 22 tons! It is clear that the astronaut cannot exert such a large force to return to the spacecraft. Even if the astronaut were being pulled toward the ship by a winch from within the spacecraft, the tether could not withstand such a large tension; at some point it would break and the astronaut would go shooting into space with whatever the tangential speed was at the time the tether broke. Moral: Space-walking astronauts should avoid acquiring tangential velocity. What could the astronaut do to move safely back to the ship?

SAMPLE PROBLEM 10-5. A turntable consisting of a disk of mass 125 g and radius 7.2 cm is spinning with an angular speed of 0.84 rev/s about a vertical axis (Fig. 10-17*a*). An identical, initially nonrotating disk is suddenly dropped onto the first. The friction between the two disks causes them eventually to rotate at the same speed. A third identical nonrotating disk is then dropped onto the combination, and eventually all three are rotating together (Fig. 10-17*b*). What is the angular speed of the combination?

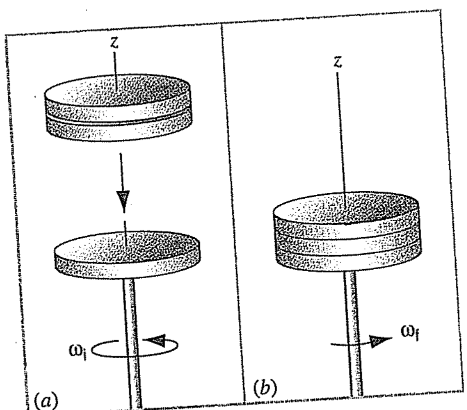


FIGURE 10-17. Sample Problem 10-5. (*a*) A disk is spinning with initial angular velocity ω_i. (*b*) Two identical disks, neither of which is initially rotating, are dropped onto the first, and the entire system then rotates with angular velocity ω_f.

Solution This problem is the rotational analogue of the completely inelastic collision, in which objects stick together (see Section 6-5). There is no net vertical external torque, so the vertical (z) component of angular momentum is constant. The frictional force between the disks is an internal force, which cannot change the angular momentum. Thus Eq. 10-17 applies, and we can write $I_i\omega_i = I_f\omega_f$, or

$$\omega_f = \omega_i \frac{I_i}{I_f}.$$

Without doing any detailed calculations, we know that the rotational inertia of three identical disks about their common axis will be three times the rotational inertia of a single disk. Thus $I_i/I_f = \frac{1}{3}$ and

$$\omega_f = (0.84\text{ rev/s})(\tfrac{1}{3}) = 0.28\text{ rev/s}.$$

Rigid Bodies

A system of particles in which the distance between any two particles do not change regardless of the forces acting is called a rigid body.

Translations and Rotations

A displacement of a rigid body is a change from one position to another

If during a displacement all points of the body on some line remain fixed, the displacement is called the rotation

If during a displacement all points of the rigid body move in a line parallel to each other

Chasle's Theorem

The general motion of a rigid body can be considered as a translation plus a rotation about a suitable point which is often taken to be the center of mass.

A rigid body in the form of a triangle ABC [Fig. 9-7] is moved in a plane to position DEF, i.e. the vertices A, B and C are carried to D, E and F respectively. Show that the motion can be considered as a translation plus a rotation about a suitable point.

Choose a point G on triangle ABC which corresponds to the point H on triangle DEF. Perform the translation in the direction GH so that triangle ABC is carried to $A'B'C'$. Using H as center of rotation perform the rotation of triangle $A'B'C'$ through the angle θ as indicated so that $A'B'C'$ is carried to DEF. Thus the motion has been accomplished by a translation plus a rotation.

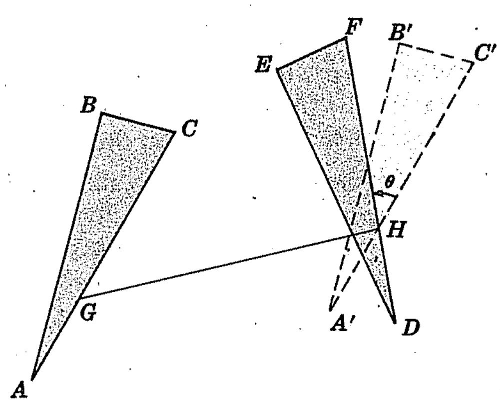


Fig. 9-7

Euler's theorem

A rotation of rigid point of the body is equivalent to a rotation about a line which passes through the point. The line is referred to as the instanteous axis of rotation.

Plane Motion of a Rigid Body

is simplified considerably when all points move parallel to a fixed plane.

In such case two type of motion, i.e., plane motion are possible

1. Rotation about a fixed axis

   The rigid body rotate about a fixed axis ⊥ to the fixed

   The system has only one degree of freedom and only one coordinate is required for describing the motion.

2. Translation parallel to the given plane.

Rotation can be finite or infinitesimal

Finite rotation cannot be represented by vectors since the commutative law fails

However, infinitesimal rotations can be represented by vectors

Give an example to show that finite rotations cannot be represented by vectors.

Let A_x represent a rotation of a body [such as the rectangular parallelepiped of Fig. 9-8(a)] about the x axis while A_y represents a rotation about the y axis. We assume that such rotations take place in a positive or counterclockwise sense according to the right hand rule.

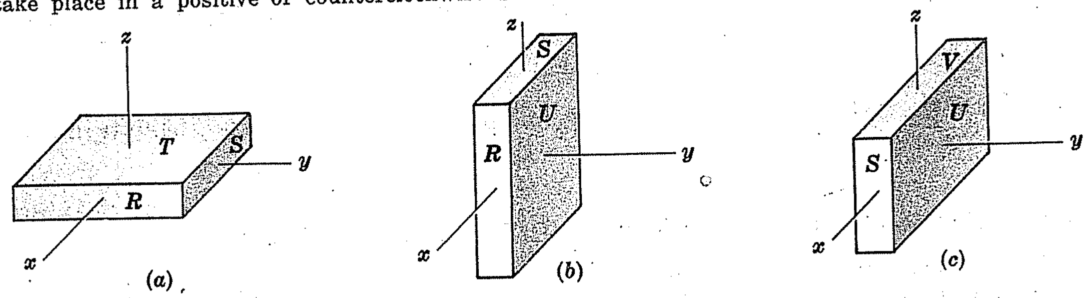


Fig. 9-8

9-11

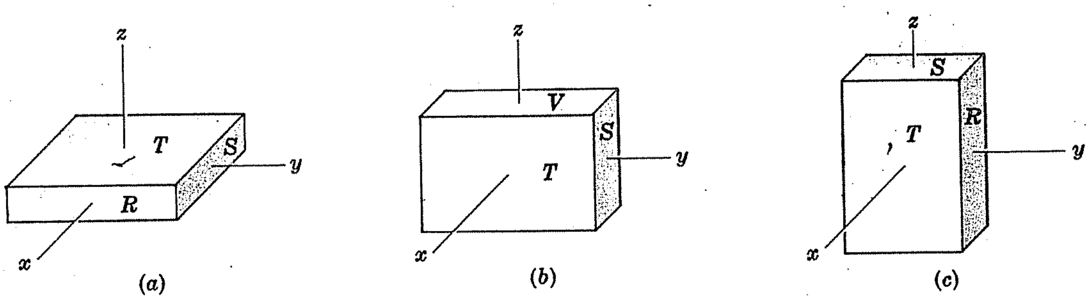


Fig. 9-9

In Fig. 9-8(a) we start with the parallelepiped in the indicated position and perform the rotation A_x about the x axis as indicated in Fig. 9-8(b) and then the rotation about the y axis as indicated in Fig. 9-8(c). Thus Fig. 9-8(c) is the result of the rotation $A_x + A_y$ on Fig. 9-8(a).

In Fig. 9-9(a) we start with the parallelepiped in the same position as in Fig. 9-8(a), but this time we first perform the rotation A_y about the y axis as indicated in Fig. 9-9(b) and then the rotation A_x about the x axis as indicated in Fig. 9-9(c). Thus Fig. 9-9(c) is the result of the rotation $A_y + A_x$ on Fig. 9-9(a).

Since the position of the parallelepiped of Fig. 9-8(c) is not the same as that of Fig. 9-9(c), we conclude that the operation $A_x + A_y$ is not the same as $A_y + A_x$. Thus the commutative law is not satisfied, so that A_x and A_y cannot possibly be represented by vectors.

Rigid Body

1. Definitions
   - Rigid Body
   - Translation
   - Rotation
2. Rotation about a fixed axis
   - $\vec{\omega}$
   - Relation between Angular and Linear velocity and acceleration
   - Rotation kinetic energy
3. Moments of Inertia

Rigid Body Kinematics

Objects in the real world are not point-like particles that we have been dealing with up to now. A real object has a mass distribution associated with its size and shape.

The motion of a real object involves both:

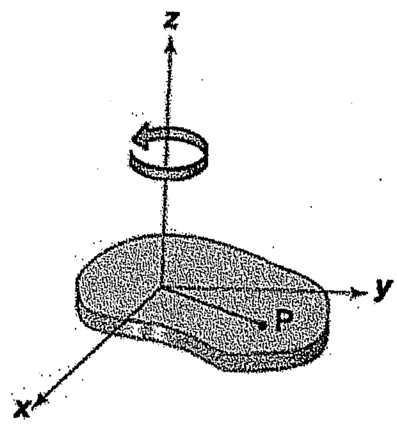
- translational motion of the cm.
- rotational motion about an axis (usually take to be an axis through the cm or some other fixed axis).

We will restrict our discussions to that of rigid bodies. A rigid body is one in which the relative coordinates connecting all the constituent particles remain constant. This is of course an idealized situation.

Rotations about a Fixed Axis

We will initially study the motion of a rigid body rotating about an axis that is fixed in an inertial frame.

Consider motion around the z-axis. Reference point P (which is not on the axis) represents the rotational motion of the body and of its angular position.

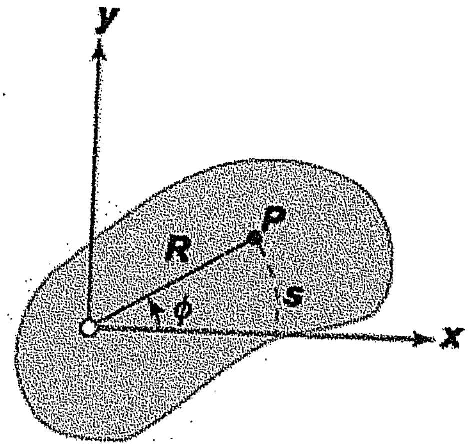


Rotation of a rigid body about a fixed axis (z axis).

Given a reference point P, its angular position is measured by the angle ϕ, between position vector $\vec{r}$ and the x-axis.

As the particle moves in a circle from the positive x-axis ($\varphi = 0$) to the point P, it moves through an arc length

$$s = R\phi$$



z out of the paper

$$\phi(\text{rad}) = \frac{\pi}{180}\,\phi(\text{deg})$$

ϕ = positive counterclockwise

$\phi = 0 \Rightarrow x$-axis

$\phi = 2\pi \Rightarrow x$-axis again.

ϕ: is not a vector [rotations do not commute]

$\vec{d\phi} = d\varphi\,\hat{k}$ [infinitesimal rotation is a vector]

The rotational motion of a body is described by the rate of change of ϕ. In general the position angle is a function of time:

$$\phi = \phi(t)$$

Suppose the particle moves from P to Q. The reference line OP makes an angle ϕ_1 at the time t_1, and an angle ϕ_2 at the time t_2. Define the average angular velocity of the body, $\bar{\omega}$, in the time interval $\Delta t = t_2 - t_1$

22-3

as the ratio of angular displacement $\Delta\phi = \phi_2 - \phi_1$ to Δt.

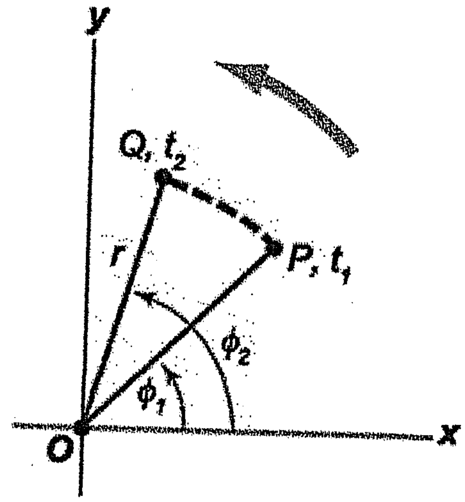
$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t} \quad \text{rad/s or } s^{-1}$$

$\hat{k}$ = unit vector along axis of rotation (z-axis)

$\vec{\omega}$ = points along axis of rotation.
[RH Rule for sign convention]

Analagous to linear velocity, the instantaneous angular velocity, is defined as the limit of this ratio as $\Delta t \to 0$. Becomes time rate-of-change of $\phi(t)$.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt}\hat{k} \quad (s^{-1})$$



A particle on a rotating rigid body moves from *P* to *Q* along the arc of a circle. In the time interval $\Delta t = t_2 - t_1$, the radius vector sweeps out an angle $\Delta\phi = \phi_2 - \phi_1$.

<u>If</u> the angular velocity, ω, is a constant $\omega = \omega_0$, the rate of rotation is often given in terms of the frequency, or number of revolutions per unit time.

1 revolution = $\Delta\phi = 2\pi$ radians

Time per revolution, or period $T = \frac{2\pi}{\omega_0}$ (s)

Frequency of revolution is $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ (Hz)

If the angular velocity of the body is changing with time (i.e. ω is not constant), then there is an angular acceleration.
If the angular velocities are ω_1 and ω_2 at the times t_1 and t_2, the average angular acceleration is

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is the limit of this ratio as $\Delta t \to 0$.

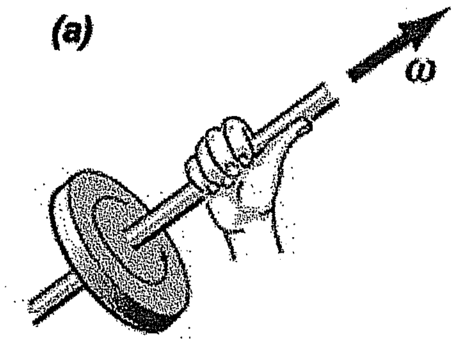
$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\vec{\omega}}{dt} . \qquad (s^{-2})$$

Since $\omega = \frac{d\phi}{dt}$, we also have

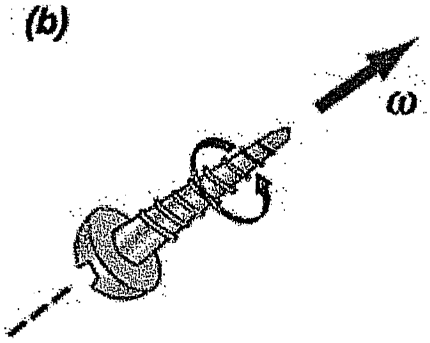
$$\vec{\alpha} = \frac{d^2\vec{\phi}}{dt^2}$$

For rotation about a fixed axis, every particle on the rigid body has the same angular velocity and the same angular acceleration.
The direction of $\vec{\alpha}$ is along the same axis as $\vec{\omega}$. If the axis of rotation is changing then $\vec{\alpha}$ is not in the same direction as $\vec{\omega}$.

Direction - Right Hand Rule.



The right-hand rule for determining the direction of the angular velocity.



The direction of ω is in the direction of advance of a right-handed screw.

Fingers of right hand are wrapped along direction of rotation. Then $\vec{\omega}$ points along thumb.

Direction of $\vec{\alpha}$ is related to $\frac{d\vec{\omega}}{dt}$.

$\frac{d\vec{\omega}}{dt} > 0$ $\quad$ $\vec{\alpha}$ same as $\vec{\omega}$

$\frac{d\vec{\omega}}{dt} < 0$ $\quad$ $\vec{\alpha}$ opposite to $\vec{\omega}$

[Fixed-Axis Rotation]

Rotational Motion with Constant Angular Acceleration

- Assume motion along a fixed axis.
- Ignore vector notation (sign designates direction)
- Results also hold for axis in linear translation

$$\frac{d\omega}{dt} = \alpha \qquad (\alpha = \text{constant})$$

$$\int d\omega = \int \alpha\, dt$$

$$\omega = \alpha t + C$$

If $\omega = \omega_0$ at $t=0$, $\Rightarrow C = \omega_0$.
and

$$\boxed{\omega = \omega_0 + \alpha t} \qquad ①$$

$$\frac{d\phi}{dt} = \omega = \omega_0 + \alpha t$$

$$\int d\phi = \int \omega_0 dt + \alpha \int t\, dt$$

$$\phi = \omega_0 t + \frac{1}{2}\alpha t^2 + C$$

If $\phi = \phi_0$ at $t=0$, $\Rightarrow C = \phi_0$.
and

$$\boxed{\phi = \phi_0 + \omega_0 t + \frac{\alpha t^2}{2}} \qquad ②$$

Solve Eq. ① for t and substitute in Eq. ②

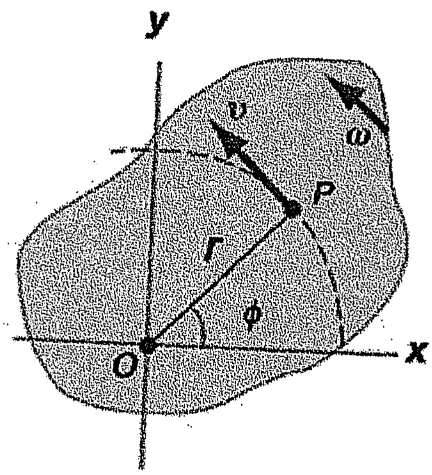
$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)} \qquad ③$$

Motion with constant linear acceleration	Motion with constant angular acceleration
$a = \text{constant}$	$\alpha = \text{constant}$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Relation between Angular and Linear Velocity and Acceleration

As a rigid body rotates about a fixed axis, every particle in the body moves in a circle the center of which is on the axis of rotation.

Consider the point P. P moves in a circle, the linear velocity vector is thus tangent to this circle. Magnitude is ds/dt, where s is distance travelled along the circular path.



As a rigid body rotates around the fixed axis through *O*, the point *P* has a linear velocity *v*, which is always tangent to the circular path of radius *r*.

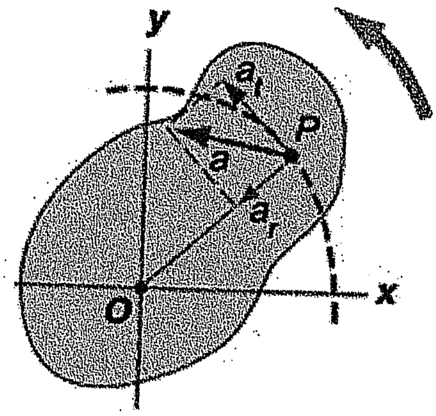
$$s = r\phi \qquad [\phi \text{ in radians}]$$

$$v = \frac{ds}{dt} = r\frac{d\phi}{dt}$$

$$v = r\omega$$

Speed of the particle is directly proportional to its distance from the axis of rotation. The further from the axis the higher its velocity.

To relate the linear acceleration of the point P to the angular acceleration of the rigid body about a fixed axis, we take the time derivative of v:



As a rigid body rotates about a fixed axis through O, the point P experiences a tangential component of acceleration, a_t, and a centripetal component of acceleration, a_r. The total acceleration of this point is $a = a_t + a_r$.

$$a_t = a_{\parallel} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

This is the tangential (parallel) component of the linear acceleration of a point at a distance r from the axis of rotation. It is related to the change in speed of the particle.

Since the particle moves in a circle, we have seen that it also has a radial or centripetal acceleration due to the changing direction of its velocity.

$$a_r = a_{\perp} = \frac{v^2}{r} = r\omega^2$$

Total linear acceleration of the particle is $\vec{a}$:

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

$$a = \sqrt{a_t^2 + a_r^2} = r\sqrt{\alpha^2 + \omega^4} \qquad m/s^2$$

Note:

All points in a rotating rigid body have the same value of ω and the same value of α. Points that are different distances from the axis have different values of v and different values of a_t and a_c.

Example - Rotating Turntable

22-9

Record player rotates at 33 rev/min and takes 20s to come to rest.

a) What is angular acceleration, assuming it is uniform?

$$\omega_0 = \left(33 \frac{\text{rev}}{\text{min}}\right)\left(2\pi \frac{\text{rad}}{\text{rev}}\right) \times \left(\frac{1\text{min}}{60\text{s}}\right) = 3.46 \text{ rad/s}.$$

$$\omega = \omega_0 + \alpha t$$

$$\omega = 0 \text{ at } t = 20\text{s}$$

$$\therefore \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 3.46}{20} = -0.173 \text{ rad/s}^2$$

(<0, decellerating)

b) How many rotations before it comes to rest?

$$\Delta\phi = \phi - \phi_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 3.46(20) - \frac{1}{2}(0.173)\,20^2$$

$$= 34.6 \text{ rad}$$

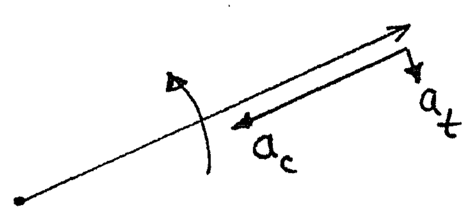
$$= 34.6/2\pi = 5.51 \text{ rev}.$$

c) If rim is at radius $r = 14\text{cm}$, what is the acceleration of a point on the rim at $t = 0$.

$$a_t = r\alpha = 14\text{ cm}\,(0.173 \text{ rad/s}^2) = 2.42 \text{ cm/s}^2$$

$$a_c = r\omega_0^2 = 14\text{cm}\,(3.46 \text{ rad/s})^2 = 168 \text{ cm/s}^2 \quad (t=0)$$

$$a = \sqrt{2.42^2 + 168^2} = 168.0 \text{ cm/s}^2$$



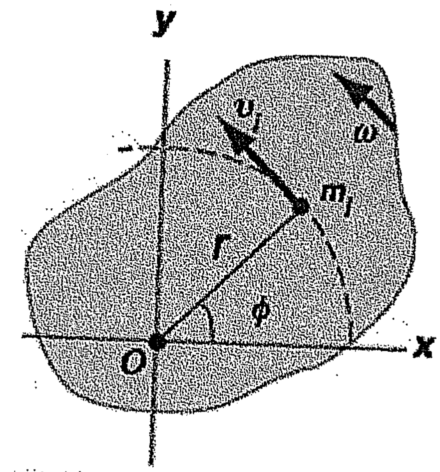
Velocity at rim (t=0):

$$v = r\omega_0 = 14\text{cm} \times 3.46 \text{ rad/s}$$

$$= 48.4 \text{ cm/s}$$

Rotational Kinetic Energy

Consider a rigid body as a collection of small particles. The KE of a rotating rigid body is the sum of the individual KE's of all the particles.



A rigid body rotating about the z axis with angular velocity ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_i v_i^2$. The total kinetic energy of the body is $\frac{1}{2} I\omega^2$.

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

Suppose the rigid body is rotating about a fixed z-axis with an angular velocity ω. All particles execute circular motion with same angular speed.

$$v_i = r_i \omega$$

r_i = particle distance from axis of rot.

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

where $I = \sum_i m_i r_i^2$ [Moment-of-Inertia]

↑ Particles at large r_i have higher speed and contribute more to KE

$[I] = kg \cdot m^2$ (SI)

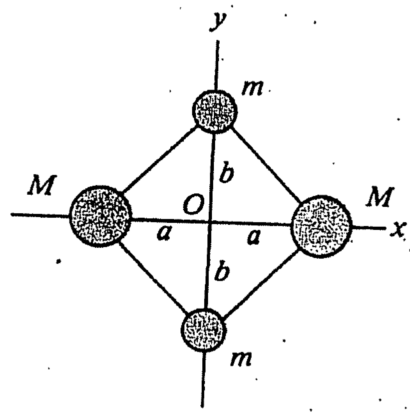
$slug \cdot ft^2$ (Br)

ω, I ← Resistance to rotational motion
v, M ← Resistance to linear motion
} Inertial quant.

<u>Example : Four Rotating Particles</u>

22-11

- Four point masses fastened to a very light frame. lying in xy-plane.



a) Rotation about y-axis with ang. velocity ω.

- masses m do not contribute since $r_i = 0$ for them. and they have no motion about y.!

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

$$K = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2$$

b) Rotation about z-axis, ⊥ to xy-plane
r_i, in each case is the ⊥ distance to axis of rot.

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$$

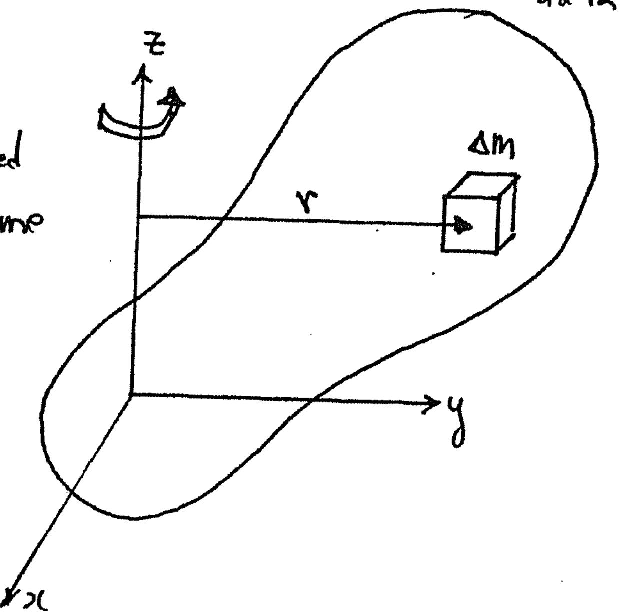
$$= 2Ma^2 + 2mb^2$$

$$K = \frac{1}{2} I_z \omega^2 = (Ma^2 + mb^2)\omega^2$$

Summary:

- Moment-of-Inertia depends on axis of rotation.
- It will take more work, for this example, to set the system into rotation about z-axis than about y-axis. Depends on the distribution of mass.

Moments of Inertia for Rigid Bodies



We evaluate I, for a rigid body rotating about a fixed axis by dividing it up into volume elements of mass Δm.
Use $I = \Sigma r^2 \Delta m$ and take the limit of this sum as $\Delta m \to 0$ we have an integral over the volume.
r: ⊥ distance from rotation axis to Δm.

$$I = \lim_{\Delta m \to 0} \Sigma r^2 \Delta m = \int_V r^2 \, dm$$

dm: must be expressed in terms of its coordinates.
- For a 3-dimensional object it is convenient to do this in terms of the local volume density, i.e mass per unit volume

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$\therefore dm = \rho \, dV$$

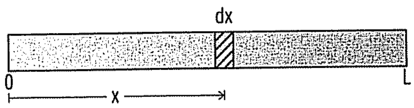
and $$\boxed{I = \int \rho r^2 \, dV}$$

↑ 2'nd moment of mass distribution.

$$I = \overset{\rho_H}{\frac{M}{V}} \int r^2 \, dV$$, for homogeneous bodies

Non-uniform rod.

$\rho(x) = bx$



A nonuniform rod of length L has a density that increases linearly from the end at $x = 0$ to the end $x = L$. The density (mass per unit length) is given by $\rho = bx$, where b is a constant and x is the distance from the end. The density at the $x = 0$ end is zero.

a. At a distance x from the $x = 0$ end, what is the mass of an increment of length dx? [G]

b. Sum up the contributions to the total mass from all of the elements of length dx to find the total mass of the rod (in terms of b and L). [O]

c. Where is the center of mass of this rod? If you can't recall the definition of this, refer to a text. [Q]

d. What is the moment of inertia, relative to an axis through $x = 0$, of the element of mass located in thickness dx at distance x from the $x = 0$ end? [X]

e. What is the total moment of inertia of this rod relative to an axis through the $x = 0$ end? [L] What is the radius of gyration of this rod relative to the $x = 0$ end? [Y]

f. What is the moment of inertia relative to an axis through the center of mass? (Use the parallel axis theorem.) [I]

g. What is the moment of inertia relative to an axis through the $x = L$ end? (Get the answer using the parallel axis theorem and by direct integration.) [W]

(a) $\rho(x)\,dx = bx\,dx$

(b) $M = \int_0^L \rho(x)\,dx = \frac{1}{2}bL^2$

(c) $M x_{c.m.} = \int_0^L x\,\rho(x)\,dx$

$= \int_0^L bx^2\,dx = \frac{1}{3}bL^3$

$\frac{1}{2}bL^2\,x_{c.m.} = \frac{1}{3}bL^3$

$\Rightarrow x_{c.m.} = \frac{2}{3}L$

(d) $dI = x^2\rho(x)\,dx$

$= bx^3\,dx$

(e) $I = \int_0^L bx^3\,dx = \frac{1}{4}bL^4$

$K^2 = \frac{I}{M} = \frac{\frac{1}{4}bL^4}{\frac{1}{2}bL^2}$

$= \frac{1}{2}L^2$

$K = 0.707\,L$

(f) $I = I_c + Md^2$

$\frac{1}{2}ML^2 = I_c + M\left(\frac{2}{3}L\right)^2$

$\frac{1}{2}ML^2 = I_c + \frac{4}{9}ML^2 \Rightarrow I_c = \frac{1M}{18}L^2$

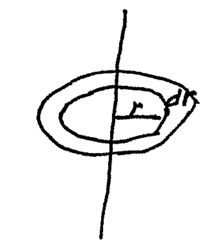
(g) $I = \frac{1}{18}ML^2 + M\left(\frac{1}{3}L\right)^2$

$= \frac{1}{18}ML^2(1+2) = \frac{3}{18}ML^2$

9-27

Moment of Inertia for a solid sphere

First we calculate for a disk



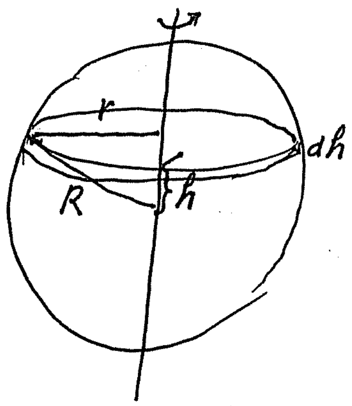
$$dm = \overset{\sigma(r)}{\rho(r)}\, 2\pi r\, dr$$

$$I = \int r^2\, dm$$

$$= \int_0^R r^2\, 2\pi r\, \overset{\sigma}{\rho}\, dr$$

$$= 2\pi\, \frac{1}{4} R^4 \rho\, dr = \frac{1}{2} M R^2$$

$$M = \overset{\sigma}{\rho} \cdot \pi R^2 \qquad \overset{\sigma}{\rho} = \frac{M}{\pi R^2}$$



$$dI = \frac{1}{2}\, dm\, r^2$$

$$= \frac{1}{2} \rho \pi r^2 dh\, r^2$$

$$r^2 = R^2 - h^2$$

$$dI = \frac{1}{2} \rho \pi (R^2 - h^2)^2 dh$$

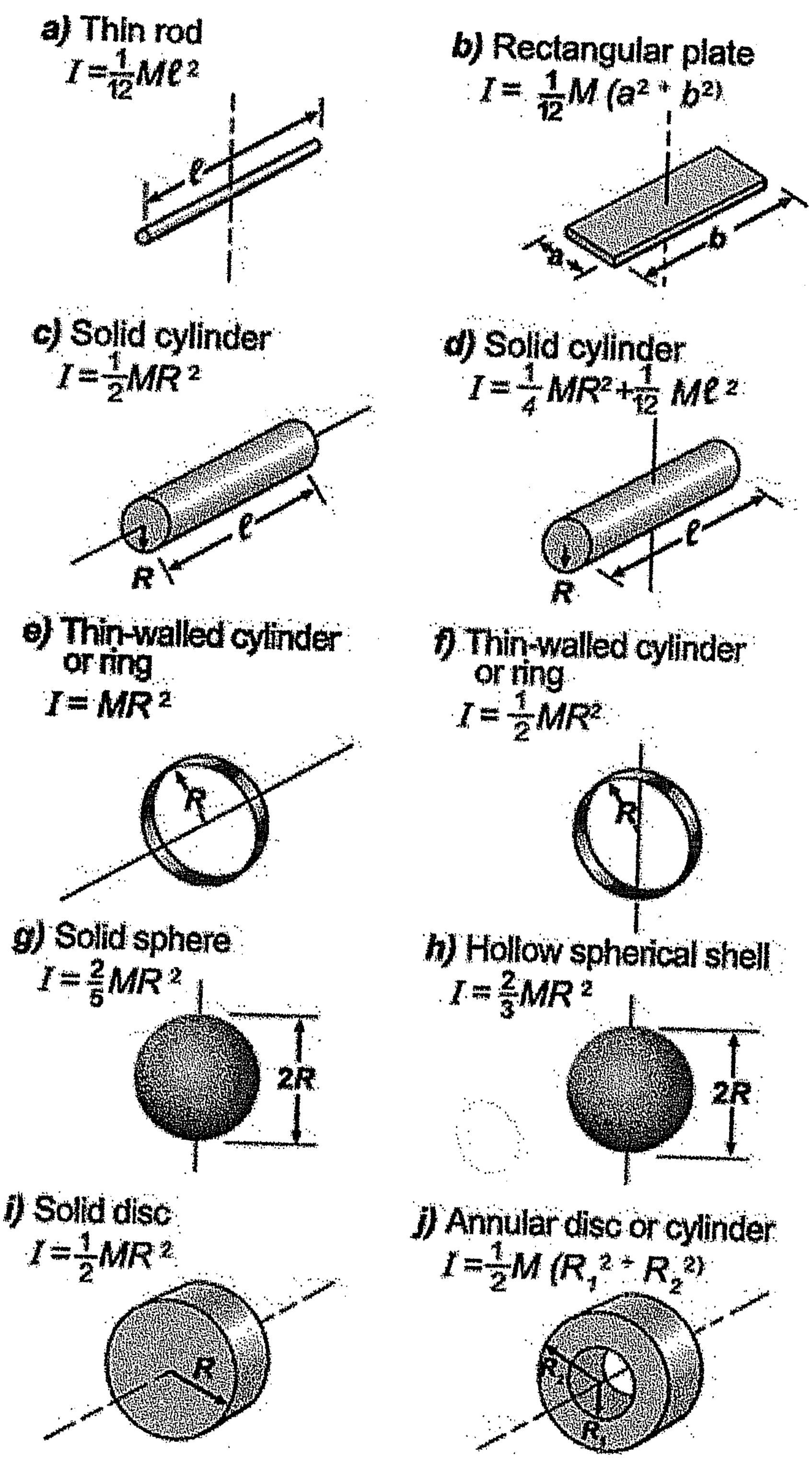
$$I = \int_{-R}^{R} \frac{1}{2} \pi \rho (R^4 - 2R^2 h^2 + h^4)\, dh$$

$$= \frac{1}{2} \pi \rho \left(R^4 h - \frac{2}{3} R^2 h^3 + \frac{1}{5} h^5 \right|_{-R}^{R}$$

$$= \frac{8}{15} \pi \rho R^5$$

$$\rho = \frac{M}{\frac{4}{3} \pi R^3}$$

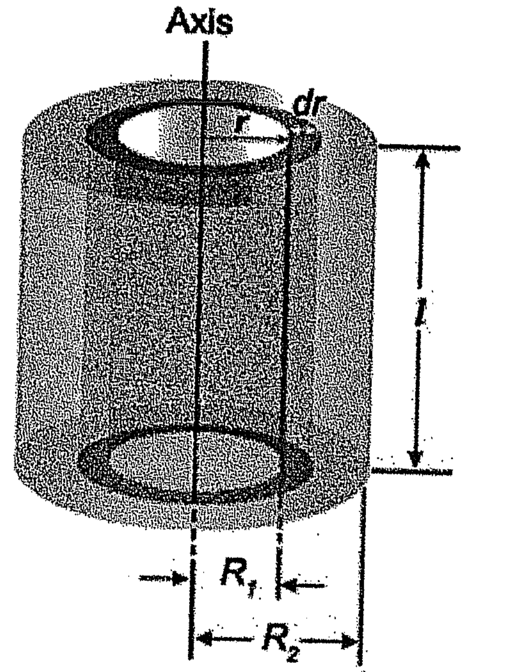
$$\Rightarrow I = \frac{8}{15} \pi R^5 \frac{M}{\frac{4}{3} \pi R^3} = \frac{2}{5} M R^2$$



Rational Inertia values for various objects for the indicated axes

Example: I – Uniform Hollow Cylinder

- Always need to choose mass elements which are a fixed radius from axis of rotation.



Moment of inertia of a hollow cylinder. The mass element is a cylindrical shell of a radius *r* and thickness *dr*.

We choose a thin cylindrical shell of radius r, thickness dr, and length ℓ.
The volume of such a shell is that of a flat sheet of length ℓ, thickness dr and width $2\pi r$.

$$dV = 2\pi \ell r \, dr$$

$$dm = \rho \, dV = 2\pi \rho \ell r \, dr$$

$$I = \rho \int r^2 dV = 2\pi \rho \ell \int_{R_1}^{R_2} r^3 dr$$

$$= \frac{\pi \rho \ell}{2} (R_2^4 - R_1^4)$$

$$= \frac{\pi \rho \ell}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Mass of Cylinder

$$M = \rho V = \pi \ell \rho (R_2^2 - R_1^2)$$

$$I = \frac{M}{2} (R_2^2 + R_1^2)$$

If cylinder is solid, $R_1 = 0$.

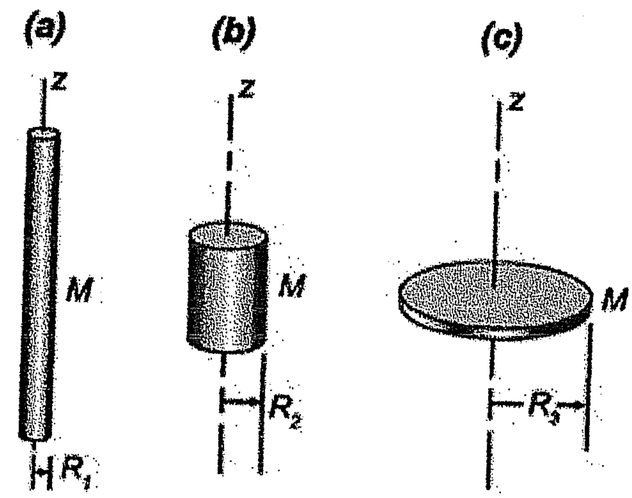
$$I = \frac{1}{2} M R^2$$

If cylinder is a very thin shell, $R_1 \sim R_2 = R$

$$I = M R^2$$

I in all cases does not depend on ℓ. The distribution along the axis does not matter. Moment-of-inertia depends on radial distribution.

Three different distributions about the z axis of the same mass M of material. The rotational inertia values are $I_z(R_1) < I_z(R_2) < I_z(R_3)$.



When using I, it is often convenient to do so in terms of a "Radius of Gyration", k

$$I = M k^2$$

It is defined such that if all of the mass of an object were located a distance k from the axis, it would have the same moment-of-inertia as the actual object.

Parallel-Axis Theorem

The moment-of-inertia depends on the location of the axis of rotation.

KE for a body about a fixed axis is

$$K = \frac{1}{2} I_z \omega^2$$

We had before that the KE is the sum of the translational energy of motion of the cm and the internal energy of motion relative to the cm.

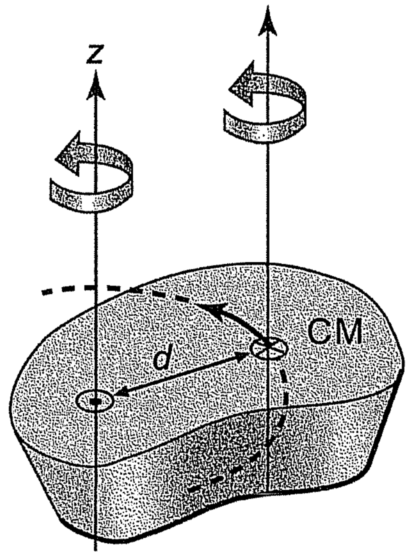
$$K = \frac{1}{2} M v_{cm}^2 + K_{Int}.$$

Consider rotation about z-axis not through cm.

cm moves in a circle of radius d around this axis.

$$\therefore v_{cm} = d\omega$$

$$\therefore \frac{1}{2} M v_{cm}^2 = \frac{1}{2} M d^2 \omega^2$$

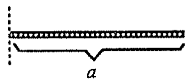
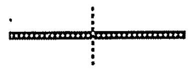
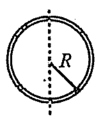
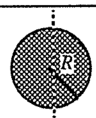
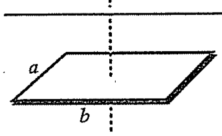


Two alternative parallel axes of rotation of a rigid body. The z axis is fixed. The center-of-mass axis moves along a circle of radius *d* around the z axis. The body is in rotational motion relative to each of these axes.

TABLE 6-1. MOMENTS OF INERTIA OF SOME SIMPLE BODIES

Body	Axis through CM	Moment of inertia
Rod, length l	Perpendicular to rod	$I_{CM} = \frac{1}{12}Ml^2$
Rectangular plate,	Parallel to side b	$I_{CM} = \frac{1}{12}Ma^2$
sides a, b	Perpendicular to plate	$I_{CM} = \frac{1}{12}M(a^2+b^2)$
Cube, sides a	Perpendicular to face	$I_{CM} = \frac{1}{6}Ma^2$
Hoop, radius a	Perpendicular to plane	$I_{CM} = Ma^2$
Disk, radius a	Perpendicular to plane	$I_{CM} = \frac{1}{2}Ma^2$
	Parallel to plane	$I_{CM} = \frac{1}{4}Ma^2$
Solid cylinder:		
radius a	Along cylinder axis	$I_{CM} = \frac{1}{2}Ma^2$
length l	Perpendicular to cylinder axis	$I_{CM} = \frac{1}{12}M(3a^2+l^2)$
Spherical shell,		
radius a	Any axis	$I_{CM} = \frac{2}{3}Ma^2$
Solid sphere,		
radius a	Any axis	$I_{CM} = \frac{3}{5}Ma^2$
Solid ellipsoid of		
semi-axes a, b, c	Along axis a	$I_{CM} = \frac{1}{5}M(b^2+c^2)$

Table 11.1. Moments of inertia for selected bodies

Body of total mass M		Moment of Inertia
	thin rod	$\frac{1}{3} M a^2$
		$\frac{1}{12} M a^2$
	circular disk	$\frac{3}{2} M R^2$
		$\frac{1}{2} M R^2$
		$\frac{1}{4} M R^2$
	thin spherical shell	$\frac{2}{3} M R^2$
	homogenous sphere	$\frac{2}{5} M R^2$
	Rectangular plate	$\frac{1}{12} M \left(a^2 + b^2\right)$

Perpendicular-Axis Theorem for a Plane Lamina

Consider a rigid body that is in the form of a plane lamina of any shape. Let us place the lamina in the xy plane (Figure 8.8). The moment of inertia about the z-axis is given by

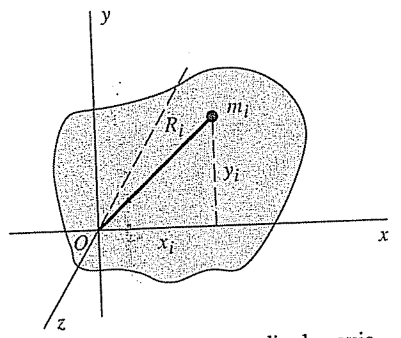


Figure 8.8 The perpendicular-axis theorem for a lamina.

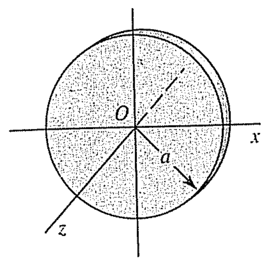


Figure 8.9 Circular disc.

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2$$

But the sum $\Sigma\, m_i x_i^2$ is just the moment of inertia I_y about the y-axis, because z_i is zero for all particles. Similarly, $\Sigma_i\, m_i y_i^2$ is the moment of inertia I_x about the x-axis. The above equation can therefore be written

$$I_z = I_x + I_y \tag{8.26}$$

This is the perpendicular-axis theorem. In words: *The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina.*

As an example of the use of this theorem, let us consider a thin circular disc in the xy plane (Figure 8.9). From Equation 8.22 we have

$$I_z = \frac{1}{2}ma^2 = I_x + I_y$$

In this case, however, we know from symmetry that $I_x = I_y$. Therefore, we must have

$$I_x = I_y = \frac{1}{4}ma^2 \tag{8.27}$$

for the moment of inertia about any axis in the plane of the disc passing through the center. The above result can also be obtained by direct integration.

11.3.2 The Perpendicular Axis Theorem

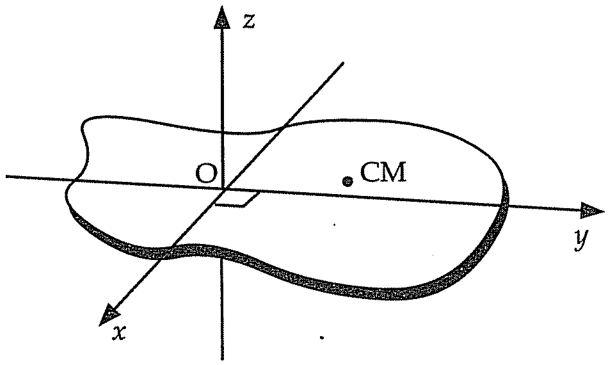


Fig. 11.4.

This theorem concerns moments of inertia about axes that are perpendicular to each other. We still consider a thin disk (see Figure 11.4). In contrast to the parallel axis theorem, this theorem is valid *only* for a planar mass distribution. Consider the moment of inertia about the x-axis:

$$I_x = \sum m_i y_i^2 \ ,$$

and similarly about the y-axis:

$$I_y = \sum m_i x_i^2 \ .$$

Furthermore:

$$I_z = \sum m_i r_i^2 = \sum m_i \left(x_i^2 + y_i^2\right) = \sum m_i x_i^2 + \sum m_i y_i^2 \ ,$$

or

$$I_z = I_x + I_y \ . \tag{11.9}$$

This is the perpendicular axes theorem:

> For a planar mass distribution (a disk), the moment of inertia about an axis perpendicular to the disk equals the sum of the moments of inertia about two mutually perpendicular axes in the plane of the disk and intersecting where the perpendicular axis passes through the disk.

The rotation of the body with angular velocity ω about a fixed z-axis is also a rotation about a parallel axis through the cm with the same angular velocity ω. One turn around z corresponds to one turn around axis through cm.

The KE associated with rotational motion around axis through cm is

$$K_{Int} = \frac{1}{2} I_{cm} \omega^2$$

$$\therefore K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M d^2 \omega^2$$

$$\frac{1}{2} I_z \omega^2 = \frac{1}{2} \left[I_{cm} + M d^2 \right] \omega^2$$

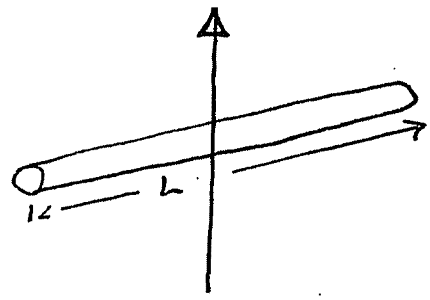
Comparing, we must have

$$I_z = I_{cm} + M d^2$$

[Parallel Axis Theorem]

Example: Rod

- Thin rod axis through its midpoint



$$I_{cm} = \frac{1}{12} ML^2$$

- what is I about an axis through its end?

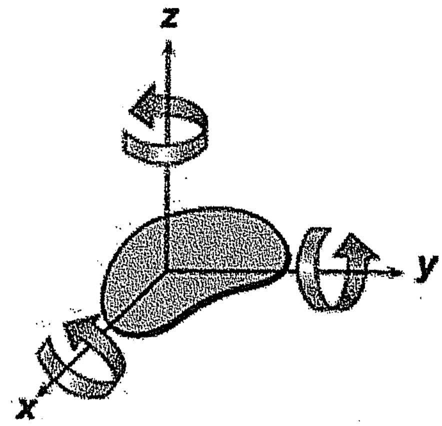
$$d = L/2$$

$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$I = \frac{ML^2}{3}$$

Perpindicular - Axis Theorem.

Relates moments-of-inertia of a thin flat plate about three mutually perpindicular axes.



A thin, flat plate. The plate may rotate about either the the x axis, the y axis, or the z axis.

Consider a thin plate which can rotate about any of three ⊥ axes

I_x, I_y, I_z } corresponding moments-of-inertia

Let plate be in xy-plane. The distance from z-axis to reference point P is

$$R = \sqrt{x^2 + y^2}$$

$$I_z = \int \rho R^2 dV = \int \rho (x^2 + y^2)\, dV$$

$$I_x = \int \rho y^2 dV$$

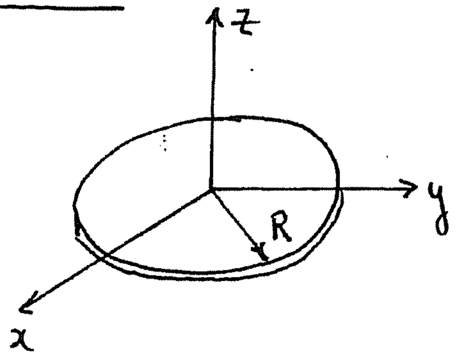
$$I_y = \int \rho x^2 dV$$

$$\therefore \boxed{I_z = I_x + I_y}$$

[Perpindicular-Axis Thm]

Example

i) Disk



• Disk in xy-plane

$$I_z = \frac{1}{2} M R^2$$

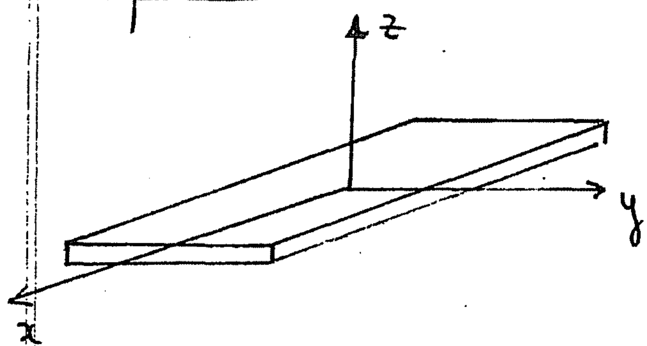
By symmetry

$$I_x = I_y$$

$$\therefore \quad I_z = 2 I_x$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{4}$$

ii) Square Plate.



Side = a

By symmetry

$$I_x = I_y.$$

$$2 I_x = I_z = \frac{1}{6} M a^2$$

Measure the moment of Inertia

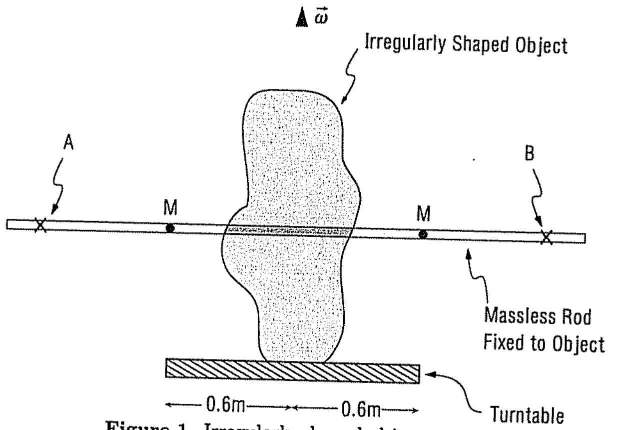


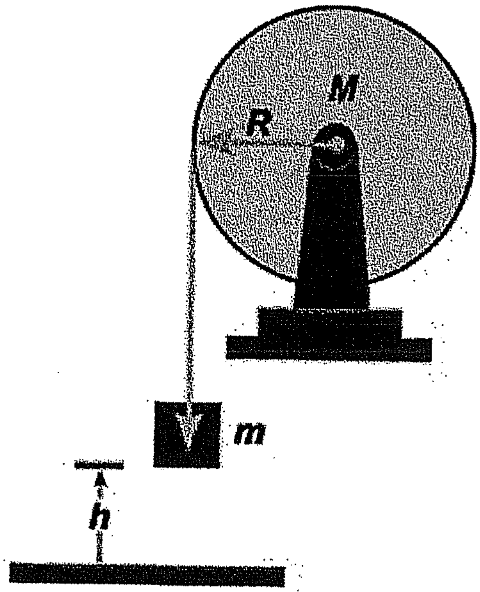
Figure 1. Irregularly shaped object rotating about vertical axis on a turntable which rotates with very little frictional loss. Two masses M are located on a massless rod held horizontally fixed to the object.

$$(I + 2 \times 4 \times 0.6^2)\,30 = (I + 2 \times 4\,(1.2)^2)\cdot 10$$

$$\Rightarrow \quad I = 1.44 \ \text{kg m.}$$

Example

A light rope is wrapped around a solid cylinder of mass M and radius R. Mass m tied to rope and released a height h above the floor. Assuming frictionless motion, what is speed of m and angular velocity of cylinder when m strikes floor?



As the cylinder rotates, the rope unwinds and mass *m* drops.

Ans: System initially has no kinetic energy but has PE. Finally both m and M have KE and the PE of m is decreased.

$$E_1 = K_1 + U_1$$
$$= 0 + mgh$$

$$E_2 = K_2 + U_2$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$v = R\omega$ [Related by geometry - see Kinematics]

$I = \frac{1}{2}MR^2$ [Solid cylinder]

$E_1 = E_2$ conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\tfrac{1}{2}MR^2)(v/R)^2 = \tfrac{1}{2}(m + M/2)v^2$$

$$v = \sqrt{2gh/(1 + M/2m)}$$

Review

System of Particle

$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i$$

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i^{ext}$$ valid in inertial frame of reference

Center of mass

$$\vec{r}_{c.m} \equiv \vec{R} \qquad \sum m_i \vec{r}_i = M\vec{R} \qquad \sum m_i(\vec{R} + \vec{\rho}_i) = M\vec{R}$$

$$\sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}} \qquad \sum m_i \rho_i = 0$$

$$\vec{r}_i = \vec{R} + \vec{\rho}_i \qquad \sum m_i \dot{\vec{\rho}}_i = 0$$

$$\vec{v}_i = \dot{\vec{R}} + \dot{\vec{\rho}}_i$$

$$\vec{L} = \sum_i (\vec{R} + \vec{\rho}_i) \times m_i(\dot{\vec{R}} + \dot{\vec{\rho}}_i)$$

$$= \sum_i \vec{R} \times m_i \dot{\vec{R}} + \sum_i m_i(\vec{R} \times \dot{\vec{\rho}}_i) + \sum_i \vec{\rho}_i \times m_i \dot{\vec{R}} + \sum_i (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i)$$

$$= \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \underbrace{\sum m_i \dot{\vec{\rho}}_i}_{0} + \underbrace{\sum m_i \rho_i}_{0} \times \dot{\vec{R}} + \sum \rho_i \times m_i \dot{\vec{\rho}}_i$$

$\vec{R} \times M\dot{\vec{R}} = \vec{R} \times \vec{P}$ → motion of the center of mass

$\sum \rho_i \times m_i \dot{\vec{\rho}}_i = \vec{L}_{c.m.}$ ↳ about the center of mass

$$\vec{\tau}_{tot} = \vec{N} = \sum \vec{r}_i \times \vec{F}_i^{ext}$$

$$= \sum_i (\vec{R} + \rho_i) \times F_i^{ext}$$

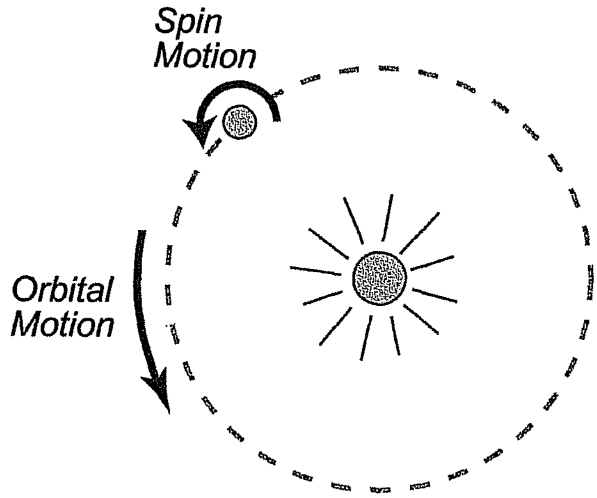
$$= \sum_i \vec{R} \times \vec{F}_i^{tot} + \sum_i \vec{\rho}_i \times \vec{F}_i^{ext}$$

$$\sum_i \vec{R} \times \vec{F}_i^{tot} = \vec{R} \times \vec{F}_{tot}^{ext}, \qquad \sum_i \vec{\rho}_i \times \vec{F}_i^{ext} = \vec{\tau}_{c.m}^{ext}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{R} \times \vec{P}) + \frac{dL_{c.m.}}{dt}$$

$$= \vec{R} \times \vec{F}_{tot}^{ext} + \underbrace{\sum_i \vec{\rho}_i \times \vec{F}_i^{ext}}_{\tau_{c.m}}$$

Earth - Sun



Example

Rod of mass M and length ℓ free to swing in a vertical plane about a fixed pivot

$$\vec{L} = \vec{R} \times \vec{P} + \vec{L}_{c.m.}$$

$$L_z = M \cdot \frac{\ell}{2} \cdot \frac{\ell}{2}\omega = \frac{1}{4} M\ell^2\omega$$

$$2 r v \, dm = 2r(r\omega)\lambda \, dr \Rightarrow 2\lambda\omega \int_0^{\ell/2} r^2 dr = \left(\frac{1}{12} \ell^2\right)\omega$$

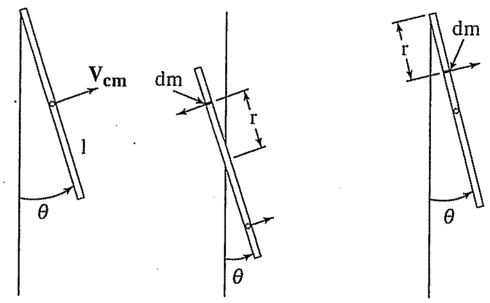
$dm = \lambda \, dr$

$\lambda = \frac{M}{\ell}$

$$= \frac{4}{12} M\ell^2 \omega = \left(\frac{1}{3} M\ell^2\right)\omega$$

Related to the parallel axis theorem

$L_{c.m.} = \frac{l}{2}|\vec{P}_{cm}| = m\frac{l}{2} v_{c.m} = m\frac{l}{2}(\frac{l}{2}\omega) = \frac{l}{4} ml^2\omega$



(a) (b) (c)

Figure 7.4 Rod of mass m and length l free to swing in a vertical plane about a fixed pivot.

Figure 7.4(b) depicts the motion of the rod as seen from the perspective of its center of mass. The angular momentum dL_{rel} of two small mass elements, each of size dm symmetrically disposed about the center of mass of the rod, is given by

$$dL_{rel} = 2rdp = 2rv\,dm = 2r(r\omega)\lambda dr$$

where λ is the mass per unit length of the rod. The total relative angular momentum is obtained by integrating this expression from $r = 0$ to $r = l/2$.

$$L_{rel} = 2\lambda\omega\int_0^{l/2} r^2 dr = \frac{1}{12}(\lambda l)l^2\omega = \left(\frac{1}{12}ml^2\right)\omega$$

We can see in the equation above that the angular momentum of the rod about its center of mass is directly proportional to the angular velocity ω of the rod. The constant of proportionality $ml^2/12$ is called the *moment of inertia* I_{cm} of the rod about its center of mass. Moment of inertia plays a role in rotational motion similar to that of inertial mass in translational motion as we shall see in the next chapter.

Finally, the total angular momentum of the rod is

$$L_{tot} = L_{cm} + L_{rel} = \frac{1}{3}ml^2\omega$$

Again, note that the total angular momentum of the rod is directly proportional to the angular velocity of the rod. Here, though, the constant of proportionality is the moment of inertia of the rod about the pivot point at the end of the rod. This moment of inertia is larger than that about the center of mass. The reason is that more of the mass of the rod is distributed farther away from its end than from its center, thus making it more difficult to rotate a rod about an end.

The total angular momentum can also be obtained by integrating down the rod, starting from the pivot point, to obtain the contribution from each mass element dm, as shown in Figure 7.4(c)

$$dL_{tot} = rdp = r(vdm) = r(r\omega)\lambda dr$$

$$L_{tot} = \lambda\omega\int_0^l r^2 dr = \frac{1}{3}ml^2\omega$$

And, indeed, the two methods yield the same result.

Kinetic Energy of the System.

$$K.E. = \sum \frac{1}{2} m_i \vec{V}_i^{\,2}$$

$$= \sum_i \frac{1}{2} m_i (\vec{V}_i \cdot \vec{V}_i)$$

$$= \sum_i \frac{1}{2} m_i (\dot{\vec{R}} + \dot{\vec{\rho}}_i) \cdot (\dot{\vec{R}} + \dot{\vec{\rho}}_i)$$

$$= \sum_i \frac{1}{2} m_i \dot{\vec{R}}^2 + \underbrace{\sum m_i (\dot{\vec{R}} \cdot \dot{\vec{\rho}}_i)}_{= \dot{\vec{R}} \cdot \underbrace{\sum m_i \dot{\vec{\rho}}_i}_{=0}} + \sum_i \frac{1}{2} m_i \dot{\rho}_i^2$$

$$= \frac{1}{2} M \vec{V}_{c.m.}^2 + \underbrace{\sum_i \frac{1}{2} m_i \dot{\rho}_i^2}_{(I)}$$

Do the same problem as before

$$\frac{1}{2} M V_{c.m} = \frac{1}{2} M \left(\frac{\ell}{2}\omega\right)^2 = \frac{1}{8} M \ell^2 \omega^2$$

$$dT = \frac{1}{2}(2\, \underset{\lambda dr}{dm})(r\omega)^2 \qquad \lambda = \frac{M}{\ell}$$

$$(I) = \lambda \omega^2 \int_0^{\ell/2} r^2\, dr$$

$$= \lambda \omega^2 \frac{1}{3}\left(\frac{\ell}{2}\right)^3$$

$$= \frac{\ell^3}{24} \frac{M}{\ell} \omega^2$$

$$= \frac{1}{2}\Big(\underbrace{\frac{1}{12} m\ell^2}_{I_{c.m.}}\Big)\omega^2$$

$$\text{Total} = \left(\frac{1}{8} + \frac{1}{24}\right) M\ell^2\omega^2 = \frac{3+1}{24} M\ell^2\omega^2 = \frac{1}{6} M\ell^2\omega^2$$

Direct calculation

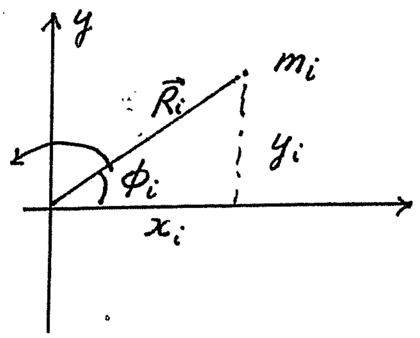
$$T = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2}\left(\frac{1}{3} M \ell^2\right)\omega^2$$

Rotation of a Rigid Body About a Fixed Axis

The simplest type of rigid body, other than pure translation, is that in which the body is constrained to rotate about a fixed axis.

$z \rightarrow$ axis of rotation



$\vec{\omega}$ direction and magnitude

$$\vec{V}_i = \vec{\omega} \times \vec{r}_i$$

$$L_z = I_z \omega$$

$$\frac{dL_z}{dt} = \frac{d(I_z \omega)}{dt} = I_z \frac{d\omega}{dt}$$

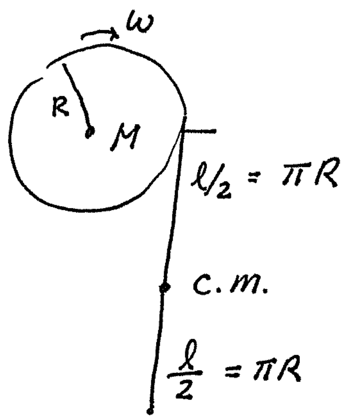
$\parallel$

τ^z_{tot}

$\downarrow$

the component

Example



center of mass of the chain

$l = 2\pi R$ mass $m = \frac{M}{2}$

Energy conservation

(M) is unchanged. $\frac{M}{2}$

$$mg\frac{l}{2} = \frac{1}{2} I \omega^2 + \boxed{\frac{1}{2} m v^2}$$

initial; $\frac{1}{2} I \omega^2$: K.E. of the disk

$$\frac{1}{2}\frac{1}{2}MR^2 \quad \frac{1}{2} m R^2 \omega^2$$

final.

$$\omega^2 = \pi \frac{g}{R}$$

Shown in Figure 8.7 is a uniform chain of length $l = 2\pi R$ and mass $m = M/2$ that was initially wrapped around a uniform, thin disc of radius R and mass M. One tiny piece of chain initially hung free, perpendicular to the horizontal axis. When the disc was released, the chain fell and unwrapped. The disc began to rotate faster and faster about its fixed z-axis, without friction. (a) Find the angular speed of the disc at the moment the chain completely unwrapped itself. (b) Solve for the case of a chain wrapped around a wheel whose mass is the same as that of the disc, but concentrated in a thin rim.

Solution:

(a) Figure 8.7 shows the disc and chain at the moment the chain unwrapped. The final angular speed of the disc is ω. Energy was conserved as the chain unwrapped.

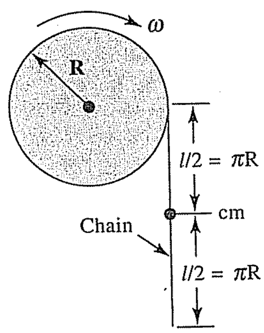


Figure 8.7 Falling chain attached to disc, free to rotate about a fixed z-axis.

Since the center of mass of the chain originally coincided with that of the disc, it fell a distance $l/2 = \pi R$, and we have

$$mg\frac{l}{2} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$\frac{l}{2} = \pi R \qquad v = \omega R \qquad I = \frac{1}{2}MR^2$$

Solving for ω^2 gives

$$\omega^2 = \frac{mg\frac{l}{2}}{\left[\frac{1}{2}\left(\frac{M}{2}\right) + \frac{1}{2}m\right]R^2} = \frac{mg\pi R}{\left(\frac{1}{2}m + \frac{1}{2}m\right)R^2}$$

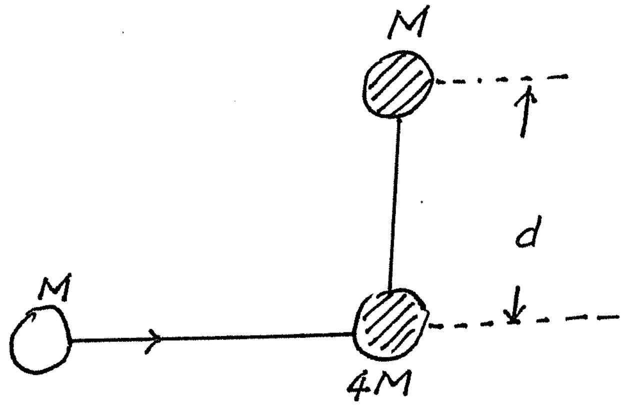
$$= \pi\frac{g}{R}$$

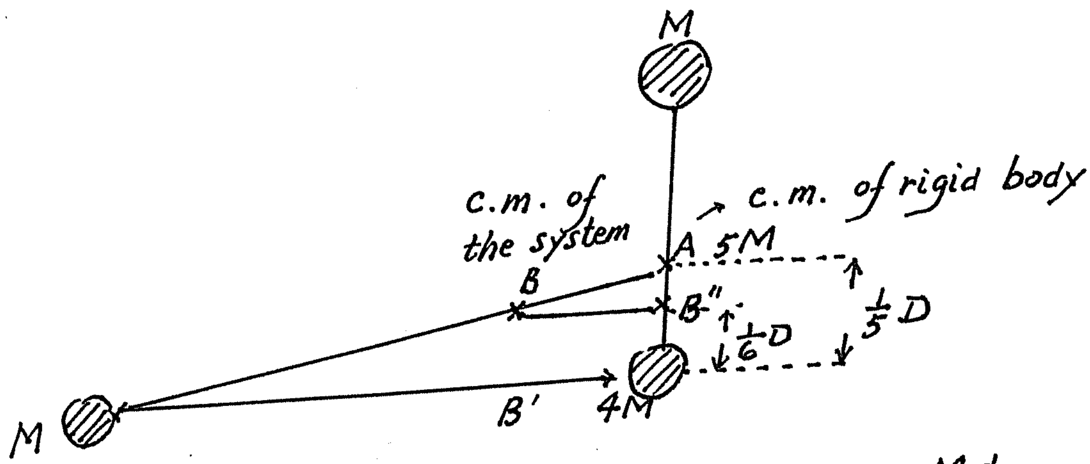
(b) The moment of inertia of a wheel is $I = MR^2$. Substituting this into the above equation yields

$$\omega^2 = \pi\frac{2g}{3R}$$

Even though the mass of the wheel is the same as that of the disc, its moment of inertia is larger, since all its mass is concentrated along the rim. Thus, its angular acceleration and final angular velocity are less than that of the disc. ■

9-45

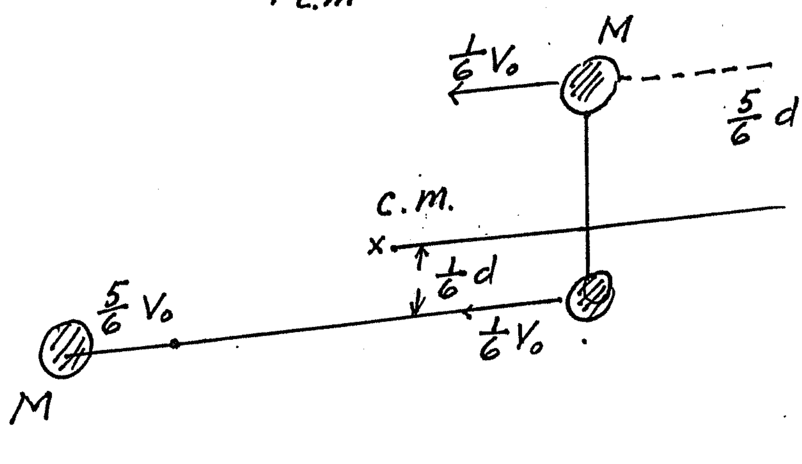




$$A \Rightarrow \frac{Md}{5M} \qquad B'' \Rightarrow \frac{Md}{6M}$$

$$\vec{V}_{c.m.} = \frac{1}{6}\vec{v}_0$$

$$\vec{P}_{c.m} = \text{constant vector}$$



$\vec{r} \times \vec{V}$

M	$\frac{5}{6} V_0$	$\cdot \frac{1}{6} d = \frac{5}{36} M V_0$ up, out of the page	$[-x\hat{i} + (-\frac{1}{6} d)\hat{j}] \times V \frac{5}{6} \hat{i} = + \hat{k}$
$4M$	$\frac{1}{6} V_0$	$\cdot \frac{1}{6} d = \frac{4}{36} M V_0$ down into the page	$-\frac{1}{6} d \hat{j} \times (-\hat{i}) = -\hat{k}$
M	$\frac{1}{6} V_0$	$\frac{5}{6} d = \frac{5}{36} M V_0 d$	$(+\hat{j}) \times (-\hat{i}) = \hat{k}$

$$\frac{10-4}{36} = \frac{1}{6} M V_0 d \quad \text{up, out of page}$$

$$I = M\left(\frac{5}{6} d\right)^2 + 5M\left(\frac{1}{6} d\right)^2 = \frac{5}{6} M d^2$$

$$\frac{1}{6} M V_0 d = \frac{5}{6} M d^2 \omega$$

$$\omega = \frac{1}{5} \frac{V_0}{d}$$

Before collision

$$\frac{1}{2} M V_0^2$$

After the collision

$$E_k = \frac{1}{2} 6 M \left(\frac{1}{6} V_0\right)^2 + \frac{1}{2} I \omega^2$$

$$\frac{1}{2} \frac{5}{6} M d^2 \cdot \frac{1}{25} \frac{V_0^2}{d^2}$$

$$= \frac{1}{2} \frac{1}{6} M V_0^2 + \frac{1}{60} M V_0^2$$

$$= \frac{5+1}{60} M V_0^2 = \frac{1}{10} M V_0^2$$

$$Q = \frac{1}{10} M V_0^2 - \frac{1}{2} M V_0^2 = \frac{1-5}{10} M V_0^2$$

$$= -\frac{4}{5}\left(\frac{1}{2} M V_0^2\right)$$

Q ↓ change in k.E.

↓ 80% of the original k.E. is lost.

$$m_{动}v + m_{静}v = m_{动}v_0, \quad 得 \quad v = \frac{m_1}{m_1 + m_2}v_0.$$

机械能损耗

$$\Delta E = \frac{1}{2}m_{动}v_0^2 - \frac{1}{2}(m_{动}+m_{静})v^2 = \frac{m_{动}m_{静}}{2(m_{动}+m_{静})}v_0^2 = \frac{m_{静}}{2(1+m_{动}/m_{静})}v_0^2.$$

故而在 $m_{动}$、v_0 一定的条件下，$m_{动}/m_{静}$ 愈大，损失的机械能愈少，也就是说，情形(1)机械能损失最少，情形(3)次之，情形(2)机械能损失最多。

3－22. 为什么茶在茶壶里容易保温，倒在茶碗里凉得快？

答： 一物体所含热量与体积 V 成正比，散热速度与表面积 S 成正比，而 $V \propto l^3$，$S \propto l^2$，这里 l 为物体的线度，l 愈大，愈容易保温。

3－23. 为什么老鼠每天摄取的食物量超过自己的体重，而猫远不要吃那么多？

答： 体重正比于体积，散热速度正比于表面面积。老鼠比猫小，单位体重散热多，需要的食物重量与体重之比大。

第四章　角动量守恒　刚体力学

4－1. 下列系统角动量守恒吗？

(1) 圆锥摆；

(2) 一端悬挂在光滑水平轴上自由摆动的米尺；

(3) 冲击摆；

(4) 阿特武德机；

(5) 荡秋千；

(6) 在空中翻筋斗的京剧演员；

(7) 在水平面上匀速滚动的车轮；

(8) 从旋转着的砂轮边缘飞出的碎屑；

(9) 绕自转轴旋转的炮弹在空中爆炸的瞬间。

答： (1) 圆锥摆在摆动过程，如果摩擦力可以忽略，则其对圆心的角动量守恒；

(2) 一端悬挂在光滑水平轴上的自由摆动的米尺，在摆动过程对轴的角动量不守恒，因为它所受的重力对轴的力矩不为 0；

(3) 冲击摆在冲击过程对轴的角动量守恒，因为在冲击过程，它所受的外力为重力和悬点的支撑力，它们对轴的力矩为 0；

(4) 阿特武德机对轴的角动量不守恒，因为两边所受的重力矩不相等，重力对轴的合力矩不为 0；

(5) 荡秋千，在摆动过程对轴的角动量不守恒，因为受到重力矩的作用；但在最低点人迅速站起的过程角动量守恒，这过程系统所受的重力对轴的力矩为 0；

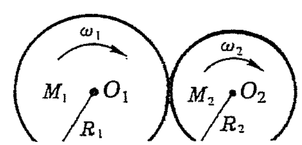
(6) 在空中翻筋斗的京剧演员，对其质心轴的角动量守恒，因为这时人受的重力对其质心轴的力矩为 0；

(7) 在水平面上匀速滚动的车轮，对其质心轴的角动量守恒；

(8) 若不计重力影响，从旋转砂轮边缘飞出的碎屑角动量守恒；

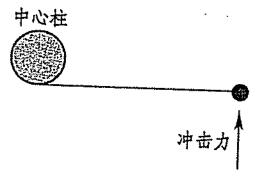
(9) 绕自转轴旋转的炮弹在空中爆炸的瞬间，对自转轴的角动量守恒。

4－2. 本章例题 10 中两个各绕自转轴旋转的圆柱构成的系统，它们的边缘接触前后系统的总角动量守恒吗？试分析守恒或不守恒的原因。在这里轴上的约束力对角动量会



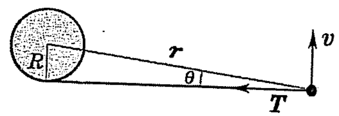
答：这两个各绕自转轴 O_1、O_2 旋转的圆柱体构成的系统，在其边缘接触前后，系统的总角动量不守恒。因两圆柱接触时接触面存在滑动摩擦力，必须外加约束力在 O_1、O_2 两轴上，才能保持两轴平行不动。若选 O_2 为轴，则加于 O_1 轴的约束力对 O_2 轴产生力矩，这对两圆柱系统来说是外力矩。或者选 O_1 为轴，则加于 O_2 轴的约束力对 O_1 轴产生力矩，这对两圆柱系统来说是也外力矩。故无论选择计算角动量的轴在何处，系统都受到外力矩，角动量都不守恒。（这里要注意的是，系统的总角动量必须指对同一轴而言。）

4－3. 如本题图，在光滑水平面上立一圆柱，在其上缠绕一根细线，线的另一头系一个质点。起初将一段线拉直，横向给质点一个冲击力，使它开始绕柱旋转。在此后的时间里线愈绕愈短，质点的角速度怎样变化？其角动量守恒吗？动能守恒吗？



思考题 4－3

答：横向给质点一个冲击力后的时间里线愈绕愈短，质点的角速度将愈来愈大。如果圆柱的半径不能忽略，则质点的角动量不守恒。因为质点在运动过程中，受到绳的张力 $\boldsymbol{T}$ 的作用，此力不是有心力，对圆柱中心的力矩 $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{T}=RT\neq 0$（如右图所示），且力矩与角动量反方向，故其角动量减少，不守恒。



动能不守恒，因为质点在运动过程中，矢径 $\boldsymbol{r}$ 与质点的速度 $\boldsymbol{v}$ 并不互相垂直，绳的张力作了功。元功 $\mathrm{d}A=\boldsymbol{T}\cdot\mathrm{d}\boldsymbol{r}=T\mathrm{d}r\cos\theta$（$\theta$ 为张力与矢径的夹角）。质点不是作圆周运动，到轴心的距离愈来愈小，其动能增加。

4－4. 骑自行车时，脚蹬子在什么位置上，人施予它的力矩最大？在什么位置上力矩最小？

答：这个问题与脚的用力的大小、方向有关。按照一般人踩车的习惯，踩车所用的力是朝下的。当脚磴子处于水平位置，并在轴的前方时，人用力往下踩，施予脚磴子的力最大，力臂也最大，故力矩最大。当脚磴子在轴的后方时，力矩是反的，当然脚不该用力了。

4－5. 经验告诉我们，推手推车上坡时，推不动了，扳车轮的上缘可省力。什么道理？

答：若对车轮的质心轴来分析，推手推车上坡时，力是作用于车轮的轴上，无力矩，只是摩擦力矩使轮转动。如果是扳车轮的上缘，则摩擦力和推力都会对质心轴产生力矩，且两者是同向的，用同样的力可以使力矩增加一倍，故省力。

如果对车轮接地面的瞬时轴来分析，则扳车轮的上缘比推车时力作

4－6. 为什么走钢丝的杂技演员手中要拿一根长竹竿来保持身体的平衡？

答：走钢丝绳的杂技演员手中拿一根长竹竿，增加了系统对钢丝轴的转动惯量，演员在行走过程中即使稍有偏离，对系统也不会产生大的角加速度，而且可以左右移动竹竿，调整质心的位置，从而容易保持身体的平衡。

4－7. 通常我们都知道，物体愈高且上面愈重，则愈不稳定。但杂技演员用手指、额头或肩膀顶一个物体时，物体愈高且上面愈重，顶起来却愈容易平衡。试解释之。

答：设杆长、质心高度、杆偏离竖直方向的角度和杆对支撑点的转动惯量分别为 l、r_C、θ 和 I，取顶点为重力势能原点，则物体和顶杆静止于竖直位置时机械能为 0．因机械能能守恒，倾角为 θ 时的能量公式为

$$\frac{1}{2}I\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2+mgr_C(\cos\theta-1)=0,$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2=\frac{2mgr_C}{I}(1-\cos\theta)\approx\frac{mgr_C}{I}\theta^2.$$

令 $I=kml^2$，$r_C=k'l$，这里 k 和 k' 都是无量纲的数。于是上式化为

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\approx\frac{gk'}{kl}\theta^2,\quad 即\quad \frac{\mathrm{d}\theta}{\mathrm{d}t}=\sqrt{\frac{gk'}{kl}}\theta,$$

顶端的线速度 $v=\mathrm{d}\theta/\mathrm{d}t$ 与线位移 $x=l\theta$ 的关系为

$$v=\sqrt{\frac{gk'}{kl}}x.$$

当顶端偏离了距离 x 时，杂技演员必须迅速把支撑点移动相应的位置以恢复平衡，上式表明，移动的速度 v 反比于杆长 l 的平方根，杆愈长，支撑点需要移动的速度 v 可以愈小，杂技演员愈容易调节。

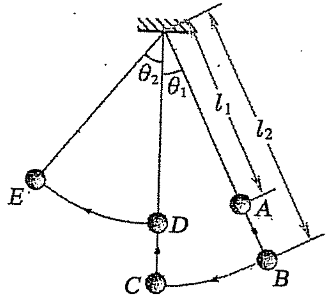
再看质量分布的影响。讨论两个极端情形，一是均匀的光杆，$k=1/3$，$k'=1/2$，$k'/k=3/2$；二是轻杆上端顶一重物，$k=1$，$k'=1$，$k'/k=1$．由上式可见，后者的调节速度是前者的 $\sqrt{2/3}\approx 82\%$，可见把重心上调是有好处的，但不太明显。

4－8. 试用角动量以及功和能的概念说明荡秋千的原理。

答：如果荡秋千者在一边（最高点）蹲下（其质心降低），在平衡位置（最低点）时站起（其质心升高），就可以愈荡愈高。下面从角动量和功、能的概念加以说明。

为简单起见，将人看成一个质点。设在摆的最高位置 A 点时人的质心

其质心降低到 B 处，质心到悬挂点的距离变为 l_2. 而后人由 B 摆到 C,在 C 位置时人突然站起来，将质心恢复到距悬挂点 l_1 处的 D 点。而后人由 D 摆到 E,在 E 位置时绳与竖直方向的夹角为 θ_2. 从 A 到 B 的过程中对悬挂点的角动量始终为 0 . 从 B 到 C 的过程机械能守恒，有



$$mv_C^2/2 = mgl_1(1-\cos\theta_1),$$

从 C 到 D 的过程外力通过悬点，角动量守恒，有

$$ml_2v_C = ml_1v_D,$$

从 D 到 E 的过程机械能守恒，有

$$mv_D^2/2 = mgl_2(1-\cos\theta_2),$$

由以上各式得

$$\cos\theta_1 = 1-v_C^2/2l_2g, \quad \cos\theta_2 = 1-v_D^2/2l_1g = 1-(l_2/l_1)^3(v_C^2/2l_1g),$$

因为 $l_2>l_1$，所以 $\cos\theta_1>\cos\theta_2$，即 $\theta_1<\theta_2$，

即摆动的偏角增大了。此后人由 E 摆向 A 点，不过摆角已从 θ_1 增大到 θ_2 了。反复这样做就可以愈荡愈高。

4 – 9. 试分析下列运动是平动还是转动：

(1) 自行车脚蹬板的运动；

(2) 月球绕地球运行。

答：(1) 自行车脚踏板不绕它的自身轴转动，只是绕轮盘的轴运行，其运动是平动。

(2) 如果月球只是绕地球运行，没有绕自身轴转动的自转运动，则月球绕地球运行是平动；但实际上月球的一面始终对着地球，说明它每月自转一周，即它又平动又转动。

4 – 10. 若滚动摩擦可以忽略，试分析自行车在加速、减速、匀速行进时，前后轮所受地面摩擦力的方向。此时摩擦力作功吗？

答：自行车在加速前进时，后轮所受摩擦力的方向向前；这是因为要使自行车加速，骑车人必须施力矩于脚踏板，再通过飞轮和链条带动后轮转动，使后轮与地的接触点有一向后运动的趋势，从而后轮受到触地处一向前的摩擦力。此力使自行车加速前进，推动前轮，使前轮与地的接触点有一向前运动的趋势，从而使前轮受到一向后的摩擦力，令它向前滚动。

当自行车在减速前进时，骑车人必须刹车。若前后轮同时刹车，则前后轮所受摩擦力的方向均向后。因为这时前后轮的转速减慢，但它向前平动

摩擦阻力。若只刹其中一轮，则刹车轮受向后的摩擦力使车子减速，另一轮所受摩擦力的方向却是向前，以减慢该轮的转速。

若上述这些摩擦力只是静摩擦力，即轮子与地的接触点相对于地没有滑动，地对车的摩擦力不作功。但车刹与车轮之间有相对滑动时，这种摩擦力是作负功的，它使车子减少的动能变为热能。若一下子把轮子刹死，车刹与车轮之间没有相对滑动，不作功了；但地面的摩擦力不足以使车马上停止，整部车必然向前滑动，地面的滑动摩擦力就作负功了。

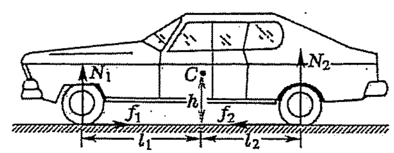
当自行车在匀速前进时，若滚动摩擦可以忽略，在前后轮与地的接触点处都没有相对运动的趋势，因而没有摩擦力，自然不存在作功的问题。

4 – 11. 汽车发动机的内力矩是不能改变汽车的总角动量的。那么，在启动的制动时，其角动量为什么能改变？

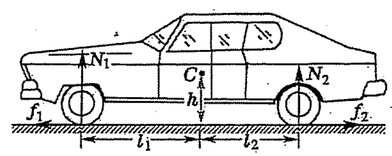
答：汽车发动机或刹车的内力矩改变车轮的转动状态，使车轮与地的接触点有一相对滑动的趋势，使之受到摩擦力。摩擦力矩是外力矩，是它改变着汽车的总角动量。

4 – 12. 为什么汽车启动时车头会稍往上抬，制动时，车头稍往下沉？

答：因为汽车的启动和制动，是靠汽车发动机或刹车的内力矩改变后轮的转动状态，使后轮与地的接触点有一相对滑动的趋势，产生摩擦力，改



a 起 动



b 制 动

变汽车的运动状态，使汽车加速或减速。启动时，后轮所受的摩擦力 f_2 向前(图 a)，它对质心 C 产生一个顺时针的力矩 f_2h；前轮所受的摩擦力 f_1 向后，它对质心 C 产生一个逆时针的力矩 f_1h. 由于 $f_2>f_1$，两力矩的合成是顺时针的，使车头稍往上抬。制动时，后轮所受的摩擦力 f_2 向后(图 b)，它对质心 C 产生一个逆时针的力矩 f_2h；前轮所受的摩擦力 f_1 向前，它对质心 C 产生一个顺时针的力矩 f_1h. 由于 $f_2<f_1$，两力矩的合成是逆时针的，使车头稍往下沉。

现作一简单的定量讨论。设汽车前后轮所受的支撑力分别为 N_1、N_2，到质心 C 的水平距离分别为 l_1、l_2. 从力矩的平衡条件有

$$N_1l_1 \mp f_1h = N_2l_2 \mp f_2h,$$

由质心运动定理有

$$N_1 l_1 = N_2 l_2 \pm (f_1 - f_2) h = N_2 l_2 + mah \begin{cases} < N_2 l_2, & \text{加速情形}(a>0), \\ > N_2 l_2, & \text{减速情形}(a<0). \end{cases}$$

地面支撑力与是对地面压力的反作用力，它们的大小反映了汽车前后轮对地面压力的大小。由上式可见，启动情形前轻后重，车头呈上抬趋势；制动情形前重后轻，车头呈下沉趋势。

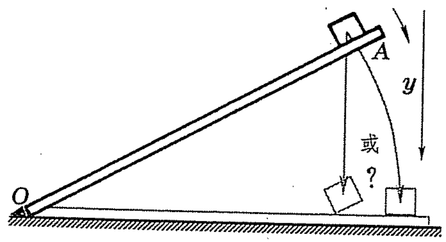
4－13. 试说明自行车刹车时前后轮给地面压力的变化。

答： 自行车刹车时，后轮对地面的压力减少，前轮对地的压力增加。分析与汽车制动时的情况相同。

4－14. 试分析拖拉机牵引农具时，前后轮对地面压力的变化。

答： 拖拉机牵引农具时，其受力情况与汽车的情况相比（思考题4－12），多一个向后的拉力。若无加速度，则无摩擦力，但拖拉机所受拉力与汽车启动时摩擦力引起的加速项 ma 类似，加大后轮对地面的压力，减小前轮对地面压力。

4－15. 通过学习物理学，我们有了这样的概念，若忽略空气的阻力，任何物体自由降落时的加速度都是一样的。如本题图所示，将一块长条木板一端抵在地面上，抬起它的另一端，在其上放一小木块。松开手后，在降落的过程中木块会离开木板吗？你可做个实验试一试。



思考题 4－15

答： 小木块若离开木板，则以重力加速度 g 自由降落，与木板没有相互作用。实现这种情况的条件是木板末端 A 点加速度的垂直分量必须大于或等于重力加速度 g. 下面考虑小木块在离开时木板的运动情况。设木板的质量为 M，长为 l，某时刻与地面的夹角为 θ. 以木板和地面的触点 O 为参考点，转动惯量为 $Ml^2/3$，重力矩为 $Mg(l/2)\cos\theta$，其转动方程为

$$\frac{1}{2}Mgl\cos\theta = \frac{1}{3}Ml^2\ddot{\theta},$$

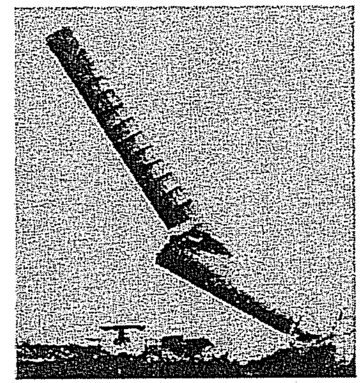
由此可得 A 点的加速度和垂直分量分为

$$a_A = l\ddot{\theta} = \frac{3}{2}g\cos\theta, \quad \text{竖直分量 } a_{Ay} = a_A\cos\theta = \frac{3}{2}g\cos^2\theta.$$

现在要求 $a_{Ay} > g$，这相当于要求 $\cos^2\theta > 2/3$，或 $\cos\theta > \sqrt{2/3}$，$\theta < 35.5°$.

换句话说，在木板与地面的夹角小于35.5°时

4－16. 工厂里很高的烟囱往往是用砖砌成的。有时为了拆除旧烟囱，可以采用从底部爆破的办法。在烟囱倾倒的过程中，往往中间偏下的部位发生断裂（见本题图）。试说明其理由。


思考题 4－16

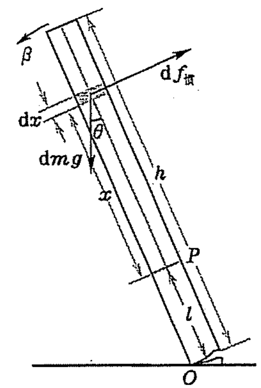
答： 取均匀柱模型。正在倒塌的烟囱未断裂前可看成刚体，设它的质量为 m，高度为 h，转动惯量为 $I = \frac{1}{3}mh^2$，质心高度 $h_C = \frac{1}{2}h$，重力矩为

$$mgh_C\sin\theta = \frac{1}{2}mgh\sin\theta.$$

设角加速度为 $\ddot{\theta}$，则烟囱倾倒过程的运动方程为

$$\frac{1}{2}mgh\sin\theta = \frac{1}{3}mh^2\ddot{\theta},$$

$$\ddot{\theta} = \frac{3g\sin\theta}{2h},$$



如右图，假设断裂点在高度为 l 的 P 处，考虑 P 点以上高度为 x 处一质元 dm 受到的外力，除重力 dmg 外，还受到两个惯性力：

$df_{惯} = -dm\,a$ [$a = (l+x)\ddot{\theta}$ 为质元的线加速度]

和惯性离心力，二者都对烟囱的断裂有作用，前者的影响是主要的。

现计算 P 点以上一段 $df_{惯}$ 和 dmg 对 P 点产生的总力矩 M：

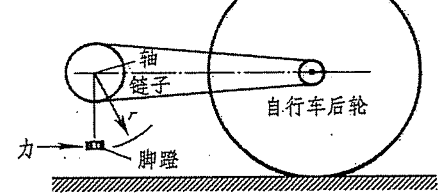
$$M = \int_0^{h-l} \eta\,dx\left[\frac{3(l+x)}{2h} - 1\right]xg\sin\theta = \frac{\eta g\sin\theta}{4h}(h-l)^2 l.$$

式中 $\eta = dm/dx$. 求 M 的极大：

$$\frac{dM}{dl} = \frac{\eta g\sin\theta}{4h}(h-l)(h-3l) = 0,$$

即 $l = h/3$ 处烟囱受到的向后弯曲的力矩最大，最易断裂。若考虑惯性离心力，断裂处会高一些。

4－17. 用手扶着静止的自行车不让倒下，把它左边的脚蹬放在朝下的位置，如本题图所示。这时用水平力向后推此脚蹬，车子向前还是向后运动？脚蹬朝哪个方向



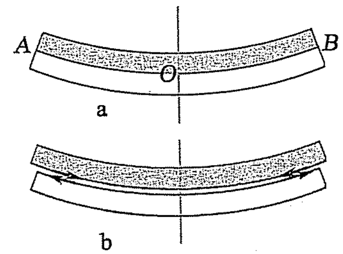
4－22. 拐弯时，骑自行车和蹬三轮车的人有不同的感觉。譬如想朝左拐，骑自行车的人只需把身体的重心偏向左边，而无需有意识地向左转动车把。如果她或他只向左转动车把，而不向左侧身，则车子就会产生朝右倾倒的趋势。若蹬三轮车的人想朝左拐的话，他必须向左转动车把，而是否向左侧身则无所谓。只要弯拐得不太急，一般用不着担心朝右倾倒。试解释之。

答： 自行车在行驶过程中，车轮具有一定的角动量，方向向左。当骑车人把身体的重心侧向左边时，则系统（人与自行车）受到重力矩的作用，其方向向后，使前轮产生进动，进动使车轮角动量向后转，即前轮向左偏。这样，自行车也就自然向左拐弯了。所以，当骑自行车的人想向左拐，只需把身体的重心侧向左边，而无需有意识地向左转动车把。但如果只向左转动车把，则车向左拐，从骑车人者看来，他受到向右的惯性离心力，车子有向右倾倒的趋势。

三轮车的后部有两个车轮，有两个支撑点。蹬三轮车的人侧身不会使车身倾斜而产生重力矩，从而也没有进动效应。他想向左拐，就得向左转动车把。因为后面有两个支撑点，只要弯拐得不太急，一般用不着担心惯性离心力把车子向右掀翻。

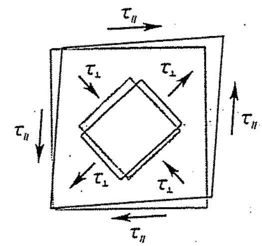
第五章　连续体力学

5－1. 在图5－4中我们分析了横梁弯曲时横断面上的正应力，你能想象其上下各层分界面上剪应力的情况吗？



答： 如图a所示，设想从中间O点作一分界曲面AOB，将弯曲横梁上下两层分开。如果上下两层没有联系，则情况将如图b所示，上层的下表面要比下层的上表面长出来。可见，在这个分界面上，上层给下层的剪应力是朝两头拉的（见图b中黑箭头），下层给上层的剪应力是朝中间挤的（见图b中灰箭头）。由于对称性，AOB面上的剪应力两头大中部小，正中央O点处为0. 此外，因为在弯曲横梁的上下表面处没有物体施加剪应力，故而那里的剪应力为0. 所以，分布在AOB面上的那种剪应力，随着此面向上或向下朝横梁的表面趋近时，减小到0.

5－2. 在图5－7中我们分析了圆柱扭转时各同轴薄层界面上的剪应力，你能想象正应力的情况吗？

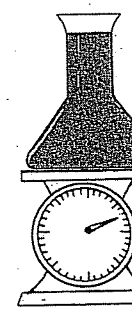


答： 圆柱扭转时各同轴柱面上都有剪应力。如图所式，凡有剪应力$\tau_{//}$的地方，转个角度看，就是正应力$\tau_\perp$（张力和压力）。但在扭转的弹性柱内，沿平行和垂直于柱轴的方向上，任何地方都没有正应力。

5－3. 用桨向后划水，在水中平行于桨面和垂直于桨面的截面上，压强哪个大？

答： 无论流体静止或流动以及有无加速度，其中压强总是各向同性的，故而用桨划水时桨附近水中任何一点平行于桨面和垂直于桨面的截面上压强一样大。

5－4. 如本题图，在一平底锥形烧瓶内盛满水银，放在台秤上。若忽略烧瓶本身的重量，水银给瓶底的压力和瓶底给秤盘的压力一样吗？哪个大？



答： 水银给瓶底的压力$F=(p_0+\rho gh)S$，瓶底给秤盘的压力$F'=p_0S+mg$，这里p_0是大气压，ρ是水银密度，h是水银高度，m是水银质量，S为瓶底面积。对于平底锥形烧瓶来说$mg<\rho ghS$，故$F'<F$，于是瓶底受秤

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第十章 剛体運動

在本章中我們將討論剛体運動，其中最基本的觀念是牛頓的轉動定律之應用。

第一節 剛体的角動量，剛体主軸及慣性矩

1. 簡介 在本節中我們將首先對剛体下一定義。剛体的運動通常是移動及轉動的組合。在描述剛体轉動的運動學中最重要的觀念是角速度，而轉動動力學中最重要的觀念仍是角動量。在本節中我們將討論此兩者間之關係。在特殊情況下（當剛体繞著其主軸時）此兩向量平行。由此我們可以引進慣性矩及迴轉半徑的觀念。在本節中我們並將討論如何計算一些簡單物体的慣性距(moment of inertia)及迴轉半徑(radii of gyration)。

2. 基本觀念

在一物体中，若其所有組成粒子間之距離保持，則此物体稱為剛体[1]。因此一剛体之形狀不變。

若剛体之各點在任何時間內之位移均相同[2]，則此剛体作純粹平移運動。

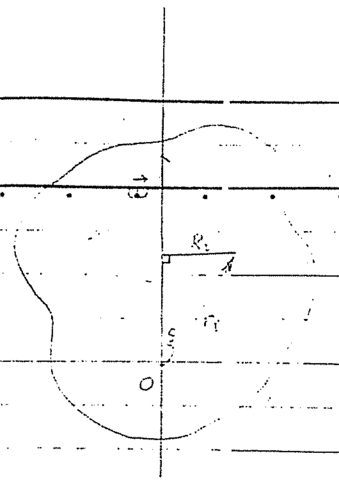
若剛体中每一點均作圓周運動，所有之圓心均在一直線上，則此直線稱為轉動軸[3]，而此剛体則對此軸作純粹轉動。

通常剛体的運動是平移和轉動之組合。我們可以質心運動來描述剛体的平移，再考慮各點對質心的轉動。

描述轉動的最重要的物理量顯然地是角速度 $\vec{\omega}$，其方向即為其轉動軸之方向，其大小即是各質點在單位時間內沿其各別圓周上所作之角位移。

而轉動動力學的基本公式是 $\frac{d\vec{L}}{dt}=\vec{\tau}$[4]。此處 $\vec{L}$ 及 $\vec{\tau}$ 分別是剛体對慣性座標中一固定點之角動量及所受之力距。 $\left(\frac{d\vec{L}^*}{dt}=\vec{\tau}^*\right)$

顯然地要了解剛体轉動我們必須了解 $\vec{L}$ 與 $\vec{\omega}$ 之間的關係。



第 i 個質點對 O 點之角動量為

$$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i \qquad (1)$$

將 $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ 代入(1)式得

$$\vec{L}_i = m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \qquad (2)$$

利用向量恒等式 $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$，(2)式可寫成

$$\vec{L}_i = m_i [(\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i] \qquad (3)$$

沿 $\vec{\omega}$ 方向之投影

$$\frac{\vec{L}_i \cdot \vec{\omega}_i}{\omega} = m_i [r_i^2 - r_i^2 \cos^2\theta]\,\omega = m_i r_i^2 \sin^2\theta\,\omega \qquad (4)$$

$$= m_i R_i^2 \omega \qquad (5)$$

R_i 是 i 質點之位置與轉動軸之垂直距離。

由第(3)式中可看出 $\vec{L}_i$ 通常不在 $\vec{\omega}$ 之方向。

若一系統中有 N 個質點，則該系統之總角動量 $\vec{L} = \sum_{i=1}^{N} \vec{L}_i$。由於 $\vec{L}_i$ 通常均不沿 $\vec{\omega}$ 之方向，通常 $\vec{L}$ 也不一定在沿 $\vec{\omega}$ 之方向。但其角動量沿 $\vec{\omega}$ 軸（取為 z 軸）之投影是

$$L_z = (\sum m_i R_i^2)\,\omega = I\omega \qquad (6)$$

$I = \sum m_i R_i^2$ [6] 是該剛体對 z 軸之慣性矩。

同時我們定義迴轉半徑為[7] radius of gyration

$$K = \sqrt{\frac{I}{M}} \qquad (7)$$

在特殊情況下，$\vec{L}$ 與 $\vec{\omega}$ 方向相同[6]，此時

$$\vec{L} = I\vec{\omega} \qquad (8)$$

而此時之轉軸稱為主軸

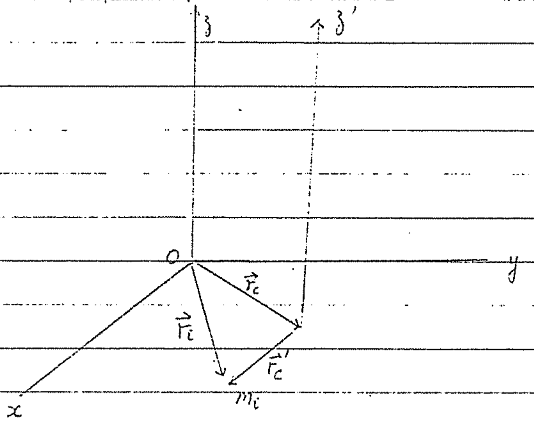
不論一物体之形狀如何，我們均可找到至少[8]三個互相垂直之主軸 X_1, Y_1, Z_1，對應於

每個主軸之慣性矩為 I_1, I_2, I_3。 I_1, I_2, I_3 通常不需要相等，但在滿足某些對稱性則可能相等。

在計算慣性矩時，以下之平行軸定理（又稱為史坦那公式）

$$I = I_c + Md^2 \tag{9}$$

此處 I 是該剛体對任何軸A所計算之慣性矩，I_c 是對與A軸平行而且通過質心之軸B所計算之慣性矩。d是A，B兩軸間之距離，M是剛体之質量



I（對O系統z軸所計算之轉動矩）

$$= \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \tag{10}$$

$$x_i = x_i' + x_c \quad , \quad y_i = y_i' + y_c \tag{11}$$

x_i, y_i 是 i 質點在O系統中之 x, y 坐標
x_c, y_c 是質心在O " " " " " "
x_i', y_i' 是 i 質點在質心系統中之 x, y 坐標

$$I = \sum_{i=1}^{N} m_i [(x_i' + x_c)^2 + (y_i' + y_c)^2]$$

$$= \sum_{i=1} m_i (x_i'^2 + y_i'^2) + 2x_c \sum m_i x_i' + 2y_c \sum m_i y_i' + (x_c^2 + y_c^2) \sum m_i \tag{12}$$

由於 x_c, y_c 是質量中心，因此 $\sum_{i=1}^{N} m_i x_i' = 0$，$\sum_{i=1}^{N} m_i y_i' = 0$，因此

$$I = I_{c.M.} + Md^2 \tag{13}$$

推廣至連續分佈之物体時

$$I = \sum_{i=1}^{N} m_i R_i^2 \rightarrow I' = \int R^2 dm \tag{14}$$

討論

(1) 剛体是一極有用的理想化的想法，實驗上並無真正的剛体。

(2) 所有的質點之速度在任何時候均相同。而此一共同速度可以是時間之函數。

(3) 在運動中轉動軸之方向可以是固定的，但也可隨時間而改變。

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(4) 也常用在質心系統中之 $\frac{d\vec{L}^*}{dt} = \vec{\tau}^*$ 公式。

(5) 由於對稱性的原因或 $\vec{r}\cdot\vec{\omega} = 0$（也即是 $\vec{r}$ 位於垂直於 $\vec{\omega}$ 之平面時）。

(6) 主要的量是質點與軸之距離而非質點與 O 之距離

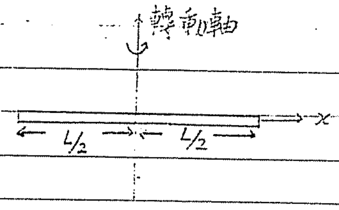
(7) 轉動半徑的優點是它只與物体之幾何有關，它的單位是長度。$L = MK^2\omega$
如果質點對一軸作等速運動則 $L = Mr^2\omega$，r是該質點與軸之距離，兩相比較，K之意義就很明顯

(8) 任何物体至少有三個主軸。當物体有對稱性時，則其主軸之數目可以而且常常大於3。如任何通過球型物体中心之直線均為主軸，因此對均勻球体而言，它有無窮多個主軸。

應用.

(1) 一維物体轉動矩求法之例子

(a)



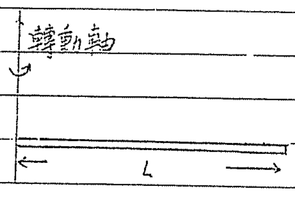
$\rho = \frac{M}{L}$ = 單位長度之質量

$dm = \rho\, dx$

$I = \int_{-L/2}^{L/2} \rho x^2 dx = \frac{M}{L}\cdot\frac{1}{3}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right]$

$= \frac{1}{12} ML^2$

(b)



(i) $I = \int_0^L \rho x^2 dx = \frac{M}{L}\cdot\frac{1}{3}L^3$

$= \frac{1}{3} ML^2$

(ii) 此一結果可由平行軸求得

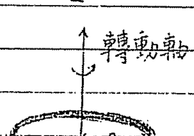
$I = I_{c.m} + Md^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$

(iii) 若是此桿之質量不均勻，如 $\rho = ax$

則 $\int_0^L \rho dx = M \Rightarrow \int_0^L ax\, dx = M \qquad a = \frac{M}{\frac{1}{2}L^2}$

$I = \int_0^L ax\, x^2 dx = a\cdot\frac{1}{4}L^4 = \frac{M}{\frac{1}{2}L^2}\cdot\frac{1}{4}L^4 = \frac{1}{2}ML^2$

(c)



(i) 圓圈之半徑為 R
顯然地 $I = MR^2$

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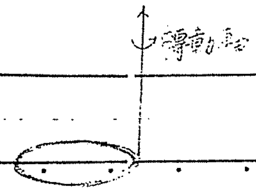
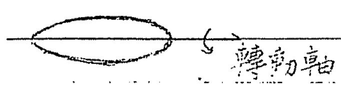


利用平行軸定理很容易的可得

$I = I_{c.M.} + Md^2 = MR^2 + MR^2 = 2MR^2$

(d)



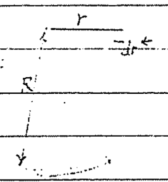
與 (ci) 比較，在 (ci) 中 $R^2 = x^2 + y^2$，x, y 兩方向之距離對 R 都有貢獻，同時由對稱可知 x, y 對其貢獻相等。而在 (d) 中 $R^2 = y^2$，只有 y 方向之距離對 R 有貢獻，因此很明顯的此時

$I = \frac{1}{2} MR^2$

(2) 二維物体轉動矩之求法

(a) 圓盤

轉動軸是通過圓心並垂直於紙面



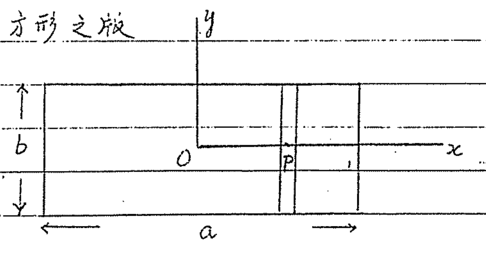
ρ = 單位面積之質量

$\rho \pi R^2 = M \Rightarrow \rho = \frac{M}{\pi R^2}$

將圓盤分成很多圓圈，在半徑為 r 與 $r + dr$ 之間的質量是 $\rho 2\pi r dr$ 其對轉動矩為 $r^2 \cdot \rho 2\pi r dr$

所以 $I = \int_0^R 2\rho \pi r^3 dr = \frac{M}{\pi R^2} 2\pi \cdot \frac{1}{4} \pi R^4 = \frac{1}{2} MR^2$

(b) 長方形之板



將長方板分成 Δx 寬之長細條

其質量為 $\frac{\Delta x}{a} M$

此一長細條對通過 P 點之垂直於紙面之軸之慣性矩為 $\frac{\Delta x}{a} M \cdot \frac{b^2}{12}$（由 (a) 得來）

此細條對通過 O 點垂直於紙面轉動軸之慣性矩，利用平行軸定理可知為

$\Delta I = \frac{\Delta x}{a} M \left(\frac{b^2}{12} + x^2 \right)$

所以 $I = \int_{-a/2}^{a/2} \frac{M}{a} \left(\frac{b^2}{12} + x^2 \right) dx = \frac{M}{12} (b^2 + a^2)$

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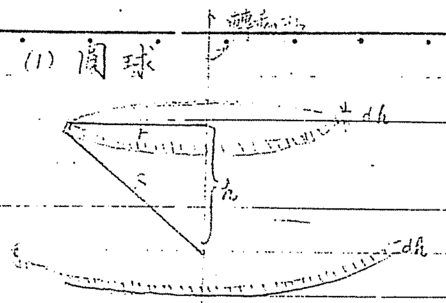
(3) 三維物体轉動距之求法

(1) 圓球



ρ = 單位体積之質量 = $\frac{M}{\frac{4}{3}\pi R^3}$

在 h 及 $h+dh$ 間之圓盤對慣性矩之貢獻為

$$dI = \frac{1}{2}\, dm \cdot r^2$$

$$dm = \rho\, \pi r^2\, dh$$

$$r^2 = R^2 - h^2$$

所以 $dI = \frac{1}{2}\rho\ \pi\ (R^2 - h^2)^2\ dh$

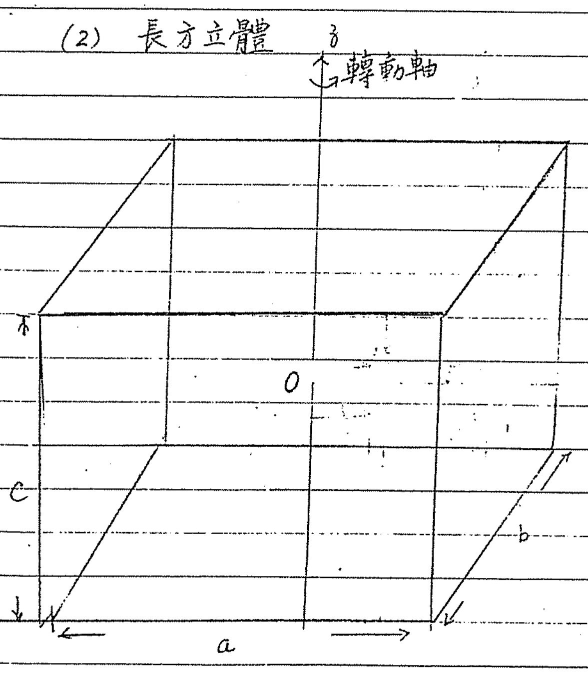
$$= \frac{1}{2}\pi\rho\,(R^4 - 2R^2h^2 + h^4)\, dh$$

將上式由 $-R$ 至 R 積分得

$$I = \frac{1}{2}\pi\rho\left(R^4 h - \frac{2}{3}R^2h^3 + \frac{1}{5}h^5\right)\Big|_{-R}^{R}$$

$$= \frac{8}{15}\pi\rho R^5 = \frac{8}{15}\pi\ \frac{M}{\frac{4}{3}\pi R^3}\cdot R^5 = \frac{2}{5}MR^2$$

(2) 長方立體



$$\rho = \frac{M}{abc}$$

$$R^2 = x^2 + y^2$$

$$I = \int \rho R^2 dV$$

$$= \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} \rho\,(x^2+y^2)\,dz$$

$$= \rho c b \int_{-a/2}^{a/2} x^2\,dx + \rho a c \int_{-b/2}^{b/2} y^2\,dy$$

$$= \rho cb\left[\frac{1}{3}\left(\frac{a}{2}\right)^3 - \frac{1}{3}\left(-\frac{a}{2}\right)^3\right] + \rho ac\left[\frac{1}{3}\left(\frac{b}{2}\right)^3 - \frac{1}{3}\left(-\frac{b}{2}\right)^3\right]$$

$$= \rho abc\left[\frac{b^2}{12} + \frac{a^2}{12}\right]$$

$$= \frac{M}{12}\left(a^2 + b^2\right)$$

顯然地,若 $a \neq b$, $a \neq c$, $b \neq c$, 則對通過 O 点以 x, y, z 為轉動軸之慣性矩均不相同.

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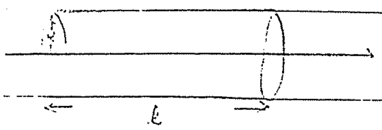
習題

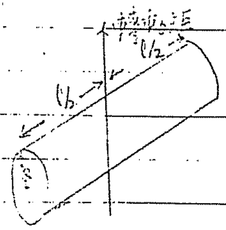
1. 求下列剛體之慣性矩

(a)



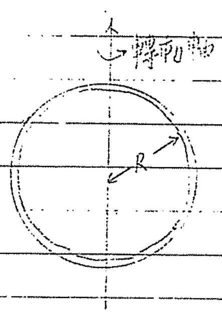
[I]

(b)



[B]

(c) 空心球



[D]

2. 三質量各為 2 kg 放在每邊長為10厘米等邊三角形之頂點，求系統之轉動慣量(慣量矩)及其迴轉半徑 相對於一垂直於三角形平面之軸並通過

(a) 一頂點 [A]

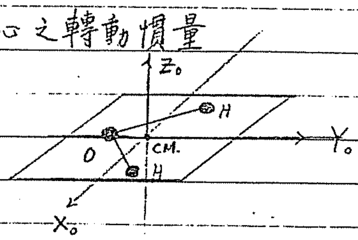
(b) 一邊之中點 [G]

(c) 質量中心 [E]

3. 在水分子中氫-氧之距離等於 0.91×10^{-10} m 且氫-氧二鍵之夾角為 105°

求分子相對於圖所示之(i) X 主軸 [C] (ii) Y 主軸 [F] 及(iii) Z 主軸 [H]

並通過質量中心之轉動慣量



答案

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[A] $I_a = 0.04\ Kg\ m^2$, $K_a = 0.0815\ m$

[B] $\frac{MR^2}{4} + \frac{Ml^2}{12}$

[C] $I_x = 9.15 \times 10^{-47}\ Kg\ m^2$

[D] $\frac{2}{3} MR^2$

[E] $I_c = 0.020\ kg\ m^2$, $K_c = 0.0576\ m$

[F] $I_y = 8.65 \times 10^{-48}\ kg\ m^2$

[G] $I_b = 0.025\ Kg\ m^2$, $K_b = 0.0645\ m$

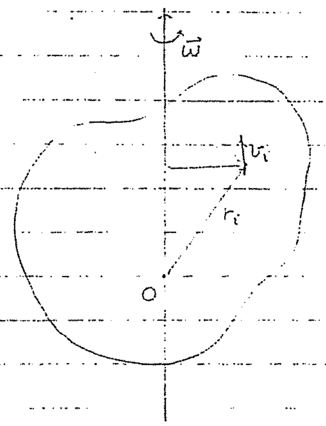
[H] $I_z = 9.01 \times 10^{-47}\ Kg\ m^2$

[I] $\frac{MR^2}{2}$

第二節 剛体之主軸

1. 簡介 我們在此節中我們將討論如何找一剛体之主軸之方法。

2. 基本觀念



我們的出發點是本章第一節中之第(3)式

$$\vec{L}_i = m_i\left[(\vec{r}_i\cdot\vec{r}_i)\vec{\omega} - (\vec{r}_i\cdot\vec{\omega})\vec{r}_i\right] \qquad (1)$$

以 O 為原點取一坐標系統 $\hat{i},\hat{j},\hat{k}$。

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\vec{L}_i = m_i\left[(x_i^2+y_i^2+z_i^2)(\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}) - (x_i\omega_x + y_i\omega_y + z_i\omega_z)(x_i\hat{i} + y_i\hat{j} + z_i\hat{k})\right] \qquad (2)$$

所以

$$L_{i,x} = m_i\left[\omega_x x_i^2 + \omega_x y_i^2 + \omega_x z_i^2 - x_i^2\omega_x - x_i y_i\omega_y - x_i z_i\omega_z\right]$$

$$= m_i(y_i^2+z_i^2)\omega_x - m_i x_i y_i\omega_y - m_i x_i z_i\omega_z \qquad (3)$$

同理可得

$$L_{i,y} = -m_i y_i x_i\omega_x + m_i(z_i^2+x_i^2)\omega_y - m_i y_i z_i\omega_z \qquad (4)$$

$$L_{i,z} = -m_i z_i x_i\omega_x - m_i z_i y_i\omega_y + m_i(x_i^2+y_i^2)\omega_z \qquad (5)$$

$I_{xx} = \sum m_i(y_i^2+z_i^2) \to \int (y^2+z^2)\rho(x,y,z)\,dx\,dy\,dz$

$m_i \to \rho_i\Delta V_i$

由 (3), (4), (5) 可得

$\to \int xy\,\rho(x,y,z)\,dx\,dy\,dz$

$$L_x = \sum_i L_{ix} = \left[\sum_i m_i(y_i^2+z_i^2)\right]\omega_x - \left[\sum_i m_i x_i y_i\right]\omega_y - \left[\sum_i m_i x_i z_i\right]\omega_z$$

$$= I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z \qquad (6)$$

同理

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國立清華大學
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	金額									用途摘要	國內差旅費第30屆光譜技術與表面科學研討會/黃耀欽/台中受款人:黃耀欽$2235		
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承辦人 組 長 單位主管				承辦人 組 長 單位主管	

$$L_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z \tag{7}$$

$$L_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z \tag{8}$$

$1 \leftrightarrow x,\ 2 \leftrightarrow y,\ 3 \leftrightarrow z$ 則 (6),(7),(8) 可寫成

$$L_j = \sum_k^3 I_{jk}\,\omega_k \qquad I_{jk} = I_{kj} \tag{9}$$

I_{jj} 叫做慣性矩，I_{jk} $(j \neq k)$ 叫做慣性乘積，I_{jk} 叫做慣性張量。

轉動的動力公式在慣性座標系統中為

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \Leftrightarrow \quad \tau_j = \frac{dL_j}{dt} = \frac{d}{dt}\sum_k I_{jk}\,\omega_k \tag{10}$$

若剛体是繞着慣性座標一定軸則以上的公式可以簡化一些。我們取 z 軸為轉動則

$\vec{\omega} = \omega_z \hat{k}$ 而動力公式則可寫成

$$\begin{aligned} L_x &= I_{xz}\,\omega_z & \tau_x &= \frac{dL_x}{dt} = \frac{d}{dt}(I_{xz}\,\omega_z) \\ L_y &= I_{yz}\,\omega_z & \tau_y &= \frac{dL_y}{dt} = \frac{d}{dt}(I_{yz}\,\omega_z) \\ L_z &= I_{zz}\,\omega_z & \tau_z &= \frac{dL_z}{dt} = \frac{d}{dt}(I_{zz}\,\omega_z) \end{aligned} \tag{11}$$

顯然地 I_{zz} 即是該剛体以 0 点沿 z 軸為轉軸之慣性矩。

若 $I_{xz} = I_{yz} = 0$ 則 $\tau_x = 0$，$\tau_y = 0$。但是若是 I_{xz} 或 I_{yz} 不為 0 則

此剛体不平衡需要外力矩 τ_x 或 τ_y 才能避免其轉軸移動。

動能之計算.

首先我們將證明此剛体之轉動動能為

$$K.E. = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\sum_j \omega_j L_j = \frac{1}{2}\sum_j\sum_k I_{jk}\,\omega_j\,\omega_k \tag{12}$$

由第 (1) 式得

$$\vec{L}_i\cdot\vec{\omega} = m_i\left[(\vec{r}_i\cdot\vec{r}_i)\,\omega^2 - (\vec{r}_i\cdot\vec{\omega})^2\right] = m_i R_i^2\,\omega^2 = m_i V_i^2 \tag{13}$$

$$K.E. = \frac{1}{2}\sum \vec{L}_i\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

在以 z 軸為轉軸時 $K.E. = \frac{1}{2} I_{zz}\,\omega_z^2$ (14)

我們現在討論如何找主軸

設 $\vec{\omega}' = \omega_x' \hat{i} + \omega_y' \hat{j} + \omega_z' \hat{k}$ 是繞着主軸轉動，則

$$\vec{L} = I\,\vec{\omega}' \tag{15}$$

也即是[4]

$$\begin{aligned} I_{xx}\,\omega_x' + I_{xy}\,\omega_y' + I_{xz}\,\omega_z' &= I\,\omega_x' \\ I_{yx}\,\omega_x' + I_{yy}\,\omega_y' + I_{yz}\,\omega_z' &= I\,\omega_y' \\ I_{zx}\,\omega_x' + I_{zy}\,\omega_y' + I_{zz}\,\omega_z' &= I\,\omega_z' \end{aligned} \tag{16}$$

整理後得

$$\begin{aligned} (I_{xx}-I)\,\omega_x' + I_{xy}\,\omega_y' + I_{xz}\,\omega_z' &= 0 \\ I_{yx}\,\omega_x' + (I_{yy}-I)\,\omega_y' + I_{yz}\,\omega_z' &= 0 \\ I_{zx}\,\omega_x' + I_{zy}\,\omega_y' + (I_{zz}-I)\,\omega_z' &= 0 \end{aligned} \tag{17}$$

以上之联立方程式在以下情況之下方有解

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-I \end{vmatrix} = 0 \tag{18}$$

將此行列式展開則得 I 之三次方程式稱為特徵方程式。因此有三個根 I_1, I_2, I_3 它們是對應於主軸之慣性矩。將 I_1 代入 (17) 式中可求得 $\omega_{x,1}' : \omega_{y,1}' : \omega_{z,1}'$ 因此我們對應於主慣性矩 I_1 之主軸之方向。同法我們可找出對應於 I_2, I_3 之主慣性軸之方向。由於 $I_{xx} \geq 0$, $I_{yy} \geq 0$, $I_{zz} \geq 0$ 及 $I_{jk} = I_{kj}$ $(j \neq k)$，我們可以證明 (1) 特徵方程式之根均為實數 (2) 若 I_1, I_2, I_3 均不相同時則其三主軸互相垂直 (3) 若 I_1, I_2, I_3 的值中有兩者以上相同時，則我們仍可找到互相垂直之主軸。若 $I_1 = I_2 = I_3$ 則稱為圓陀螺，若 $I_1 = I_2 \neq I_3$ 則稱為對稱陀螺，若三個慣性矩若不相同則稱為不對稱陀螺，若 $I_1 = 0$，$I_2 = I_3$ 則稱為是一個轉体。

討論

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(1) 由慣性矩的定義 $I_{xx}=\sum m_i(y_i^2+z_i^2)$

$I_{yy}=\sum m_i(x_i^2+z_i^2)$

$I_{zz}=\sum m_i(x_i^2+y_i^2)$ (19)

中可看出 $I_{xx}+I_{yy}=\sum m_i(x_i^2+y_i^2+2z_i^2)\geq I_{zz}$ (20)

同理 $I_{xx}+I_{zz}\geq I_{yy}$, $I_{yy}+I_{zz}\geq I_{xx}$ (21)

若一剛体分佈在 $z=0$ 之平面上，也即是 $z_i=0$ 時 $I_{xx}+I_{yy}=I_{zz}$ (22)

(2) 若 $I_{xz}=I_{yz}=0$ 則若 $\vec{\omega}=\omega_z\hat{z}$ 時，$L_x=L_y=0$ ，$L_z=I_{zz}\omega_z$ ，所以此時 $\vec{L}=I\vec{\omega}$ 成立，而 z 軸是此剛体之主軸。

(3) 此式可以矩陣之方式寫出為

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{23}$$

↓ 行矩陣　　↓ 正方矩陣　　↓ 行矩陣

由於 $I_{xy}=I_{yx}$
$I_{xz}=I_{zx}$
$I_{yz}=I_{zy}$

∴ 此一正方矩陣是一對稱矩陣

(4) 此式以矩陣之方式可寫成

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{pmatrix} = I \begin{pmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{pmatrix} \tag{24}$$

$$\underset{=}{I} \qquad \underset{\sim}{\omega}' \qquad \underset{\sim}{\omega}'$$

它可簡寫成 $\underset{=}{I}\,\underset{\sim}{\omega}'=I\,\underset{\sim}{\omega}'$ 。此類公式稱為本徵方程式，$\underset{\sim}{\omega}'$ 稱為本徵向量，I 稱為本徵值。

4. 應用

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若 I_1, I_2, I_3 是一剛体之三個主慣性矩
$\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$ 是其對應主軸之角速度，其方向即為其主軸之方向。

我們以 m, n 來分辨三個主軸
i, j, k 來分辨其 x, y, z 之分量

$$\vec{\omega}_1 = \omega_{x,1}\hat{i} + \omega_{y,1}\hat{j} + \omega_{z,1}\hat{k} \quad , \quad \vec{\omega}_2 = \omega_{x,2}\hat{i} + \omega_{y,2}\hat{j} + \omega_{z,2}\hat{k}$$
$$\vec{\omega}_3 = \omega_{x,3}\hat{i} + \omega_{y,3}\hat{j} + \omega_{z,3}\hat{k}$$
$$\omega_{1,1} \leftrightarrow \omega_{x,1} \qquad \omega_{3,2} = \omega_{z,2} \tag{25}$$

由於 $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$ 是沿主軸方向轉動

$$\vec{L}_1 = I_1\vec{\omega}_1 \Rightarrow L_{x1} = I_1\omega_{x1} \;,\; L_{y1} = I_1\omega_{y1} \;,\; L_{z1} = I_1\omega_{z1}$$
$$\vec{L}_2 = I_2\vec{\omega}_2 \Rightarrow L_{x2} = I_2\omega_{x2} \;,\; L_{y2} = I_2\omega_{y2} \;,\; L_{z2} = I_3\omega_{z3}$$
$$\vec{L}_3 = I_3\vec{\omega}_3 \Rightarrow L_{x3} = I_3\omega_{x3} \;,\; L_{y3} = I_3\omega_{y3} \;,\; L_{z3} = I_3\omega_{z3} \tag{26}$$

$$L_{im} = I_m\omega_{im} \tag{27}$$

$L_i = \sum_k I_{ik}\omega_k$ 此一公式對任何軸均成立 (28)

$$L_{im} = \sum_k I_{ik}\omega_{km} \tag{29}$$

由 (21) 及 (23) 兩式得

$$\sum_k I_{ik}\omega_{km} = I_m\omega_{im} \tag{30}$$

將 $\sum_i I_{ki}\omega_{in} = I_n\omega_{kn}$ 兩邊取共軛複數 (31)

$$\sum_i I_{ki}\omega_{in}^* = I_n^*\omega_{kn}^* \tag{32}$$

將 (30) 乘以 ω_{in}^* 再把 $i=1,2,3$ 加起來得

$$\sum_i\sum_k I_{ik}\omega_{km}\omega_{in}^* = \sum_i I_m\omega_{im}\omega_{in}^* \tag{33}$$

將 (32) 式乘以 ω_{km} 再把 $k=1,2,3$ 加起來得

$$\sum_k\sum_i I_{ki}\omega_{km}\omega_{in}^* = \sum_k I_n^*\omega_{kn}^*\omega_{km} \tag{34}$$

由於 $I_{ik} = I_{ki}$

$$(I_m - I_n^*)\sum_l \omega_{ln}^*\omega_{lm} = 0 \tag{35}$$

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$m=n$ 則 $I_m = I_m^*$ 因此 I_m 是實數

由 (21) 式可知 L_{im} 是角動量的分量因此必須是實數，I_m 是實數 因此 ω_{im} 均為實數

$m \neq n$ 時 (29) 式現在可寫成

$$(I_m - I_n)\ \vec{\omega}_m \cdot \vec{\omega}_n = 0 \qquad (36)$$

若 $I_m \neq I_n$ 則 $\vec{\omega}_m \cdot \vec{\omega}_n = 0$ 也即是 $\vec{\omega}_m$ 與 $\vec{\omega}_n$ 垂直。

我們現在舉例來說明此情形

(1) $I_{xx} = I_{yy} = I_{zz} = 1$, $I_{yz} = I_{zy} = 1$, $I_{xy} = I_{yx} = 0$, $I_{xz} = I_{zx} = 0$

第(17)式變成

$$\begin{aligned} (1-I)\ \omega_x' &= 0 \\ (1-I)\ \omega_y' + \omega_z' &= 0 \\ \omega_y' + (1-I)\omega_z' &= 0 \end{aligned} \qquad (37)$$

而其特徵方程式是

$$\begin{vmatrix} 1-I & 0 & 0 \\ 0 & 1-I & 1 \\ 0 & 1 & 1-I \end{vmatrix} = 0 \qquad (38)$$

$$(1-I)\ [\ (1-I)^2 - 1] = 0$$

$$(1-I)\ (\ 1-2I+I^2) = 0 \quad \Rightarrow \quad (1-I)(2-I)\ I = 0 \qquad (39)$$

所以 $I = 0,\ 1,\ 2$

(a) 當 $I=0$ 時由 (37) 得

$$\omega_{x,1}' = 0 \qquad -\omega_{y,1}' + \omega_{z,1}' = 0 \qquad (40)$$

$$\hat{u}_1 = (0, +\tfrac{1}{\sqrt{2}}, +\tfrac{1}{\sqrt{2}}) \qquad (41)$$

(b) 當 $I=1$ 時，由 (37) 式得

$\omega_{x,2}'$ 沒有條件，$\omega_{z,2}'=0$; $\omega_{y,2}'=0$ 因此 $\hat{u}_2 = (1, 0, 0)$ (42)

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(c) 當 $I=2$ 時，由(31)式得

$$\omega'_{x,3}=0,\quad \omega'_{y,3}+\omega'_{z,3}=0 \qquad (0,\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}) \tag{43}$$

因此 $\hat{u}_1, \hat{u}_2, \hat{u}_3$ 互相垂直

(2) $I_{jk}=1 \quad (j,k=1,2,3)$

第(17)式變成

$$\begin{aligned}(1-I)\,\omega_x' + \omega_y' + \omega_z' &= 0\\ \omega_x' + (1-I)\,\omega_y' + \omega_z' &= 0\\ \omega_x' + \omega_y' + (1-I)\,\omega_z' &= 0\end{aligned} \tag{44}$$

而其特徵方程式可寫成

$$\begin{vmatrix} 1-I & 1 & 1\\ 1 & 1-I & 1\\ 1 & 1 & 1-I \end{vmatrix}=0 \tag{45}$$

也即是

$$(1-I)\left[(1-I)^2-1\right]-1\left(1-I-1\right)+1\left(1-(1-I)\right)=0$$

$$(1-I)\left[-2+I\right]I+2I=0$$

$$I\left[(1-I)(-2+I)+2\right]=0$$

$$I\,(3-I)\,I=0 \tag{46}$$

$$I=3,\ 0,\ 0$$

(a) 當 $I=3$ 時，(38)式變成

$$\begin{aligned}-2\omega'_{x,1} + \omega_y' + \omega_z' &= 0\\ \omega'_{x,1} - 2\omega_y' + \omega_z' &= 0\\ \omega'_{x,1} + \omega_y' - 2\omega_z' &= 0\end{aligned} \tag{47}$$

由觀察可得知 $\omega_x' : \omega_y' : \omega_z' = 1:1:1$ 會滿足以上三式

$$\hat{u}_1 \text{ 可取 } \tfrac{1}{\sqrt{3}}(1,1,1) \tag{48}$$

(b) 當 $I=0$ 時，(38)式變成

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$$\omega_x' + \omega_y' + \omega_z' = 0 \qquad (49)$$

一式，因此可有無窮多個解。例如取 $\omega_x' : \omega_y' : \omega_z' = 2 : -1 : -1$ 即可滿足(43)式

$\hat{u}_2$ 可取 $\hat{u}_2 = \frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k})$，顯然地，$\hat{u}_1$ 與 $\hat{u}_2$ 垂直

$\hat{u}_1 \times \hat{u}_2 = \frac{1}{\sqrt{2}}(\hat{j} - \hat{k})$。此一向量之 $\omega_x' : \omega_y' : \omega_z' = 0 : 1 : -1$，也滿足(43)

式。所以它也是一慣性主軸。因此我們找到三個互相垂直之主軸 $\hat{u}_1, \hat{u}_2, \hat{u}_3$

其主慣性矩分別是 3, 0, 0。

習題

1. 有一剛体可用下列的三個質點來代表

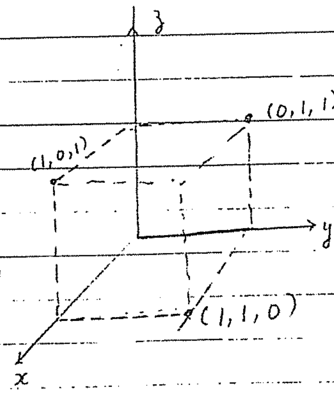
   $m_1 = 1$ 位於 $(1, 1, -2)$，$m_2 = 2$ 位於 $(-1, -1, 0)$，$m_3 = 1$ 位於 $(1, 1, 2)$

   (a) 求 I_{xx} [F]，I_{yy} [M] 及 I_{zz} [H]

   (b) 求 I_{xy} [B]，I_{xz} [I] 及 I_{yz} [K]

   (c) 求此剛体三主軸之方向及其對應之慣性矩。

2. 單位質量置於如圖所示處



   (a) 求 I_{xx} [R]，I_{yy} [J] 及 I_{zz} [N]

   (b) 求 I_{xy} [C]，I_{xz} [O] 及 I_{yz} [G]

   (c) 求此剛体三主軸之方向及其對應之慣性矩 [D]

3. 一薄而均勻置於 $x_1 - x_2$ 平面上之平板之慣性矩陣可寫成

$$\begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

若將座標軸以 x_3 軸轉 θ 角，則新的矩陣可寫成

$$\begin{pmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{pmatrix}$$

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(a) 求 A' 與 A, B, C 及 θ 之關係 [E]

(b) 求 B' 與 A, B, C 及 θ 之關係 [L]

(c) 求 C' 與 A, B, C 及 θ 之關係 [A]

(d) 以 x_3 為軸轉角為何時其座標軸才成為該剛体之主軸。 [Q]

答案

[A] $C' = C\cos 2\theta - \frac{1}{2}(B-A)\sin 2\theta$

[B] -4

[C] -1

[D] $I_1 = 2$ $\hat{u}_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $I_2 = I_3 = 5$。 $\hat{u}_2$, $\hat{u}_3$ 為在垂直於 $\hat{u}_1$ 平面互相垂直之兩單位向量

[E] $A' = A\cos^2\theta - C\sin 2\theta + B\sin^2\theta$

[F] 12

[G] -1

[H] 8

[I] 0

[J] 4

[K] 0

[L] $B' = A\sin^2\theta + C\sin 2\theta + B\cos^2\theta$

[M] 12

[N] 4

[O] -1

[P] $I_1 = 16$ $\vec{u}_1 = \frac{1}{\sqrt{2}}(1,-1,0)$ $I_2, I_3 = 8$。 $\hat{u}_2$, $\hat{u}_3$ 為在垂直於 $\hat{u}_1$ 之平面上之兩互相垂直之單位向量

[Q] $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C}{B-A}\right)$

[R] 4

第三節 剛体在平面上之運動

簡介：剛体運動一般而言，較為複雜，但是若是剛体之運動是局限於一平面時，則較為容易。在此節中我們將討論剛体在平面上之運動。為了更清楚一點我們將先討論純移動及純轉動，然後再討論一般性的問題

基本觀念

(1) 平面移動

(a) 定義　剛体中之所有質點在任何瞬間之速度均相同，其共同速度在運動中可以是時間的函数，但它維持與一固定平面 K 平行。

$$\vec{v}_i(t) = \vec{v}_{C.M.}(t) \qquad i = 1, \cdots N \qquad (1)$$

若取固定平面 K 為 xy 平面，則 $\vec{v}_{CM}(t) = v_{CM,x}(t)\,\hat{i} + v_{CM,y}(t)\,\hat{j}$ (2)

(b) 動能

$$K.E. = \frac{1}{2} M \vec{v}_{C.M.}^{\,2} \qquad (3)$$

此處 M 是剛体之質量，$\vec{v}_{C.M.}$ 是其質心的速度

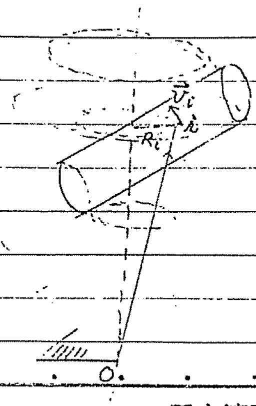
(c) 對一定点 O 之角動量 $\vec{L}_o = M\vec{r}_{C.M.} \times \vec{v}_{C.M.}$ (4)

此處 $\vec{r}$ 是質心在以 O 点為座標系統之位置向量

(d) 動力公式 [1,2]

$$\Sigma \vec{F}_{ext} = M \frac{d\vec{v}_{C.M.}}{dt} = M\vec{a}_{C.M.} \qquad (5)$$

(2) 對一定軸轉動



(a) 剛体對一垂直於固定平面 K 之定軸作轉動，其角速度 $\vec{\omega}$ 之方向是沿軸之方向

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \qquad (6)$$

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(b) 動能[3]

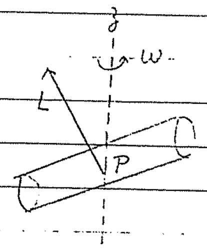
$$K.E. = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_i^{\,2} = \frac{1}{2}\sum_{i=1}^{N} m_i (\vec{\omega}\times\vec{r}_i)\cdot(\omega\times\vec{r}_i)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i^2\right)\omega^2 = \frac{1}{2} I \omega^2 \qquad (7)$$

I 是對定軸之慣性矩

(c) 若令轉軸之方向為 z 軸[4]，則

$$L_z = I\omega \qquad (8)$$



L_z 是對轉軸上一點所計算之角動量在轉軸方向之投影

(d) 若轉軸是該剛体的主軸則[5]

$$\vec{L} = I\vec{\omega} \qquad (9)$$

(e) 動力公式[6]

由 $\frac{d\vec{L}}{dt} = \sum \vec{\tau}_{ext}$ 可得

$$\frac{dL_z}{dt} = \sum \tau_{z,ext} \qquad (10)$$

將第(8)式及(10)式合在一起可得

$$\sum \tau_{z,ext} = I\frac{d\omega}{dt} \qquad (11)$$

(f) 由於 $dW = d\left[\frac{1}{2}I\omega^2\right]$ （動能與功之關係）

$$= I\omega\, d\omega$$

$$= I\omega\, dt \frac{d\omega}{dt}$$

$$= I\frac{d\omega}{dt}\, d\theta$$

$$= \sum \tau_{z,ext}\, d\theta \quad \text{（利用第(11)式）} \qquad (12)$$

由(12)式得 $P = \sum \tau_{z,ext} \cdot \omega \qquad (13)$

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及 $W = \int_{\theta_1}^{\theta_2} (\Sigma \tau_{z,ext})\, d\theta$ [7] (14)

(3) 一般性的剛体在平面上之運動

(a) 一剛体其平移運動平行於一固定平面而其轉動之轉軸與此定平面垂直。[8]

(b) 動能 $K.E. = \frac{1}{2} M V_{C.M.}^2 + \frac{1}{2} I_{C.M.} \omega^2$ [9] (15)

(c) 對一定點 O 所計算之角動量在沿轉軸(z)方向之投影

$$L_z = L_{oz} + I_{C.M.}\,\omega \qquad (16)$$

L_{oz} 是質心對定点 O 所計算角動量在沿轉軸(z)方向之投影，I 是剛体對質心及以z軸為轉所計算之慣性矩。

(d) 動力公式

$$\Sigma \vec{F}_{ext} = M \frac{d\vec{V}_{C.M.}}{dt} = M \vec{a}_{C.M.} \qquad (17)$$

$$\Sigma \tau^*_{z,ext} = I_{C.M.} \frac{d\omega}{dt} = I_{CM}\,\alpha \qquad (18)$$

此處 $\Sigma \tau^*_{z,ext}$ 是對質心所計算之外力距在轉動軸z方向之投影的代數和。α 是對通過質心轉軸所計算之角加速度。而 $I_{C.M}$ 是對通過質心轉軸所計算之慣性距。

討論

(1) 在慣性系統中才成立。

(2) $\Sigma \vec{F}_{ext}$ 只包括外力。

(3) 与平移動能 $\frac{1}{2} M V^2$ 相似。

(4) 為了方便起見我們取轉動軸為z軸

(5) 注意第(8)式是一純量公式而第(9)式是一向量公式。第(9)式只有轉動軸是該剛体之主軸時才成立。

(6) $\vec{L}$, $\vec{\tau}_{ext}$ 必需是對同一点來計算。而此式在慣性座標才成立。

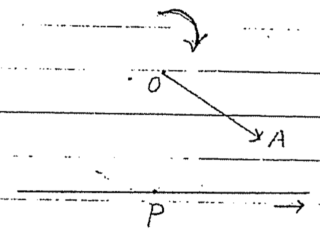
(7) 當剛体對轉軸由 θ_1 轉至 θ_2 中外力對剛体所作之功。

(8) 在很多情況下，此一條件成立。

(9) $\frac{1}{2}MV_{C.M}^2$ = 移動動能，$\frac{1}{2}I_{C.M.}\omega^2$ = 對質心的轉動動能。

應用：

我們首先討論輪子轉動時之一些定義及觀念



$\vec{V}_A = \vec{V}_{C.M.} + \vec{\omega} \times \vec{r}_A^*$

$\vec{V}_A$ 是位於 A 点之速度

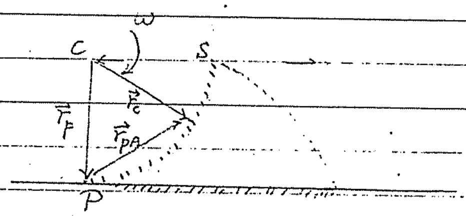
$\vec{\omega}$ 是轉動之角速度我們定義順時針旋轉時為正

$\vec{V}_{C.M.}$ 向右時為正，$\vec{r}_A^*$ 是由 O 至 A 之向量

(i) 滑動 $\vec{\omega} = 0$，$\vec{V}_A = \vec{V}_{C.M}$（所有的点之速度均相同，平移運動） (19)

(ii) 轉動 $\vec{V}_{C.M} = 0$，$\vec{V}_A = \vec{\omega} \times \vec{r}_A^*$ (20)

(iii) 滾動



當剛体轉動一角度 θ 時，接觸点移動了 Rθ

$$S = R\theta$$

$$\frac{dS}{dt} = R\frac{d\theta}{dt} \Rightarrow V_{C.M} = \omega R \Rightarrow \omega = \frac{V_{C.M.}}{R} \quad (21)$$

因此在滾動運動中其接觸点之瞬時速度為 0.

因為在接觸點P之瞬時速度為 0，因此 P 点可以想成瞬時固定，因此，在此瞬間剛体對此点以角速度 ω 作轉動。P 点稱為瞬時轉動中心，通過 P 点

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沿 $\vec{\omega}$ 方向之軸即為瞬時轉軸。要證明以上的說法我們需證明剛体上每一点 A 速度均為 $\vec{v}_A = \vec{\omega} \times \vec{r}_{PA}$ 此處 $\vec{r}$ 是連接 P 为 A 点之向量。

證明：

$$\vec{v}_A = \vec{V}_{CM} + (\vec{\omega} \times \vec{r}_{CA}) \qquad (22)$$

$\vec{r}_C$ 是由質心至 A 点之向量

$$\vec{r}_{CA} = \vec{r}_{CP} + \vec{r}_{PA} \qquad (23)$$

將 (23) 式代入 (22) 式得

$$\vec{V} = \vec{V}_{CM} + \vec{\omega} \times \vec{r}_{CP} + \vec{\omega} \times \vec{r}_{PA} \qquad (24)$$

前兩項是在 P 之瞬時速度 在滾動時為零，因此

$$\vec{V}_A = \vec{\omega} \times \vec{r}_{PA} \qquad (25)$$

我們現在來計算一半徑為 R 之圓筒之動能

(A) 對 C 點來算

$$K.E. = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$$

（$\frac{1}{2}MR^2$：對 C 之慣性矩）

$$= \frac{3}{4}MR^2\omega^2 \qquad (26)$$

(B) 對 P 點來算

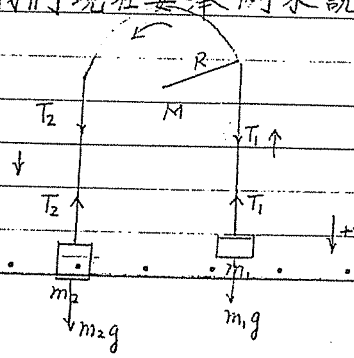
$$K.E. = \frac{1}{2}I'\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2 + MR^2\right)\omega^2$$

（I'：對 P 之慣性矩；利用平行軸定理）

$$= \frac{3}{4}MR^2\omega^2 \qquad (27)$$

(26) 式及 (27) 式相符。

我們現在要舉例來說明這些觀念



(a) 兩物体之質量分別為 m_1 及 m_2 ($m_1 < m_2$)，懸於一質量為 M，半徑為 R 之滑輪之兩邊，求在重力下 m_2 之運動加速度

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$$m_1 g - T_1 = -m_1 a_1 \qquad (28)$$

$$m_2 g - T_2 = m_2 a_2 \qquad (29)$$

$\tau_1 = R T_1$ 向紙內 (30)

$\tau_2 = R T_2$ 向紙外 (31)

$$\tau = \tau_2 - \tau_1 = I\alpha = R(T_2 - T_1) \qquad (32)$$

由於繩子是緊而不滑動 $R\alpha = a_1 = a_2 = a$ (33)

將 (33) 式代入 (28), (29), (32) 得

$$\begin{cases} T_1 - m_1 g = m_1 a & (34) \\ m_2 g - T_2 = m_2 a & (35) \\ I \dfrac{a}{R} = R(T_2 - T_1) & (36) \end{cases}$$

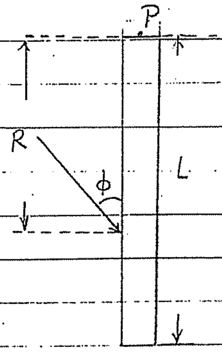
滑輪之慣性矩為 $I = \frac{1}{2}MR^2$，帶入 (36) 式得

$$\frac{1}{2} M a = T_2 - T_1 \qquad (37)$$

(34), (35) 及 (36) 三式中有三個未知數 T_1 T_2 及 a. 解此聯立方程式後得

$$a = \frac{(m_2 - m_1) g}{(m_1 + m_2 + \frac{M}{2})} \qquad (38)$$

由 (36) 式，可見若 $I = 0$ 時則 $T_2 = T_1$。在簡單的阿特午機時即常常將滑輪之慣性矩略去不算，那時 $T_2 = T_1$。（滑輪只改變張力之方向但不更改其大小只有在以上所討論之情況下方成立）。



(b) 在一水平而又平滑之表面上，有一均勻之棍子其質量為 M 長度為 L 用釘子固定於其一端 P 點。棍子在此平面上繞 P 點自由轉動。一大小不變之外力對棍子施力並始終保持其方棍子之交角為 ϕ。施力點離 P 之距離為 R

在此力之作用下此棍子由靜止開始轉動。

I（棍子對 P 點之慣性矩） $= \frac{1}{3}ML^2$ (38)

$\vec{\tau} = \vec{r} \times \vec{F}$ 對 P 點來算 (39)

$\vec{\tau}$ 之大小為 $FR\sin\phi$ 其方向為向外

$\tau = I\alpha \Rightarrow FR\sin\phi = \frac{1}{3}ML^2\alpha$ (40)

$$\alpha = \frac{3FR\sin\phi}{ML^2} \quad (41)$$

若 $t=0$ 時棍子是如圖所示，以 θ 量度其垂直方向之夾角，則

$\theta(t) = \frac{1}{2}\alpha t^2$ （此處 $\theta(t=0) = \omega(t=0) = 0$） (42)

該棍子轉第一周所需之時間 T 可由下式中求出

$$2\pi = \frac{1}{2}\alpha T^2$$

$$T = \sqrt{\frac{4\pi}{\alpha}} \quad (43)$$

轉第一周其動能之增加為

$$K.E. = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \cdot \frac{1}{3}ML^2 \cdot \alpha^2 T^2$$

$$= \frac{1}{2} \cdot \frac{1}{3} ML^2 \alpha^2 \cdot \frac{4\pi}{\alpha}$$

$$= 2\pi \cdot \frac{1}{3} ML^2 \cdot \frac{3FR\sin\phi}{ML^2}$$

$$= 2\pi FR\sin\phi \quad (44)$$

轉一周外力對其所做之功

$$W = \int_0^{2\pi} \tau \, d\theta$$

$$= FR\sin\phi \cdot 2\pi \quad (45)$$

(44) 式與 (45) 式之結果相同。這即是動能與功之關係式。

我們現在計算釘子對該棍子所施之力。

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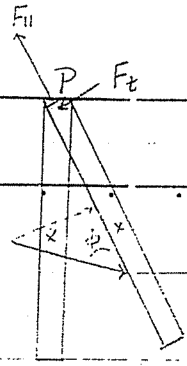


$$F\sin\phi - F_t = Ma_t \qquad (46)$$

$$a_t = \frac{dv}{dt} = \frac{L}{2}\frac{d\omega}{dt} = \frac{L}{2}\alpha \qquad (47)$$

$$F_t = F\sin\phi - M\frac{L}{2}\cdot\frac{3FR\sin\phi}{ML^2}$$

$$= F\sin\phi\left[1 - \frac{3R}{2L}\right) \qquad (48)$$

$$F_{\parallel} - F\cos\phi = Ma_{\parallel} \qquad (49)$$

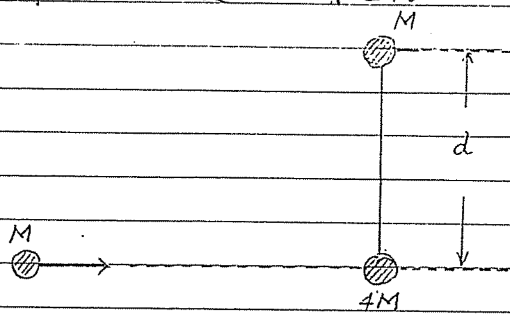
$$a_{\parallel} = \frac{v^2}{L/2} = \frac{L}{2}\omega^2 \qquad (50)$$

在 $t = T$ 時 $\quad a_{\parallel} = \frac{L}{2}\alpha^2T^2 = \frac{L}{2}\alpha^2\cdot\frac{4\pi}{\alpha} = 2\pi\alpha L$

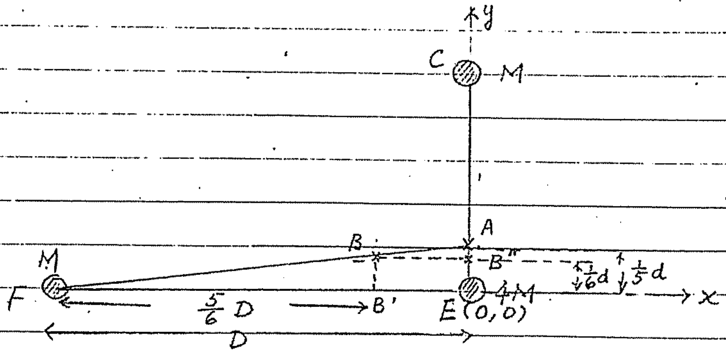
$$= 2\pi\cdot L\,\frac{3FR\sin\phi}{ML^2} \qquad (51)$$

$$F_{\parallel} = F\cos\phi + \frac{6\pi FR\sin\phi}{L} \qquad (52)$$

(c) 在外太空中，不受重力之情况下。



若一剛体是由 M 及 4M 之兩質點以一無質量之鋼棍相連。另一質量為 M 之質點以速度 $\vec{V_0}$ 沿垂直於連接兩質點鋼棍的方向趨向質量 4M 之質點相撞而与它黏在一起。描述此系統在碰撞後之運動情形。



A 是剛体之質量中心 位於 $(0, \frac{1}{5}d)$

當 F 位於 $(-D, 0)$ 時

該系統之質量中心 位於 $B(-\frac{1}{6}D, \frac{1}{6}d)$

當 M 与 4M 碰撞時該系統之質量中心將位於 B''点 $(0, \frac{1}{6}d)$，由圖可以很容易的看出

$$\vec{V}_{C.M.} = \frac{1}{6}\vec{V_0} \qquad (53)$$

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由於此系統不受外力，所以碰撞後，其質量中心仍以 $\frac{1}{6}V_0$ 之速度進行。

當 M 位於 (-D, 0) 時

位於 C 處質量為 M 之質點對質量中心 B 之相對速度為 $-\frac{1}{6}V_0$ 則它對質心之角動量為 $M\frac{5}{6}d\frac{1}{6}V_0$ 向外

位於 E 處質量為 4M 之質點對質量中心 B 之相對速度為 $-\frac{1}{6}V_0$ 則它對質心之角動量為 $4M\frac{1}{6}d\frac{1}{6}V_0$ 向內

位於 E 處質量為 M 之質點對質量中心 B 之相對速度為 $\frac{5}{6}V_0$ 則它對質心之角動量為 $M\frac{1}{6}d\frac{5}{6}V_0$ 向外

所以對質心該系統之總角動量為 $\frac{1}{6}MV_0d$ (向外)。由於此系統不受外力。因此它的角動量守恒。碰撞後其角動量仍是 $\frac{1}{6}MV_0d$ (向外)。碰撞後，該系統變成一剛体。對其質心其慣性矩 $I = M(\frac{5}{6}d)^2 + 5M(\frac{1}{6}d)^2 = \frac{5}{6}Md^2$ (54)

由 $L = I\omega$，我們求得 $\omega = \frac{V_0}{5d}$ (55)

我們對碰撞後該系統的運動有了如下完整的描述，

(1) 質心以等速 $\frac{1}{6}V_0$ 沿 x 軸進行

(2) 鋼棍以通過質心垂直於紙面之轉軸以等角速度 $\omega = \frac{V_0}{5d}$ 反時鐘方向轉動。

碰撞前之動能 $K.E. = \frac{1}{2}MV_0^2$ (56)

碰撞後之動能

$$K.E. = \frac{1}{2}(M+4M+M)(\frac{1}{6}V_0)^2 + \frac{1}{2}\underset{I}{(\frac{5}{6}Md^2)}\cdot\underset{\omega}{(\frac{V_0}{5d})^2}$$

$$= \frac{1}{12}MV_0^2 + \frac{1}{60}MV_0^2 \quad (57)$$

碰撞之 Q 值 $Q = \frac{1}{12}MV_0^2 + \frac{1}{60}MV_0^2 - \frac{1}{2}MV_0^2 = -\frac{4}{5}(\frac{1}{2}MV_0^2)$ (58)

由於碰撞該系統損失了它原來動能之百分之八十。

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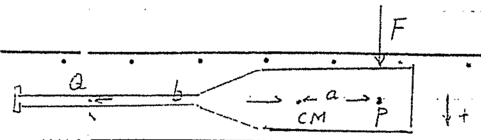
(d)



一棒球之長度為 L 質量為 M，置於一平滑表面上，在很短的一時間 ΔT 受一外力 F 產生一衝量 $F\Delta T$。F 作用於 P 點（位於離質心 a 處）

力作用 $F\Delta T$ 後質心之速度為 $V_{C.M.} = \frac{F\Delta T}{M}$

$$\tau = \frac{dL}{dt} \quad (\tau, L \text{ 均在 } z \text{ 方向}) \tag{59}$$

$$\tau = I\frac{d\omega}{dt} \Rightarrow \text{在作用以後之角速度為 } I\omega = \tau\Delta T = Fa\Delta T \tag{60}$$

$$\omega = \frac{Fa\Delta T}{I} = \frac{Fa\Delta T}{MK^2} \tag{61}$$

在 Q 之速度 $= V_{C.M.} - b\omega$

$$= \frac{F\Delta T}{M} - \frac{bFa\Delta T}{MK^2} \tag{62}$$

若 Q 之速度為 0 則 $K^2 = ab \Rightarrow b$ 位於 $\frac{K^2}{a}$ 處。

Q 點稱為打擊中心。

反過來若力作用於 Q 點則 P 點是其打擊中心之證明可由上式中很容易看出（以上公式中 $a \leftrightarrow b$）。

另一法解此体，我們可以再 Q 處加一假想力 F_P 而使 Q 點保持不變。

當 F 作用時，此棒球棍對 Q 點有一角加速 α，其大小為

$$\underset{\tau}{F(a+b)} = \underset{\text{對 Q 點之慣性矩}}{I}\alpha \tag{63}$$

$$I = I_{C.M.} + Mb^2 = M(K^2+b^2) \tag{64}$$

$$\alpha = \frac{F(b+a)}{M(K^2+b^2)} \quad \text{(順時鐘)} \tag{65}$$

此時質心的加速度為

$$a_{C.M.} = b \cdot \frac{F(b+a)}{M(K^2+b^2)} \quad (+) \tag{66}$$

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由牛頓定律得

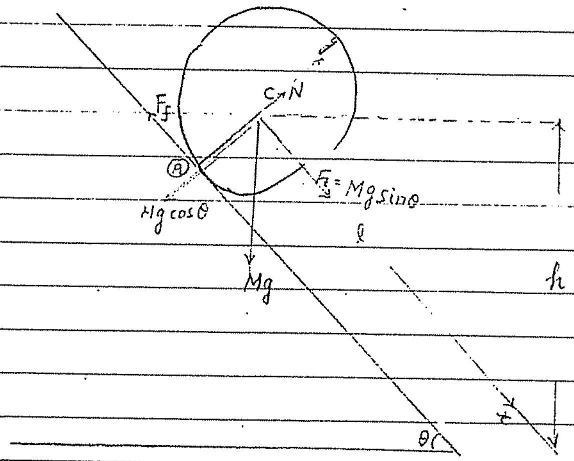
$$F - F_p = Ma_{c.M.} = Mb\frac{F(b+a)}{M(K^2+b^2)} \quad (67)$$

因此 $F_p = F\frac{K^2-ab}{K^2+b^2}$ (68)

若 $K^2 = ab$ 則 $F_p = 0$（真正的情形）也就是說此時在 Q 點不需加任何力來使 Q 點保持不動。也即是說當 $b = \frac{K^2}{a}$ 處是打擊中心。

在打擊時若球員握棒於 b 處則若要使 Q 不動則對該棍必須施一力 F_p。由於牛頓第三定律球棒對球員也施以同樣大小之力。

(e) 我們現在討論一圓筒由斜面滑下之問題



首先計算 $\vec{a}_{c.M}$ 之大小

因接觸點是瞬時轉動中心

$$\tau_A = RMg\sin\theta$$

$$= I_A\alpha \quad (69)$$

$$I_A = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2 \quad (70)$$

$$\alpha = \frac{RMg\sin\theta}{3/2\ MR^2} = \frac{2}{3}\frac{g}{R}\sin\theta \quad (71)$$

$\vec{V}_{c.M.} = \vec{\omega}\times\vec{r}_A \Rightarrow \vec{a}_{c.M.} = \vec{\alpha}\times\vec{r}_A$ （$\vec{r}_A$ 是固定之向量）

$$|a_{c.M.}| = \alpha R = \frac{2}{3}g\sin\theta \quad (72)$$

$$|\vec{N}| = |Mg\cos\theta| \quad (73)$$

$$\overbrace{Mg\sin\theta - F_f}^{F} = \overbrace{M\frac{2}{3}g\sin\theta}^{Ma}$$

$$\Rightarrow F_f = \frac{1}{3}Mg\sin\theta \quad (74)$$

$$F_f \le \mu N = \mu Mg\cos\theta \quad (75)$$

（μ：靜摩擦力，因為在接觸點 $v=0$）

要保持純轆動運動要求 $F_f = \frac{1}{3}Mg\sin\theta$ (76)

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因此 $\frac{1}{3}Mg\sin\theta \le \mu Mg\cos\theta$

也即是 $\tan\theta \le 3\mu$ 時圓筒才能保持純滾動運動。

我們現在求該圓筒到達底部時之速度 ($v_0=0$) 由於是等加速度運動

$$v^2 = 2al = 2\frac{2}{3}g\sin\theta \cdot l$$

因此 $v^2 = \frac{4}{3}gl\sin\theta$ (77)

利用能量守恆來作這個題目

在開始時 位能為 $Mgh = Mgl\sin\theta$ (78)

動能為 0

總能量為 $Mgl\sin\theta$

在結束時 位能為 0

動能為 $\frac{1}{2}Mv^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{MR^2}{2}\cdot\frac{v^2}{R^2} = \frac{3}{4}Mv^2$

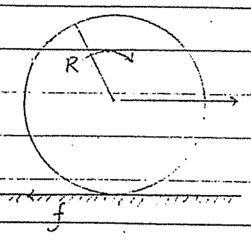
總能量為 $\frac{3}{4}Mv^2$

能量守恆 $\frac{3}{4}Mv^2 = Mgl\sin\theta \Rightarrow v^2 = \frac{4}{3}gl\sin\theta$ (79)

在此時我們仍能用能量守恆的理由是在純滾動時摩擦力對剛体不作功。這可由

$P = \vec{F}_f \cdot \vec{v}_{\text{接觸點之速度}} = 0$ 很清楚的看出。

(f) 一質量為 M，半徑為 R 之撞球，以一撞桿給予一通過質心之衝量使其獲得初速 v_0。問該撞球於滑動多遠後才開始滾動？



$t=0$ 時 $V_{c.M} = V_0$，$\omega = 0$ (80)

由於摩擦力的關係

$\tau_c = fR = \mu MgR$　　f 是摩擦力 (81)

$I\alpha = \mu MgR$ (82)

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$$I = \frac{2}{5} MR^2 \quad (83)$$

因此 $\alpha = \frac{5}{2}\mu\frac{g}{R}$ (84)

$$\omega = \frac{5}{2}\mu\frac{g}{R}t \quad (85)$$

$$Ma_c = -f \quad (86)$$

$$a_c = -\frac{\mu g M}{M} = -g\mu \quad (87)$$

$$V_c = v_0 - \mu g t \quad (88)$$

滾動的條件

$$V_c = \omega R \quad (89)$$

也即是

$$v_0 - \mu g t = \frac{5}{2}\mu\frac{g}{R}t \quad (90)$$

$$t = \frac{2}{7}\frac{v_0}{\mu g} \quad (91)$$

所走之距離為

$$S = v_0 t - \frac{1}{2}\mu g t^2 = \frac{12}{49}\frac{v_0^2}{\mu g} \quad (92)$$

也即是走了 $S = \frac{12}{49}\frac{v_0^2}{\mu g}$ 後撞球開始滾動。

若撞桿作用於高出質心 h 處，則於 $t=0$ 時 $\omega = \frac{\overbrace{Mv_0h}^{L}}{I} = \frac{5}{2}\frac{v_0h}{R^2}$ (93)

(i) 若 $\frac{5}{2}\frac{v_0h}{R^2}R = v_0 \Leftrightarrow h = \frac{2}{5}R$ 則馬上即開始滾動

(ii) 若 $h < \frac{2}{5}R$ 則 $\omega < \frac{V_0}{R}$，需一段時候才開始滾動

(iii) 若 $h > \frac{2}{5}R$ 則 $\omega > \frac{V_0}{R}$，此時摩擦力之方向是向前的，因為在接觸点其速度是向後

在滾動時，由於接觸点之速度為零，所以摩擦力是靜摩擦力，它在某一範圍內可以自行調節使剛体之運動保持為滾動運動。

習題

(1) 在外太空中而不受外力時，一均勻長度為 L 及質量為 M 之棍子在一慣性坐標系統中靜止不動。一質量亦為 M 之質點沿垂直方向以速度 V_0 趨向該棍子，質量在質心下處擊中該棍子。碰撞後該質點以 $\frac{1}{2}V_0$ 之速度繼續前進

(a) 當入射質點離棍子 D 時，此系統之質心位於何處？ [C] 那時質心之速度為何？ [O]

(b) 在棍子原來靜止之系統中，碰撞後棍子之質心速度為何？ [H]

(c) 在該系統的質量中心座標系統中，碰撞後棍子質心的速度為何？ [L]

(d) 對棍子之質心而言，在碰撞前該系統之角動量為何？ [S]

(e) 求棍子在碰撞後對其質心的角速度為何！ [F]

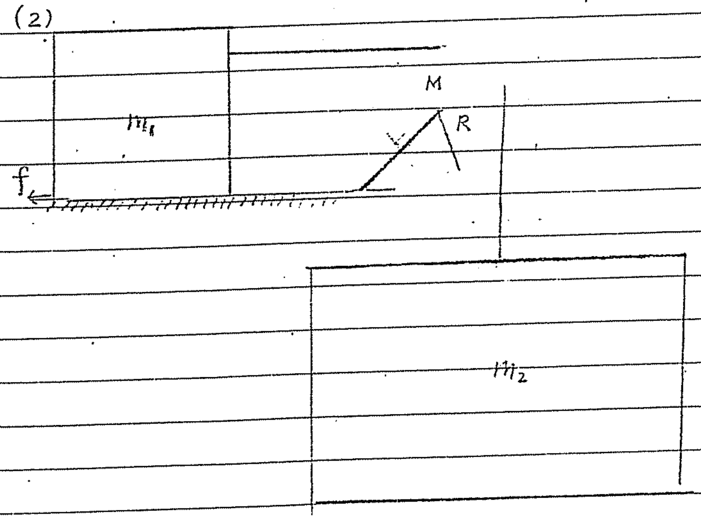
(f) 由原來的慣性系統來觀察，在碰撞前該系統之動能為何？ [X]

(g) 在以上之座標系統中，碰撞後該系統之動能為何？ [Q]

(h) 此反應之 Q 值為何？ [T]

(i) 若是此反應為一彈性反應，則 d 與 L 有何關係？ [W]

(2)



繩子在滑輪上沒有滑動。m_1 與表面之摩擦力為 12 牛頓，輪之半徑為 0.2 米，其迴轉半徑為 0.15 米，

$M = 20\ kg$, $m_1 = 5\ kg$, $m_2 = 40\ kg$

(a) 求 m_2 之加速度 [D]

(b) 輪子之角加速度 [V]

(c) 與 m_1 相連之繩上之張力為何？ [K]

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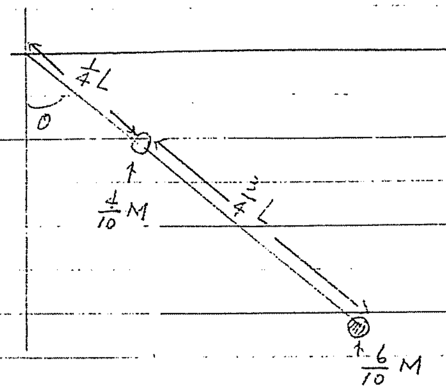
(d) 與 m_2 相連之繩上之張力為何？ [U]

(3)



一擺是由與一無質量之剛桿相連之兩質點所組成。此桿在一端固定，它們受重力在一垂直之平面上運動。0.4 M 之質點位於離固定之端點 $\frac{2}{5}L$ 處，0.6 M 之質點則位於桿之另一端

(a) 該系統之質心位於何處？ [A]

(b) 對固定點求該系統所受之力距和，並求描述桿之角運動方程式。 [E]

(c) 求此系統之慣性矩 [M]

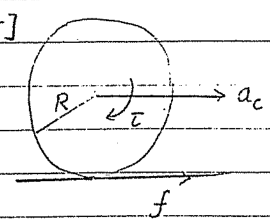
(d) 求此系統在小幅振盪時之振盪週期 [P]

(e) 若是固定的支點現在置於該兩質點之中點時，該系統在小幅振盪時其週期為何？[R]

(f) 若是固定的支點漸趨向質心時，其週期如何變化？ [G]

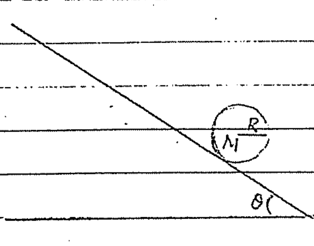
(4) 若一汽車其引擎對其輪軸施一 2τ 之力矩，車輪之半徑為 R，慣性矩為 $I=mR^2$

(a) 求在平坦路面上行駛時，求作用於每個車輪上摩擦力之大小 [I]



(b) 當 $2\tau = 10^3 J$， $M=2\times10^3 kg$，$R=0.5m$，及 $m=12.5kg$ 時，汽車之加速度為何？ [N]

(5) 一質量為 M，半徑為 R 之圓筒投向一角度為 θ 之斜面。圓筒是滾動的，而無滑動。其初速為 v_0



(a) 求圓筒質心之加速度為何？ [B]

(b) 求它走多遠後會暫時靜止？ [J]

答案

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[A] 離固定點 $\frac{7}{10}L$ 處

[B] $-\frac{2}{3}g\sin\theta$

[C] 入射質量與棍子中心連線之中點

[D] $(m_2 g - f)(m_1+m_2+\frac{MK^2}{R^2})^{-1}$

[E] $\vec{\tau} = -Mg\cdot\frac{7}{10}L\sin\theta\ \hat{k}$

$\frac{d^2\theta}{dt^2} + \frac{Mg\frac{7}{10}L\sin\theta}{I} = 0$

[F] $6V_0 d L^{-2}$

[G] 變大，在質心處週期趨向無窮大

[H] $\frac{1}{2}\vec{V_0}$，方向是沿入射質點之方向。

[I] $f = \frac{\tau/R}{1+(4m/M)}$

[J] $\frac{3V_0^2}{4g\sin\theta}$

[K] ~~45.8~~ Newton

[L] 0

[M] $\frac{5}{8}ML^2$

[N] $1\ m/sec^2$

[O] $\frac{1}{2}\vec{V_0}$

[P] $2\pi(\frac{50L}{56g})^{\frac{1}{2}}$

[Q] $\frac{1}{2}MV_0^2(\frac{1}{2}+\frac{3d^2}{L^2})$

[R] $2\pi(\frac{30L}{16g})^{1/2}$

[S] $MV_0 d$ 向外

[T] $\frac{1}{2}MV_0^2(\frac{3d^2}{L^2}-\frac{1}{2})$

[U] 121.8 Newtons

[V] $33.8\ sec^{-1}$

[W] $(6)^{-\frac{1}{2}}L$

[X] $\frac{1}{2}MV_0^2$

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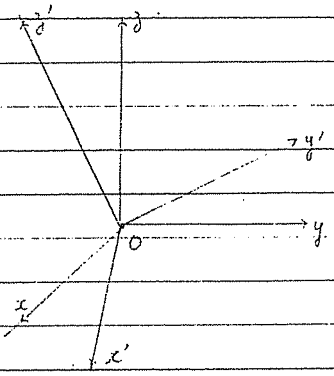
第四節 奧尤勒公式及其應用

簡介：當一剛体在轉動時，其對在一慣性座標軸的慣性矩也隨時間而改變，為了簡化對剛体之描述，我們將換用一與剛体同時轉動之座標系統。在此座標系統中，慣性矩與時間無關。由於剛体之三個主軸互相垂直，我們可以選擇它們為座標系統，這樣使得對剛体的運動的描述更簡單。因此法得到的轉動動力公式叫做奧尤勒公式。我們在這一節中我們將導演奧尤勒公式並討論其應用。

基本觀念



令 $\hat i, \hat j, \hat k$ 為一固定之慣性座標系統。而 $\hat i', \hat j', \hat k'$ 為一固定在物体上之座標系統。若剛体以 $\vec\omega$（對固定之座標系統）之角速度轉動則 $\hat i', \hat j', \hat k'$ 也以 $\vec\omega$（對固定之座標系統）之角速度轉動。[1,2] 因此 $\hat i'(t), \hat j'(t), \hat k'(t)$ 是時間之函數。

$$\vec\omega = \omega_x(t)\,\hat i + \omega_y(t)\,\hat j + \omega_z(t)\,\hat k$$

$$= \omega_x'(t)\,\hat i'(t) + \omega_y'(t)\,\hat j'(t) + \omega_z'(t)\,\hat k'(t) \qquad (1)$$

$$\vec L = L_x(t)\,\hat i + L_y(t)\,\hat j + L_z(t)\,\hat k$$

$$= L_x'(t)\,\hat i'(t) + L_y'(t)\,\hat j'(t) + L_z'(t)\,\hat k'(t) \qquad (2)$$

L_x', L_y', L_z' 是在 $\hat i', \hat j', \hat k'$ 中所觀察之角動量。

$$\vec\tau = \tau_x(t)\,\hat i + \tau_y(t)\,\hat j + \tau_z(t)\,\hat k$$

$$= \tau_x'(t)\,\hat i'(t) + \tau_y'(t)\,\hat j'(t) + \tau_z'(t)\,\hat k' \qquad (3)$$

$$\frac{d\vec L}{dt} = \frac{dL_x'}{dt}\hat i' + \frac{dL_y'}{dt}\hat j' + \frac{dL_z'}{dt}\hat k' + L_x'(t)\frac{d\hat i'(t)}{dt} + L_y'(t)\frac{d\hat j'(t)}{dt} + L_z'(t)\frac{d\hat k'(t)}{dt}$$

$$= \left(\frac{d\vec L}{dt}\right)' + L_x'(t)\,(\vec\omega\times\hat i') + L_y'(t)\,(\vec\omega\times\hat j') + L_z'(t)\,(\vec\omega\times\hat k')$$ [3]

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$$= \left(\frac{d\vec{L}}{dt}\right)' + \vec{\omega} \times \vec{L} \quad (4)$$

$\left(\frac{d\vec{L}}{dt}\right)'$ 是在 $\hat{i}', \hat{j}', \hat{k}'$ 中角速度之時變率

$\left(\frac{d\vec{L}}{dt}\right)$ 是在 $\hat{i}, \hat{j}, \hat{k}$ 中角速度之時變率

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)' + \vec{\omega} \times \vec{L} \quad (5)$$

$$\tau_x'(t)\,\hat{i}'(t) + \tau_y'(t)\,\hat{j}'(t) + \tau_z'(t)\,\hat{k}'(t) = \frac{dL_x'}{dt}\hat{i}' + \frac{dL_y'}{dt}\hat{j}' + \frac{dL_z'}{dt}\hat{k}'$$

$$+ L_x'(\omega_z'\hat{j}' - \omega_y'\hat{k}') + L_y'(\omega_x'\hat{k}' - \omega_z'\hat{i}') + L_z'(\omega_y'\hat{i}' - \omega_x'\hat{j}') \quad (6)$$

因此，

$$\tau_x' = \frac{dL_x'}{dt} - L_y'\omega_z' + L_z'\omega_y'$$

$$\tau_y' = \frac{dL_y'}{dt} + L_x'\omega_z' - L_z'\omega_x'$$

$$\tau_z' = \frac{dL_z'}{dt} - L_x'\omega_y' + L_y'\omega_x' \quad (7)$$

若我們選擇 $\hat{i}', \hat{j}', \hat{k}'$ 為其主軸。令 I_1, I_2, I_3 為其對應主軸慣矩則

$$L_x' = I_1\omega_x', \quad L_y' = I_2\omega_y', \quad L_z' = I_3\omega_z' \quad (8)$$

代入(7)式得

$$\tau_x' = I_1\frac{d\omega_x'}{dt} + (I_3 - I_2)\,\omega_y'\omega_z'$$

$$\tau_y' = I_2\frac{d\omega_y'}{dt} + (I_1 - I_3)\,\omega_x'\omega_z'$$

$$\tau_z' = I_3\frac{d\omega_z'}{dt} + (I_2 - I_1)\,\omega_x'\omega_y' \quad (9)$$

這即是著名的奧尤勒公式。

討論，

(1) 我們將兩個系統之原點重疊

(2) 此節中所用之方法借重於轉動座標系統處甚多，閱讀此節式應參看有關轉動座標之筆記。

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(3) $\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}'$ ， $\frac{d\hat{j}'}{dt} = \vec{\omega}\times\hat{j}'$ ， $\frac{d\hat{k}'}{dt} = \vec{\omega}\times\hat{k}'$ 曾於轉動座標系統一節中導過。

(亦 $\vec{v} = \vec{\omega}\times\vec{r}$ 相似)

(4) 此公式与 $\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)' + \vec{\omega}\times\vec{r}$ 相似

應用

(1) 迴轉器

$\vec{\tau} = \frac{d\vec{L}}{dt}$ 若 $\vec{\tau}=0$ 則 $\vec{L}$ 不變，若它又是對剛体之主軸轉動，則其角動量 $\vec{\omega}_0$ 也不變，也即是其轉軸的方向也保持不變。以上所講的均是對一固定空標而言。

現在我們在一對固定座標系統以 $\vec{\omega}$ 之角速度轉動之座標系統描述以上之情形時，由於

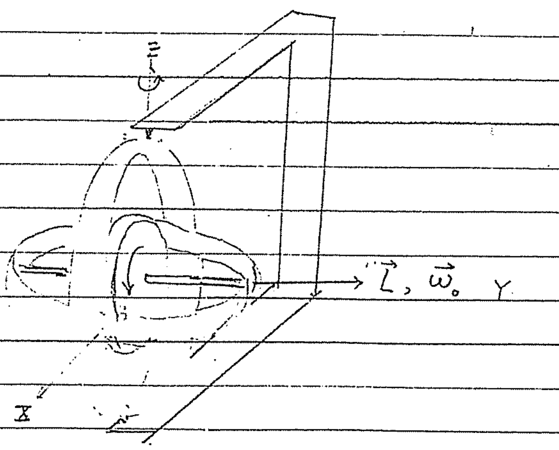
$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)' + \vec{\omega}\times\vec{L} \quad (10)$$

當 $\frac{d\vec{L}}{dt}=0$ 時 $\left(\frac{d\vec{L}}{dt}\right)' = -\vec{\omega}\times\vec{L}'$ (11)

此處 $\left(\frac{d\vec{L}}{dt}\right)'$ 是在轉動座標系統中所觀察的角動量的變化。若是它是對主軸轉動則其角動量与角速度之方向一致。它們在轉動坐標系統中看起來是以 $-\vec{\omega}$ 之角速度在轉動

迴轉器的基本要件

(a) 該系統所受之外力矩恆為 0 ，(b) 該旋轉器對其主軸旋轉



用以上之裝置若其球軸承之摩擦很少時，飛輪可以被認為不受任何外力矩之影响。以垂直於均勻圓盤且通過其中心之軸為轉軸則它顯然地是對主軸轉動。

若將此旋轉器置於北極則在以地球上之座標系統中 $\vec{\omega}_0$ 之方向在 XY 面上作等速轉動，其週期是一天。也即是若在 $t=0$ 時 $\vec{\omega}_0$ 是指向 +Y 軸時，六小時以後 $\vec{\omega}_0$ 則會沿 X 之方向。

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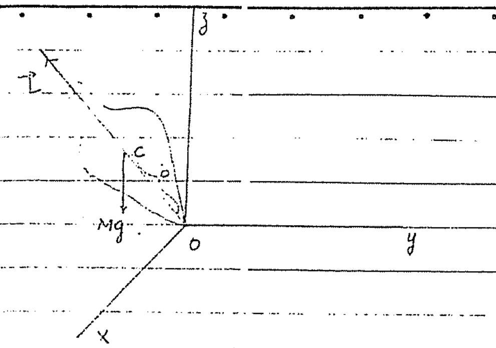
(2) 對稱陀螺在重力下之運動

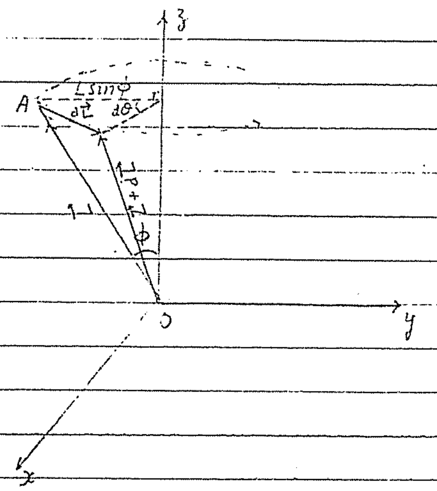


$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{OC} \times Mg(-\hat{k}) \qquad (12)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad (\vec{\tau}\text{，與 }\vec{L}\text{ 均對 O 點來計算})$$

$$d\vec{L} = \vec{\tau}\, dt \qquad (13)$$



我們現在來研討 $d\vec{L}$ 之性質

$\vec{OC} \parallel \vec{L}$ 因此 $\vec{\tau} \perp \vec{L}$，所以

$$\vec{L}\cdot d\vec{L} = \vec{L}\cdot\vec{\tau}\, dt = 0 \Rightarrow d|\vec{L}|^2 = 0 \qquad (14)$$

也即是 $|\vec{L}|$ 之大小保持不變

同理 $\vec{\tau}$ 也與 z 軸相垂直，所以

$$dL_z = \hat{k}\cdot d\vec{L} = 0 \qquad (15)$$

也即是 L_z 之大小保持不變

所以 $\vec{L}$ 之大小及其與 z 軸之夾角保持不變，因此 $\vec{L}$ 向量之頂點在以 O 點與 z 軸成錐形的圓周上運動。 $\vec{OA}$ 在 y-z 平面上，y 為負時則 $\vec{\tau}$ 之方向為向 +x 方向。所以其運動方向為反時鐘方向。如圖所示，也即是其角速度之方向沿 z 軸。

$|d\vec{L}|$ 之大小為 $Mgb\sin\phi\, dt$，此處 b 是 O 點與質心之距離。

由圖二中很易看出 $L\sin\phi\, d\theta = |d\vec{L}| = Mgb\sin\phi\, dt$ (16)

$$L = I\omega \qquad (17)$$

ω ── 陀螺之角速度

I ── 陀螺對其對稱軸之慣性矩

由 (16)，(17) 可得

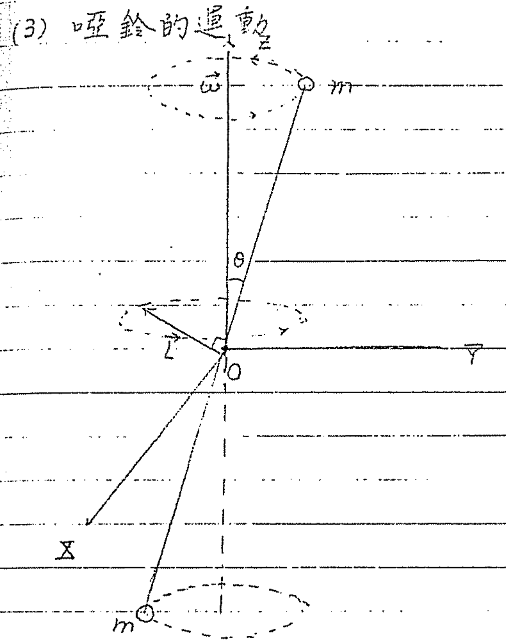
$$\frac{d\theta}{dt} = \Omega = \frac{Mgb}{I\omega} \qquad (18)$$

因此在重力之作用下，其轉軸對一定軸轉動，此一現象稱為旋進，其旋進角速度即為上式所示。

以上的結果只有在 $\omega >> \Omega$ 時才成立。理由如下，當物体在旋進時其角速度為 $\vec{\Omega}+\vec{\omega}$

因此其角動量即不是我們在上式所假設之 $I\vec{\omega}$。

(3) 啞鈴的運動



在固定座標系統中

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{19}$$

$\vec{\tau}$ 是支點對棍子所施之力矩

$$\vec{L} = m(\vec{r}_1\times\vec{V}_1 + \vec{r}_2\times\vec{V}_2)$$

$$= 2m\vec{r}_1\times\vec{V}_1 \quad (\vec{r}_1=-\vec{r}_2,\ \vec{V}_1=-\vec{V}_2) \tag{20}$$

由於 $\vec{V}_1 = \vec{\omega}\times\vec{r}_1$

$$\vec{L} = 2m\vec{r}_1\times(\vec{\omega}\times\vec{r}_1) = 2m[\vec{\omega}r_1^2 - \vec{r}_1(\vec{\omega}\cdot\vec{r}_1)] \tag{21}$$
$$= \tfrac{1}{2}ml^2\omega(\hat{\omega} - \cos\theta\,\hat{r}_1)$$

因此 $\vec{L}$ 与 $\vec{r}_1$ 垂直，而位於由 $\vec{\omega}$ 及 $\vec{r}_1$ 所形成之平面上。所以 $\vec{L}$ 以角速度 $\vec{\omega}$ 轉動

$$\frac{d\vec{L}}{dt} = \vec{\omega}\times\vec{L} \tag{22}$$
$\uparrow$
$\vec{\tau}$

由 (20), (21) 及 (22) 式得

$$\vec{\tau} = 2m(\vec{r}_1\times\vec{\omega})(\vec{r}_1\cdot\vec{\omega}) = \tfrac{1}{2}ml^2\omega\sin\theta\cos\theta\,\hat{n} \tag{23}$$

是要維持此啞鈴以角速度轉動 $\vec{\omega}$ 在 O 点所需施之力距。此處 $\hat{n} = \frac{\vec{r}_1\times\vec{\omega}}{|\vec{r}_1\times\vec{\omega}|}$ (24)

當 $\theta = \frac{\pi}{2}$ 時，$\vec{L} = \frac{1}{2}ml^2\vec{\omega}$，$\vec{\tau}=0$。此時不需加外力矩來保持以 $\vec{\omega}$ 角速度轉動。

由上式中很顯然的可看出 $\hat{r}_1$, $\hat{L}$ 及 $\hat{n}$ 是互相垂直的三個單位向量。

現在我們用奧尤勒公式來處理此公式。顯然的 $\hat{r}_1$ 是該啞鈴之主軸，令其為第三主軸，它對應的慣矩 $I_3=0$。由於對稱性，任何在垂直於 $\hat{r}_1$ 平面上之方向均為主軸。取第一軸為沿 $\hat{L}$ 方向，第二主軸為沿 $\hat{n}$ 方向，其所對應之慣性矩 $I_2 = I_3 = \frac{1}{2}ml^2$

$\vec{\omega}$ 很容易的可看出可寫成 $\vec{\omega} = \omega \sin\theta\, \hat{i} + \omega \cos\theta\, \hat{n}$ (25)

也即是 $\omega_x' = \omega \sin\theta$, $\omega_y' = 0$, $\omega_z' = \omega \cos\theta$ (26)

將 (25), (26) 之結果代入第 (9) 式奧尤勒公式得

$$\tau_x' = 0\ ,\quad \tau_y' = \tfrac{1}{2} m l^2 \omega^2 \sin\theta \cos\theta\ ,\quad \tau_z' = 0 \tag{27}$$

結果与 (23) 式相符。

(4) 地球之轉動

地球是橢圓形的，顯然地連接南北極的線是地球轉動主軸之一，取其為第三主軸，其對應之主慣性矩為 I_3。由對稱性可知，另外兩個主軸可選擇赤道面之兩垂直軸，對應於它們之兩主慣性矩 $I_1 = I_2 = I$。

由於太陽及地球所施之重力矩甚小，因此地球之轉動可以約略地以沒有外力矩之奧尤勒公式來描述。此時之奧尤勒公式變成

$$\frac{d\omega_x'}{dt} + \frac{I_3 - I_1}{I}\omega_z'\omega_y' = 0 \tag{28.a}$$

$$\frac{d\omega_y'}{dt} - \frac{I_3 - I_1}{I}\omega_x'\omega_z' = 0 \tag{28.b}$$

$$\frac{d\omega_z'}{dt} = 0 \tag{28.c}$$

由 (28c) 式得 $\omega_z'(t) = \omega_z'(0) = \omega_z'$ (29)

令 $\Omega = \frac{I_3 - I}{I}\omega_z'$ ，代入 (28a), (28b) 得

$$\frac{d\omega_x'}{dt} + \Omega\,\omega_y' = 0 \tag{30a}$$

$$\frac{d\omega_y'}{dt} - \Omega\,\omega_x' = 0 \tag{30b}$$

將 (30a) 對 t 微分得

$$\frac{d^2\omega_x'}{dt^2} = -\Omega\frac{d\omega_y'}{dt} \tag{31}$$

將 (30b) 代入 (31) 式得 $\frac{d^2\omega_x'}{dt^2} = -\Omega^2\omega_x'$ (32)

其解為 $\omega_x'(t) = A\cos(\Omega t+\alpha)$ (33)

將 (33) 代入 (30a) 得 $\omega_y'(t) = A\sin(\Omega t+\alpha)$ (34)

(33),(34) 之常數 A,α 是由 $\omega_1(0)$ 及 $\omega_2(0)$ 來決定。 $\omega=\sqrt{A^2+\omega_3^2}$ = 常數

若 $\omega_x(0)$ 及 $\omega_y(0)$ 不為 0，則 $\vec{\omega}$（在地球轉動座標中）中繞着第三主軸以角速度 Ω 旋進。

$\vec{\omega}$ 旋進之週期是

$$T = \frac{2\pi}{\Omega} = \frac{I}{I_3-I}\ \frac{2\pi}{\omega_3'}$$

由實驗中得 $\frac{I}{I_3-I}\approx 300,\ \frac{2\pi}{\omega_3'}\simeq 1$天

所以 T 大約為 300天。

實驗結果是 $\vec{\omega}'$ 的旋進不太規則，其週期約為 427天。其不規則的原因是由很多我們忽略了地震等對地球轉動的影响。其週期與預計不符主要是由於地球並非剛体。

(5) 剛体轉動的穩定性。

當一剛体不受外力時，奧尤勒公式可寫成

$$I_1\frac{d\omega_x'}{dt} + (I_3-I_2)\,\omega_y'\omega_z' = 0$$

$$I_2\frac{d\omega_y'}{dt} + (I_1-I_3)\,\omega_x'\omega_z' = 0$$

$$I_3\frac{d\omega_z'}{dt} + (I_2-I_1)\,\omega_x'\omega_y' = 0 \qquad (35)$$

我們將假設 $I_3>I_2>I_1$。

若在 $t=0$ 時 $\vec{\omega}' = \omega_x'\hat{i}' + \lambda\hat{j}' + \mu\hat{k}'$ 而 $\lambda,\mu \ll \omega_x'$ 我們想了解 ω_y'，ω_z' 會不會變得很大？若是不會，則該剛体仍將大致沿 $\hat{i}'$ 軸轉動，我們稱此轉動為穩定。否則，轉動稱為不穩定。

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$$I_1 \frac{d\omega_x'}{dt} - (I_2 - I_3)\lambda\mu = 0 \quad \rightarrow \quad I_1 \frac{d\omega_x'}{dt} \simeq 0 \quad \omega_x' = constant \tag{36a}$$
$$I_2 \frac{d\lambda}{dt} - (I_3 - I_1)\mu\omega_x' = 0 \quad \rightarrow \quad I_2 \frac{d\lambda}{dt} - (I_3 - I_1)\mu\omega_x' = 0 \tag{36b}$$
$$I_3 \frac{d\mu}{dt} - (I_1 - I_2)\lambda\omega_x' = 0 \quad \quad I_3 \frac{d\mu}{dt} - (I_1 - I_2)\lambda\omega_x' = 0 \tag{36c}$$

$\lambda\mu \simeq 0$ (我們用微擾法) $\Rightarrow$ $\omega_x' = constant$ ． 代入上式後利用前討論第(4)小節之方法得
第(36b)對t微分，以(36c)代入

$$\frac{d^2\lambda}{dt^2} + \left(\frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3} \omega_x'^2 \right) \lambda = 0 \tag{37}$$

$$\lambda(t) = A \sin(\Omega_{1\lambda} + \alpha) \qquad (I_1 - I_2)(I_1 - I_3) > 0 \tag{38}$$

$$\text{而} \quad \Omega_{1\lambda} = \omega_x' \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{39}$$

因此 $\lambda(t)$ 是一個振盪函數，因此不會長大

將 (38) 代入 (36b) 中得

$$I_2 A \Omega_{1\lambda} \cos(\Omega_{12} + \alpha) = (I_3 - I_1)\omega_x' \mu \tag{40}$$

因此 μ 也是振盪函數，因此也不會長大。因此在此情況下若剛体在 $t=0$ 時非常接近沿 $\hat{i}'$ 軸轉動則以後仍大致沿 $\hat{i}'$ 軸轉動。$\Rightarrow$ 沿 $\hat{i}'$ 軸之轉動是穩定的

若在 $t=0$ 時 $\vec{\omega}' = \lambda\hat{i} + \omega_y'\hat{j} + \mu\hat{k}$ ， $\lambda\mu \ll \omega_y'$ ，則用以上相同之方法可得

$$\frac{d^2\lambda}{dt^2} + \left(\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3} \omega_y' \right) \lambda = 0 \tag{41}$$

但此時由於 $I_1 > I_2 > I_3$ ，第二項前面之係數是負的，其解為

$$\lambda = A e^{\Omega t} + B e^{-\Omega t} \tag{42}$$

$$\Omega = \left(\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3} \right)^{1/2}$$

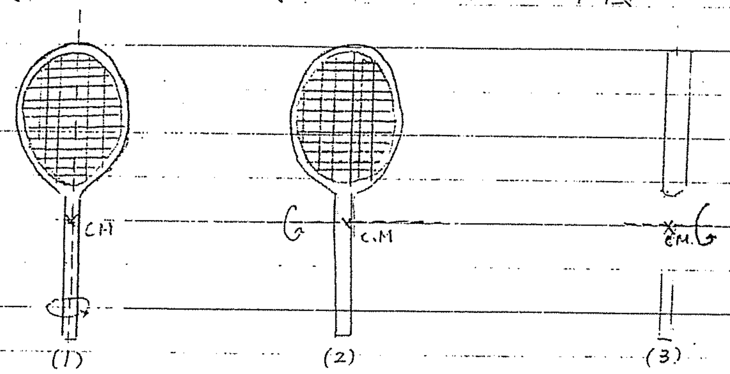
由於通常 $A \neq 0$ ， λ 會變大（但當 λ 變得不比 ω_y' 小很多時，微擾法就不能使用了，而 (42) 式即不再是奧尤勒公式之解，因此不會變成無窮大），也即是說若剛開始時是沿 $\hat{j}'$ 轉動，以後並不一定沿 $\hat{j}'$ 軸轉動 $\Rightarrow$ 沿 $\hat{j}'$ 軸之轉動是不穩定的

同法，我們可證明對 $\hat{k}'$ 軸之轉動是穩定的。

分類：
編號： 9
總號：

例. 網球拍

網球拍之三個主軸可以很容易看出如下圖



$I_1 < I_2 < I_3$

將球拍拋起時若開始時是對(1)或(3)軸轉動則球拍大略仍對原來之軸轉動,因此很容易接住。但若是開始是對(2)軸轉動,由於會產生對其它軸的轉動,運動變得很複雜,因此不容易接住。

Review:

Definitions

System

Internal , External

Coordinates

The coordinates of a physical system are the numbers (possibly dimensional) that describe the system at a given fixed time

Configuration

The configuration $\{q(t)\}$ of a mechanical system is a number (or vectors) consisting of values of a set of <u>independent</u> coordinate that <u>completely</u> describe the system.

Independenty ⇒ the coordinates in the configuration are independent in the sense that each can be changed independently, and each different set of va i describes something physically different

Given a set of values of the coordinates at some particular time ⇒ can figure out what the system looks like at that time.

↓

a configuration is just a mathematical snapshot of the system at a given time problem of classical mechanics

↓

want to understand how the configuration of the system evolves with time

↓

want to put the snapshots together into a mathematical movie to describe how the system moves

Degree of freedom

The number of independent components of the configuration $\{q\}$ is called the number of degree of freedom of the mechanical system

Examples

system	coordinate	# of DOFs
point mass on a track	ℓ (distance along the track)	1
point mass on a flat surface	(x, y)	2
point mass in 3-d	$\vec{r} = (x, y, z)$	3
rigid body in 3-d	$\vec{r}$ of center + 3 angles	6
2 masses + massless spring	$\vec{r}_1$ and $\vec{r}_2$	6
2 masses + massive spring	$\vec{r}_1, \vec{r}_2$ and spring	"∞"

For a system of N point mass in 3 dimension the degree of freedom is $3N$.

A continuous massive spring formally has an infinite number of degrees of freedom, because to specify its configuration we would have to give a continuous function describing how much every point on the spring is stretched. Really, of course, a physical spring has a finite but very large number of degrees of freedom, because it is not actually continuous, but is made up of atoms. But the difference between ∞ and Avogadro's number is often not very important.

This brings up an important philosophical point. What the heck is a "point" mass? What is a "rigid" body? What is a "massless" spring? Most of you have probably been dealing with physics problems for so long that you are used to these phrases. But it is important to remember that these are mathematical idealizations. Real physical systems are complicated, and in fact, what we choose for q may depend on what kind of physical questions we want to ask and what level of accuracy we need in the answer. So for example, for a hockey puck sliding on the ice at the Boston Garden, we might decide that the configuration is specified by giving the x and y coordinates that determine the puck's position in the plane of the ice. Then q would stand for the two dimensional vector, (x, y). But if we do this, we have ignored many details. For example, for a shot that comes off the ice, we would need to include the z coordinate to describe the motion of the puck. For some purposes, we would also need to include descriptions of the puck itself. For example, we have not included an angular variable that would allow us to specify how the puck is turned about its vertical axis. This is probably good enough for most problems. But sometimes, more information is required to give a good description of the physics. For example, if we wanted to understand how a rapidly rotating puck moves, we might need this more detailed information. We could also go on and describe how the puck might deform when hit by the stick, and so on. We could include more and more information until we got down to the level where we begin to see the molecular structure of the rubber of the puck. At this point, we begin to see quantum mechanical effects, and classical mechanics is no longer enough to give an accurate description.

The Art of Theoretical Physics

This is a good lesson. The coordinates that we use to describe the system may depend on what kind of information we want to get out of our mechanical model of the system, and how accurately we want our model to reproduce reality. We usually will not go over these niceties each time we discuss a system, but they are important to remember. There is really a very important point here. In physics courses, we frequently discuss "toy" systems which are obviously oversimplified, in which we have clearly left out features that are important in the "real world." This is not something to apologize for. This is precisely the art of theoretical physics. We work hard to abstract the essential physics of a system, without including things that don't matter at the level of description that we are interested in. This down-to-earth ability to focus on the crucial parameters is far more important than fancy mathematical gymnastics.

In fact, I believe, getting better at this art is one of the most useful things you can get out of this course. It is generally useful far beyond this or future physics courses. The ability to build mathematical models of phenomena is crucial to many fields. But models can be as misleading as they can be useful unless they focus on the right parameters, and unless the modeler is aware of the model's limitations. Physics is the paradigm for this kind of thinking. This is one of the reasons why, over the years, trained physicists have been so much in demand in very different fields.

System of Particles

Discrete

$\vec{r}_i(t)$

Degree of freedom $= 3N$

Continuous System

Density as a function of points at a given time

$\rho = \frac{\Delta M}{\Delta V}$
‖
volume density

$\sigma = \frac{\Delta M}{\Delta A}$
‖
surface density

$\lambda = \frac{\Delta M}{\Delta L}$
↓
linear density

For many purpose a discrete system having a very large but finit number of particles can be considered as a continuous system.
Conversely a continuous system can be considered as a discrete system consisting of a large but finite number of particles.

Center of Mass

Discrete

$$\vec{r}_{c.m.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

$$= \frac{1}{M}\sum m_i \vec{r}_i$$

Continuous

$$\vec{r}_{c.m} = \frac{\int \rho \vec{r}\, dV}{\int \rho\, dV}$$

In practicle it is fairly simple to go from discrete to continuous systems by merely replacing summation by integration.

⇓

All theorems will be presented for discrete system

Momentum of a System of Particle

$$\vec{P} \equiv \sum_i m_i \dot{\vec{r}}_i = \sum_i m_i \vec{v}_i = M\dot{\vec{r}}_{c.m} = M\vec{V}_{c.m}$$

↓

total momentum of the system

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{\,ext}$$

(Newton's third law has been used)

Conservation of Momentum

If the resultant external force acting on a system of particles is zero then the total momentum remains constant ⇒ total momentum is conserved.

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{\,ext}$$

↓

Equation of Motion (Translational)

Angular Momentum of a System of Particles

This has been discussed in P. 9-1 to 9-8

Only the notation has been rewritten.

$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i = \sum \vec{L}_i$$

$$\vec{r}_i \equiv \vec{r}_{c.m.} + \vec{\rho}_i \quad \text{(definition)}$$

$$= \sum (\vec{r}_i - \vec{r}_{c.m}) \times m_i \vec{v}_i + \sum \vec{r}_{c.m} \times \underbrace{m_i \vec{v}_i}_{\vec{p}_i}$$

$$= \sum \vec{\rho}_i \times m_i \vec{v}_i + \vec{r}_{c.m} \times \underbrace{\sum \vec{p}_i}_{\vec{P}}$$

$$= \vec{L}_s + \vec{r}_{c.m} \times \vec{P}_{\text{total momentum}}$$

$\vec{L}_s$ ↓ spin angular momentum; $\vec{r}_{c.m} \times \vec{P}$ ↓ orbital angular momentum.

$$\sum m_i \vec{r}_i = \vec{r}_{c.m} \sum m_i + \sum m_i \vec{\rho}_i = M \vec{r}_{c.m} + \sum m_i \vec{\rho}_i$$

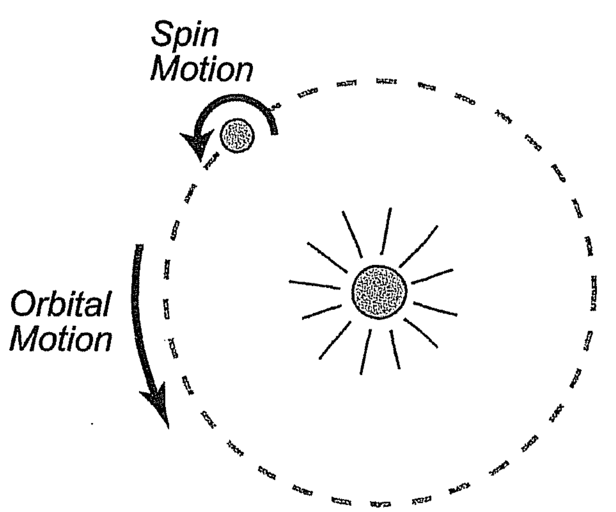
$$\Downarrow$$

$$\sum m_i \rho_i = 0 \Rightarrow \sum m_i \dot{\rho}_i = 0$$

$$\sum \rho_i \times m_i \vec{v}_i, \quad \vec{v}_i = (\vec{v}_{c.m.} + \dot{\vec{\rho}}_i)$$

$$= (\underbrace{\sum m_i \rho_i}_{0}) \times \vec{v}_{c.m} + \sum \rho_i \, m_i \times \dot{\vec{\rho}}_i$$

Earth - Sun.



Equation of Motion

$$\frac{dL}{dt} = \frac{d\vec{L}_s}{dt} + \frac{d}{dt}(\vec{r}_{c.m.} \times \vec{p})$$

$$\vec{\tau}_{tot} = \sum \vec{r}_i \times \vec{F}_i^{ext} = \sum_i (\vec{r}_{cm} + \vec{\rho}_i) \times \vec{F}_i^{ext}$$

↓

total torque

$$= \sum \vec{r}_{c.m} \times \vec{F}_i^{ext} + \sum \vec{\rho}^{\,i} \times \vec{F}_i^{ext}$$

$$\vec{r}_{c.m} \times \vec{F}_{tot}^{ext} + \sum \vec{\rho}^{\,i} \times \vec{F}_i^{ext}$$

$$\frac{dL}{dt} = \frac{d\vec{L}_s}{dt} + \frac{d}{dt}(\vec{r}_{c.m} \times \vec{p})$$

$$= \vec{r}_{cm} \times \vec{F}_{tot}^{ext} + \sum \vec{\rho}^{\,i} \times \vec{F}_i^{ext}$$

$$\frac{d\vec{L}_s}{dt} = \sum \vec{\rho}^{\,i} \times F_i = \qquad (i)$$

$$\frac{d}{dt}(\vec{r}_{c.m} \times \vec{p}) = \vec{r}_{c.m} \times \vec{F}_{tot} \ . \quad (ii)$$

Total Kietic Energy of a System of Particles

$$K.E. = \sum \frac{1}{2} m_i \vec{v}_i^{\,2}$$

$$= \sum \frac{1}{2} m_i (\vec{v}_{c.m} + \vec{\rho}_i)^2$$

$$= \sum \frac{1}{2} m_i V_{c.m}^2 + \sum 2 \times \frac{1}{2} m_i \vec{V}_{c.m} \cdot \rho_i + \sum \frac{1}{2} m_i \vec{\rho}_i^{\,2}$$

$$= \frac{1}{2} M \vec{V}_{c.m}^2 + \frac{1}{2} \sum m_i \vec{\rho}_i^{\,2}$$

Furthermore

The total external torque about the center of mass equals the time rate of change in angular momentum about the center of mass, i.e., $\frac{d\vec{L}_s}{dt} = \vec{\tau}^{\,ext}_{c.m.}$ holds not only for internal coordinate systems but also for system moving with the center of mass.

If motion is described relative to points other than the center of mass, the results in the above theorem become more complicated.

Impulse

If $\vec{F}$ is the total external force acting on a system of particles, then $\int_{t_1}^{t_2} F\, dt$ is the total linear impulse. As in the case of one particle, we can prove $\int_1^2 \vec{F}\, dt = \vec{P}_2 - \vec{P}_1$

↓ change in linear momentum

If $\vec{\tau}$ is the total external torque (about origin O) applied to a system of particles the

$$\int_1^2 \vec{\tau}\, dt = \vec{L}_2 - \vec{L}_1$$

$\int_1^2 \vec{\tau}\, dt$ ↓ total angular impulse

$\vec{L}_2 - \vec{L}_1$ ↓ change in angular momentum

9-7 COMBINED ROTATIONAL AND TRANSLATIONAL MOTION

Figure 9-28 shows a time-exposure photograph of a rolling wheel. This is one example of a possibly complex motion in which an object simultaneously undergoes both rotational and translational displacements.

In general, the translational and rotational motions are completely independent. For example, consider a puck sliding across a horizontal surface (perhaps a sheet of ice). You can start the puck in translational motion only (no rotation), or you can spin it in one place so that it has only rotational and no translational motion. Alternatively, you can simultaneously push the puck (with any linear velocity) and rotate it (with any angular velocity), so it moves across the ice with both translational and rotational motion. The center of mass moves in a straight line (even in the presence of an external force such as friction), but the motion of any other point of the puck may be a complicated combination of the rotational and translational motions, like the point on the rim of the wheel in Fig. 9-28.

As represented by the sliding puck or the rolling wheel, we restrict our discussion of this combined motion to cases satisfying two conditions: (1) the axis of rotation passes through the center of mass (which serves as the reference point for calculating torque and angular momentum), and (2) the axis always has the same direction in space (that is, the axis at one instant is parallel to the axis at any other instant). If these two conditions are valid, we may apply Eq. 9-11 ($\Sigma\, \tau_z = I\alpha_z$, using only *external* torques) to the rotational motion. Independent of the rotational motion, we may apply Eq. 7-16 ($\Sigma\, \vec{\mathbf{F}} = M\vec{\mathbf{a}}_{\text{cm}}$, using only *external* forces) to the translational motion.

There is one special case of this type of motion that we often observe; this case is illustrated by the rolling wheel of Fig. 9-28. Note that where the illuminated point on the rim

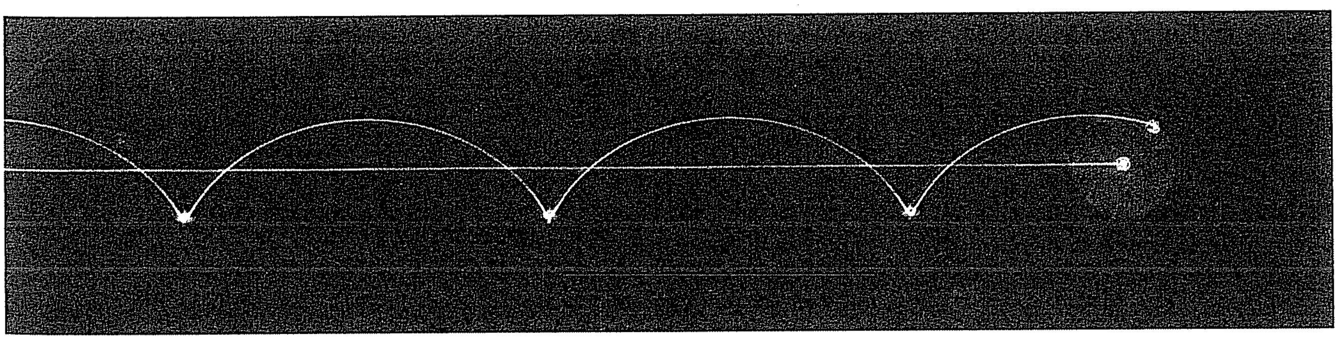


FIGURE 9-28. A time-exposure photo of a rolling wheel. Small lights have been attached to the wheel, one at its center and another at its edge. The latter traces out a curve called a *cycloid.*

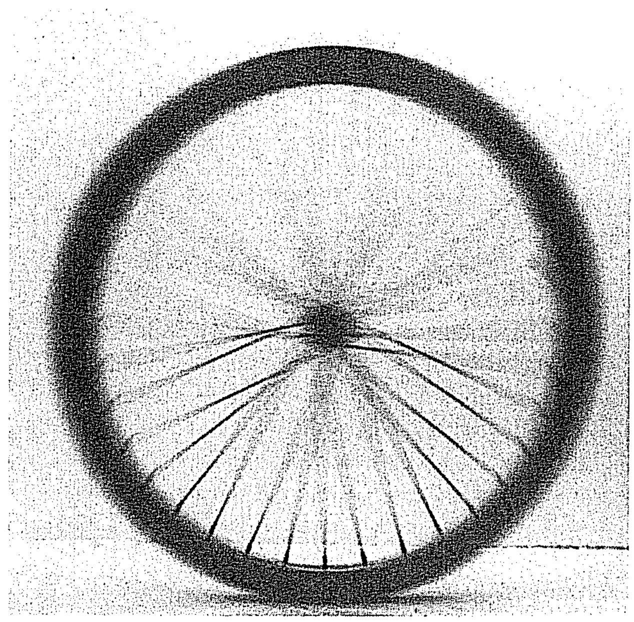


FIGURE 9-29. A photo of a rolling bicycle wheel. Note that the spokes near the top of the wheel are more blurred than those near the bottom. This is because the top has a greater linear velocity.

contacts the surface, the light seems especially bright, corresponding to a long exposure of the film. At these instants, that point is moving very slowly relative to the surface, or may perhaps be instantaneously at rest. This special case, in which an object rolls across a surface in such a way that there is no relative motion between the object and the surface at the instantaneous point of contact, is called *rolling without slipping.*

Figure 9-29 shows another example of rolling without slipping. Note that the spokes of the bicycle wheel near the bottom are in sharper focus than the spokes at the top, which appear blurred. The top of the wheel is clearly moving faster than the bottom! In rolling without slipping, the frictional force between the wheel and the surface is responsible for preventing the relative motion at the point of contact. Even though the wheel is moving, it is the force of *static* friction that applies.

Not all cases of rolling on a frictional surface result in rolling without slipping. For example, imagine a car trying to start on an icy street. At first, perhaps the wheels spin in place, so we have pure rotation with no translation. If sand is placed on the ice, the wheels still spin rapidly, but the car begins to inch forward. There is still some slipping between the tires and the ice, but we now have some translational motion. Eventually the tires stop slipping on the ice, so there is no relative motion between them; this is the condition of rolling without slipping.

Figure 9-30 shows one way to view rolling without slipping as a combination of rotational and translational motions. In pure translational motion (Fig. 9-30*a*), the center of mass C (along with every point on the wheel) moves with velocity v_{cm} to the right. In pure rotational motion (Fig. 9-30*b*) at angular speed ω, every point on the rim has tangential speed ωR. When the two motions are combined, the resulting velocity of point B (at the bottom of the wheel) is $v_{cm} - \omega R$. For rolling without slipping, the point where the wheel contacts the surface must be at rest; thus $v_{cm} - \omega R = 0$, or

$$v_{cm} = \omega R. \tag{9-36}$$

Superimposing the resulting translational and rotational motions, we obtain Fig. 9-30*c*. Note that the linear speed at the top of the wheel (point T) is exactly twice that at the center.

Equation 9-36 applies *only* in the case of rolling without slipping; in the general case of combined rotational and translational motion, v_{cm} does not equal ωR.

There is yet another instructive way to analyze rolling without slipping: we consider the point of contact B to be an instantaneous axis of rotation, as illustrated in Fig. 9-31. At each instant there is a new point of contact B and therefore a new axis of rotation, but instantaneously the motion consists of a pure rotation about B. The angular velocity of this rotation about B is the same as the angular velocity ω of the rotation about the center of mass. Since the distance from B to T is twice the distance from B to C, once again we conclude that the linear speed at T is twice that at C.

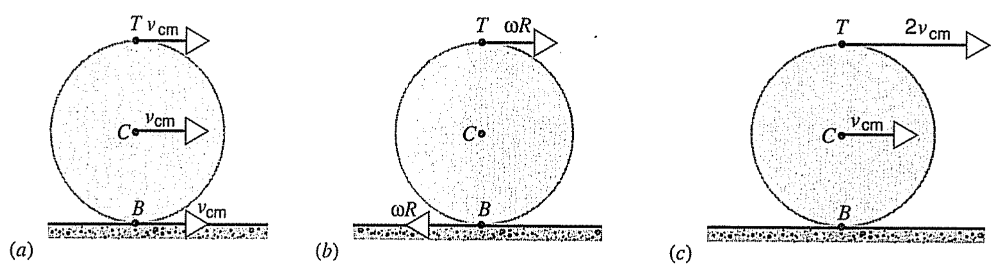


FIGURE 9-30. Rolling can be viewed as a superposition of pure translation and rotation about the center of mass. (*a*) The translational motion, in which all points move with the same linear velocity. (*b*) The rotational motion, in which all points move with the same angular velocity about the central axis. (*c*) The superposition of (*a*) and (*b*), in which the velocities at T, C, and B have been obtained by vector addition of the translational and rotational components.

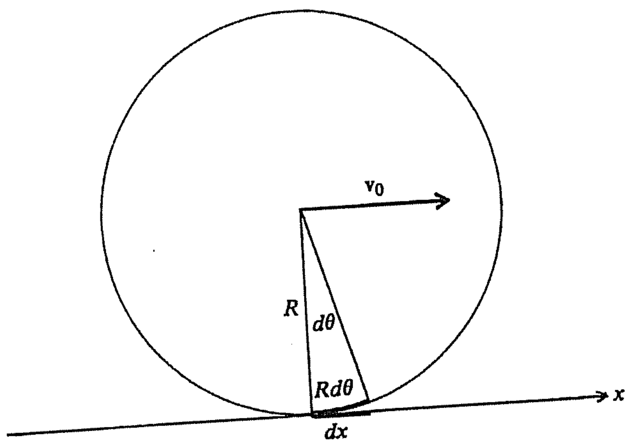


FIGURE 6-4. Wheel rolling without slipping on a level surface.

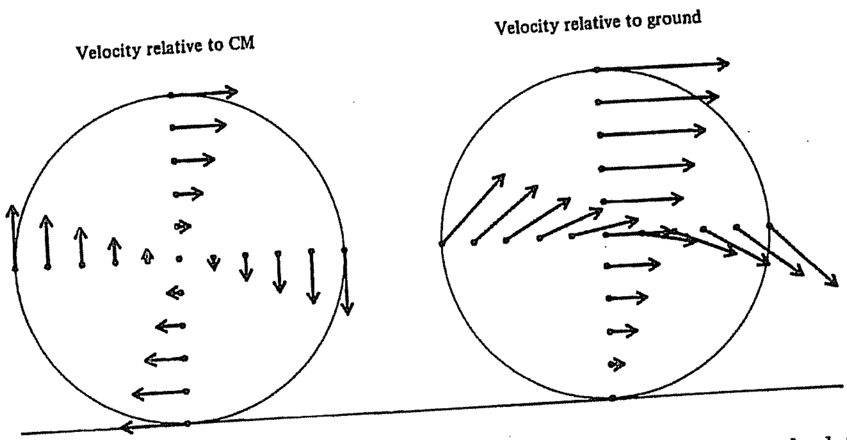


FIGURE 6-5. Velocity of points on a wheel relative to the center of mass and relative to the ground.

6.10 Impulses and Billiard Shots

For forces that act only during a very short time, it is convenient to use an integrated form of the laws of motion. The translational motion of the center-of-mass point is determined by

$$\dot{\mathbf{P}} = \mathbf{F} \tag{6.134}$$

If we multiply both sides of this equation by dt and integrate over the short time interval $\Delta t = t_1 - t_0$, during which the force acts, we obtain

$$\Delta \mathbf{P} = \mathbf{P}^1 - \mathbf{P}^0 = \int_{t_0}^{t_1} \mathbf{F} dt \tag{6.135}$$

The time integral of the force on the right is called the *impulse.* For angular motion the integrated form of the equation of motion in (6.48) is

$$\Delta \mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \int_{t_0}^{t_1} \mathbf{N} dt \tag{6.136}$$

The time integral of the torque is called the *angular impulse.* For rigid-body rotations about a fixed z axis, we can use (6.119) to rewrite the angular-impulse equation (6.136) as

$$\Delta L_z = I_{zz} \Delta \omega_z = I_{zz}(\omega_z^1 - \omega_z^0) = \int_{t_0}^{t_1} N_z dt \tag{6.137}$$

As an example of the usefulness of the impulse formulation of the equations of motion, we discuss the dynamics of billiard shots. For simplicity we consider only shots in which the cue hits the ball in its vertical median plane in a horizontal direction. In billiard jargon these are shots without "English".

The cue imparts an impulse to the stationary ball at a vertical distance h above the table, as illustrated in Fig. 6-16. The linear impulse from (6.135) is

$$M \Delta V_x = M V_x^1 = \int_{t_0}^{t_1} F_x dt \tag{6.138}$$

where V_x^1 is the velocity of the CM just after impact. The angular impulse of (6.137) about the z axis in Fig. 6-17 which passes through the CM of

10-14

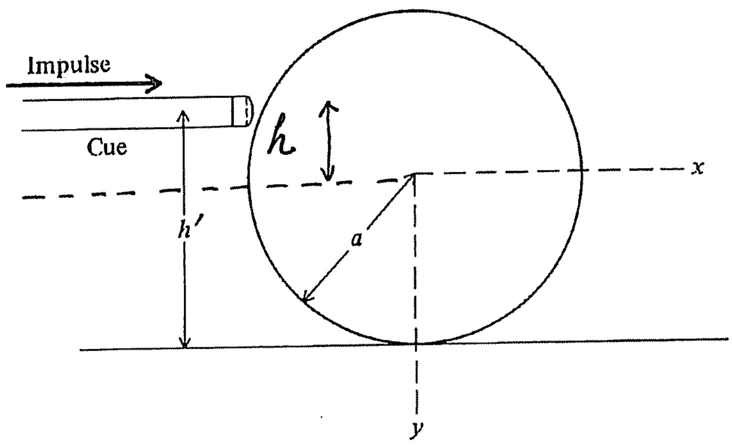


FIGURE 6-16. Impulse imparted to a billiard ball by the cue stick.

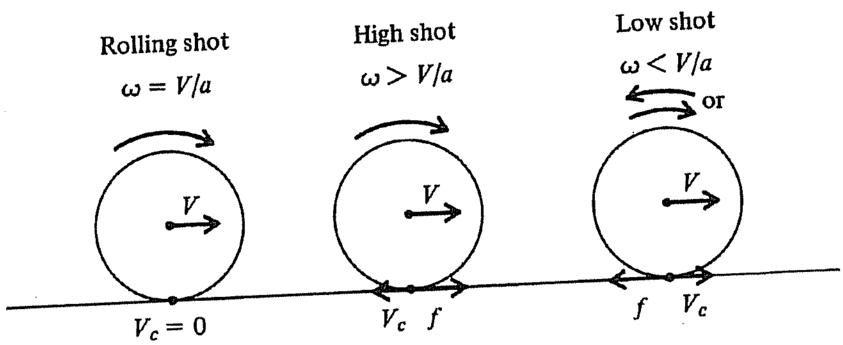
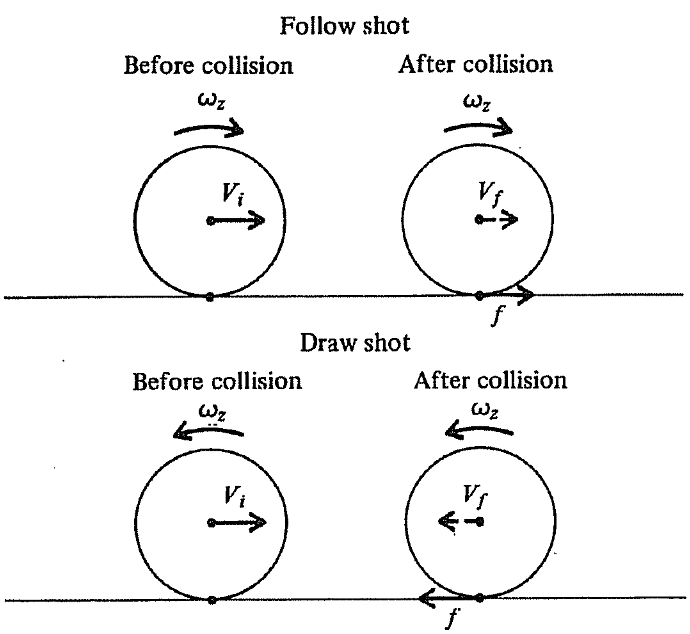


FIGURE 6-17. Rolling, high, and low shots in billiards.



the ball is

$$\Delta L_z = I_{zz}\omega_z^1 = \int_{t_0}^{t_1} (h' - a)F_x dt \qquad (6.139)$$

where a is the radius of the ball. By elimination of the force integral between (6.138) and (6.139) and substitution of the moment of inertia from (6.133), we arrive at the following relation between the spin and velocity of the ball immediately after the impulse:

$$\omega_z^1 = \frac{5}{2}\left(\frac{h' - a}{a^2}\right) V_x^1 \qquad (6.140)$$

The velocity of the ball at the point of contact with the table is

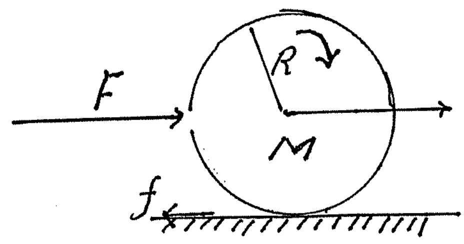
$$V_c = V_x^1 - a\omega_z^1 = V_x^1 \left(\frac{7a - 5h'}{2a}\right) \qquad (6.141)$$

If the ball is to roll without slipping ($V_c = 0$), we find that

$$h' = \tfrac{7}{5}a \quad \Rightarrow h = \frac{2}{5}a \qquad (6.142)$$

Only if the ball is hit exactly at this height does pure rolling take place from the very start. For a *high shot* with $h > \frac{7}{5}a$, V_c is opposite in direction to V_x^1. Since the friction at the billiard cloth opposes V_c, it

$a = R$



Impulse through the center of mass

$\Rightarrow$ initial velocity

$$V_{cm} = V_0 \ , \quad \omega = 0$$

Due to friction force

$$\tau = fR = \mu M g R = I\alpha$$

$$I = \frac{2}{5} M R^2$$

$$\Rightarrow \quad \alpha = \frac{5}{2}\mu \frac{g}{R}$$

$$\omega = \frac{5}{2}\mu \frac{g}{R} t$$

$$M a_c = -f$$

$$\Rightarrow \quad a_c = -\frac{\mu M g}{M} = -\mu g$$

$$V_{c.m.} = V_0 - \mu g t = \omega R = \frac{5}{2}\mu \frac{g}{R} t R$$

Condition for rolling

$$V_{c.m.} = \omega R$$

$$t = \frac{2}{7}\frac{V_0}{\mu g}$$

time elapsed before rolling

$$S = v_0 t - \frac{1}{2}\mu g t^2$$

$$= \frac{12}{49} \cdot \frac{v_0^2}{\mu g}$$

$F \Delta t = M V_0$ Impulse

$F h \Delta t = I \omega$ Angular Impulse

$$\frac{M V_0}{\Delta t} h \Delta t = I \omega$$

$$\Rightarrow \boxed{M V_0 h} = I \omega$$

h Impulse approximation

$$\omega = \frac{M V_0 h}{I} = \frac{5}{2} \frac{V_0 h}{R^2}$$

If $\omega R = V_0$, then $h = \frac{2}{5} R$, immediately start rolling

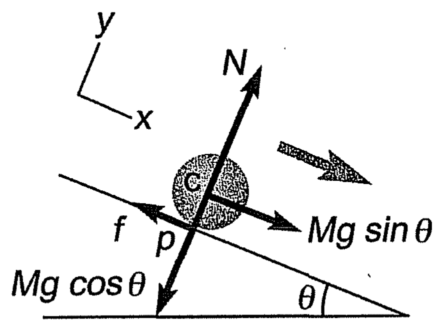
$h < \frac{2}{5} R$ $\omega < \frac{V_0}{R}$, take some time before rolling

$h > \frac{2}{5} R$ $\omega > \frac{V_0}{R}$ friction is moving forward

At the contact point, the velocity is going backward.

Example : Rolling without slipping

Any round object of radius R rolls about its cm as it translates down plane of angle θ.



Free-body diagram for a solid sphere rolling down an incline.

Mass = M

Inertia = $I = \beta M R^2$

① $\sum \tau = I\alpha$ (aboot cm)

√② $\tau_f + \tau_{mg} + \tau_N = Rf + 0 + 0 = I\alpha$

√③ $\sum f_x = Mg\sin\theta - f = Ma_{cm}$

If motion is rolling without slipping

$v_{cm} = R\omega$ and $a_{cm} = R\alpha$ → condition

$$Mg\sin\theta - \frac{I}{R}\alpha = Mg\sin\theta - \frac{\beta MR^2}{R}\frac{a_{cm}}{R}$$

$$= Mg\sin\theta - \beta M a_{cm} = M a_{cm}$$

$$a_{cm} = \frac{g\sin\theta}{1+\beta}$$

Friction is static friction

$\therefore f_s \leq \mu_s N$

$$f_s = \frac{I\alpha}{R} = \frac{\beta M R^2}{R} \cdot \frac{1}{R} \frac{g \sin\theta}{1+\beta} \leq \mu_s M g \cos\theta$$

$$\therefore \tan\theta \leq \mu_s \frac{1+\beta}{\beta}$$

condition for angle above which object will slide as it rolls down the plane.

If object slides: $\left.\begin{array}{l} \omega R \neq v \\ \alpha R \neq a \end{array}\right\}$!!!

Hoop	$\beta = 1$
Cylinder	$\beta = 1/2$
Sphere	$\beta = 2/5$

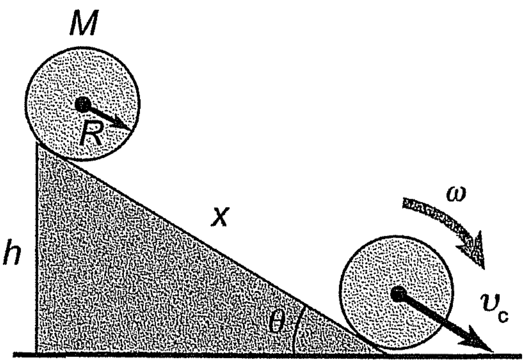
$$(a_{cm})_{sphere} = \frac{5}{7} g \sin\theta$$

$$(a_{cm})_{cyl} = \frac{2}{3} g \sin\theta$$

$$(a_{cm})_{Hoop} = \frac{1}{2} g \sin\theta$$

Example: Rolling down Incline

- Release from rest at top.
- No slipping.
- Rolling possible only if friction present to produce torque about cm.
- No energy lost since contact point does not move relative to surface.



A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

$$v_c = R\omega$$

$$K = \frac{1}{2} I_c \left(\frac{v_c}{R}\right)^2 + \frac{1}{2} M v_c^2$$

$$= \frac{1}{2}\left[\frac{I_c}{R^2} + M\right] v_c^2$$

Potential energy lost if object drops a height h:

$$\Delta U = Mgh$$

$$\Delta K = \Delta U$$

$$\frac{1}{2}\left(\frac{I_c}{R^2} + M\right) v_c^2 = Mgh$$

$$v_c = \sqrt{\frac{2gh}{1 + I_c/MR^2}}$$

Example: Sphere down Plane

$$I_c = \frac{2}{5} MR^2$$

$$v_c = \sqrt{\frac{2gh}{1+\frac{2}{5}\frac{MR^2}{MR^2}}} = \sqrt{\frac{10}{7} gh}$$

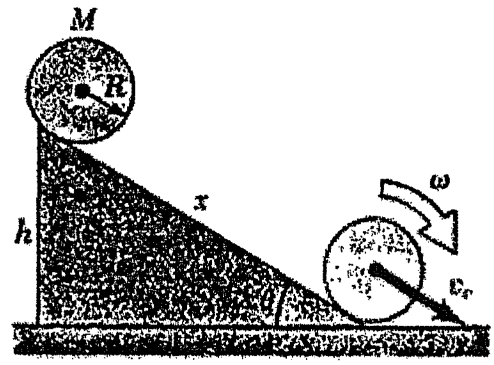


Figure A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

x = distance along incline
$h = x \sin\theta$

$$v_c^2 = \frac{10}{7} g x \sin\theta$$

$$v_c^2 = 2 a_c x \quad \text{[constant acceleration]}$$

$$a_c = \frac{5}{7} g \sin\theta$$

Note:
Velocity and acceleration are independent of mass and radius of sphere. All homogeneous solid spheres would have the same velocity and acceleration on a given incline.
Hollow spheres, cylinders + hoops would give similar results. Constants in expressions for v_c and a_c would be different.
Acceleration is less than for an object which does not roll.