Chapter 9 PLANE MOTION of RIGID BODIES

RIGID BODIES

A system of particles in which the distance between any two particles does not change regardless of the forces acting is called a *rigid body*. Since a rigid body is a special case of a system of particles, all theorems developed in Chapter 7 are also valid for rigid bodies.

TRANSLATIONS AND ROTATIONS

A *displacement* of a rigid body is a change from one position to another. If during a displacement all points of the body on some line remain fixed, the displacement is called a *rotation* about the line. If during a displacement all points of the rigid body move in lines parallel to each other the displacement is called a *translation*.

EULER'S THEOREM. INSTANTANEOUS AXIS OF ROTATION

The following theorem, called *Euler's theorem*, is fundamental in the motion of rigid bodies.

Theorem 9.1. A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the point.

The line referred to is called the *instantaneous axis of rotation*.

Rotations can be considered as finite or infinitesimal. Finite rotations cannot be represented by vectors since the commutative law fails. However, infinitesimal rotations can be represented by vectors.

GENERAL MOTION OF A RIGID BODY. CHASLE'S THEOREM

In the general motion of a rigid body, no point of the body may be fixed. In such case the following theorem, called *Chasle's theorem*, is fundamental.

Theorem 9.2. The general motion of a rigid body can be considered as a translation plus a rotation about a suitable point which is often taken to be the center of mass.

PLANE MOTION OF A RIGID BODY

The motion of a rigid body is simplified considerably when all points move parallel to a given fixed plane. In such case two types of motion, called *plane motion*, are possible.

1. **Rotation about a fixed axis.** In this case the rigid body rotates about a fixed axis perpendicular to the fixed plane. The system has only one degree of freedom [see Chapter 7, page 165] and thus only one coordinate is required for describing the motion.

2. **General plane motion.** In this case the motion can be considered as a translation parallel to the given fixed plane plus a rotation about a suitable axis perpendicular to the plane. This axis is often chosen so as to pass through the center of mass. The number of degrees of freedom for such motion is 3: two coordinates being used to describe the translation and one to describe the rotation.

The axis referred to is the *instantaneous axis* and the point where the instantaneous axis intersects the fixed plane is called the *instantaneous center of rotation* [see page 229].

We shall consider these two types of plane motion in this chapter. The motion of a rigid body in three dimensional space is more complicated and will be considered in Chapter 10.

MOMENT OF INERTIA

A geometric quantity which is of great importance in discussing the motion of rigid bodies is called the *moment of inertia.*

The *moment of inertia of a particle* of mass m about a line or *axis AB* is defined as

$$I = mr^2 \tag{1}$$

where r is the distance from the mass to the line.

The *moment of inertia of a system of particles,* with masses $m_1, m_2, \ldots, m_N$ about the line or axis AB is defined as

$$I = \sum_{\nu=1}^{N} m_\nu r_\nu^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N^2 \tag{2}$$

where $r_1, r_2, \ldots, r_N$ are their respective distances from AB.

The *moment of inertia of a continuous distribution of mass,* such as the solid rigid body $\mathcal{R}$ of Fig. 9-1, is given by

$$I = \int_{\mathcal{R}} r^2\, dm \tag{3}$$

where r is the distance of the element of mass dm from AB.

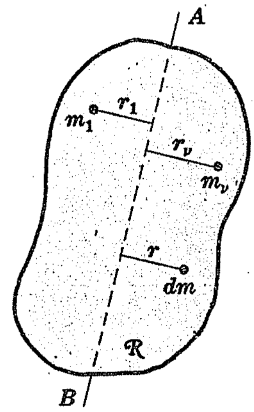


Fig. 9-1

RADIUS OF GYRATION

Let $I = \sum_{\nu=1}^{N} m_\nu r_\nu^2$ be the moment of inertia of a system of particles about AB, and $M = \sum_{\nu=1}^{N} m_\nu$ be the total mass of the system. Then the quantity K such that

$$K^2 = \frac{I}{M} = \frac{\sum_\nu m_\nu r_\nu^2}{\sum_\nu m_\nu} \tag{4}$$

is called the *radius of gyration* of the system about AB.

For continuous mass distributions (4) is replaced by

$$K^2 = \frac{I}{M} = \frac{\int_{\mathcal{R}} r^2\, dm}{\int_{\mathcal{R}} dm} \tag{5}$$

THEOREMS ON MOMENTS OF INERTIA

1. ***Theorem 9.3*: *Parallel Axis Theorem.*** Let I be the moment of inertia of a system about axis AB and let I_C be the moment of inertia of the system about an axis parallel to AB and passing through the center of mass of the system. Then if b is the distance between the axes and M is the total mass of the system, we have

$$I = I_C + Mb^2 \tag{6}$$

2. ***Theorem 9.4*: *Perpendicular Axes Theorem.*** Consider a mass distribution in the xy plane of an xyz coordinate system. Let I_x, I_y and I_z denote the moments of inertia about the x, y and z axes respectively. Then

$$I_z = I_x + I_y \tag{7}$$

SPECIAL MOMENTS OF INERTIA

The following table shows the moments of inertia of various rigid bodies which arise in practice. In all cases it is assumed that the body has uniform [i.e. constant] density.

Rigid Body	Moment of Inertia
1. *Solid Circular Cylinder* of radius a and mass M about axis of cylinder.	$\frac{1}{2}Ma^2$
2. *Hollow Circular Cylinder* of radius a and mass M about axis of cylinder. Wall thickness is negligible.	Ma^2
3. *Solid Sphere* of radius a and mass M about a diameter.	$\frac{2}{5}Ma^2$
4. *Hollow Sphere* of radius a and mass M about a diameter. Sphere thickness is negligible.	Ma^2
5. *Rectangular Plate* of sides a and b and mass M about an axis perpendicular to the plate through the center of mass.	$\frac{1}{12}M(a^2+b^2)$
6. *Thin Rod* of length a and mass M about an axis perpendicular to the rod through the center of mass.	$\frac{1}{12}Ma^2$

COUPLES

A set of two equal and parallel forces which act in opposite directions but do not have the same line of action [see Fig. 9-2] is called a *couple*. Such a couple has a turning effect, and the *moment* or *torque of the couple* is given by $\mathbf{r} \times \mathbf{F}$.

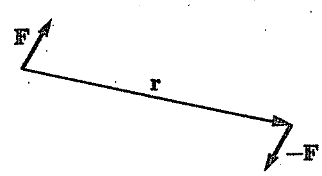


Fig. 9-2

The following theorem is important.

Theorem 9.5. Any system of forces which acts on a rigid body can be equivalently replaced by a single force which acts at some specified point together with a suitable couple.

KINETIC ENERGY AND ANGULAR MOMENTUM ABOUT A FIXED AXIS

Suppose a rigid body is rotating about a fixed axis with angular velocity $\boldsymbol{\omega}$ which has the direction of the axis AB [see Fig. 9-3]. Then the *kinetic energy of rotation* is given by

$$T = \tfrac{1}{2}I\omega^2 \qquad (8)$$

where I is the moment of inertia of the rigid body about the axis.

Similarly the *angular momentum* is given by

$$\boldsymbol{\Omega} = I\boldsymbol{\omega} \qquad (9)$$

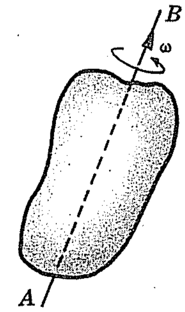


Fig. 9-3

MOTION OF A RIGID BODY ABOUT A FIXED AXIS

Two important methods for treating the motion of a rigid body about a fixed axis are given by the following theorems.

Theorem 9.6: Principle of Angular Momentum. If $\boldsymbol{\Lambda}$ is the *torque* or the moment of all external forces about the axis and $\boldsymbol{\Omega} = I\boldsymbol{\omega}$ is the angular momentum, then

$$\boldsymbol{\Lambda} = \frac{d}{dt}(I\boldsymbol{\omega}) = I\dot{\boldsymbol{\omega}} = I\boldsymbol{\alpha} \qquad (10)$$

where $\boldsymbol{\alpha}$ is the angular acceleration.

Theorem 9.7: Principle of Conservation of Energy. If the forces acting on the rigid body are conservative so that the rigid body has a potential energy V, then

$$T + V = \tfrac{1}{2}I\omega^2 + V = E = \text{constant} \qquad (11)$$

WORK AND POWER

Consider a rigid body $\mathcal{R}$ capable of rotating in a plane about an axis O perpendicular to the plane, as indicated in Fig. 9-4. If Λ is the magnitude of the torque applied to the body under the influence of force $\mathbf{F}$ at point $\mathbf{A}$, the *work done in rotating the body* through angle $d\theta$ is

$$dW = \Lambda\, d\theta \qquad (12)$$

and the instantaneous *power* developed is

$$\mathcal{P} = \frac{dW}{dt} = \Lambda\omega \qquad (13)$$

where ω is the angular speed.

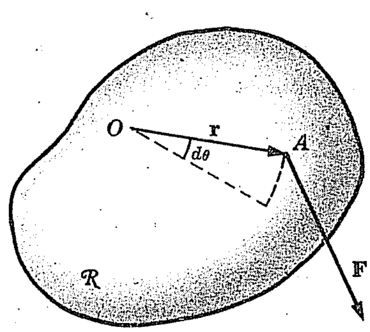


Fig. 9-4

We have the following

Theorem 9.8. The total work done in rotating a rigid body from an angle θ_1 where the angular speed is ω_1 to angle θ_2 where the angular speed is ω_2 is the difference in the kinetic energy of rotation at ω_1 and ω_2. In symbols,

$$\int_{\theta_1}^{\theta_2} \Lambda\, d\theta = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \qquad (14)$$

IMPULSE. CONSERVATION OF ANGULAR MOMENTUM

The time integral of the torque

$$\mathcal{J} = \int_{t_1}^{t_2} \mathbf{\Lambda}\, dt \tag{15}$$

is called the *angular impulse* from time t_1 to t_2.

We have the following theorems.

Theorem 9.9. The angular impulse is equal to the change in angular momentum. In symbols

$$\int_{t_1}^{t_2} \mathbf{\Lambda}\, dt = \mathbf{\Omega}_2 - \mathbf{\Omega}_1 \tag{16}$$

Theorem 9.10**: **Conservation of Angular Momentum. If the net torque applied to a rigid body is zero, then the angular momentum is constant, i.e. is conserved.

THE COMPOUND PENDULUM

Let $\mathcal{R}$ [Fig. 9-5] be a rigid body which is free to oscillate in a vertical plane about a fixed horizontal axis through O under the influence of gravity. We call such a rigid body a *compound pendulum*.

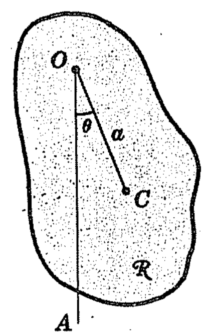


Fig. 9-5

Let C be the center of mass and suppose that the angle between OC and the vertical OA is θ. Then if I_0 is the moment of inertia of $\mathcal{R}$ about the horizontal axis through O, M is the mass of the rigid body and a is the distance OC, we have for the equation of motion,

$$\ddot{\theta} + \frac{Mga}{I_0}\sin\theta = 0 \tag{17}$$

For small oscillations the period of vibration is

$$P = 2\pi\sqrt{I_0/Mga} \tag{18}$$

The length of the equivalent simple pendulum is

$$l = I_0/Ma \tag{19}$$

The following theorem is of interest.

Theorem 9.11. The period of vibration of a compound pendulum is a minimum when the distance $OC = a$ is equal to the radius of gyration of the body about the horizontal axis through the center of mass.

GENERAL PLANE MOTION OF A RIGID BODY

The general plane motion of a rigid body can be considered as a translation parallel to the plane plus a rotation about a suitable axis perpendicular to the plane. Two important methods for treating general plane motion of a rigid body are given by the following theorems.

Theorem 9.12**: **Principle of Linear Momentum. If $\mathbf{r}$ is the position vector of the center of mass of a rigid body relative to an origin O, then

$$\frac{d}{dt}(M\dot{\mathbf{r}}) = M\ddot{\mathbf{r}} = \mathbf{F} \tag{20}$$

where M is the total mass, assumed constant, and $\mathbf{F}$ is the net external force acting on the body.

Theorem 9.13*: *Principle of Angular Momentum. If I_C is the moment of inertia of the rigid body about the center of mass, ω is the angular velocity and $\mathbf{\Lambda}_C$ is the torque or total moment of the external forces about the center of mass, then

$$\mathbf{\Lambda}_C = \frac{d}{dt}(I_C\,\boldsymbol{\omega}) = I_C\,\dot{\boldsymbol{\omega}} \tag{21}$$

Theorem 9.14*: *Principle of Conservation of Energy. If the external forces are conservative so that the potential energy of the rigid body is V, then

$$T + V = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}I_C\omega^2 + V = E = \text{constant} \tag{22}$$

Note that $\frac{1}{2}m\dot{r}^2 = \frac{1}{2}mv^2$ is the *kinetic energy of translation* and $\frac{1}{2}I_C\omega^2$ is the *kinetic energy of rotation* of the rigid body about the center of mass.

INSTANTANEOUS CENTER. SPACE AND BODY CENTRODES

Suppose a rigid body $\mathcal{R}$ moves parallel to a given fixed plane, say the xy plane of Fig. 9-6. Consider an $x'y'$ plane parallel to the xy plane and rigidly attached to the body.

As the body moves there will be at any time t a point of the moving $x'y'$ plane which is instantaneously at rest relative to the fixed xy plane. This point, which may or may not be in the body, is called the *instantaneous center*. The line perpendicular to the plane and passing through the instantaneous center is called the *instantaneous axis*.

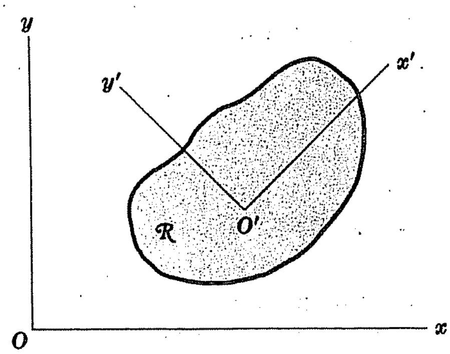


Fig. 9-6

As the body moves, the instantaneous center also moves. The locus or path of the instantaneous center relative to the fixed plane is called the *space locus* or *space centrode*. The locus relative to the moving plane is called the *body locus* or *body centrode*. The motion of the rigid body can be described as a rolling of the body centrode on the space centrode.

The instantaneous center can be thought of as that point about which there is rotation without translation. In a pure translation of a rigid body the instantaneous center is at infinity.

STATICS OF A RIGID BODY

The statics or equilibrium of a rigid body is the special case where there is no motion. The following theorem is fundamental.

Theorem 9.15. A necessary and sufficient condition for a rigid body to be in equilibrium is that

$$\mathbf{F} = \mathbf{0}, \quad \mathbf{\Lambda} = \mathbf{0} \tag{23}$$

where $\mathbf{F}$ is the net external force acting on the body and $\mathbf{\Lambda}$ is the net external torque.

PRINCIPLE OF VIRTUAL WORK AND D'ALEMBERT'S PRINCIPLE

Since a rigid body is but a special case of a system of particles, the principle of virtual work and D'Alembert's principle [see page 171] apply to rigid bodies as well.

Then since the mass is

$$M = \int_{r=0}^{a} 2\pi\sigma rh\,dr = \sigma\pi a^2 h \qquad (2)$$

we find $I = \frac{1}{2}Ma^2$.

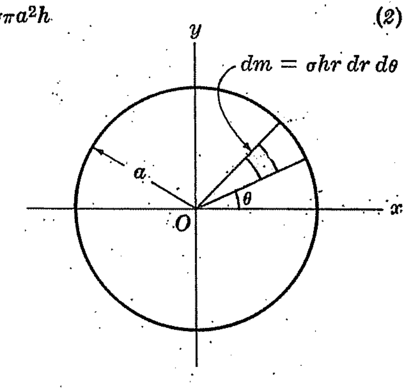


Fig. 9-12

Method 2, using double integration.

Using polar coordinates (r, θ), we see from Fig. 9-12 that the moment of inertia of the element of mass dm distant r from the axis is

$$r^2\,dm = r^2\sigma hr\,dr\,d\theta = \sigma hr^3\,dr\,d\theta$$

since $hr\,dr\,d\theta$ is the volume element and σ is the mass per unit volume (density). Then the total moment of inertia is

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \sigma hr^3\,dr\,d\theta = \tfrac{1}{2}\pi\sigma ha^4 \qquad (1)$$

The mass of the cylinder is given by

$$M = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \sigma hr\,dr\,d\theta = \sigma\pi a^2 h \qquad (2)$$

which can also be found directly by noting that the volume of the cylinder is $\pi a^2 h$. Dividing equation (1) by (2), we find $I/M = \frac{1}{2}a^2$ or $I = \frac{1}{2}Ma^2$.

9.5. Find the radius of gyration, K, of the cylinder of Problem 9.4.

Since $K^2 = I/M = \frac{1}{2}a^2$, $K = a/\sqrt{2} = \frac{1}{2}a\sqrt{2}$.

9.6. Find the (a) moment of inertia and (b) radius of gyration of a rectangular plate with sides a and b about a side.

Method 1, using single integration.

(a) The element of mass shaded in Fig. 9-13 is $\sigma b\,dx$, and its moment of inertia about the y axis is $(\sigma b\,dx)x^2 = \sigma bx^2\,dx$. Thus the total moment of inertia is

$$I = \int_{x=0}^{a} \sigma bx^2\,dx = \tfrac{1}{3}\sigma ba^3$$

Since the total mass of the plate is $M = ab\sigma$, we have $I/M = \frac{1}{3}a^2$ or $I = \frac{1}{3}Ma^2$.

(b) $K^2 = I/M = \frac{1}{3}a^2$ or $K = a/\sqrt{3} = \frac{1}{3}a\sqrt{3}$.

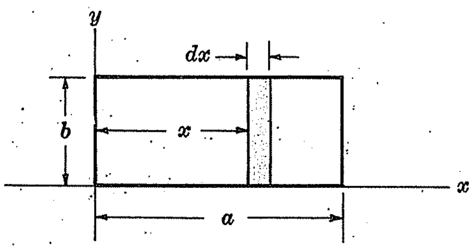


Fig. 9-13

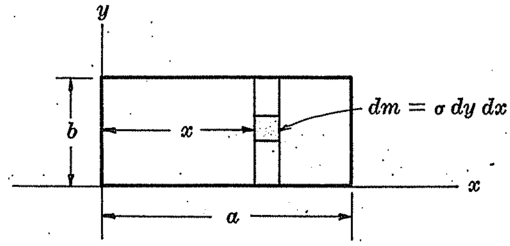


Fig. 9-14

Method 2, using double integration.

Assume the plate has unit thickness. If $dm = \sigma\,dy\,dx$ is an element of mass [see Fig. 9-14], the moment of inertia of dm about the side which is chosen to be on the y axis is $x^2\,dm = \sigma x^2\,dy\,dx$. Then the total moment of inertia is

$$I = \int_{x=0}^{a}\int_{y=0}^{b} \sigma x^2\,dy\,dx = \tfrac{1}{3}\sigma ba^3$$

The total mass of the plate is $M = ab\sigma$. Then, as in Method 1, we find $I = \frac{1}{3}Ma^2$ and $K = \frac{1}{3}a\sqrt{3}$.

9.7. Find the moment of inertia of a right circular cone of height h and radius a about its axis.

Method 1, using single integration.

The moment of inertia of the circular cylindrical disc one quarter of which is represented by PQR in Fig. 9-15 is, by Problem 9.4,

$$\tfrac{1}{2}(\pi r^2 \sigma\, dz)(r^2) \;=\; \tfrac{1}{2}\pi\sigma r^4\, dz$$

since this disc has volume $\pi r^2\, dz$ and radius r.

From Fig. 9-15, $\dfrac{h-z}{h} = \dfrac{r}{a}$ or $r = a\left(\dfrac{h-z}{h}\right)$.

Then the total moment of inertia about the z axis is

$$I \;=\; \tfrac{1}{2}\pi\sigma \int_{z=0}^{h} \left\{a\left(\frac{h-z}{h}\right)\right\}^4 dz \;=\; \tfrac{1}{10}\pi a^4 \sigma h$$

Also,

$$M \;=\; \pi\sigma \int_{z=0}^{h} \left\{a\left(\frac{h-z}{h}\right)\right\}^2 dz \;=\; \tfrac{1}{3}\pi a^2 h\sigma$$

Thus $I = \tfrac{3}{10}Ma^2$.

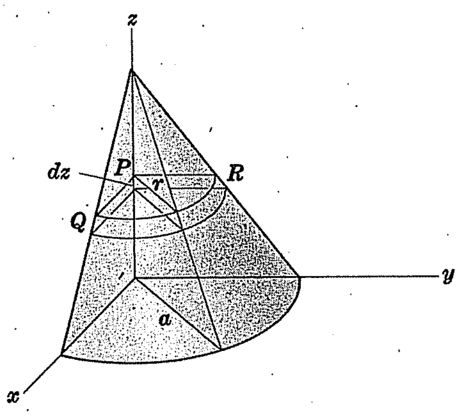


Fig. 9-15

Method 2, using triple integration.

Subdivide the cone, one quarter of which is shown in Fig. 9-16, into elements of mass dm as indicated in the figure.

In cylindrical coordinates (r, θ, z) the element of mass dm of the cylinder is $dm = \sigma r\, dr\, d\theta\, dz$ where σ is the density.

The moment of inertia of dm about the z axis is

$$r^2\, dm \;=\; \sigma r^3\, dr\, d\theta\, dz$$

As in Method 1, $\dfrac{h-z}{h} = \dfrac{r}{a}$ or $z = h\left(\dfrac{a-r}{a}\right)$.

Then the total moment of inertia about the z axis is

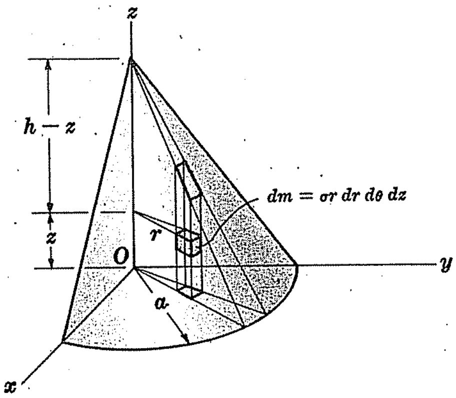


Fig. 9-16

$$I \;=\; \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h(a-r)/a} \sigma r^3\, dr\, d\theta\, dz \;=\; \tfrac{1}{10}\pi a^4 \sigma h$$

The total mass of the cone is

$$M \;=\; \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h(a-r)/r} \sigma r\, dr\, d\theta\, dz \;=\; \tfrac{1}{3}\pi a^2 h\sigma$$

which can be obtained directly by noting that the volume of the cone is $\tfrac{1}{3}\pi a^2 h$.

Thus $I = \tfrac{3}{10}Ma^2$.

9.8. Find the radius of gyration K of the cone of Problem 9.7.

$K^2 = I/M = \tfrac{3}{10}a^2$ and $K = a\sqrt{\tfrac{3}{10}} = \tfrac{1}{10}a\sqrt{30}$.

THEOREMS ON MOMENTS OF INERTIA

9.9. Prove the parallel axis theorem [Theorem 9.3, page 226].

Let OQ be any axis and ACP a parallel axis through the centroid C and distant b from OQ. In Fig. 9-17 below, OQ has been chosen as the z axis so that AP is perpendicular to the xy plane at P.

If $\mathbf{b}_1$ is a unit vector in the direction OP, then the vector OP is given by

$$\mathbf{b} = b\mathbf{b}_1 \tag{1}$$

where b is constant and is the distance between axes.

Let $\mathbf{r}_\nu$ and $\mathbf{r}'_\nu$ be the position vectors of mass m_ν relative to O and C respectively. If $\bar{\mathbf{r}}$ is the position vector of C relative to O then we have

$$\mathbf{r}_\nu = \mathbf{r}'_\nu + \bar{\mathbf{r}} \tag{2}$$

The total moment of inertia of all masses m_ν about axis OQ is

$$I = \sum_{\nu=1}^{N} m_\nu(\mathbf{r}_\nu \cdot \mathbf{b}_1)^2 \tag{3}$$

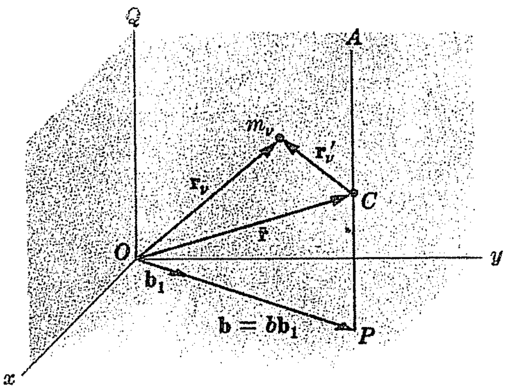


Fig. 9-17

The total moment of inertia of all masses m_ν about axis ACP is

$$I_C = \sum_{\nu=1}^{N} m_\nu(\mathbf{r}'_\nu \cdot \mathbf{b}_1)^2 \tag{4}$$

Then using (2) we find

$$I = \sum_{\nu=1}^{N} m_\nu(\mathbf{r}_\nu \cdot \mathbf{b}_1)^2 = \sum_{\nu=1}^{N} m_\nu(\mathbf{r}'_\nu \cdot \mathbf{b}_1 + \bar{\mathbf{r}} \cdot \mathbf{b}_1)^2$$

$$= \sum_{\nu=1}^{N} m_\nu(\mathbf{r}'_\nu \cdot \mathbf{b}_1)^2 + 2\sum_{\nu=1}^{N} m_\nu(\mathbf{r}'_\nu \cdot \mathbf{b}_1)(\bar{\mathbf{r}} \cdot \mathbf{b}_1) + \sum_{\nu=1}^{N} m_\nu(\bar{\mathbf{r}} \cdot \mathbf{b}_1)^2$$

$$= I_C + 2b\left(\sum_{\nu=1}^{N} m_\nu \mathbf{r}'_\nu\right) \cdot \mathbf{b}_1 + b^2 \sum_{\nu=1}^{N} m_\nu = I_C + Mb^2$$

since $\bar{\mathbf{r}} \cdot \mathbf{b}_1 = b$, $\sum_{\nu=1}^{N} m_\nu = M$ and $\sum_{\nu=1}^{N} m_\nu \mathbf{r}'_\nu = \mathbf{0}$ [Problem 7.16, page 178].

The result is easily extended to continuous mass systems by using integration in place of summation.

9.10. Use the parallel axis theorem to find the moment of inertia of a solid circular cylinder about a line on the surface of the cylinder and parallel to the axis of the cylinder.

Suppose the cross section of the cylinder is represented as in Fig. 9-18. Then the axis is represented by C, while the line on the surface of the cylinder is represented by A.

If a is the radius of the cylinder, then by Problem 9.4 and the parallel axis theorem we have

$$I_A = I_C + Ma^2 = \tfrac{1}{2}Ma^2 + Ma^2 = \tfrac{3}{2}Ma^2$$

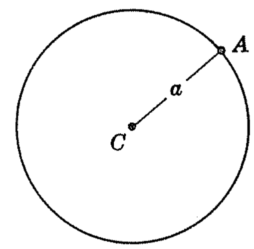


Fig. 9-18

9.11. Prove the perpendicular axes theorem [Theorem 9.4, page 226].

Let the position vector of the particle with mass m_ν in the xy plane be

$$\mathbf{r}_\nu = x_\nu\mathbf{i} + y_\nu\mathbf{j}$$

[see Fig. 9-19]. The moment of inertia of m_ν about the z axis is $m_\nu|\mathbf{r}_\nu|^2$.

Then the total moment of inertia of all particles about the z axis is

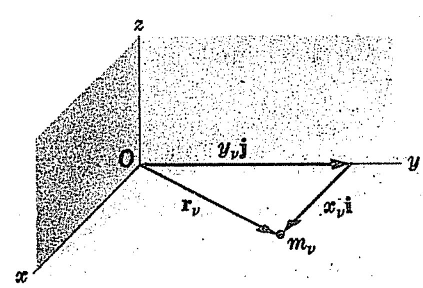


Fig. 9-19

$$I_z = \sum_{\nu=1}^{N} m_\nu |\mathbf{r}_\nu|^2 = \sum_{\nu=1}^{N} m_\nu (x_\nu^2 + y_\nu^2)$$
$$= \sum_{\nu=1}^{N} m_\nu x_\nu^2 + \sum_{\nu=1}^{N} m_\nu y_\nu^2 = I_x + I_y$$

where I_x and I_y are the total moments of inertia about the x axis and y axis respectively.

The result is easily extended to continuous systems.

9.12. Find the moment of inertia of a rectangular plate with sides a and b about an axis perpendicular to the plate and passing through a vertex.

Choose the rectangular plate [see Fig. 9-20] in the xy plane with sides on the x and y axes. Choose the z axis perpendicular to the plate at a vertex.

From Problem 9.6 we have for the moments of inertia about the x and y axes,

$$I_x = \tfrac{1}{3}Mb^2, \quad I_y = \tfrac{1}{3}Ma^2$$

Then by the perpendicular axes theorem the moment of inertia about the z axis is

$$I_z = I_x + I_y = \tfrac{1}{3}M(b^2 + a^2)$$
$$= \tfrac{1}{3}M(a^2 + b^2)$$

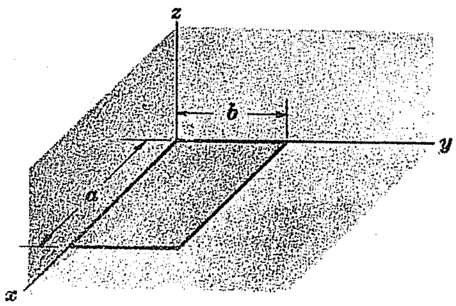


Fig. 9-20

COUPLES

9.13. Prove that a force acting at a point of a rigid body can be equivalently replaced by a single force acting at some specified point together with a suitable couple.

Let the force be $\mathbf{F}_1$ acting at point P_1 as in Fig. 9-21. If Q is any specified point, it is seen that the effect of $\mathbf{F}_1$ alone is the same if we apply two forces $\mathbf{f}_1$ and $-\mathbf{f}_1$ at Q.

In particular if we choose $\mathbf{f}_1 = -\mathbf{F}_1$, i.e. if $\mathbf{f}_1$ has the same magnitude as $\mathbf{F}_1$ but is opposite in direction, we see that the effect of $\mathbf{F}_1$ alone is the same as the effect of the couple formed by $\mathbf{F}_1$ and $\mathbf{f}_1 = -\mathbf{F}_1$ [which has moment $\mathbf{r}_1 \times \mathbf{F}_1$] together with the force $-\mathbf{f}_1 = \mathbf{F}_1$.

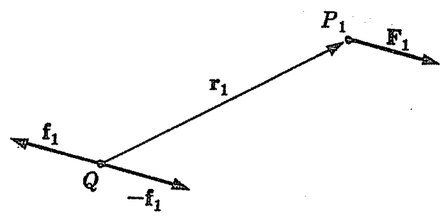


Fig. 9-21

9.14. Prove Theorem 9.5, page 227: Any system of forces which acts on a rigid body can be equivalently replaced by a single force which acts at some specified point together with a suitable couple.

By Problem 9.13 we can replace the force $\mathbf{F}_\nu$ at P_ν by the force $\mathbf{F}_\nu$ at Q plus a couple of moment $\mathbf{r}_\nu \times \mathbf{F}_\nu$. Then the system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N$ at points $P_1, P_2, \ldots, P_N$ can be combined into forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N$ at Q having resultant

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_N$$

together with couples having moments

$$\mathbf{r}_1 \times \mathbf{F}_1, \ \mathbf{r}_2 \times \mathbf{F}_2, \ \ldots, \ \mathbf{r}_N \times \mathbf{F}_N$$

which may be added to yield a single couple. Thus the system of forces can be equivalently replaced by the single force $\mathbf{F}$ acting at Q together with a couple.

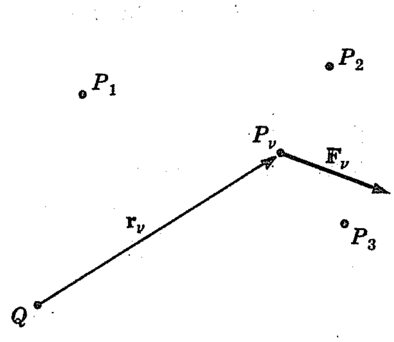


Fig. 9-22

KINETIC ENERGY AND ANGULAR MOMENTUM

9.15. If a rigid body rotates about a fixed axis with angular velocity ω, prove that the kinetic energy of rotation is $T = \frac{1}{2}I\omega^2$ where I is the moment of inertia about the axis.

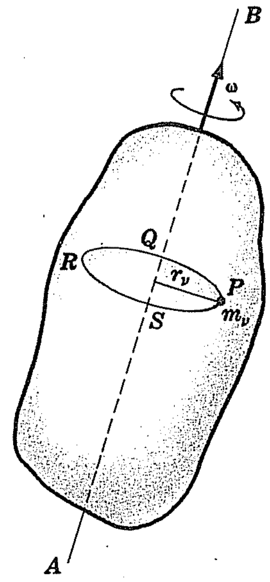


Fig. 9-23

Choose the axis as AB in Fig. 9-23. A particle P of mass m_ν will rotate about the axis with angular speed ω. Then it will describe a circle $PQRSP$ with linear speed $v_\nu = \omega r_\nu$ where r_ν is its distance from axis AB. Thus its kinetic energy of rotation about AB is $\frac{1}{2}m_\nu v_\nu^2 = \frac{1}{2}m_\nu \omega^2 r_\nu^2$, and the total kinetic energy of all particles is

$$T = \sum_{\nu=1}^{N} \frac{1}{2}m_\nu \omega^2 r_\nu^2 = \frac{1}{2}\left(\sum_{\nu=1}^{N} m_\nu r_\nu^2\right)\omega^2$$

$$= \frac{1}{2}I\omega^2$$

where $I = \sum_{\nu=1}^{N} m_\nu r_\nu^2$ is the moment of inertia about AB.

The result could also be proved by using integration in place of summation.

9.16. Prove that the angular momentum of the rigid body of Problem 9.15 is $\Omega = I\omega$.

The angular momentum of particle P about axis AB is $m_\nu r_\nu^2 \omega$. Then the total angular momentum of all particles about axis AB is

$$\Omega = \sum_{\nu=1}^{N} m_\nu r_\nu^2 \omega = \left(\sum_{\nu=1}^{N} m_\nu r_\nu^2\right)\omega = I\omega$$

where $I = \sum_{\nu=1}^{N} m_\nu r_\nu^2$ is the moment of inertia about AB.

The result could also be proved by using integration in place of summation.

MOTION OF A RIGID BODY ABOUT A FIXED AXIS

9.17. Prove the principle of angular momentum for a rigid body rotating about a fixed axis [Theorem 9.6, page 227].

By Problem 7.12, page 176, since a rigid body is a special case of a system of particles $\Lambda = d\Omega/dt$ where Λ is the torque or moment of all external forces about the axis and Ω is the total angular momentum about the axis.

Since $\Omega = I\omega$ by Problem 9.16, $\Lambda = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\dot{\omega}$.

9.18. Prove the principle of conservation of energy for a rigid body rotating about a fixed axis [Theorem 9.7, page 227] provided the forces acting are conservative.

The principle of conservation of energy applies to any system of particles in which the forces acting are conservative. Hence in particular it applies to the special case of a rigid body rotating about a fixed axis. If T and V are the total kinetic energy and the potential energy, we thus have

$$T + V = \text{constant} = E$$

Using the result of Problem 9.15, this can be written $\frac{1}{2}I\omega^2 + V = E$.

WORK, POWER AND IMPULSE

9.19. Prove equation (*12*), page 227, for the work done in rotating a rigid body about a fixed axis.

Refer to Fig. 9-4, page 227. Let the angular velocity of the body be $\boldsymbol{\omega} = \omega\mathbf{k}$ where $\mathbf{k}$ is a unit vector in the direction of the axis of rotation. The work done by $\mathbf{F}$ is

$$dW = \mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \mathbf{F}\cdot\mathbf{v}\,dt = \mathbf{F}\cdot(\boldsymbol{\omega}\times\mathbf{r})\,dt$$
$$= (\mathbf{r}\times\mathbf{F})\cdot\boldsymbol{\omega}\,dt = \boldsymbol{\Lambda}\cdot\boldsymbol{\omega}\,dt = \Lambda\omega\,dt = \Lambda\,d\theta$$

where in the last two steps we use $\boldsymbol{\Lambda} = \Lambda\mathbf{k}$, $\boldsymbol{\omega} = \omega\mathbf{k}$ and $\omega = d\theta/dt$.

9.20. Prove equation (*13*), page 227, for the power developed.

From Problem 9.19 and the fact that $d\theta/dt = \omega$,

$$\mathcal{P} = dW/dt = \Lambda\,d\theta/dt = \Lambda\omega$$

9.21. Prove Theorem 9.8, page 227.

We have $\boldsymbol{\Lambda} = I\,d\boldsymbol{\omega}/dt$ so that $\Lambda = I\,d\omega/dt$. Then from Problem 9.19 and the fact that $d\theta = \omega\,dt$, we have

$$\text{Work done} = \int_{\theta_1}^{\theta_2}\Lambda\,d\theta = \int_{t_1}^{t_2} I\frac{d\omega}{dt}\omega\,dt = \int_{\omega_1}^{\omega_2} I\omega\,d\omega = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

9.22. Prove Theorem 9.9, page 228: The angular impulse is equal to the change in angular momentum.

$$\int_{t_1}^{t_2}\boldsymbol{\Lambda}\,dt = \int_{t_1}^{t_2}\frac{d\boldsymbol{\Omega}}{dt}dt = \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1$$

9.23. Prove Theorem 9.10, page 229, on the conservation of angular momentum if the net torque is zero.

From Problem 9.22, if $\boldsymbol{\Lambda} = \mathbf{0}$ then $\boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_1$.

THE COMPOUND PENDULUM

9.24. Obtain the equation of motion (*17*), page 228, for a compound pendulum.

Method 1.

Suppose that the vertical plane of vibration of the pendulum is chosen as the xy plane [Fig. 9-24] where the z axis through origin O is the horizontal axis of suspension.

Let point C have the position vector $\mathbf{a}$ relative to O. Since the body is rigid, $|\mathbf{a}| = a$ is constant and is the distance from O to C.

The only external force acting on the body is its weight $M\mathbf{g} = -Mg\mathbf{j}$ acting vertically downward. Thus we have

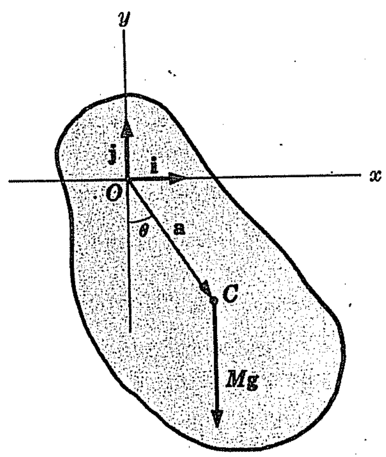


Fig. 9-24

$$\boldsymbol{\Lambda} = \text{total external torque about } z \text{ axis}$$
$$= \mathbf{a}\times M\mathbf{g} = -\mathbf{a}\times Mg\mathbf{j} = aMg\sin\theta\,\mathbf{k} \quad (1)$$

where $\mathbf{k}$ is a unit vector in the positive z direction [out of the plane of the paper toward the reader].

Also, the instantaneous angular velocity is

$$\boldsymbol{\omega} = -\omega\mathbf{k} = -\frac{d\theta}{dt}\mathbf{k} = -\dot{\theta}\,\mathbf{k} \qquad (2)$$

so that if I_0 is the moment of inertia about the z axis

$$\boldsymbol{\Omega} = \text{angular momentum about } z \text{ axis} = I_0\omega = -I_0\dot{\theta}\,\mathbf{k}$$

Substituting from (1) and (2) into $\boldsymbol{\Lambda} = d\boldsymbol{\Omega}/dt$,

$$aMg\sin\theta\,\mathbf{k} = \frac{d}{dt}(-I_0\dot{\theta}\,\mathbf{k}) \qquad \text{or} \qquad \ddot{\theta} + \frac{Mga}{I_0}\sin\theta = 0 \tag{3}$$

Method 2.

The force $M\mathbf{g} = -Mg\mathbf{j}$ is conservative, so that the potential energy V is such that

$$-\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k} = -Mg\mathbf{j} \qquad \text{or} \qquad \frac{\partial V}{\partial x} = 0,\ \frac{\partial V}{\partial y} = Mg,\ \frac{\partial V}{\partial z} = 0$$

from which

$$V = Mgy + c = -Mga\cos\theta + c \tag{4}$$

since $y = -a\cos\theta$. This could be seen directly since $y = -a\cos\theta$ is the height of C below the x axis taken as the reference level.

By Problem 9.15, the kinetic energy of rotation is $\frac{1}{2}I_0\omega^2 = \frac{1}{2}I_0\dot{\theta}^2$. Then the principle of conservation of energy gives

$$T + V = \tfrac{1}{2}I_0\dot{\theta}^2 - Mga\cos\theta = \text{constant} = E \tag{5}$$

Differentiating equation (5) with respect to t,

$$I_0\dot{\theta}\ddot{\theta} + Mga\sin\theta\,\dot{\theta} = 0$$

or, since $\dot{\theta}$ is not identically zero, $I_0\ddot{\theta} + Mga\sin\theta = 0$ as required.

9.25. Show that for small vibrations the pendulum of Problem 9.24 has period $P = 2\pi\sqrt{Mga/I_0}$.

For small vibrations we can make the approximation $\sin\theta = \theta$ so that the equation of motion becomes

$$\ddot{\theta} + \frac{Mga}{I_0}\theta = 0 \tag{1}$$

Then, as in Problem 4.23, page 102, we find that the period is $P = 2\pi\sqrt{I_0/Mga}$.

9.26. Show that the length l of a simple pendulum equivalent to the compound pendulum of Problem 9.24 is $l = I_0/Ma$.

The equation of motion corresponding to a simple pendulum of length l suspended vertically from O is [see Problem 4.23, equation (2), page 102]

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{1}$$

Comparing this equation with (1) of Problem 9.25, we see that $l = I_0/Ma$.

GENERAL PLANE MOTION OF A RIGID BODY

9.27. Prove the principle of linear momentum, Theorem 9.12, page 228, for the general plane motion of a rigid body.

This follows at once from the corresponding theorem for systems of particles [Theorem 7-1, page 167], since rigid bodies are special cases.

9.28. Prove the principle of angular momentum, Theorem 9.13, page 229, for general plane motion of a rigid body.

This follows at once from the corresponding theorem for systems of particles [Theorem 7-4, page 168], since rigid bodies are special cases.

Chapter 10

SPACE MOTION of RIGID BODIES

GENERAL MOTION OF RIGID BODIES IN SPACE

In Chapter 9 we specialized the motion of rigid bodies to one of translation of the center of mass plus rotation about an axis through the center of mass and perpendicular to a *fixed* plane. In this chapter we treat the general motion of a rigid body in space. Such general motion is composed of a translation of a fixed point of the body [usually the center of mass] plus rotation about an axis through the fixed point which is not necessarily restricted in direction.

DEGREES OF FREEDOM

The number of degrees of freedom [see page 165] for the general motion of a rigid body in space is 6, i.e. 6 coordinates are needed to specify the motion. We usually choose 3 of these to be the coordinates of a point in the body [usually the center of mass] and the remaining 3 to be angles [for example, the Euler angles, page 257] which describe the rotation of the rigid body about the point.

If a rigid body is constrained in any way, as for example by keeping one point fixed, the number of degrees of freedom is of course reduced accordingly.

PURE ROTATION OF RIGID BODIES

Since the general motion of a rigid body can also be expressed in terms of translation of a fixed point of the rigid body plus rotation of the rigid body about an axis through the point, it is natural for us to consider first the case of pure rotation and later to add the effects of translation. To do this we shall first assume that one point of the rigid body is fixed in space. The effects of translation are relatively easy to handle and can be obtained by using the result (*10*), page 167.

VELOCITY AND ANGULAR VELOCITY OF A RIGID BODY WITH ONE POINT FIXED

Suppose that point O of the rigid body $\mathcal{R}$ of Fig. 10-1 is fixed. Then at a given instant of time the body will be rotating with *angular velocity* $\boldsymbol{\omega}$ about the instantaneous axis through O. A particle P of the body having position vector $\mathbf{r}_\nu$ with respect to O will have an *instantaneous velocity* $\mathbf{v}_\nu$ given by

$$\mathbf{v}_\nu = \dot{\mathbf{r}}_\nu = \boldsymbol{\omega} \times \mathbf{r}_\nu \qquad (1)$$

See Problem 10.2.

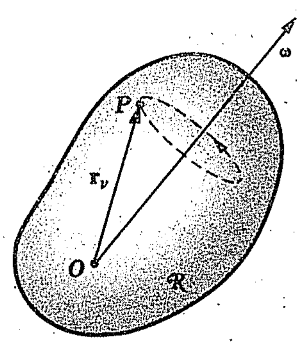


Fig. 10-1

9.29. A solid cylinder of radius a and mass M rolls without slipping down an inclined plane of angle α. Show that the acceleration is constant and equal to $\frac{2}{3}g \sin\alpha$.

Suppose that initially the cylinder has point O in contact with the plane and that after time t the cylinder has rotated through angle θ [see Fig. 9-25].

The forces acting on the cylinder at time t are: (i) the weight $M\mathbf{g}$ acting vertically downward at the center of mass C; (ii) the reaction $\mathbf{R}$ of the inclined plane acting perpendicular to the plane; (iii) the frictional force $\mathbf{f}$ acting upward along the incline.

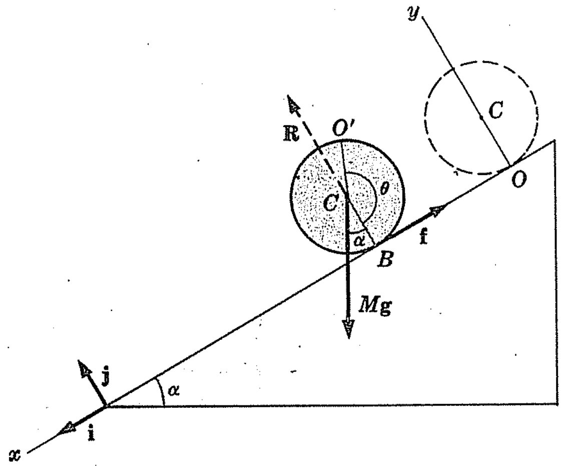


Fig. 9-25

Choose the plane in which motion takes place as the xy plane, where the x axis is taken as positive down the incline and the origin is at O.

If $\mathbf{r}$ is the position of the center of mass at time t, then by the principle of linear momentum,

$$M\ddot{\mathbf{r}} = M\mathbf{g} + \mathbf{R} + \mathbf{f} \tag{1}$$

But $\mathbf{g} = g \sin\alpha\,\mathbf{i} - g\cos\alpha\,\mathbf{j}$, $\mathbf{R} = R\mathbf{j}$, $\mathbf{f} = -f\mathbf{i}$. Hence (1) can be written

$$M\ddot{\mathbf{r}} = (Mg\sin\alpha - f)\mathbf{i} + (R - Mg\cos\alpha)\mathbf{j} \tag{2}$$

The total external torque about the horizontal axis through the center of mass is

$$\mathbf{\Lambda} = \mathbf{0}\times M\mathbf{g} + \mathbf{0}\times\mathbf{R} + \mathbf{CB}\times\mathbf{f} = \mathbf{CB}\times\mathbf{f} = (-a\mathbf{j})\times(-f\mathbf{i}) = -af\mathbf{k} \tag{3}$$

The total angular momentum about the horizontal axis through the center of mass is

$$\mathbf{\Omega} = I_C\boldsymbol{\omega} = I_C(-\dot{\theta}\mathbf{k}) = -I_C\dot{\theta}\,\mathbf{k} \tag{4}$$

where I_C is the moment of inertia of the cylinder about this axis.

Substituting (3) and (4) into $\mathbf{\Lambda} = d\mathbf{\Omega}/dt$, we find $-af\mathbf{k} = -I_C\ddot{\theta}\,\mathbf{k}$ or $I_C\ddot{\theta} = af$.

Using $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ in (2), we obtain

$$M\ddot{x} = Mg\sin\alpha - f, \qquad M\ddot{y} = R - Mg\cos\alpha \tag{5}$$

Now if there is no slipping, $x = a\theta$ or $\theta = x/a$. Similarly, since the cylinder remains on the incline, $\ddot{y} = 0$; hence from (5), $R = Mg\cos\alpha$.

Using $\theta = x/a$ in $I_C\ddot{\theta} = af$, we have $f = I_C\ddot{x}/a^2$. From Problem 9.4, $I_C = \frac{1}{2}Ma^2$. Then substituting $f = \frac{1}{2}M\ddot{x}$ into the first equation of (5), we obtain $\ddot{x} = \frac{2}{3}g\sin\alpha$ as required.

9.30. Prove that in Problem 9.29 the coefficient of friction must be at least $\frac{1}{3}\tan\alpha$.

The coefficient of friction is $\mu = f/R$.

From Problem 9.29 we have $f = \frac{1}{2}M\ddot{x} = \frac{1}{3}Mg\sin\alpha$ and $R = Mg\cos\alpha$. Thus in order that slipping will not occur, μ must be at least $f/R = \frac{1}{3}\tan\alpha$.

9.31. (*a*) Work Problem 9.29 if the coefficient of friction between the cylinder and inclined plane is μ and (*b*) discuss the motion for different values of μ.

(*a*) In equation (5) of Problem 9.29, substitute $f = \mu R = \mu Mg\cos\alpha$ and obtain

$$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$$

Note that in this case the center of mass of the cylinder moves in the same manner as a particle sliding down an inclined plane. However, the cylinder may slip as well as roll.

The acceleration due to rolling is $$a\ddot{\theta} = \frac{a^2 f}{I_C} = \frac{a^2\mu Mg\cos\alpha}{\frac{1}{2}Ma^2} = 2\mu g\cos\alpha.$$

The acceleration due to slipping is $$\ddot{x} - a\ddot{\theta} = g(\sin\alpha - 3\mu\cos\alpha).$$

(*b*) If $(\sin\alpha - 3\mu\cos\alpha) > 0$, i.e. $\mu < \frac{1}{3}\tan\alpha$, then slipping will occur. If $(\sin\alpha - 3\mu\cos\alpha) \leqq 0$, i.e. $\mu \geqq \frac{1}{3}\tan\alpha$, then rolling but no slipping will occur. These results are consistent with those of Problem 9.30.

9.32. Prove the principle of conservation of energy [Theorem 9.14, page 229].

This follows from the corresponding theorem for systems of particles, Theorem 7-7, page 16[…] The total kinetic energy T is the sum of the kinetic energy of translation of the center of mass plus the kinetic energy of rotation about the center of mass, i.e.,

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{1}{2}I_C\omega^2$$

If V is the potential energy, then the principle of conservation of energy states that if […] is a constant,

$$T + V = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{1}{2}I_C\omega^2 + V = E$$

9.33. Work Problem 9.29 by using the principle of conservation of energy.

The potential energy is composed of the potential energy due to the external forces [in th[e] case gravity] and the potential energy due to internal forces [which is a constant and can […] omitted]. Taking the reference level as the base of the plane and assuming that the height [of] the center of mass above this plane initially and at any time t to be H and h respectively, we ha[ve]

$$\frac{1}{2}M\dot{\mathbf{r}}^2 + \frac{1}{2}I_C\omega^2 + Mgh = MgH$$

or, using $H - h = x\sin\alpha$ and $\dot{\mathbf{r}}^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2$ since $\dot{y} = 0$,

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_C\omega^2 = Mgx\sin\alpha$$

Substituting $\omega = \dot{\theta} = \dot{x}/a$ and $I_C = \frac{1}{2}Ma^2$, we find $\dot{x}^2 = \frac{4}{3}gx\sin\alpha$. Differentiating with respect [to] t, we obtain

$$2\dot{x}\ddot{x} = \tfrac{4}{3}g\dot{x}\sin\alpha \quad \text{or} \quad \ddot{x} = \tfrac{2}{3}g\sin\alpha$$

INSTANTANEOUS CENTER. SPACE AND BODY CENTRODES

9.34. Find the position vector of the instantaneous center for a rigid body movi[ng] parallel to a given fixed plane.

Choose the XY plane of Fig. 9-26 as the fixed plane and the xy plane as the plane attached to and moving with the rigid body $\mathcal{R}$. Let point P of the xy plane [which may or may not be in the rigid body] have position vectors $\mathbf{R}$ and $\mathbf{r}$ relative to the XY and xy planes respectively. If $\mathbf{v}$ and $\mathbf{v}_A$ are the respective velocities of P and A relative to the XY system,

$$\mathbf{v} = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r} = \mathbf{v}_A + \boldsymbol{\omega}\times(\mathbf{R}-\mathbf{R}_A) \qquad (1)$$

where $\mathbf{R}_A$ is the position vector of A relative to O. If P is to be the instantaneous center, then $\mathbf{v} = \mathbf{0}$ so that

$$\boldsymbol{\omega}\times(\mathbf{R}-\mathbf{R}_A) = -\mathbf{v}_A \qquad (2)$$

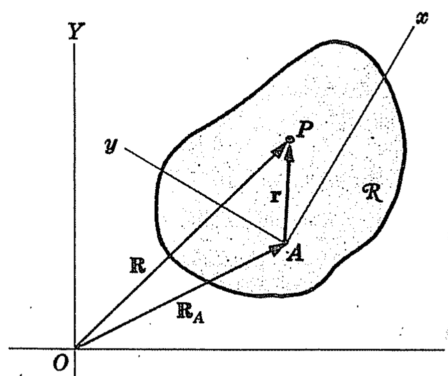


Fig. 9-26

Multiplying both sides of (2) by $\boldsymbol{\omega}\times$ and using (7), page 5,

$$\boldsymbol{\omega}\{\boldsymbol{\omega}\cdot(\mathbf{R}-\mathbf{R}_A)\} - (\mathbf{R}-\mathbf{R}_A)(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = -\boldsymbol{\omega}\times\mathbf{v}_A$$

Then since $\boldsymbol{\omega}$ is perpendicular to $\mathbf{R}-\mathbf{R}_A$, this becomes

$$(\mathbf{R}-\mathbf{R}_A)\omega^2 = \boldsymbol{\omega}\times\mathbf{v}_A \quad \text{or} \quad \mathbf{R} = \mathbf{R}_A + \frac{\boldsymbol{\omega}\times\mathbf{v}_A}{\omega^2}$$

9.35. A cylinder moves along a horizontal plane. Find the (*a*) space centrode, (*b*) body centrode. Discuss the case where slipping may occur.

(*a*) The general motion is one where both rolling and slipping may occur. Suppose the cylinder is moving to the right with velocity $\mathbf{v}_A$ [the velocity of its center of mass] and is rotating about A with angular velocity $\boldsymbol{\omega}$.

Since $\boldsymbol{\omega} = -\omega\mathbf{k}$ and $\mathbf{v}_A = v_A\mathbf{i}$, we have $\boldsymbol{\omega} \times \mathbf{v}_A = -\omega v_A\mathbf{j}$ so that (*3*) of Problem 9.34 becomes

$$\mathbf{R} = \mathbf{R}_A - \frac{(\omega v_A)\mathbf{j}}{\omega^2} = \mathbf{R}_A - \frac{v_A}{\omega}\mathbf{j}$$

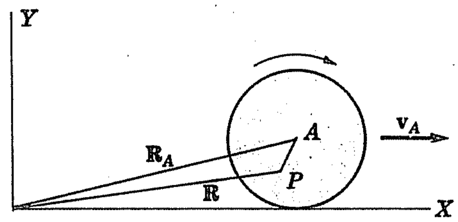


Fig. 9-27

In component form,

$$X\mathbf{i} + Y\mathbf{j} = X_A\mathbf{i} + a\mathbf{j} - (v_A/\omega)\mathbf{j} \qquad \text{or} \qquad X = X_A, \quad Y = a - v_A/\omega$$

Thus the instantaneous center is located vertically above the point of contact of the cylinder with the ground and at height $a - v_A/\omega$ above it.

Then the space centrode is a line parallel to the horizontal and at distance $a - v_A/\omega$ above it. If there is no slipping, then $v_A = a\omega$ and the space centrode is the X axis while the instantaneous center is the point of contact of the cylinder with the X axis.

(*b*) The body centrode is given by $|\mathbf{r}| = v_0/\omega$, or a circle of radius v_0/ω. In case of no slipping, $v_0 = a\omega$ and the body centrode is the circumference of the cylinder.

9.36. Solve Problem 9.29 by using the instantaneous center.

By Problem 9.35, if there is no slipping then the point of contact P of the cylinder with the plane is the instantaneous center. The motion of P is parallel to the motion of the center of mass, so that we can use the result of Problem 7.86(*c*), page 191.

The moment of inertia of the cylinder about P is, by the parallel axis theorem, $\frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$. The torque about the horizontal axis through P is $Mga \sin\alpha$. Thus

$$\frac{d}{dt}\left(\tfrac{3}{2}Ma^2\dot{\theta}\right) = Mga \sin\theta$$

or

$$\ddot{\theta} = \frac{2g}{3a}\sin\theta$$

Since $x = a\theta$, the acceleration is $\ddot{x} = \frac{2}{3}g \sin\theta$.

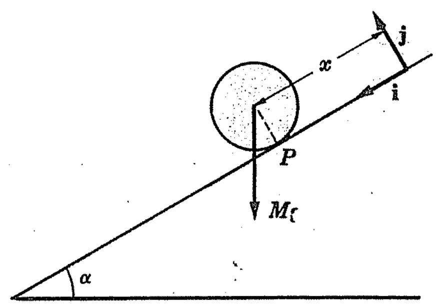


Fig. 9-28

STATICS OF A RIGID BODY

9.37. A ladder of length l and weight W_l has one end against a vertical wall which is frictionless and the other end on the ground assumed horizontal. The ladder makes an angle α with the ground. Prove that a man of weight W_m will be able to climb the ladder without having it slip if the coefficient of friction μ between the ladder and the ground is at least $\dfrac{W_m + \frac{1}{2}W_l}{W_m + W_l}\cot\alpha$.

Let the ladder be represented by AB in Fig. 9-29 and choose an xy coordinate system as indicated.

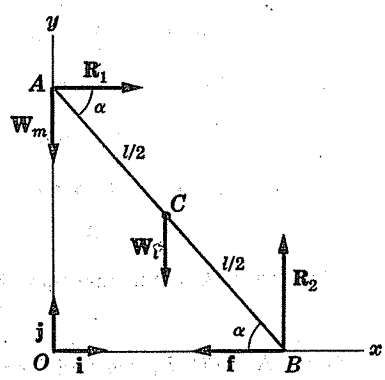


Fig. 9-29

The most dangerous situation in which the ladder would slip occurs when the man is at the top of the ladder. Hence we would require that the ladder be in equilibrium in such case.

The forces acting on the ladder are: (i) the reaction $\mathbf{R}_1 = R_1\mathbf{i}$ of the wall; (ii) the weight $\mathbf{W}_m = -W_m\mathbf{j}$ of the man; (iii) the weight $\mathbf{W}_l = -W_l\mathbf{j}$ of the ladder concentrated at C, the center of gravity; (iv) the reaction $\mathbf{R}_2 = R_2\mathbf{j}$ of the ground; (v) the friction force $\mathbf{f} = -f\mathbf{i}$.

For equilibrium we require that

$$\mathbf{F} = \mathbf{0}, \qquad \mathbf{\Lambda} = \mathbf{0} \tag{1}$$

where $\mathbf{F}$ is the total external force on the ladder and $\mathbf{\Lambda}$ the total external torque taken about a suitable axis which we shall take as the horizontal axis through A perpendicular to the xy plane. We have

$$\mathbf{F} = \mathbf{R}_1 + \mathbf{W}_m + \mathbf{W}_l + \mathbf{R}_2 + \mathbf{f} = (R_1 - f)\mathbf{i} + (-W_m - W_l + R_2)\mathbf{j} = \mathbf{0}$$

if

$$R_1 - f = 0 \qquad \text{and} \qquad -W_m - W_l + R_2 = 0 \tag{2}$$

Also,

$$\begin{aligned}\mathbf{\Lambda} &= (\mathbf{0}) \times \mathbf{R}_1 + (\mathbf{0}) \times \mathbf{W}_m + (\mathbf{AC}) \times \mathbf{W}_l + (\mathbf{AB}) \times \mathbf{R}_2 + (\mathbf{AB}) \times \mathbf{f} \\ &= (\mathbf{0}) \times (R_1\mathbf{i}) + (\mathbf{0}) \times (-W_m\mathbf{j}) + (\tfrac{1}{2}l\cos\alpha\,\mathbf{i} - \tfrac{1}{2}l\sin\alpha\,\mathbf{j}) \times (-W_l\mathbf{j}) \\ &\quad + (l\cos\alpha\,\mathbf{i} - l\sin\alpha\,\mathbf{j}) \times (R_2\mathbf{j}) + (l\cos\alpha\,\mathbf{i} - l\sin\alpha\,\mathbf{j}) \times (-f\mathbf{i}) \\ &= -\tfrac{1}{2}lW_l\cos\alpha\,\mathbf{k} + lR_2\cos\alpha\,\mathbf{k} - lf\sin\alpha\,\mathbf{k} = \mathbf{0}\end{aligned}$$

if

$$-\tfrac{1}{2}W_l\cos\alpha + R_2\cos\alpha - f\sin\alpha = 0 \tag{3}$$

Solving simultaneously equations (2) and (3), we find

$$f = R_1 = (W_m + \tfrac{1}{2}W_l)\cot\alpha \qquad \text{and} \qquad R_2 = W_m + W_l$$

Then the minimum coefficient of friction necessary to prevent slipping of the ladder is

$$\mu = \frac{f}{R_2} = \frac{W_m + \frac{1}{2}W_l}{W_m + W_l}\cot\alpha$$

MISCELLANEOUS PROBLEMS

9.38. Two masses m_1 and m_2 are connected by an inextensible string of negligible mass which passes over a frictionless pulley of mass M, radius a and radius of gyration K which can rotate about a horizontal axis through C perpendicular to the pulley. Discuss the motion.

Choose unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the plane of rotation as shown in Fig. 9-30.

If we represent the acceleration of mass m_1 by $A\mathbf{j}$, then the acceleration of mass m_2 is $-A\mathbf{j}$.

Choose the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the string as shown in the figure. By Newton's second law,

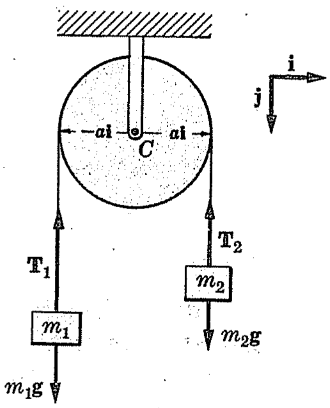


Fig. 9-30

$$m_1A\mathbf{j} = \mathbf{T}_1 + m_1\mathbf{g} = -T_1\mathbf{j} + m_1g\mathbf{j} \tag{1}$$

$$-m_2A\mathbf{j} = \mathbf{T}_2 + m_2\mathbf{g} = -T_2\mathbf{j} + m_2g\mathbf{j} \tag{2}$$

Thus $\quad m_1A = m_1g - T_1, \qquad m_2A = T_2 - m_2g \qquad (3)$

or $\quad T_1 = m_1(g - A), \qquad T_2 = m_2(g + A) \qquad (4)$

The net external torque about the axis through C is

$$\mathbf{\Lambda} = (-a\mathbf{i}) \times (-T_1\mathbf{j}) + (a\mathbf{i}) \times (-T_2\mathbf{j}) = a(T_1 - T_2)\mathbf{k} \tag{5}$$

The total angular momentum about O is

$$\mathbf{\Omega} = I_C\boldsymbol{\omega} = I_C\omega\mathbf{k} = I_C\dot{\theta}\mathbf{k} \tag{6}$$

Since $\mathbf{\Lambda} = d\mathbf{\Omega}/dt$, we find from (5) and (6),

$$a(T_1 - T_2) = I_C\ddot{\theta} = MK^2\ddot{\theta} \tag{7}$$

If there is no slipping about the pulley, we also have

$$A = a\ddot{\theta} \tag{8}$$

Using (8) in (7),
$$T_1 - T_2 = \frac{MK^2}{a^2}A \tag{9}$$

Using (4) in (9),
$$A = \frac{(m_1 - m_2)g}{m_1 + m_2 + Mk^2/a^2} \tag{10}$$

Thus the masses move with constant acceleration given in magnitude by (10). Note that if $M = 0$, the result (10) reduces to that of Problem 3.22, page 76.

9.39. Find the moment of inertia of a solid sphere about a diameter.

Let O be the center of the sphere and AOB be the diameter about which the moment of inertia is taken [Fig. 9-31]. Divide the sphere into discs such as $QRSTQ$ perpendicular to AOB and having center on AOB at P.

Take the radius of the sphere equal to a, $OP = z$, $SP = r$ and the thickness of the disc equal to dz. Then by Problem 9.4 the moment of inertia of the disc about AOB is

$$\tfrac{1}{2}(\pi r^2 \sigma\, dz)r^2 = \tfrac{1}{2}\pi\sigma r^4\, dz \tag{1}$$

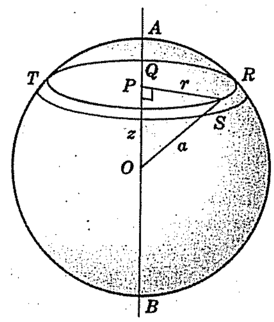


Fig. 9-31

From triangle OSP, $r^2 = a^2 - z^2$. Substituting into (1), the total moment of inertia is

$$I = \int_{z=-a}^{a} \tfrac{1}{2}\pi\sigma(a^2 - z^2)^2\, dz = \tfrac{8}{15}\pi\sigma a^5 \tag{2}$$

The mass of the sphere is

$$M = \int_{z=-a}^{a} \pi\sigma(a^2 - z^2)\, dz = \tfrac{4}{3}\pi a^3 \sigma \tag{3}$$

which could also be seen by noting that the volume of the sphere is $\frac{4}{3}\pi a^3$.

From (2) and (3) we have $I/M = \frac{2}{5}a^2$ or $I = \frac{2}{5}Ma^2$.

9.40. A cube of edge s and mass M is suspended vertically from one of its edges. (a) Show that the period for small vibrations is $P = 2\pi\sqrt[4]{2}\sqrt{s/3g}$. (b) What is the length of the equivalent simple pendulum?

(a) Since the diagonal of a square of side s has length $\sqrt{s^2 + s^2} = s\sqrt{2}$, the distance OC from axis O to the center of mass is $\frac{1}{2}s\sqrt{2}$.

The moment of inertia I of a cube about an edge is the same as that of a square plate about a side. Thus by Problem 9.6, $I = \frac{1}{3}M(s^2 + s^2) = \frac{2}{3}Ms^2$.

Then the period for small vibrations is, by Problem 9.25,

$$P = 2\pi\sqrt{\tfrac{2}{3}Ms^2/[Mg(\tfrac{1}{2}s\sqrt{2})]} = 2\pi\sqrt[4]{2}\sqrt{s/3g}$$

(b) The length of the equivalent simple pendulum is, by Problem 9.26,

$$l = \tfrac{2}{3}Ms^2/[M(\tfrac{1}{2}s\sqrt{2})] = \tfrac{2}{3}\sqrt{2}\,s$$

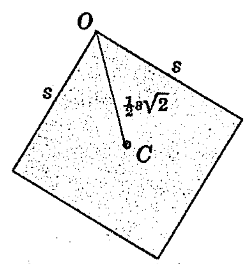


Fig. 9-32

9.41. Prove Theorem 9.11, page 228: The period of small vibrations of a compound pendulum is a minimum when the distance $OC = a$ is equal to the radius of gyration of the body about a horizontal axis through the center of mass.

If I_C is the moment of inertia about the center of mass axis and I_0 is the moment of inertia about the axis of suspension, then by the parallel axis theorem we have

$$I_0 = I_C + Ma^2$$

Then the square of the period for small vibrations is given by

$$P^2 = \frac{4\pi^2 I_0}{Mga} = \frac{4\pi^2}{g}\left(\frac{I_C}{Ma} + a\right) = \frac{4\pi^2}{g}\left(\frac{K_C^2}{a} + a\right)$$

where $K_C^2 = I_C/M$ is the square of the radius of gyration about the center of mass axis.

Setting the derivative of P^2 with respect to a equal to zero, we find

$$\frac{d}{da}(P^2) = \frac{4\pi^2}{g}\left(-\frac{K_C^2}{a^2} + 1\right) = 0$$

from which $a = K_C$. This can be shown to give the minimum value since $d^2(P^2)/da^2 < 0$. Thus the theorem is proved.

The theorem is also true even if the vibrations are not assumed small. See Problem 9.147.

9.42. A sphere of radius a and mass m rests on top of a fixed rough sphere of radius b. The first sphere is slightly displaced so that it rolls without slipping down the second sphere. Where will the first sphere leave the second sphere?

Let the xy plane be chosen so as to pass through the centers of the two spheres, with the center of the fixed sphere as origin O [see Fig. 9-33]. Let the position of the center of mass C of the first sphere be measured by angle θ, and suppose that the position vector of this center of mass C with respect to O is $\mathbf{r}$. Let $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$ be unit vectors as indicated in Fig. 9-33.

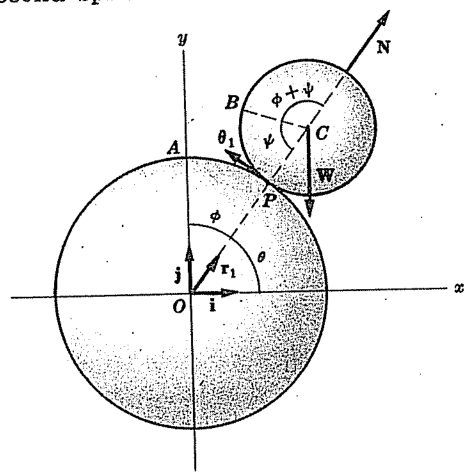


Fig. 9-33

Resolving the weight $\mathbf{W} = -mg\mathbf{j}$ into components in directions $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$, we have [compare Problem 1.43, page 24]

$$\begin{aligned}\mathbf{W} &= (\mathbf{W}\cdot\mathbf{r}_1)\mathbf{r}_1 + (\mathbf{W}\cdot\boldsymbol{\theta}_1)\boldsymbol{\theta}_1 \\ &= (-mg\mathbf{j}\cdot\mathbf{r}_1)\mathbf{r}_1 + (-mg\mathbf{j}\cdot\boldsymbol{\theta}_1)\boldsymbol{\theta}_1 \\ &= -mg\sin\theta\,\mathbf{r}_1 - mg\cos\theta\,\boldsymbol{\theta}_1\end{aligned}$$

The reaction force $\mathbf{N}$ and frictional force $\mathbf{f}$ are $\mathbf{N} = N\mathbf{r}_1$, $\mathbf{f} = f\boldsymbol{\theta}_1$. Using Theorem 9.12, page 228, together with the result of Problem 1.49, page 26, we have

$$\begin{aligned}\mathbf{F} = m\mathbf{a} &= m[(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1] \\ &= \mathbf{W} + \mathbf{N} + \mathbf{f} \\ &= (N - mg\sin\theta)\mathbf{r}_1 + (f - mg\cos\theta)\boldsymbol{\theta}_1\end{aligned}$$

from which $\quad m(\ddot{r} - r\dot{\theta}^2) = N - mg\sin\theta, \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = f - mg\cos\theta \quad$ (

Since $r = a + b$ [the distance of C from O], these equations become

$$-m(a+b)\dot{\theta}^2 = N - mg\sin\theta, \qquad m(a+b)\ddot{\theta} = f - mg\cos\theta$$

We now apply Theorem 9.13, page 229. The total external torque of all forces about t center of mass C is [since $\mathbf{W}$ and $\mathbf{N}$ pass through C],

$$\mathbf{\Lambda} = (-a\mathbf{r}_1)\times\mathbf{f} = (-a\mathbf{r}_1)\times(f\boldsymbol{\theta}_1) = -af\mathbf{k}$$

Also, the angular acceleration of the first sphere about C is

$$\boldsymbol{\alpha} = -\frac{d^2}{dt^2}(\phi + \psi)\mathbf{k} = -(\ddot{\phi} + \ddot{\psi})\mathbf{k}$$

Since there is only rolling and no slipping it follows that arc AP equals arc BP, or $b\phi =$ Then $\phi = \pi/2 - \theta$ and $\psi = (b/a)(\pi/2 - \theta)$, so that

$$\boldsymbol{\alpha} = -(\ddot{\phi} + \ddot{\psi})\mathbf{k} = -\left(-\ddot{\theta} - \frac{b}{a}\ddot{\theta}\right)\mathbf{k} = \left(\frac{a+b}{a}\right)\ddot{\theta}\mathbf{k}$$

Since the moment of inertia of the first sphere about the horizontal axis of rotation through C is $I = \frac{2}{5}ma^2$, we have by Theorem 9.13,

$$\mathbf{\Lambda} = I\boldsymbol{\alpha}, \qquad -af\mathbf{k} = \tfrac{2}{5}ma^2\left(\frac{a+b}{a}\right)\ddot{\theta}\mathbf{k} \qquad \text{or} \qquad f = -\tfrac{2}{5}m(a+b)\ddot{\theta}$$

Using this value of f in the second equation of (1), we find

$$\ddot{\theta} = -\frac{5g}{7(a+b)}\cos\theta \tag{2}$$

Multiplying both sides by $\dot{\theta}$ and integrating, we find after using the fact that $\dot{\theta} = 0$ at $t = 0$ or $\theta = \pi/2$,

$$\dot{\theta}^2 = \frac{10g}{7(a+b)}(1 - \sin\theta) \tag{3}$$

Using (3) in the first of equations (1), we find $N = \frac{1}{7}mg(17\sin\theta - 10)$. Then the first sphere leaves the second sphere where $N = 0$, i.e. where $\theta = \sin^{-1} 10/17$.

Supplementary Problems

RIGID BODIES

9.43. Show that the motion of region $\mathcal{R}$ of Fig. 9-34 can be carried into region $\mathcal{R}'$ by means of a translation plus a rotation about a suitable point.

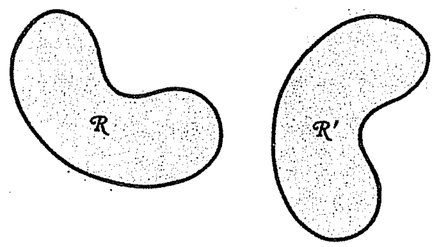


Fig. 9-34

9.44. Work Problem 9.1, page 230, by first applying a translation of the point A of triangle ABC.

9.45. If A_x, A_y, A_z represent rotations of a rigid body about the x, y and z axes respectively, is it true that the associative law applies, i.e. is $A_x + (A_y + A_z) = (A_x + A_y) + A_z$? Justify your answer.

MOMENTS OF INERTIA

9.46. Three particles of masses 3, 5 and 2 are located at the points $(-1,0,1)$, $(2,-1,3)$ and $(-2,2,1)$ respectively. Find (a) the moment of inertia and (b) the radius of gyration about the x axis. *Ans.* 71

9.47. Find the moment of inertia of the system of particles in Problem 9.46 about (a) the y axis, (b) the z axis. *Ans.* (a) 81, (b) 44

9.48. Find the moment of inertia of a uniform rod of length l about an axis perpendicular to it and passing through (a) the center of mass, (b) an end, (c) a point at distance $l/4$ from an end. *Ans.* (a) $\frac{1}{12}Ml^2$, (b) $\frac{1}{3}Ml^2$, (c) $\frac{7}{48}Ml^2$

9.49. Find the (a) moment of inertia and (b) radius of gyration of a square of side a about a diagonal. *Ans.* (a) $\frac{1}{12}Ma^2$, (b) $\frac{1}{6}a\sqrt{3}$

9.50. Find the moment of inertia of a cube of edge a about an edge. *Ans.* $\frac{2}{3}Ma^2$

9.51. Find the moment of inertia of a rectangular plate of sides a and b about a diagonal. *Ans.* $\frac{1}{6}Ma^2b^2/(a^2+b^2)$

9.52. Find the moment of inertia of a uniform parallelogram of sides a and b and included angle α about an axis perpendicular to it and passing through its center. *Ans.* $\frac{1}{12}M(a^2+b^2)\sin^2\alpha$

9.53. Find the moment of inertia of a cube of side a about a diagonal.

9.54. Find the moment of inertia of a cylinder of radius a and height h about an axis parallel to the axis of the cylinder and distant b from its center. *Ans.* $\frac{1}{2}M(a^2+2b^2)$

9.55. A solid of constant density is formed from a cylinder of radius a and height h and a hemisphere of radius a as shown in Fig. 9-35. Find its moment of inertia about a vertical axis through their centers. *Ans.* $M(2a^3+15a^2h)/(10a+15h)$

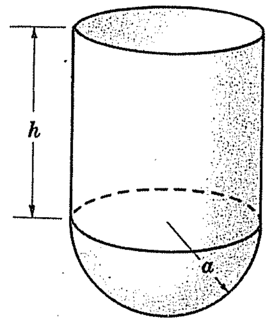


Fig. 9-35

9.56. Work Problem 9.55 if the cylinder is replaced by a cone of radius a and height h.

9.57. Find the moment of inertia of the uniform solid region bounded by the paraboloid $cz = x^2+y^2$ and the plane $z=h$ about the z axis. *Ans.* $\frac{1}{3}Mch$

9.58. How might you define the moment of inertia of a solid about (a) a point, (b) a plane? Is there any physical significance to these results? Explain.

9.59. Use your definitions in Problem 9.58 to find the moment of inertia of a cube of side a about (a) a vertex and (b) a face. *Ans.* (a) Ma^2, (b) $\frac{1}{3}Ma^2$

KINETIC ENERGY AND ANGULAR MOMENTUM

9.60. A uniform rod of length 2 ft and mass 6 lb rotates with angular speed 10 radians per second about an axis perpendicular to it and passing through its center. Find the kinetic energy of rotation. *Ans.* 100 lb ft^2/sec^2

9.61. Work Problem 9.60 if the axis of rotation is perpendicular to the rod and passes through an end. *Ans.* 400 lb ft^2/sec^2

9.62. A hollow cylindrical disk of radius a and mass M rolls along a horizontal plane with speed v. Find the total kinetic energy. *Ans.* Mv^2

9.63. Work Problem 9.62 for a solid cylindrical disk of radius a. *Ans.* $\frac{3}{4}Mv^2$

9.64. A flywheel having radius of gyration 2 meters and mass 10 kilograms rotates at angular speed of 5 radians/sec about an axis perpendicular to it through its center. Find the kinetic energy of rotation. *Ans.* 1000 joules

9.65. Find the angular momentum of (a) the rod of Problem 9.60 (b) the flywheel of Problem 9.64. *Ans.* (a) 5 lb ft^2/sec, (b) 200 kg m^2/sec

9.66. Prove the result of (a) Problem 9.15, page 236, (b) Problem 9.16, page 236, by using integration in place of summation.

9.67. Derive a "parallel axis theorem" for (a) kinetic energy and (b) angular momentum and explain the physical significance.

MOTION OF A RIGID BODY. THE COMPOUND PENDULUM. WORK, POWER AND IMPULSE

9.68. A constant force of magnitude F_0 is applied tangentially to a flywheel which can rotate about a fixed axis perpendicular to it and passing through its center. If the flywheel has radius a, radius of gyration K and mass M, prove that the angular acceleration is given by F_0a/MK^2.

9.69. How long will it be before the flywheel of Problem 9.68 reaches an angular speed ω_0 if it starts from rest? *Ans.* $MK^2\omega_0/F_0a$

9.70. Assuming that the flywheel of Problem 9.68 starts from rest, find (a) the total work done, (b) the total power developed and (c) the total impulse applied in getting the angular speed up to ω_0. *Ans.* (a) $\frac{1}{2}MK^2\omega_0^2$, (b) $F_0a\omega_0$, (c) $MK^2\omega_0$

9.71. Work (*a*) Problem 9.68, (*b*) Problem 9.69 and (*c*) Problem 9.70 if $F_0 = 10$ newtons, $a = 1$ meter, $K = 0.5$ meter, $M = 20$ kilograms and $\omega_0 = 20$ radians/sec.

Ans. (*a*) 2 rad/sec²; (*b*) 10 sec; (*c*) 250 joules, 200 joules/sec, 100 newton sec

9.72. Find the period of small vibrations for a simple pendulum assuming that the string supporting the bob is replaced by a uniform rod of length l and mass M while the bob has mass m.

Ans. $2\pi\sqrt{\dfrac{2(M+3m)l}{3(M+2m)g}}$

9.73. Discuss the cases (*a*) $M = 0$ and (*b*) $m = 0$ in Problem 9.72.

9.74. A rectangular plate having edges of lengths a and b respectively hangs vertically from the edge of length a. (*a*) Find the period for small oscillations and (*b*) the length of the equivalent simple pendulum. *Ans.* (*a*) $2\pi\sqrt{2b/3g}$, (*b*) $\frac{2}{3}b$

9.75. A uniform solid sphere of radius a and mass M is suspended vertically downward from a point on its surface. (*a*) Find the period for small oscillations in a plane and (*b*) the length of the equivalent simple pendulum. *Ans.* (*a*) $2\pi\sqrt{7a/5g}$, (*b*) $7a/5$

9.76. A yo-yo consists of a cylinder of mass 80 gm around which a string of length 60 cm is wound. If the end of the string is kept fixed and the yo-yo is allowed to fall vertically starting from rest, find its speed when it reaches the end of the string. *Ans.* 280 cm/sec

9.77. Find the tension in the string of Problem 9.76.

Ans. 19,600 dynes

9.78. A hollow cylindrical disk of mass M moving with constant speed v_0 comes to an incline of angle α. Prove that if there is no slipping it will rise a distance $v_0^2/(g \sin\alpha)$ up the incline.

9.79. If the hollow disk of Problem 9.78 is replaced by a solid disk, how high will it rise up the incline? *Ans.* $3v_0^2/(4g \sin\alpha)$

9.80. In Fig. 9-36 the pulley, assumed frictionless, has radius 0.2 meter and its radius of gyration is 0.1 meter. What is the acceleration of the 5 kg mass? *Ans.* 2.45 m/sec²

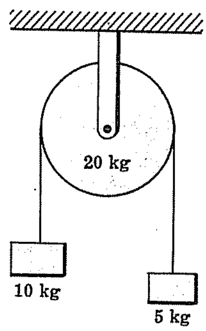


Fig. 9-36

INSTANTANEOUS CENTER. SPACE AND BODY CENTRODES

9.81. A ladder of length l moves so that one end is on a vertical wall and the other on a horizontal floor. Find (*a*) the space centrode and (*b*) the body centrode.

Ans. (*a*) A circle having radius l and center at point O where the floor and wall meet.
(*b*) A circle with the ladder as diameter

9.82. A long rod AB moves so that it remains in contact with the top of a post of height h while its foot B moves on a horizontal line CD [Fig. 9-37]. Assuming the motion to be in one plane, find the locus of instantaneous centers.

9.83. What is the (*a*) body centrode and (*b*) space centrode in Problem 9.82?

9.84. Work Problems 9.82 and 9.83 if the post is replaced by a fixed cylinder of radius a.

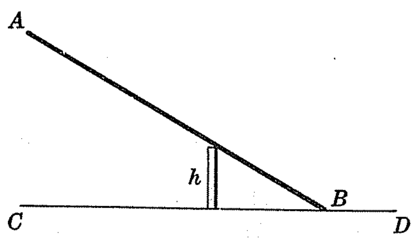


Fig. 9-37

STATICS OF A RIGID BODY

9.85. A uniform ladder of weight W and length l has its top against a smooth wall and its foot on a floor having coefficient of friction μ. (*a*) Find the smallest angle α which the ladder can make with the horizontal and still be in equilibrium. (*b*) Can equilibrium occur if $\mu = 0$? Explain.

9.86. Work Problem 9.85 if the wall has coefficient of friction μ_1.

9.87. In Fig. 9-38, AB is a uniform bar of length l and weight W supported at C. It carries weight at A and W_2 at D so that $AC = a$ and $CD = b$. Where must a weight W_3 be placed o so that the system will be in equilibrium?

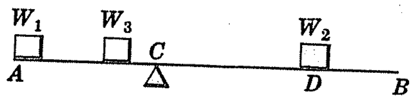


Fig. 9-38

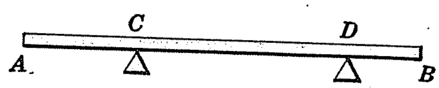


Fig. 9-39

9.88. A uniform triangular thin plate hangs from a fixed point O by strings OA, OB and O lengths a, b and c respectively. Prove that the tensions T_1, T_2 and T_3 in the strings are such $T_1/a = T_2/b = T_3/c$.

9.89. A uniform plank AB of length l and weight W is supported at points C and D distant a from A and b from B respectively [Fig. 9-39]. Determine the reaction forces at C and D respectively.

9.90. In Fig. 9-40, OA and OB are uniform rods having the same density and connected at O so that AOB is a right angle. The system is supported at O so that AOB is in a vertical plane. Find the angles α and β for which equilibrium occurs.
Ans. $\alpha = \tan^{-1}(a/b)$, $\beta = \pi/2 - \tan^{-1}(a/b)$

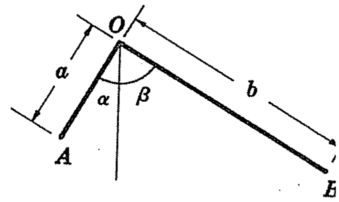


Fig. 9-40

MISCELLANEOUS PROBLEMS

9.91. A circular cylinder has radius a and height h. Prove that the moment of inertia about an a perpendicular to the axis of the cylinder and passing through the centroid is $\frac{1}{12}M(h^2 + 3a^2)$.

9.92. Prove that the effect of a force on a rigid body is not changed by shifting the force along its l of action.

9.93. A cylinder of radius a and radius of gyration K rolls without slipping down an inclined plane angle α and length l, starting from rest at the top of the incline. Prove that when it reaches t bottom of the incline is speed will be $\sqrt{(2gla^2 \sin\alpha)/(a^2 + K^2)}$.

9.94. A cylinder resting on top of a fixed cylinder is given a slight displacement so that it rolls witho slipping. Determine where it leaves the fixed cylinder.
Ans. $\theta = \sin^{-1} 4/7$ where θ is measured as in Fig. 9-33, page 244.

9.95. Work Problem 9.42 if the sphere is given an initial speed v_0.

9.96. Work Problem 9.94 if the cylinder is given an initial speed v_0.

9.97. A sphere of radius a and radius of gyration K about a diameter rolls without slipping dow an incline of angle α. Prove that it descends with constant acceleration given by $(ga^2 \sin\alpha)/(a^2 + K^2$

9.98. Work Problem 9.97 if the sphere is (*a*) solid, (*b*) hollow and of negligible thickness.
Ans. (*a*) $\frac{5}{7}g \sin\alpha$, (*b*) $\frac{3}{5}g \sin\alpha$

9.99. A hollow sphere has inner radius a and outer radius b. Prove that if M is its mass, then th moment of inertia about an axis through its center is

$$\frac{2}{5}M\left(\frac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{a^2 + ab + b^2}\right)$$

Discuss the cases $b = 0$ and $a = b$.

9.100. Wooden plates, all having the same rectangular shape are stacked one above the other as indicated in Fig. 9-41. (*a*) If the length of each plate is $2a$, prove that equilibrium conditions will prevail if the $(n+1)$th plate extends a maximum distance of a/n beyond the nth plate where $n = 1, 2, 3, \ldots$. (*b*) What is the maximum horizontal distance which can be reached if more and more plates are added?

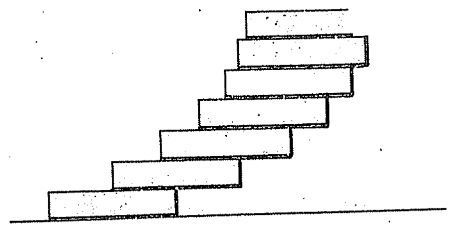


Fig. 9-41

9.101. Work Problem 9.100 if the plates are stacked on a sphere of radius R instead of on a flat surface as assumed in that problem.

9.102. A cylinder of radius a rolls on the inner surface of a smooth cylinder of radius $2a$. Prove that the period of small oscillations is $2\pi\sqrt{3a/2g}$.

9.103. A ladder of length l and negligible weight rests with one end against a wall having coefficient of friction μ_1 and the other end against a floor having coefficient of friction μ_2. It makes an angle α with the floor. (*a*) How far up the ladder can a man climb before the ladder slips? (*b*) What is the condition that the ladder not slip at all regardless of where the man is located?
Ans. (*a*) $\mu_2 l(\mu_1 + \tan\alpha)/(\mu_1\mu_2 + 1)$, (*b*) $\tan\alpha > 1/\mu_2$

9.104. Work Problem 9.103 if the weight of the ladder is not negligible.

9.105. A ladder AB of length l [Fig. 9-42] has one end A on an incline of angle α and the other end B on a vertical wall. The ladder is at rest and makes an angle β with the incline. If the wall is smooth and the incline has coefficient of friction μ, find the smallest value of μ so that a man of weight W_m will be able to climb the ladder without having it slip. Check your answer by obtaining the result of Problem 9.37, page 241, as a special case.

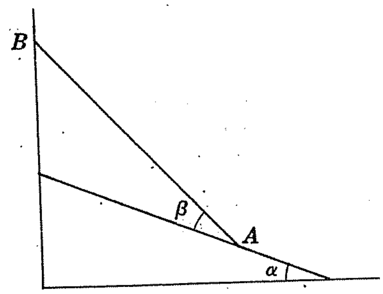


Fig. 9-42

9.106. Work Problem 9.105 if the wall has coefficient of friction μ_1.

9.107. A uniform rod AB with point A fixed rotates about a vertical axis so that it makes a constant angle α with the vertical [Fig. 9-43]. If the length of the rod is l, prove that the angular speed needed to do this is $\omega = \sqrt{(3g\sec\alpha)/2l}$.

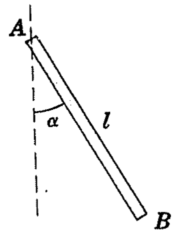


Fig. 9-43

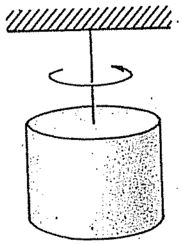


Fig. 9-44

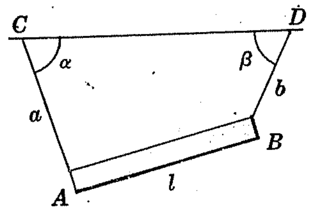


Fig. 9-45

9.108. A circular cylinder of mass m and radius a is suspended from the ceiling by a wire as shown in Fig. 9-44. The cylinder is given an angular twist θ_0 and is then released. If the torque is assumed proportional to the angle through which the cylinder is turned and the constant of proportionality is λ, prove that the cylinder will undergo simple harmonic motion with period $2\pi a\sqrt{m/2\lambda}$.

9.109. Find the period in Problem 9.108 if the cylinder is replaced by a sphere of radius a.
Ans. $2\pi a\sqrt{2m/5\lambda}$

9.110. Work (*a*) Problem 9.108 and (*b*) Problem 9.109 if damping proportional to the instantaneous angular velocity is present. Discuss physically.

9.111. A uniform beam AB of length l and weight W [Fig. 9-45] is supported by ropes AC and BD of lengths a and b respectively making angles α and β with the ceiling CD to which the ropes are fixed. If equilibrium conditions prevail, find the tensions in the ropes.

9.112. In Fig. 9-46 the mass m is attached to a rope which is wound around a fixed pulley of mass M and radius of gyration K which can rotate freely about O. If the mass is released from rest, find (a) the angular speed of the pulley after time t and (b) the tension in the rope.

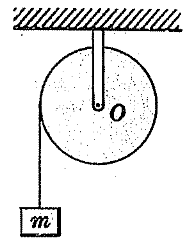


Fig. 9-46

9.113. Prove that the acceleration of the mass m in Problem 9.112 is $ga^2/(a^2+K^2)$.

9.114. Describe how Problem 9.112 can be used to determine the radius of gyration of a pulley.

9.115. A uniform rod AB [Fig. 9-47] of length l and weight W having its ends on a frictionless wall OA and floor OB respectively, slides starting from rest when its foot B is at a distance d from O. Prove that the other end A will leave the wall when the foot B is at a distance from O given by $\frac{1}{3}\sqrt{5l^2+4d^2}$.

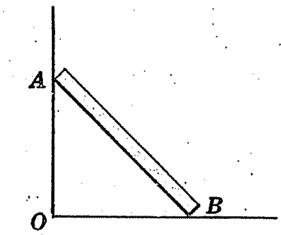


Fig. 9-47

9.116. A cylinder of mass 10 lb rotates about a fixed horizontal axis through its center and perpendicular to it. A rope wound around it carries a mass of 20 lb. Assuming that the mass starts from rest, find its speed after 5 seconds. *Ans.* 128 ft/sec

9.117. What must be the length of a rod suspended from one end so that it will be a seconds pendulum on making small vibrations in a plane? *Ans.* 149 cm

9.118. A solid sphere and a hollow sphere of the same radius both start from rest at the top of an inclined plane of angle α and roll without slipping down the incline. Which one gets to the bottom first? Explain. *Ans.* The solid sphere

9.119. A compound pendulum of mass M and radius of gyration K about a horizontal axis is displaced so that it makes an angle θ_0 with the vertical and is then released. Prove that if the center of mass is at distance a from the axis, then the reaction force on the axis is given by

$$\frac{Mg}{K^2+a^2}\sqrt{([K^2+2a^2]\cos\theta - a^2\cos\theta_0)^2 + (K^2\sin\theta)^2}$$

9.120. A rectangular parallelepiped of sides a, b, and c is suspended vertically from the side of length a. Find the period of small oscillations.

9.121. Find the least coefficient of friction needed to prevent the sliding of a circular hoop down an incline of angle α. *Ans.* $\frac{2}{3}\tan\alpha$

9.122. Find the period of small vibrations of a rod of length l suspended vertically about a point $\frac{1}{3}$ from one end.

9.123. A pulley system consists of two solid disks of radius r_1 and r_2 respectively rigidly attached to each other and capable of rotating freely about a fixed horizontal axis through the center O. A weight W is suspended from a string wound around the smaller disk as shown in Fig. 9-48. If the radius of gyration of the pulley system is K and its weight is w, find (a) the angular acceleration with which the weight descends and (b) the tension in the string.
Ans. (a) $Wgr_1/(Wr_1^2+wK^2)$, (b) $WwK^2/(Wr_1^2+wK^2)$

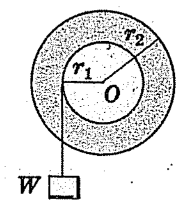


Fig. 9-48

9.124. A solid sphere of radius b rolls on the inside of a smooth hollow sphere of radius a. Prove that the period for small oscillations is given by $2\pi\sqrt{7(a-b)/5g}$.

9.125. A thin circular solid plate of radius a is suspended vertically from a horizontal axis passing through a chord AB [see Fig. 9-49]. If it makes small oscillations about this axis, prove that the frequency of such oscillations is greatest when AB is at distance $a/2$ from the center.

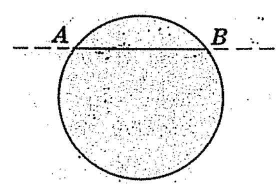


Fig. 9-49

9.126. A uniform rod of length $5l$ is suspended vertically from a string of length $2l$ which has its other end fixed. Prove that the normal frequencies for small oscillations in a plane are $\frac{1}{2\pi}\sqrt{\frac{g}{5l}}$ and $\frac{1}{2\pi}\sqrt{\frac{3g}{l}}$, and describe the normal modes.

9.127. A uniform rod of mass m and length l is suspended from one of its ends. What is the minimum speed with which the other end should be hit so that it will describe a complete vertical circle?

9.128. (a) If the bob of a simple pendulum is a uniform solid sphere of radius a rather than a point mass, prove that the period for small oscillations is $2\pi\sqrt{l/g + 2a^2/5gl}$.

(b) For what value of l is the period in (a) a minimum?

9.129. A sphere of radius a and mass M rolls along a horizontal plane with constant speed v_0. It comes to an incline of angle α. Assuming that it rolls without slipping, how far up the incline will it travel? *Ans.* $10v_0^2/(7g \sin \alpha)$

9.130. Prove that the doughnut shaped solid or *torus* of Fig. 9-50 has a moment of inertia about its axis given by $\frac{1}{4}M(3a^2 + 4b^2)$.

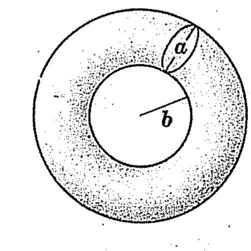


Fig. 9-50

9.131. A cylinder of mass m and radius a rolls without slipping down a 45° inclined plane of mass M which is on a horizontal frictionless table. Prove that while the rolling takes place the incline will move with an acceleration given by $mg/(3M + 2m)$.

9.132. Work Problem 9.131 if the incline is of angle α. *Ans.* $(mg \sin 2\alpha)/(3M + 2m - m \cos 2\alpha)$

9.133. Find the (a) tension in the rope and (b) acceleration of the system shown in Fig. 9-51 if the radius of gyration of the pulley is 0.5 m and its mass is 20 kg.

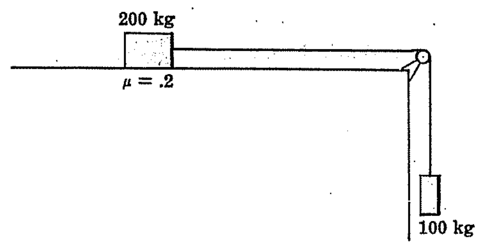


Fig. 9-51

9.134. Compare the result of Problem 9.133 with that obtained assuming the pulley to have negligible mass.

9.135. Prove that if the net external torque about an axis is zero, then it is also zero about any other axis.

9.136. A solid cylindrical disk of radius a has a circular hole of radius b whose center is at distance c from the center of the disk. If the disk rolls down an inclined plane of angle α, find its acceleration. [See Fig. 9-52.]

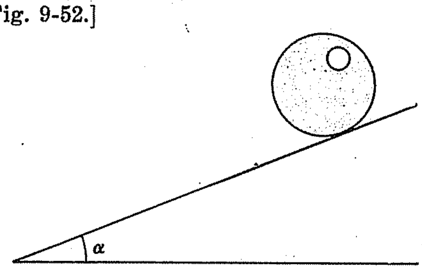


Fig. 9-52

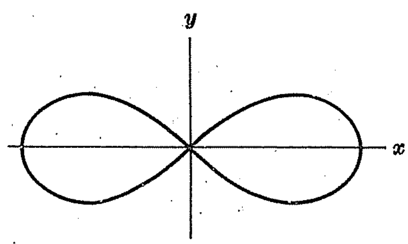


Fig. 9-53

9.137. Find the moment of inertia of the region bounded by the lemniscate $r^2 = a^2 \cos 2\theta$ [see Fig. 9-53] about the x axis. *Ans.* $Ma^2(3\pi - 8)/48$

9.138. Find the largest angle of an inclined plane down which a solid cylinder will roll without slipping if the coefficient of friction is μ.

9.139. Work Problem 9.138 for a solid sphere.

9.140. Discuss the motion of a hollow cylinder of inner radius a and outer radius b as it rolls down an inclined plane of angle α.

9.141. A table top of negligible weight has the form of an equilateral triangle ABC of side s. The legs of the table are perpendicular to the table top at the vertices. A heavy weight W is placed on the table top at a point which is distant a from side BC and b from side AC. Find that part of the weight supported by the legs at A, B and C respectively.

Ans. $\dfrac{2Wa}{s\sqrt{3}},\ \dfrac{2Wb}{s\sqrt{3}},\ W\left(1-\dfrac{2a+2b}{s\sqrt{3}}\right)$

9.142. Discuss the motion of the disk of Problem 9.136 down the inclined plane if the coefficient of friction is μ.

9.143. A hill has a cross section in the form of a cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1-\cos\theta)$$

as indicated in Fig. 9-54. A solid sphere of radius b starting from rest at the top of the hill is given a slight displacement so that it rolls without slipping down the hill. Find the speed of its center when it reaches the bottom of the hill.
Ans. $\sqrt{10g(2a-b)/7}$

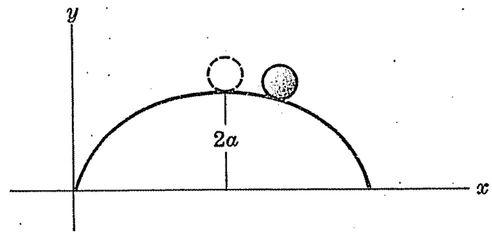


Fig. 9-54

9.144. Work Problem 3.108, page 85, if the masses and moments of inertia of the pulleys are taken into account.

9.145. Work Problem 9.38, page 242, if friction is taken into account.

9.146. A uniform rod of length l is placed upright on a table and then allowed to fall. Assuming that its point of contact with the table does not move, prove that its angular velocity at the instant when it makes an angle θ with the vertical is given in magnitude by $\sqrt{3g(1-\cos\theta)/2l}$.

9.147. Prove Theorem 9.11, page 228, for the case where the vibrations are not necessarily small. Compare Problem 9.41, page 243.

9.148. A rigid body moves parallel to a given fixed plane. Prove that there is one and only one point of the rigid body where the instantaneous acceleration is zero.

9.149. A solid hemisphere of radius a rests with its convex surface on a horizontal table. If it is displaced slightly, prove that it will undergo oscillations with period equal to that of a simple pendulum of equivalent length $4a/3$.

9.150. A solid cylinder of radius a and height h is suspended from axis AB as indicated in Fig. 9-55. Find the period of small oscillations about this axis.

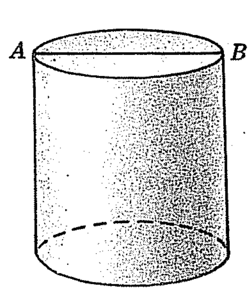


Fig. 9-55

9.151. Prove that a solid sphere will roll without slipping down an inclined plane of angle α if the coefficient of friction is at least $\frac{2}{7}\tan\alpha$.

9.152. Find the least coefficient of friction for an inclined plane of angle α in order that a solid cylinder will roll down it without slipping.
Ans. $\frac{1}{3}\tan\alpha$

Symmetrical Versus Asymmetrical Bodies (Optional)

How does the situation differ for symmetrical and asymmetrical rotating bodies? Suppose the rod connecting the two particles in the symmetrical body of Fig. 10-7 were inclined at an arbitrary angle β with respect to the central shaft. Figure 10-11 shows the connecting rod, the shaft, and the two bearings (assumed frictionless) that holds the shaft along the z axis. The shaft rotates at a constant angular speed ω about this axis, the vector $\vec{\omega}$ thus pointing along this axis. Experience tells us that such a system is "unbalanced" or "lopsided," and if the connecting rod were not rigidly fastened to the vertical shaft near O, it would tend to move until the angle β became 90°, in which position the system would then be symmetrical about the shaft.

At the instant shown in Fig. 10-11, the upper particle is moving into the page at right angles to it, and the lower particle is moving out of the page at right angles to it. The linear momentum vectors of the two particles are therefore equal but opposite, and so are their position vectors with respect to O. Hence, by application of the right-hand rule in $\vec{r} \times \vec{p}$, we find that $\vec{l}$ is the same for each particle and that their sum, the total angular momentum vector $\vec{L}$ of the system, is as shown in the figure, at right angles to the connecting rod and in the plane of the page. Hence $\vec{L}$ and $\vec{\omega}$ are not parallel at this instant. As the system rotates, the angular momentum vector, while constant in magnitude, rotates around the fixed axis of rotation.

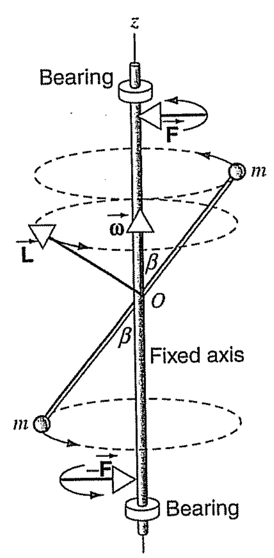


FIGURE 10-11. A rotating two-particle system, similar to Fig. 10-7, but with the axis of rotation making an angle β with the connecting rod. The angular momentum vector $\vec{L}$ rotates with the system, as do the forces $\vec{F}$ and $-\vec{F}$ exerted by the bearings.

The rotation of $\vec{L}$ about the fixed axis of Fig. 10-11 is perfectly consistent with the fundamental relation $\vec{\tau} = d\vec{L}/dt$. The external torque on the entire system arises from the unbalanced sideways forces exerted by the bearings on the shaft and transmitted by the shaft to the connecting rod. At the instant shown in the figure, the upper particle would tend to move outward to the right. The shaft would be pulled to the right against the upper bearing, which in turn exerts a force $\vec{F}$ on the shaft that points to the left. Similarly, the lower particle tends to move outward to the left. The shaft would be pulled to the left against the lower bearing, which in turn exerts a force $-\vec{F}$ on the shaft that points to the right. The torque $\vec{\tau}$ about O as a result of these forces points perpendicularly out of the page, at right angles to the plane formed by $\vec{L}$ and $\vec{\omega}$, and in the right direction to account for the rotary motion of $\vec{L}$. (Compare with Fig. 10-6*c*, in which $\vec{\tau}$ was also perpendicular to the plane formed by $\vec{l}$ and $\vec{\omega}$.) Note that because $\vec{\tau}$ is perpendicular to $\vec{\omega}$, there is no component of the angular acceleration $\vec{\alpha}$ in the direction of $\vec{\omega}$, and so the angular velocity remains constant. In the absence of friction, the system will spin forever. Friction in the bearings would give rise to a torque directed along the shaft (parallel to $\vec{\omega}$), which *would* have an angular acceleration component along $\vec{\omega}$ and thus would change the angular velocity.

The forces $\vec{F}$ and $-\vec{F}$ lie in the plane of Fig. 10-11 at the instant shown. As the system rotates, these forces, and therefore the torque τ, rotate with it, so that $\vec{\tau}$ always remains at right angles to the plane formed by $\vec{\omega}$ and $\vec{L}$. The rotating forces $\vec{F}$ and $-\vec{F}$ cause a wobble in the upper and lower bearings. The bearings and their supports must be made strong enough to provide these forces. For a symmetrical rotating body there is no bearing wobble, and the shaft rotates smoothly. ■

7.5 Dynamical Balance of a Rigid Body

The formulation of the equations of motion in a rotating reference system is also quite valuable in the description of rigid-body motion. As an introduction to the general treatment of rigid-body rotational motion, we discuss a simple example of a dumbbell formed by two point masses m at the ends of a massless rod of length l. The dumbbell rotates at a fixed inclination θ with constant angular velocity ω about a pivot at the center of the rod, as shown in Fig. 7-11. The equation of motion (6.48) in a fixed reference frame for rotation about the pivot of the rod is

$$\mathrm{N} = \frac{d\mathbf{L}}{dt} \tag{7.57}$$

where N is the external torque on the rod applied at the pivot. The angular momentum is given by

$$\mathbf{L} = m(\mathbf{r}_1 \times \mathbf{v}_1 + \mathbf{r}_2 \times \mathbf{v}_2) \tag{7.58}$$

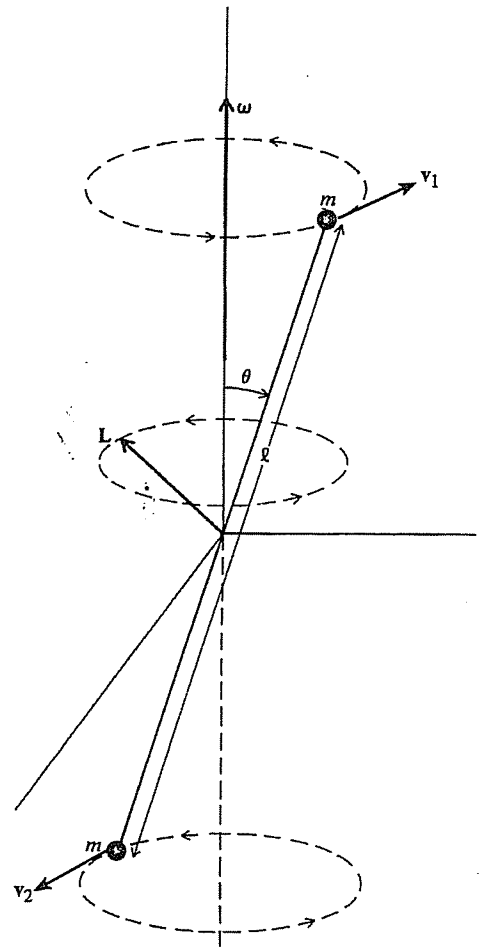


FIGURE 7-11. Dumbbell rotating about a pivot at center of the rod at a fixed inclination angle θ.

Since $\mathbf{r}_2 = -\mathbf{r}_1$ and $\mathbf{v}_1 = \boldsymbol{\omega} \times \mathbf{r}_1$ and $\mathbf{v}_2 = \boldsymbol{\omega} \times \mathbf{r}_2$ we have

$$\mathbf{v}_2 = -\mathbf{v}_1 \tag{7.59}$$

Thus $\mathbf{L}$ can be expressed in terms of $\boldsymbol{\omega}$ and $\mathbf{r}_1$ as

$$\mathbf{L} = 2m\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{r}_1) = 2m\left[\boldsymbol{\omega} r_1^2 - \mathbf{r}_1(\boldsymbol{\omega} \cdot \mathbf{r}_1)\right] \tag{7.60}$$

Since $\mathbf{L}$ is perpendicular to $\mathbf{r}_1$ and lies in the plane determined by $\mathbf{r}_1$ and $\boldsymbol{\omega}$, it also rotates with angular velocity $\boldsymbol{\omega}$. From (7.8), we then have

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega}\times\mathbf{L} \tag{7.61}$$

The external torque from (7.57), (7.60), and (7.61) necessary to maintain the rotation is

$$\mathrm{N} = \boldsymbol{\omega}\times\mathbf{L} = 2m(\mathbf{r}_1\times\boldsymbol{\omega})(\mathbf{r}_1\cdot\boldsymbol{\omega}) \tag{7.62}$$

We can alternatively derive the result in (7.62) in a coordinate frame which rotates with the dumbbell. In a rotating reference frame the following rigid-body equation of motion can be derived from (7.16) to (7.21):

$$\frac{\delta\mathbf{L}}{\delta t} = \mathrm{N} + \sum_i \mathbf{r}_i\times(\mathbf{F}^i_{cf} + \mathbf{F}^i_{Cor} + \mathbf{F}^i_{az} + \mathbf{F}^i_{tr}) \tag{7.63}$$

In a coordinate frame rotating with the dumbbell, $\delta\mathbf{L}/\delta t = 0$ and $\mathbf{F}^i_{Cor} = \mathbf{F}^i_{az} = \mathbf{F}^i_{tr} = 0$. Hence, to maintain the rotation, the torque applied at the pivot must balance the torque due to the centrifugal forces.

$$\mathrm{N} = -(\mathbf{r}_1\times\mathbf{F}^1_{cf} + \mathbf{r}_2\times\mathbf{F}^2_{cf}) \tag{7.64}$$

From (7.18) the centrifugal forces are given by

$$\begin{aligned}\mathbf{F}^1_{cf} &= -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1)\\ \mathbf{F}^2_{cf} &= -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2)\end{aligned} \tag{7.65}$$

Using $\mathbf{r}_1 = -\mathbf{r}_2$, the torque reduces to

$$\begin{aligned}\mathrm{N} &= 2m\mathbf{r}_1\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1)]\\ &= 2m\mathbf{r}_1\times\left[\boldsymbol{\omega}(\boldsymbol{\omega}\cdot\mathbf{r}_1) - \mathbf{r}_1\omega^2\right]\\ &= 2m(\mathbf{r}_1\times\boldsymbol{\omega})(\mathbf{r}_1\cdot\boldsymbol{\omega})\end{aligned} \tag{7.66}$$

in agreement with the result in (7.62).

In terms of the angle θ between $\boldsymbol{\omega}$ and $\mathbf{r}_1$, the angular momentum and torque in (7.60) and (7.62) of the rotating dumbbell can be written

$$\mathbf{L} = \tfrac{1}{2}ml^2\omega(\hat{\boldsymbol{\omega}} - \cos\theta\,\hat{\mathbf{r}}_1) \tag{7.67}$$

$$\mathrm{N} = \tfrac{1}{2}ml^2\omega\sin\theta\cos\theta\,\hat{\mathbf{n}} \tag{7.68}$$

where $\hat{\mathbf{n}} = \mathbf{r}_1\times\boldsymbol{\omega}/|\mathbf{r}_1\times\boldsymbol{\omega}|$. For $\theta = \pi/2$, we find

$$\begin{aligned}\mathbf{L} &= (\tfrac{1}{2}ml^2)\boldsymbol{\omega}\\ \mathrm{N} &= 0\end{aligned} \tag{7.69}$$

In this orientation the motion does not require an imposed torque.

From (7.67) and (7.68), we see that torques on the rod are present whenever the angular momentum $\mathbf{L}$ does not lie along the axis of rotation $\boldsymbol{\omega}$. This result is generally true for rigid-body rotations.

A practical application in which it is important that $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel is the dynamic balance of automobile tires. If a wheel is not balanced, noise and vibration result in the car and excessive wear occurs on the tire. There are two criteria for complete balance of a wheel:

(1) *Static balance:* Unless the CM of the wheel lies on the rotation axis, a time-varying centrifugal force is present. This acts to make the axle oscillate and imparts vibration to the car. In a static balance the wheel is removed from the car and mounted on a vertical axis. Weights are attached around the rim of the wheel until the wheel is in equilibrium in a horizontal plane.

(2) *Dynamic balance:* Even when the CM lies on the wheel axis, it is possible that in rotation the angular momentum does not lie along the axis. If we specify the x axis as the rotation axis, $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$, the angular-momentum vector from (6.105) is

$$\mathbf{L} = (I_{xx}\hat{\mathbf{x}} + I_{yx}\hat{\mathbf{y}} + I_{zx}\hat{\mathbf{z}})\omega \tag{7.70}$$

Unless the products of inertia I_{yx} and I_{zx} vanish, $\mathbf{L}$ does not lie along $\boldsymbol{\omega}$. The time variation of $\mathbf{L}$ then leads to a time-varying torque, causing the wheel to wobble. A dynamic balance consists of the application of weights until the wheel spins smoothly with no wobble. Since modern tires are usually very nearly symmetrical, a static balance alone is often sufficient to ensure good driving results.

7.6 Principal Axes and Euler's Equations

For a rigid body of arbitrary shape, the rotational equation of motion (6.48) in a fixed coordinate system or with origin at the center-of-mass point is

$$N_j = \frac{dL_j}{dt} = \frac{d}{dt}(I_{jk}\omega_k) \tag{7.71}$$

where a sum over the index k is implied. Since the moments and products of inertia I_{jk} relative to the fixed coordinate system change as a function of time as the body rotates, the description of the motion through (7.71) can be cumbersome and difficult. The analysis of the motion can often be greatly simplified by choosing instead a *body-fixed* coordinate system that rotates with the body. In this reference frame the moments and products of inertia are time-independent. Using (7.8) and (7.71), the equation of motion with respect to the moving body axes is

$$N_j = \frac{\delta L_j}{\delta t} + (\boldsymbol{\omega}\times\mathbf{L})_j \tag{7.72}$$

A further simplification can be made by a judicious choice of the orientations of the rotating axes with respect to the rigid body. As we shall shortly prove, it is always possible to make a choice of axes in the body for which all the products of inertia vanish.

$$I_{ij} = 0 \qquad \text{for } i \neq j \tag{7.73}$$

The axes for which (7.73) holds are called the *principal axes* of the rigid body. For these axes the angular-momentum components in (6.105) reduce to

$$\begin{aligned} L_1 &= I_{11}\omega_1 \equiv I_1\omega_1 \\ L_2 &= I_{22}\omega_2 \equiv I_2\omega_2 \\ L_3 &= I_{33}\omega_3 \equiv I_3\omega_3 \end{aligned} \tag{7.74}$$

where I_1, I_2, I_3 denote the principal moments of inertia. From (7.72), expressing the cross product in cartesian coordinates, we obtain *Euler's equations of motion* for a rigid body in terms of the coordinate system aligned with the principal axes of the body.

$$\begin{aligned} N_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \\ N_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ N_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 \end{aligned} \tag{7.75}$$

It should be emphasized that the angular velocity and torque components appearing above refer to the $\boldsymbol{\omega}$ and $\mathbf{N}$ vectors of the inertial system projected onto the principal body axes. The Euler equations are a convenient starting point for many discussions of rigid body rotations.

To illustrate the application of Euler's equations, we return to the rotating rod of the preceding section. The principal axes of the body lie along and perpendicular to the rod, as illustrated in Fig. 7-12. With the z axis along the rod and the x axis in the plane of the rod and $\boldsymbol{\omega}$, the components of $\boldsymbol{\omega}$ are

$$\begin{aligned} \omega_1 &= \omega\sin\theta \\ \omega_2 &= 0 \\ \omega_3 &= \omega\cos\theta \end{aligned} \tag{7.76}$$

where θ is the angle between $\boldsymbol{\omega}$ and the rod. The principal moments of inertia are

$$\begin{aligned} I_1 = I_2 &= m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = \tfrac{1}{2}m\ell^2 \\ I_3 &= 0 \end{aligned} \tag{7.77}$$

Using (7.76) and (7.77) in (7.75), we find

$$\begin{aligned} N_1 &= 0 \\ N_2 &= \left(\tfrac{1}{2}m\ell^2\right)\omega^2\sin\theta\cos\theta \\ N_3 &= 0 \end{aligned} \tag{7.78}$$

where $\dot{\omega} = 0$ has been used. This result obtained from Euler's equations is the same as (7.68).

In the derivation of (7.75) we have used the diagonal property in (7.73) of the inertia tensor in the principal-axes coordinate system. We will now establish this property. Suppose that there exists a direction in space $\boldsymbol{\omega}$ for which $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$

$$\mathbf{L} = I\boldsymbol{\omega} \tag{7.79}$$

If such a direction can be found it will by definition be a principal axis since the products of inertia vanish and the principal moment is I. In the original coordinate system, $\boldsymbol{\omega}$ will in general have three components:

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}} \tag{7.80}$$

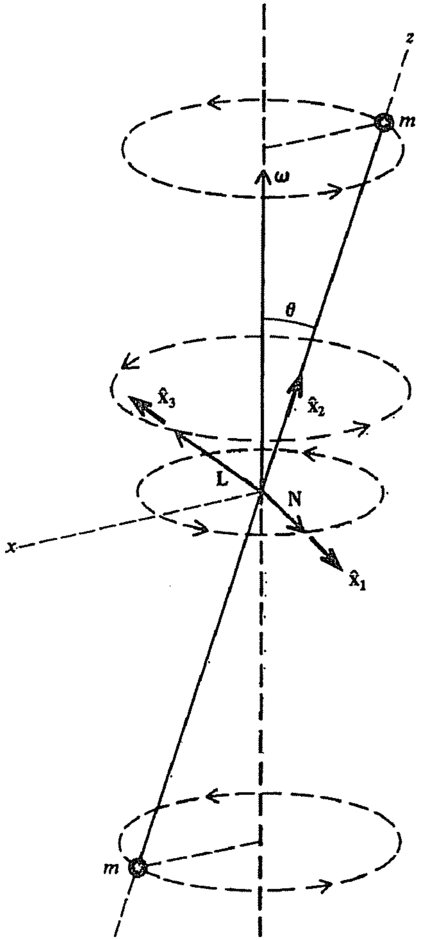


FIGURE 7-12. Principal axes $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ of the dumbbell.

From (7.79) the components of **L** along the inertial axes are

$$\begin{aligned} L_1 &= I\omega_1 \\ L_2 &= I\omega_2 \\ L_3 &= I\omega_3 \end{aligned} \tag{7.81}$$

These components of **L** must be equivalent to the expression for **L** given in (6.105), namely,

$$\begin{aligned} L_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned} \tag{7.82}$$

Equating the components in (7.81) and (7.82), we find

$$\begin{aligned} (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 &= 0 \end{aligned} \tag{7.83}$$

which in vector notation is

$$\sum_{j=1}^{3} I_{ij}\omega_j = I\omega_i \tag{7.84}$$

or

$$\mathbb{I} \cdot \boldsymbol{\omega} = I\boldsymbol{\omega} \tag{7.85}$$

For $\boldsymbol{\omega} \neq 0$, this system of homogeneous equations for $(\omega_1, \omega_2, \omega_3)$ has solutions only if the determinant of the coefficients of the $\boldsymbol{\omega}$ components vanishes.

$$\begin{vmatrix} (I_{11} - I) & I_{12} & I_{13} \\ I_{21} & (I_{22} - I) & I_{23} \\ I_{31} & I_{32} & (I_{33} - I) \end{vmatrix} = 0 \tag{7.86}$$

This leads to a cubic equation in I_1 of the form

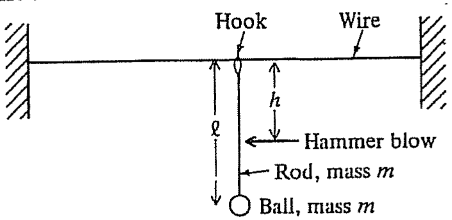
$$I^3 + aI^2 + bI + c = 0 \tag{7.87}$$

where a, b, c are products of the inertia tensor elements I_{ij}. There are three real solutions for I from (7.87), any is appropriate to (7.79). The other two solutions refer to the principal moments about two other orthogonal principal axes. The detailed proofs are outlined in the exercises. By construction we have therefore shown that it is always possible to find a principal-axis system for any rigid body. In many applications the choice of principal axes is obvious from the symmetry of the body. If two of the principal moments are equal the body is called *rotation symmetric* about the third axis in that plane. If all three principal moments are equal the body is *rotation isotropic*.

6.10 Impulses and Billiard Shots

6-28. A pencil of length ℓ and mass m lying flat on a frictionless horizontal tabletop receives an impulse on one end at a right angle to the pencil. What is the orientation and position of the center of mass of the pencil at a time t after the impulse?

6-29. A rod of mass m and length ℓ hangs vertically from a horizontal frictionless wire, as shown. Attached to the end of the rod is a small ball also of mass m. The rod is free to move along the wire.



a) Find the location of the center of mass for rod plus ball, taking the hook as the origin of the coordinate system.

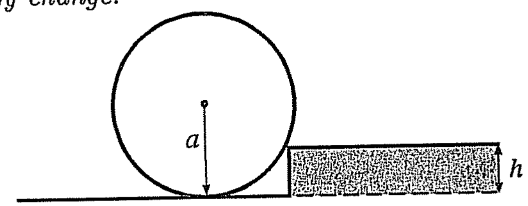
b) Find the moment of inertia for rod plus ball about the hook.

c) Use the parallel-axis theorem and the result in part *b* to find the moment of inertia about the center of mass.

d) The rod-ball system is struck by an impulsive hammer blow a distance h from the hook. Set up equations for the linear and angular motion of the system.

e) Find h such that the hook does not move along the wire at the instant of blow. This point is known as the *center of percussion.*

6-30. A ball of radius a rolling with velocity v on a level surface collides inelastically with a step of height $h < a$, as shown. Find the minimum velocity for which the ball will "trip" up over the step. Assume that no slipping occurs at the impact point. *Hint: compute the total angular momentum about the point of impulsive contact. This angular momentum is conserved and can be used to compute the energy change.*



6-31. In the sport of bowling, if the ball is rolled straight down the middle of the alley, pins on the sides will often be left standing (wide splits). A good right-handed bowler will impart a spin to the ball on release, causing it to curve to the left as it goes down the alley and strike the pins somewhat to the side. Describe the required spin and show that the trajectory of the ball is approximately parabolic in shape.

6.11 Super-Ball Bounces

6-32. One of the Super-Ball examples discussed in the text concerned a ball dropped straight down with spin. Discuss the subsequent motion through several bounces.

6-33. Under what conditions will a Super-Ball bounce back and forth as illustrated? How does the spin change?

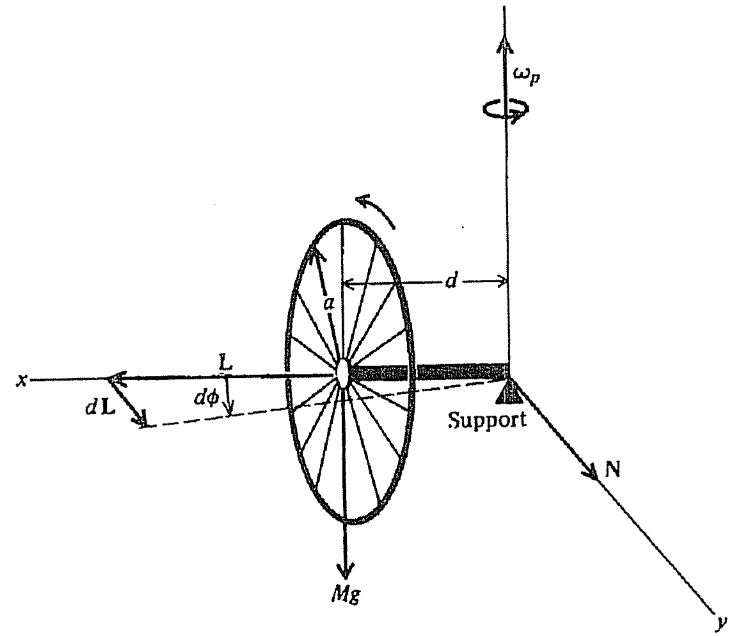


FIGURE 6-6. Gyroscope effect for a wheel with massive rim.

to first order dt. The direction of **L** is rotated counterclockwise, viewed from above in the x, y plane, through an angle $d\phi$ given by

$$d\phi = \frac{dL}{L} = \frac{N}{L}dt \tag{6.76}$$

If N remains perpendicular to **L** and in the x, y plane, the angular velocity of precession about the z axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{N}{L} \tag{6.77}$$

This result known as *simple precession*, since we neglected the angular momentum associated with the precession motion. Whenever the applied torque is small or the spin large, simple precession is a good approximation. When the precession angular momentum is taken into account in the description of the motion, an oscillation called *nutation* about the x, y plane may be present, in addition to the precessional motion.

A popular lecture demonstration experiment that illustrates simple precession uses a bicycle rim loaded with lead. The wheel is oriented in a vertical plane as in Fig. 6-6. The suspension point is located a distance d from the plane of the wheel along the wheel axis. The weight of the wheel then supplies the torque.

$$\mathbf{N} = (Mgd)\hat{\mathbf{y}} \tag{6.78}$$

If the wheel has radius a and mass M and spins with angular velocity ω, the angular momentum is

$$\mathbf{L} = (M\omega a^2)\hat{\mathbf{x}} \tag{6.79}$$

The resulting angular velocity of precession about the z axis is

$$\omega_P = \frac{gd}{\omega a^2} \tag{6.80}$$

where we have used (6.77) to (6.79). For $a = d = 0.3$m and a spin rate of $\omega/2\pi = 200$ r/min, we find precession rate of

$$\frac{\omega_P}{2\pi} = \frac{9.8 \times 3,600}{200(2\pi)^2(0.3)} = 15\text{r/min} \tag{6.81}$$

6.6 The Boomerang

An explanation of why a boomerang returns can be given in terms of the gyroscope effect. The boomerang can take on a variety of shapes. In its most common form it appears as two airfoil-shaped blades meeting at an angle near 90°, as illustrated in Fig. 6-7. However, the characteristic banana-like shape of most boomerangs has little to do with their ability to return. Another version consists of two crossed blades, as shown in Fig. 6-8. The boomerang is thrown overhand in a nearly vertical plane in the manner of Fig. 6-9. As it leaves the hand, the blades are rapidly rotating about the CM, and the CM is moving parallel to the ground. Due to its spin, the boomerang has an angular momentum about the CM that is initially directed to the left, as shown in Fig. 6-10.

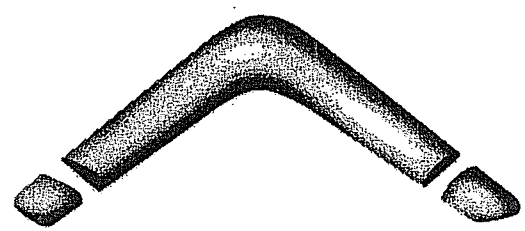


FIGURE 6-7. Common boomerang.

The aerodynamic "lift" forces on the airfoils act perpendicular to the plane of rotation, as indicated in Fig. 6-11. The total aerodynamic force on the boomerang accelerates it perpendicular to the plane of rotation, in the direction of **L**. An upper blade of the boomerang moves more rapidly through the air than a lower blade because the rotation and translation velocities add on the upper blade and subtract on the lower blade. Since the aerodynamic force is larger for higher blade velocities, the upper blade experiences a greater force, and an external torque about the CM is generated by the forces on the airfoils. The torque points opposite the CM velocity direction. Thus the initial directions of **N** and **L** are identical with those for the wheel in Fig. 6-6. From the gyroscope effect discussed in § 6.5 we predict that the plane of rotation precesses counterclockwise about the vertical axis. This precession of the rotational plane accompanied by translational acceleration perpendicular to the rotational plane allows the boomerang to travel in a circular orbit and return to the thrower; see the illustration in Fig. 6-12.

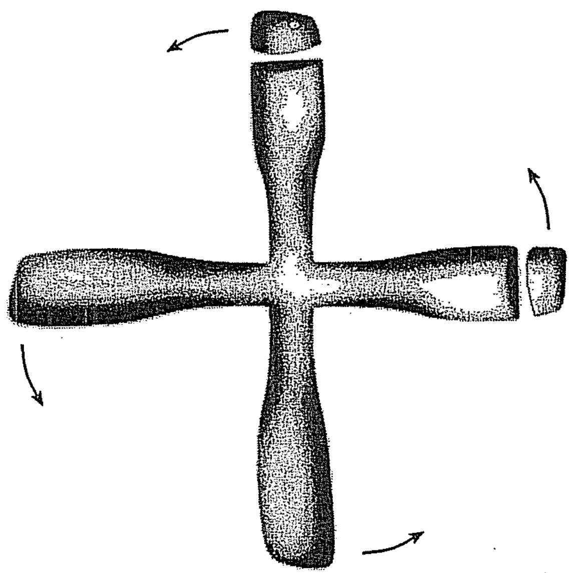


FIGURE 6-8. Cross-blade boomerang.

To discuss the flight of the boomerang in a more quantitative fashion, we consider the crossed-blade boomerang of Fig. 6-8. In this case the CM lies at the blade hub, which simplifies the analysis considerably. We choose the origin of our coordinate system at the hub. Initially, we take

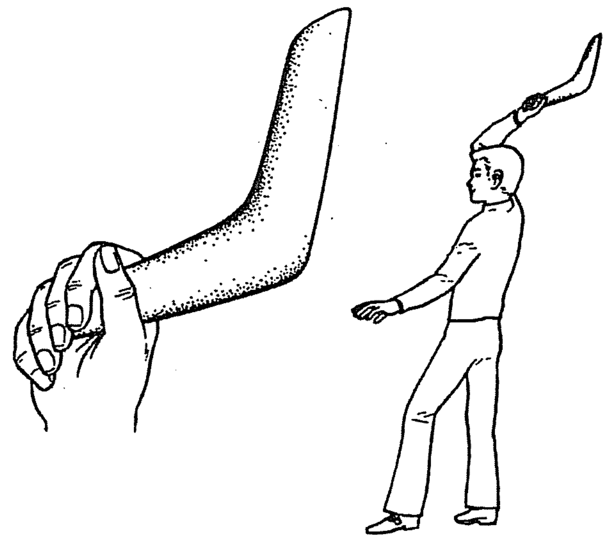


FIGURE 6-9. Proper method of throwing a boomerang.

the CM motion along the negative y axis with velocity $\mathbf{V} = -V\hat{\mathbf{y}}$, as in Fig. 6-10. Rotation occurs around the x axis in a counterclockwise sense with angular velocity ω. One of the blades with length l, mass $\frac{1}{4}M$, and linear mass density $\mu = \frac{1}{4}(M/l)$ is depicted in Fig. 6-13. A point on the blade at a distance r from the CM is specified as a function of time by

$$\mathbf{r} = r(\hat{\mathbf{y}}\cos\omega t + \hat{\mathbf{z}}\sin\omega t) \tag{6.82}$$

The aerodynamic force is dependent on the transverse component v_t of the air velocity over the airfoil in a direction perpendicular to the long edge of the blade. The force will be approximated by a quadratic dependence on v_t. The force on an element of blade at a distance r is

$$d\mathbf{F} = \hat{\mathbf{x}} c v_t^2 dr \tag{6.83}$$

The perpendicular air-velocity component v_t is due to the rotational motion of the blade and to the translation motion of the boomerang CM at velocity V. This tangential velocity component is given by

$$v_t = \omega r + V\sin\omega t \tag{6.84}$$

Using (6.83) and (6.84), we find

$$d\mathbf{F}(r,t) = \hat{\mathbf{x}}c(\omega^2 r^2 + 2\omega V r\sin\omega t + V^2\sin^2\omega t)dr \tag{6.85}$$

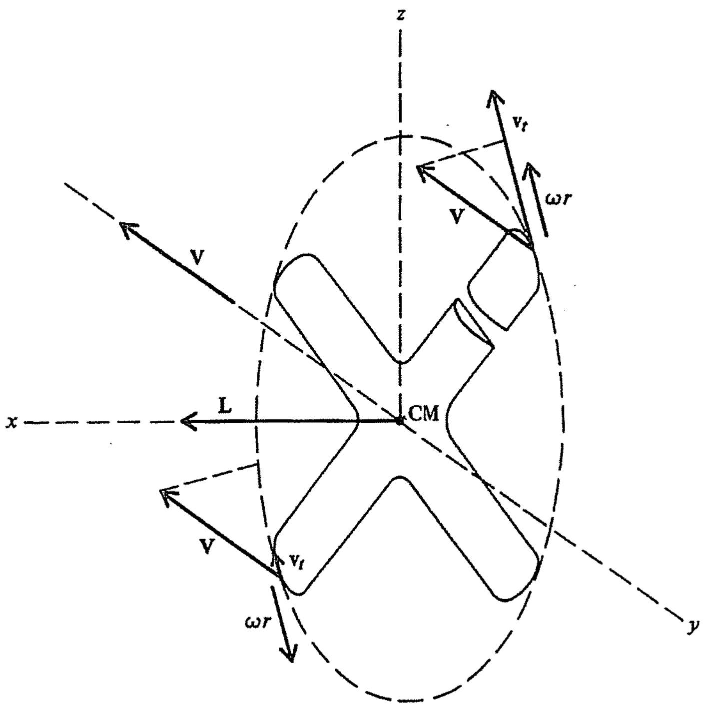


FIGURE 6-10. Boomerang-blade velocities.

As the blade rotates, $d\mathbf{F}$ varies in magnitude. The three remaining blades contribute forces similar to (6.85), but with ωt replaced by $\omega t+\pi/2$, $\omega t+\pi$, and $\omega t+3\pi/2$, respectively. The net force on the boomerang from all four blades due to the elements of length dr at distance r is

$$d\mathbf{F}(r) = 4\hat{\mathbf{x}}c\left(\omega^2 r^2 + \frac{V^2}{2}\right) \tag{6.86}$$

Adding the elements by integration over r, we find the total force normal to the plane of rotation is

$$\mathbf{F} = 4\hat{\mathbf{x}}cl\left(\frac{\omega^2 l^2}{3} + \frac{V^2}{2}\right) \tag{6.87}$$

From (6.82) and (6.83) the torque about the CM from an element on one of the blades is

$$d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = r(\hat{\mathbf{y}}\cos\omega t + \hat{\mathbf{z}}\sin\omega t) \times \hat{\mathbf{x}}cv_t^2 dr \tag{6.88}$$

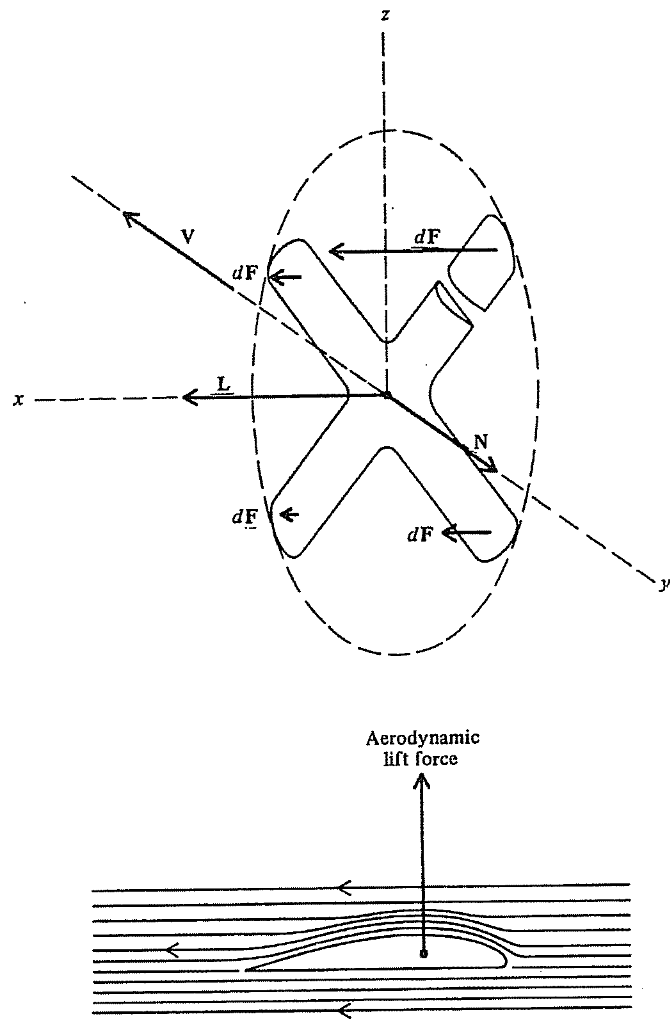


FIGURE 6-11. Aerodynamic forces on a cross-blade boomerang.

Expansion of this result, using (6.84), gives

$$d\mathbf{N} = cr(\omega^2 r^2 + 2\omega V r \sin\omega t + V^2 \sin^2\omega t)(\hat{\mathbf{y}}\sin\omega t - \hat{\mathbf{z}}\cos\omega t)dr \tag{6.89}$$

We add to this the torques from the other three blades' elements to find the net torque

$$d\mathbf{N} = 4c\omega V r^2 \hat{\mathbf{y}} dr \tag{6.90}$$

The torque due to all elements is then obtained by integrating (6.90) over

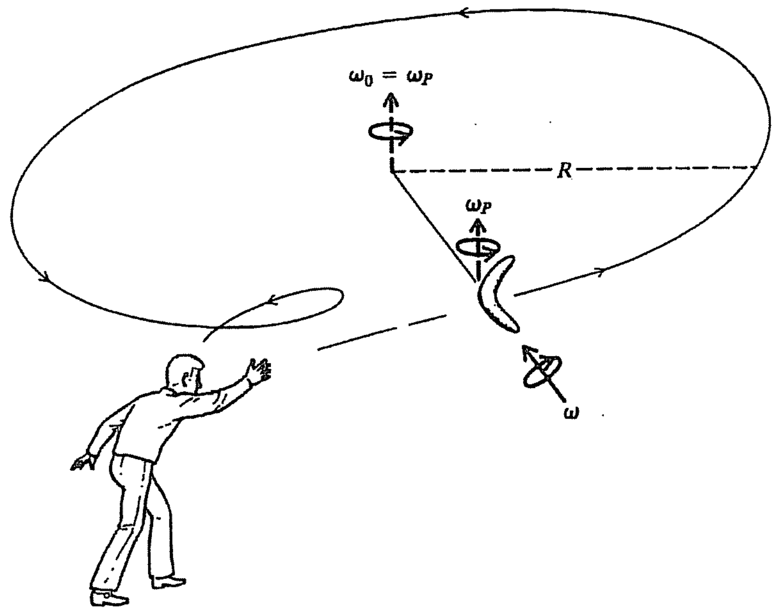


FIGURE 6-12. Typical boomerang orbit.

the length of a blade.

$$\mathbf{N} = \tfrac{4}{3} c\omega V l^3 \hat{\mathbf{y}} \tag{6.91}$$

The angular momentum $\mathbf{L}$ about the CM of the boomerang can be computed as follows. A blade element at distance r has mass $dm = \mu dr = \frac{1}{4}(M/l)dr$. As the blade rotates with angular velocity ω, the angular momentum of the element is $d\mathbf{L} = dmr^2\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \hat{\mathbf{x}}\omega$. The angular momentum of the whole blade is then

$$\mathbf{L} = \frac{M}{4l}\boldsymbol{\omega}\int_0^l r^2 dr = \tfrac{1}{12} M l^2 \boldsymbol{\omega} \tag{6.92}$$

The complete boomerang has angular momentum

$$\mathbf{L} = \tfrac{1}{3} M l^2 \boldsymbol{\omega} \tag{6.93}$$

The constant of proportionality between $\mathbf{L}$ and $\boldsymbol{\omega}$ is called the *moment of inertia.*

$$I = \tfrac{1}{3} M l^2 \tag{6.94}$$

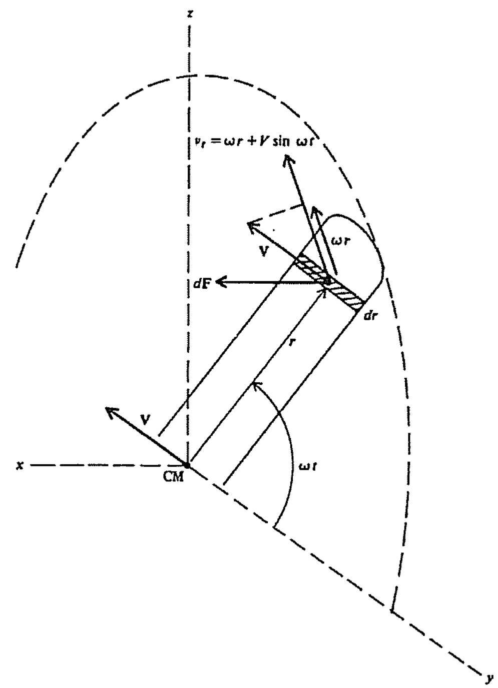


FIGURE 6-13. Diagram of one blade in a four-blade boomerang.

The torque in the $\hat{\mathbf{y}}$ direction induces a precession of the $\mathbf{L}$ vector as given by (6.77). The precession angular velocity is

$$\omega_P = \frac{N}{L} = \frac{\frac{4}{3}c\omega V l^3}{\frac{1}{3}M l^2 \omega} = \frac{4cVl}{M} \tag{6.95}$$

The motion of the CM is influenced by the aerodynamic force normal to the plane of rotation, gravity, and a drag force due to air resistance. For actual boomerangs the gravity and drag forces are not negligible but can be counterbalanced by a small tilt of the boomerang plane from vertical.

Since a boomerang is supposed to return to the thrower, we investigate under what conditions the CM will travel in a circular orbit of radius R with angular velocity ω_0. In a circular orbit the aerodynamic force exactly balances the centrifugal force. In order for the aerodynamic force to be always radial inward toward the center of a circle, the orbital angular velocity ω_0 must match the precession rate ω_P.

$$\omega_0 = \frac{V}{R} = \omega_P \tag{6.96}$$

From (6.95) and (6.96) we can determine the radius of the orbit as

$$R = \frac{M}{4lc} = \frac{\mu}{c} \tag{6.97}$$

where μ is the linear mass density and c is the lift constant determined by the airfoil shape and air properties. By equating the magnitude of the lift force in (6.87) to the mass times the centripetal acceleration,

$$F = 4cl\left(\frac{\omega^2 l^2}{3} + \frac{V^2}{2}\right) = \frac{MV^2}{R} \tag{6.98}$$

we obtain

$$V = \sqrt{\tfrac{2}{3}}\omega l \tag{6.99}$$

For a simple circular return flight the CM velocity V and spin ω must be related as in (6.99).

From (6.97) we see that the boomerang has a flight radius which is independent of how hard it is thrown. Of course, if it is thrown very slowly, the effects of gravity will become important, and our theory breaks down. If an indoor boomerang is desired, it should have an exaggerated airfoil shape to obtain a small orbit radius in (6.97). For long flights a boomerang made of dense material is needed, and of course the design should minimize drag. It is said that some native Australians can throw the boomerang 90 m and have it return to their feet. Such a record-setting boomerang would be useless to someone without a very strong arm since it could not be thrown with a smaller radius of orbit. A typical outdoor boomerang orbit may have a diameter of about 25 m. The boomerang starts its flight with a CM velocity of about 25 m/s and a rotation rate of about 100 r/s. It stays in the air for about 5 s.

6.7 Moments and Products of Inertia

The dynamics of rigid-body rotations are contained in (6.48), which relates the time rate of change of the total angular momentum to the external torque. The angular momentum $\mathbf{L}$ about a point O can be computed in terms of the angular velocity $\boldsymbol{\omega}$ in (6.64). We denote the location relative to O of a point mass m_i in the rigid body by $\mathbf{r}_i$. Then from (6.64) the velocity of the mass m_i relative to O is

$$\mathbf{v}_i = \boldsymbol{\omega}\times\mathbf{r}_i \tag{6.100}$$

The angular momentum about O is

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i\times\mathbf{v}_i) = \sum_i m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i) \tag{6.101}$$

The summation is over all mass points in the body. Using (2-44a) to expand the triple cross product, we obtain

$$\mathbf{L} = \boldsymbol{\omega}\left(\sum_i m_i|\mathbf{r}_i|^2\right) - \sum_i m_i\mathbf{r}_i(\mathbf{r}_i\cdot\boldsymbol{\omega}) \tag{6.102}$$

We observe that the angular-momentum vector $\mathbf{L}$ will not necessarily be parallel to the angular-velocity vector $\boldsymbol{\omega}$. We can write (6.102) in cartesian components as

$$\begin{aligned}
L_x &= \omega_x\sum_i m_i(y_i^2+z_i^2) - \omega_y\sum_i m_i x_i y_i - \omega_z\sum_i m_i x_i z_i \\
L_y &= -\omega_x\sum_i m_i y_i x_i + \omega_y\sum_i m_i(x_i^2+z_i^2) - \omega_z\sum_i m_i y_i z_i \\
L_z &= -\omega_x\sum_i m_i z_i x_i - \omega_y\sum_i m_i z_i y_i + \omega_z\sum_i m_i(x_i^2+y_i^2)
\end{aligned} \tag{6.103}$$

For the coefficients of the angular-velocity components we introduce the notation

$$\begin{array}{lll}
I_{xx} = \sum_i m_i(y_i^2+z_i^2) & I_{xy} = -\sum_i m_i x_i y_i & I_{xz} = -\sum_i m_i x_i z_i \\
I_{yx} = -\sum_i m_i y_i x_i & I_{yy} = \sum_i m_i(x_i^2+z_i^2) & I_{yz} = -\sum_i m_i y_i z_i \\
I_{zx} = -\sum_i m_i z_i x_i & I_{zy} = -\sum_i m_i z_i y_i & I_{zz} = \sum_i m_i(x_i^2+y_i^2)
\end{array} \tag{6.104}$$

In terms of these quantities we have

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \tag{6.105}$$

The components of angular momentum in (6.105) are then compactly written

$$L_j = \sum_k I_{jk}\omega_k \tag{6.106}$$

In this expression the superscripts j, k take on the values x, y, z. The three quantities I_{jj} are known as *moments of inertia*, and the six I_{jk} with $j \neq k$ are called *products of inertia*. From (6.104) we note that the products of inertia are symmetric.

$$I_{jk} = I_{kj} \tag{6.107}$$

The nine quantities I_{jk} form a *symmetric tensor* and can be written as a 3×3 matrix.

$$\mathbb{I} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{6.108}$$

In vector notation (6.106) can be written

$$\mathbf{L} = \mathbb{I} \cdot \boldsymbol{\omega} \tag{6.109}$$

From (6.48) and (6.106) the equation of motion for general rotations of a rigid body about its CM point or about a fixed point in space is

$$N_j = \frac{dL_j}{dt} = \sum_k \frac{d}{dt}(I_{jk}\omega_k) \tag{6.110}$$

or more compactly, in vector notation,

$$\mathbf{N} = \dot{\mathbf{L}} = \frac{d}{dt}(\mathbb{I} \cdot \boldsymbol{\omega}) \tag{6.111}$$

The kinetic energy of a rigid body can likewise be expressed in terms of the moments and products of inertia. The kinetic energy is given by

$$K = \tfrac{1}{2}\sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \tfrac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot(\boldsymbol{\omega}\times\mathbf{r}_i) = \tfrac{1}{2}\boldsymbol{\omega}\cdot\left[\sum_i m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i)\right] \tag{6.112}$$

where we have interchanged dot and cross products in the final step. Using (6.101) and (6.106), we find

$$K = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \tfrac{1}{2}\sum_{jk} I_{jk}\omega_j\omega_k \tag{6.113}$$

In vector notation K can be written

$$K = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\cdot\boldsymbol{\omega} \tag{6.114}$$

6.8 Single-Axis Rotations

The equation of motion in (6.110) simplifies considerably for the case of rotation about a single fixed axis. For definiteness we choose the z axis as the axis of rotation, $\boldsymbol{\omega} = \omega_z\hat{\mathbf{z}}$. The components of the angular momentum in (6.105) are then

$$\begin{aligned} L_x &= I_{xz}\omega_z \\ L_y &= I_{yz}\omega_z \\ L_z &= I_{zz}\omega_z \end{aligned} \tag{6.115}$$

The equations of motion from (6.110) are

$$\begin{aligned} N_x &= \dot{L}_x = \frac{d}{dt}(I_{xz}\omega_z) \\ N_y &= \dot{L}_y = \frac{d}{dt}(I_{yz}\omega_z) \\ N_z &= \dot{L}_z = \frac{d}{dt}(I_{zz}\omega_z) \end{aligned} \tag{6.116}$$

For a rigid body that is symmetrical about the z axis, we find from (6.104) that

$$I_{xz} = I_{yz} = 0 \tag{6.117}$$

In this case the torques N_x and N_y in (6.116) are zero. On the other hand, if one of the products of inertia, I_{xz} or I_{yz} is nonzero, the body is unbalanced and the bearings must provide the torques N_x or N_y to keep the axis of rotation from moving. For single axis rotation the principal moment of inertia is usually known as the *moment of inertia*.

The rotational motion about the z axis is accelerated by the external torque N_z. From (6.104) we observe that the moment of inertia

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) \tag{6.118}$$

is time independent, since the perpendicular distance $\sqrt{x_i^2 + y_i^2}$ from the rotation axis of the mass point m_i is fixed in a rigid body. Thus the equation of motion for the z component in (6.116) is

$$N_z = \dot{L}_z = I_{zz}\dot{\omega}_z \tag{6.119}$$

The kinetic energy for rigid-body rotation about the fixed z axis is found from (6.113) to be given by

$$K = \tfrac{1}{2} I_{zz}\omega_z^2 \tag{6.120}$$

The equation for rotational motion about a fixed axis in (6.119) has the same mathematical structure as the equation for linear motion in one direction.

$$F_z = M\dot{v}_z \tag{6.121}$$

In fact, the following direct correspondences can be made between the physical quantities of angular and linear motion:

Angular motion	Linear motion
Moment of inertia, I_{zz}	Mass, M
Angular acceleration, $\alpha_z = \dot{\omega}_z = \frac{d^2\phi}{dt^2}$	Linear acceleration, $a_z = \dot{v}_z = \frac{d^2 z}{dt^2}$
Torque, N_z	Force, F_z
Angular velocity, $\omega_z = \frac{d\phi}{dt}$	Linear velocity, $v_z = \frac{dz}{dt}$
Angular position, ϕ	Linear position, z
Angular momentum, $L_z = I_{zz}\omega_z$	Linear momentum, $P_z = Mv_z$
Kinetic energy, $K = \frac{1}{2} I_{zz}\omega_z^2$	Kinetic energy, $K = \frac{1}{2} M v_z^2$

As a consequence, we can directly apply the techniques for solving one-dimensional problems in Chapters 1 and 2 to solve (6.113) for rotations about a single axis. For example, if the torque is conservative (function only of the angle ϕ), we can define a potential energy

$$V(\phi) = -\int_{\phi_S}^{\phi} N_z(\phi')\,d\phi' \tag{6.122}$$

in correspondence with (2-6).

6.9 Moments-of-Inertia Calculations

The moment of inertia I_O of a rigid body about a given axis through the point O is related to the moment of inertia I_{CM} about a parallel axis which passes the center of mass by the *parallel-axis rule*

$$I_O = I_{CM} + Md^2 \tag{6.123}$$

where d is the perpendicular distance between the two axes; see Fig. 6-14. In practice I_{CM} is often easier to compute than I_O, making it advantageous to use the parallel-axis rule. In any case only one moment of inertia about a set of parallel axes needs to be computed.

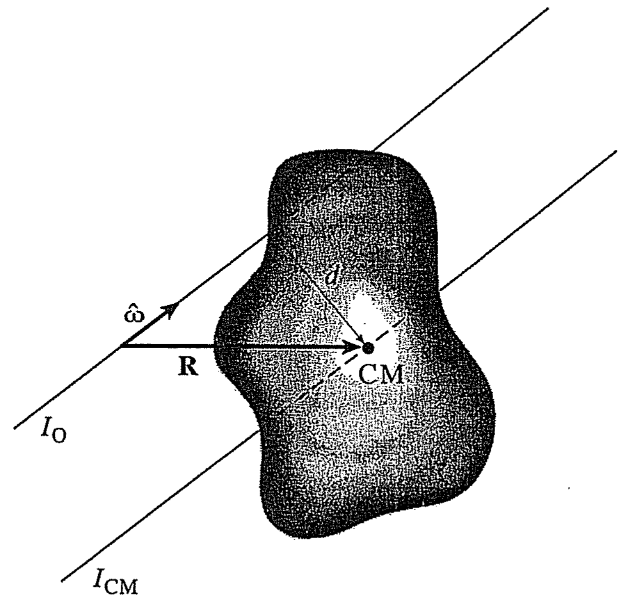


FIGURE 6-14. Parallel-axis rule for moments of inertia.

To prove the parallel axis theorem we use the expression (6.17) for the system angular momentum about O

$$\mathbf{L} = M\mathbf{R} \times \mathbf{V} + \sum_i m_i \mathbf{r}'_i \times \mathbf{v}'_i \tag{6.124}$$

For the moment of inertia we just need the component of $\mathbf{L}$ along $\boldsymbol{\omega}$

$$\mathbf{L} \cdot \hat{\omega} \equiv I_{\text{O}}\, \omega = M\mathbf{R} \times \mathbf{V} \cdot \hat{\omega} + I_{\text{CM}}\, \omega \tag{6.125}$$

The first term on the right side simplifies to

$$\begin{aligned} M\mathbf{R} \times \mathbf{V} \cdot \hat{\omega} &= M\hat{\omega} \cdot \mathbf{R} \times \mathbf{V} \\ &= M\hat{\omega} \times \mathbf{R} \cdot \mathbf{V} \\ &= M(\hat{\omega} \times \mathbf{R})^2 \omega \end{aligned} \tag{6.126}$$

By referring to Fig. 6-14 we see that the length of $\hat{\omega} \times \mathbf{R}$ is the perpendicular distance d between the two parallel axes, and $I_{\text{O}} = Md^2 + I_{\text{CM}}$ follows.

Another useful rule for moments of inertia, known as the *perpendicular axis rule*, applies to bodies whose mass is distributed in a single plane. For a body in the x, y plane with mass density per unit area σ, the moments of inertia about the three axes are

$$\begin{aligned} I_{xx} &= \int y^2 \sigma dA \\ I_{yy} &= \int x^2 \sigma dA \\ I_{zz} &= \int (x^2 + y^2) \sigma dA \end{aligned} \tag{6.127}$$

From these we derive the perpendicular-axis rule

$$I_{xx} + I_{yy} = I_{zz} \tag{6.128}$$

illustrated in Fig. 6-15. When the mass distribution is azimuthally symmetrical about the z axis, the two moments in the plane are equal and we obtain

$$I_{xx} = I_{yy} = \tfrac{1}{2} I_{zz} \tag{6.129}$$

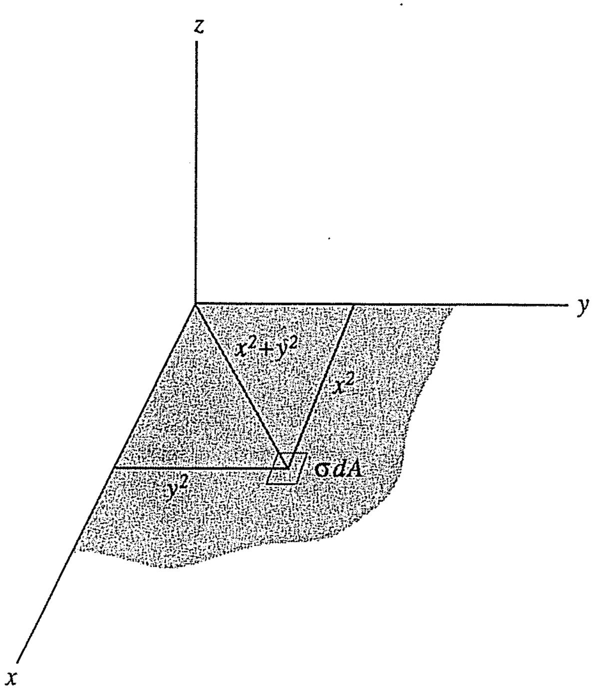


FIGURE 6-15. Perpendicular-axis rule for a body whose mass is distributed only in the x, y plane.

In the applications to be considered in the following two sections, we shall need the moment of inertia of a spherical body about an axis through its center of mass. For a sphere of radius a the mass density is

$$\rho = \frac{M}{\frac{4}{3}\pi a^3} \tag{6.130}$$

where M is the total mass. The integral for the moment of inertia about the z axis,

$$I_{\text{CM}} = \frac{M}{\frac{4}{3}\pi a^3} \int (x'^2 + y'^2) dV' \tag{6.131}$$

can be carried out simply in cylindrical coordinates.

$$\begin{aligned} x' &= r' \cos \phi' \\ y' &= r' \sin \phi' \\ dV' &= (r' d\phi') dr' dz' \end{aligned} \tag{6.132}$$

When the moving cue ball makes a head-on collision with a target ball at rest, the CM of the cue ball momentarily stops, and the target ball moves forward with the CM velocity of the cue ball, as shown earlier, in § 3.4. Since the balls are assumed smooth, the cue ball retains its spin ω_z^1 in the collision. Consequently, the contacting point on the cue ball moves with velocity $V_c = -a\omega_z^1$ immediately after collision. If $\omega_z^1 > 0$ at the moment of collision, the friction force acting opposite to the direction of V_c accelerates the cue ball forward, as illustrated in Fig. 6-18. This is the so-called *follow shot*. If $\omega_z^1 < 0$, the friction force accelerates the cue ball backward until pure rolling motion sets in. This is the *draw shot*. These shots play an important part in the tactics of pool or billiards.

6.11 Super-Ball Bounces

The bizarre behavior observed in bounces of the Wham-O Super-Ball (Registered trademark of Wham-O Corporation, San Gabriel, CA.) can be predicted from the rigid-body equations of motion. The Super-Ball is a hard spherical rubber ball. The bounces of a Super-Ball on a hard surface are almost elastic (*i.e.*, energy-conserving) and essentially nonslip at the point of contact. As an idealization, we shall also neglect gravity in our calculations, though its inclusion does not change the principal results.

We begin with an analysis of a single bounce from the floor. We denote the initial components of the CM velocity by v_x^0 and v_y^0 and the initial spin of the ball about the z axis through the CM by ω_z^0, as pictured in Fig. 6-19. The frictional force f_x and the normal force f_y act on the ball only for a very short time duration, Δt. We can determine the changes in the velocities Δv_x, Δv_y from the linear-impulse equation (6.135), and the change in spin from the angular-impulse equation (6.137). We obtain

$$\begin{aligned} M\Delta v_x &= -\int_{t_0}^{t_1} f_x dt \\ M\Delta v_y &= -\int_{t_0}^{t_1} f_y dt \\ I_{zz}\Delta\omega_z &= a\int_{t_0}^{t_1} f_x dt \end{aligned} \tag{6.143}$$

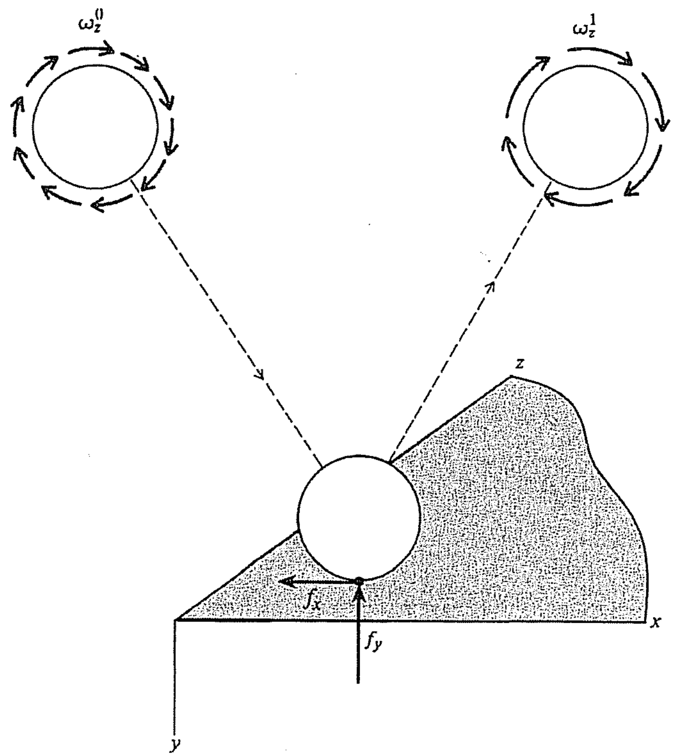


FIGURE 6-19. Super-Ball bounce from a hard surface.

By elimination of the frictional force f_x from the first and third equations, we obtain a relation between Δv_x and $\Delta\omega_z$ caused by f_x.

$$M(v_x^1 - v_x^0) = -\frac{I_{zz}}{a}(\omega_z^1 - \omega_z^0) \tag{6.144}$$

The assumption that f_x and f_y are independent (*i.e.*, that the deformations of the superball result in stresses in the x and y directions, which are independent of one another) requires that the energies associated with the x and y motions be separately conserved. In other words, both f_x and f_y are conservative forces. The conservative nature of f_y leads to

$$\tfrac{1}{2}M(v_y^1)^2 = \tfrac{1}{2}M(v_y^0)^2 \tag{6.145}$$

We conclude that the vertical component of velocity must be reversed by the action of the normal force f_y:

$$v_y^1 = -v_y^0 \tag{6.146}$$

The stipulation that f_x be energy-conserving (*i.e.*, no slipping) yields the condition

$$\tfrac{1}{2}I_{zz}(\omega_z^1)^2 + \tfrac{1}{2}M(v_x^1)^2 = \tfrac{1}{2}I_{zz}(\omega_z^0)^2 + \tfrac{1}{2}M(v_x^0)^2 \tag{6.147}$$

Equations (6.144), (6.145), and (6.147) govern the dynamics of a Super-Ball bounce. One possible solution to (6.144) and (6.147) is

$$\begin{aligned} v_x^1 &= v_x^0 \\ \omega_z^1 &= \omega_z^0 \end{aligned} \tag{6.148}$$

This solution corresponds to zero frictional force in (6.143) and is therefore relevant only for a smooth ball. The solution appropriate for a Super-Ball can be obtained by division of (6.144) and (6.147). We find

$$v_x^1 + v_x^0 = a(\omega_z^1 + \omega_z^0) \tag{6.149}$$

or by rearrangement

$$(v_x^1 - a\omega_z^1) = -(v_z^0 - a\omega_z^0) \tag{6.150}$$

The quantity $(v_x - a\omega_z)$ is just the horizontal velocity at the point on the ball that makes contact with the floor. Hence the velocity at the point of contact is exactly reversed by a bounce. From (6.144) and (6.150) we can solve for the spin and horizontal velocity immediately after the bounce in terms of the initial spin and velocity. With the moment of inertia of the Super-Ball given by (6.133), we arrive at

$$\begin{aligned} v_x^1 &= \tfrac{3}{7}V_x^0 + \tfrac{4}{7}\omega_z^0 a \\ \omega_z^1 &= -\tfrac{3}{7}\omega_z^0 + \tfrac{10}{7}\frac{v_x^0}{a} \end{aligned} \tag{6.151}$$

as the general solution for a Super-Ball bounce.

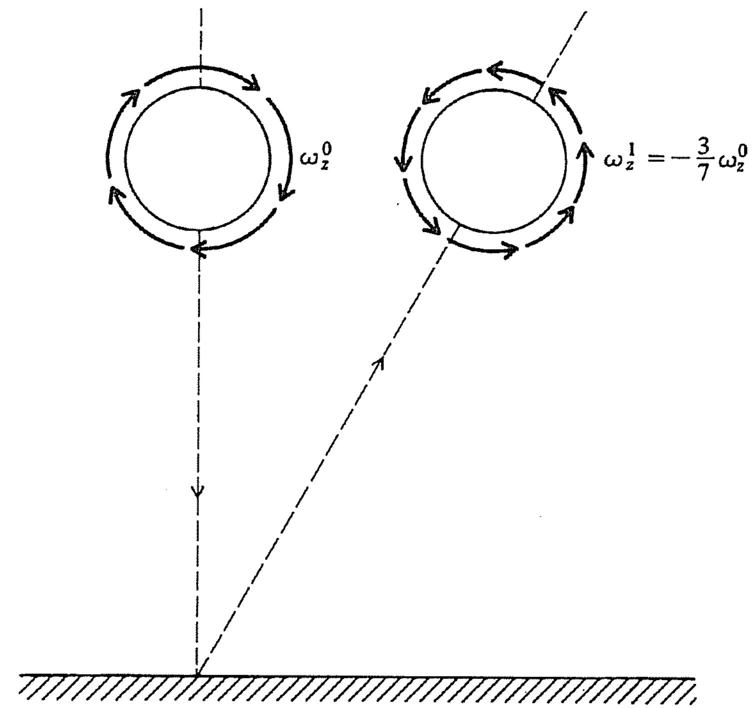


FIGURE 6-20. Deflection of a Super-Ball from a vertical bounce.

As an example of the result in (6.151), a Super-Ball which approaches the floor from a vertical direction ($v_x^0 = 0$) with initial spin ω_z^0 will leave the floor with

$$\begin{aligned} v_x^1 &= \tfrac{4}{7}\omega_z^0 a \\ \omega_z^1 &= -\tfrac{3}{7}\omega_z^0 \\ v_y^1 &= -V_y^0 \end{aligned} \tag{6.152}$$

as illustrated in Fig. 6-20. A smooth ball with the same initial velocity and spin would bounce back in the vertical direction.

The unexpected behavior of a Super-Ball is even more dramatically exhibited in successive bounces. As indicated in Fig. 6-21, a Super-Ball thrown to the floor in such a way that it bounces from the underside of a table will return to the hand. We can show this quite simply from repeated applications of (6.151). If the initial spin of the ball is zero, the

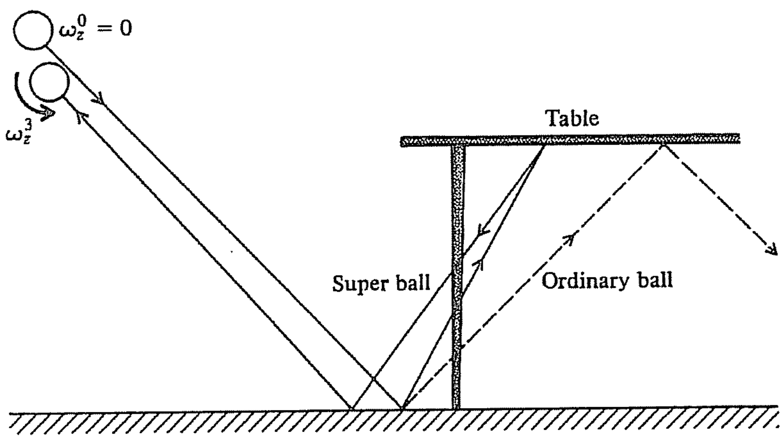


FIGURE 6-21. Return of a Super-Ball when bounced from the underside of a table.

velocity and spin after the first bounce from the floor are

$$\begin{aligned} v_x^1 &= \tfrac{3}{7} v_x^0 \\ \omega_z^1 &= \tfrac{10}{7} \frac{v_x^0}{a} \end{aligned} \tag{6.153}$$

For the bounce off the underside of the table, the angular impulse is opposite in sign to the impulse in (6.143). With the angular impulse reversed, the appropriate modifications of (6.151) for the second bounce are

$$\begin{aligned} v_x^2 &= \tfrac{3}{7} v_x^1 - \tfrac{4}{7} \omega_z^1 a \\ \omega_z^2 &= -\tfrac{3}{7} \omega_z^1 - \tfrac{10}{7} \frac{v_x^1}{a} \end{aligned} \tag{6.154}$$

When we substitute (6.153) into (6.154), we get

$$\begin{aligned} v_x^2 &= -\tfrac{31}{49} v_x^0 \\ \omega_z^2 &= -\tfrac{60}{49} \frac{v_x^0}{a} \end{aligned} \tag{6.155}$$

Thus the horizontal direction of motion has been reversed. For the final bounce off the floor we again apply (6.151), with (6.155) as initial values. We find

$$\begin{aligned} v_x^3 &= -\tfrac{333}{343} v_x^0 \\ \omega_z^3 &= -\tfrac{130}{343} v_x^0 \end{aligned} \tag{6.156}$$

Thus the Super-Ball returns after the three bounces with a slightly lower velocity than when it started, although the total kinetic energy remains the same. A smooth ball would not return, but would continue bouncing between the floor and the table, as indicated by the dashed line in Fig. 6-21.

PROBLEMS

6.1 Center of Mass and the Two-Body Problem

6-1. Find the distance of the center of mass of the earth-moon system from the center of the earth.

6-2. For a two-particle system in a region of uniform gravitational acceleration g, show that the net gravitational torque about the CM point of the system is zero.

6-3. A boat of mass 60 kg and length 4 m is at rest in quiet water. If a man of mass 80 kg walks from the bow to the stern, what distance will the boat move? Neglect water resistance.

6-4. Two particles on a line are mutually attracted by a force

$$F = -fr$$

where f is a constant and r is the distance of separation. At time $t = 0$, particle A of mass M is located at $x = 5$ cm, and particle B of mass $\frac{1}{4}M$ is located at $x = 10$ cm. If the particles are at rest at time $t = 0$, at what value of x do they collide? What is the relative velocity of the two particles at the moment the collision occurs?

6-5. Compare the magnitude of the gravity forces on the moon due to the earth and sun. Despite this result show from (6.22) that the sun is not very important in determining the moon's motion relative to the earth. If the moon's distance from the earth were greater would your conclusion remain valid?

6-6. The two atoms in a diatomic molecule (masses m_1 and m_2) interact through a potential energy

$$V(r) = \frac{a^2}{4r^4} - \frac{b^2}{3r^3}$$

where r is the separation of the atoms.

a) Find the equilibrium separation of the atoms and the frequency of small oscillations about the equilibrium assuming that the molecule does not rotate. How much energy must be supplied to the molecule in order to break it up?

b) Determine the maximum angular momentum which the molecule can have without breaking up, assuming that the motion is in circular orbits. Find the particle separation at the break up angular momentum.

c) Calculate the velocity of each particle in the laboratory system at break up, assuming that the center of mass is at rest. *Hint: break up occurs when V_{eff} no longer has a minimum.*

6-7. Two point masses are connected by a spring with spring constant k but are otherwise free to move in space. The equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = -k(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{l}) \qquad m_2\ddot{\mathbf{r}}_2 = k(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{l})$$

where $\mathbf{l} = l(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$.

a) Find the equilibrium separation of the masses and the frequency of oscillation of the masses about equilibrium assuming that the system does not rotate.

b) How will the equilibrium separation of the masses and the frequency of small oscillations about equilibrium change as the system rotates about an axis through the CM perpendicular to the axis of the oscillator?

c) Show that the total energy of the system is conserved.

6-8. Two particles with masses m_1 and m_2 collide head on. Particle 1 has an initial velocity v_1 and particle 2 is initially at rest in the laboratory system. The particles interact through a potential energy

$$V = V_0\left(\frac{a}{r_{12}}\right)^2$$

where $V_0 > 0$ and $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$.

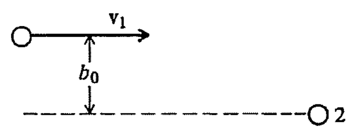
a) Compute the total energy and angular momentum of the two particles in the CM system. Express the results in terms of m_1, m_2, and v_1.

b) Describe the motion qualitatively as it appears in the CM system and in the lab system.

c) Find the distance of closest approach (minimum separation between the particles).

d) Find the velocity of particle 2 in the lab system after the collision.

6-9. Two particles of masses m_1 and m_2 collide. The initial velocity of particle 1 in the lab system is $\mathbf{v}_1$, while particle 2 is initially at rest. The initial impact parameter is b_0, as shown. The particles interact through a repulsive potential $V = V_0/|\mathbf{r}_1 - \mathbf{r}_2|^4$.



a) Calculate the total energy and angular momentum in the CM system in terms of the particle masses and velocities.

b) Derive the equation of motion for the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Find the distance of closest approach.

c) Show how the angle β between the final velocities of the particles in the lab can be calculated if the magnitude $|\mathbf{v}_{1f}|$ of the final lab velocity of particle 1 is measured.

6.2 Rotational Equation of Motion

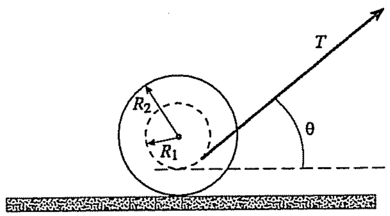
6-10. If the force on m_1 by m_2 is $\mathbf{F}_1^{[2]}$ and the force on m_2 by m_1 is $\mathbf{F}_2^{[1]}$, the *extended third law* requires that not only $\mathbf{F}_1^{[2]} + \mathbf{F}_2^{[1]} = 0$ but also that the forces act on a line connecting m_1 and m_2. Show that the total torque due to this pair of forces about a point p at $\mathbf{r}_p$ is zero, thus demonstrating that the extended third law implies zero total internal torque.

6-11. If the potential energy between two particles of a system depends only on their separation show that this potential energy depends on the angle of rotation about a fixed axis only through differences in particle angle coordinates. Then show that the resulting internal torque is zero.

6.3 Rigid Bodies: Static Equilibrium

6-12. A circular tabletop of radius 1 m and mass 3 kg is supported by three equally spaced legs on the circumference. When a vase is placed on the table, the legs support 1, 2, and 3 kg, respectively. How heavy is the vase, and where is it located on the table? What is the lightest vase which might upset the table?

6-13. A spool rests on a rough table as shown. A thread wound on the spool is pulled with force T at angle θ.



a) If $\theta = 0$ will the spool move to the left or right?

b) Show that there is an angle θ for which the spool remains at rest.

c) At this critical angle find the maximum T for equilibrium to be maintained. Assume a coefficient of friction μ.

6-14. A cylindrical glass full of ice weighs four times as much as when empty. At what intermediate level of filling is the glass least likely to tip? Neglect the mass of the bottom of the glass. Would the result change if the glass contained water (of the same density)? *Hint: show that the maximum angle of tip corresponds to the minimum CM height.*

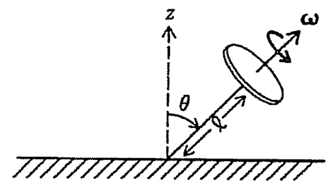
6.4 Rotations of Rigid Bodies

6-15. Consider again the drag racer of Fig. 1-1 which, while accelerating horizontally, is in vertical and rotational equilibrium. In § 1.3 it was found that the maximum acceleration is μg where μ is the coefficient of friction between the rear tires and the track. Show that in order to realize this optimal acceleration the CM must be located such that the ratio of its height h above the ground to the distance b_2 forward of the rear wheels satisfies $b_2 = \mu h$.

6-16. A vehicle has brakes on all four wheels. At rest the weight supported by each wheel is the same. Find the deceleration which corresponds to maximum possible braking. Calculate the normal forces the on front and back wheels when the brakes are applied. Why are the front brakes the most important in braking?

6.5 Gyroscopic Effect

6-17. A heavy axially symmetric gyroscope is supported at a pivot, as shown. The mass of the gyroscope is M, and the moment of inertia about its symmetry axis is I. The initial angular velocity about its symmetry axis is ω.



Give a suitable approximate equation of motion for the system, assuming that ω is very large. Find the angular frequency of the gyroscopic precession. Show that the above approximation is justified for

$$\omega \gg \sqrt{\frac{g}{\ell}}$$

where all moments of inertia are taken to be roughly $M\ell^2$.

6.6 The Boomerang

6-18. Show that if the aerodynamic force on a boomerang blade is proportional to v_t (not v_t^2 as in the text), the ratio of spin to CM velocity must still be related as $V = \sqrt{\frac{2}{3}}\,\omega l$ for a successful return. Show that the radius of the orbit is now proportional to the velocity.

6.8 Single-Axis Rotations

6-19. A disk of radius R is oriented in a vertical plane and spinning about its axis with angular velocity ω. If the spinning disk is set down on a horizontal surface, with what translational CM velocity will it roll away? *Hint: friction accelerates the CM and slows rotation until rolling begins.*

6-20. A spherical asteroid of uniform density 3 g/cm^3 and radius 100 m is rotating once per minute. It gradually acquires meteoritic material of the same density until a few billion years later its radius has doubled. If, on the average, this matter has arrived radially, what is the final rate of rotation?

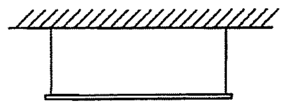
6-21. Due to tidal friction the earth's day increases in length by 4.4×10^{-8} s each day.

a) Compute the accumulated error between time based on the earth's rotation and absolute time (say by an atomic clock) after one century and after 3000 years. This accumulated error would be evident in the observation versus prediction of eclipses. *Hint: compare the angle through which an accelerating sphere turns with that of a sphere rotating uniformly.*

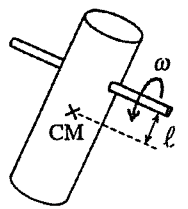
b) Estimate the power dissipated assuming a uniform earth. Compare this to a 10^9 W electrical generation facility.

6.9 Moments-of-Inertia Calculations

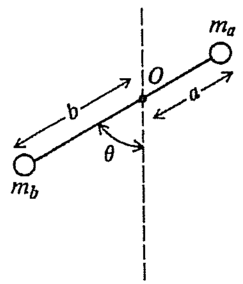
6-22. A thin, uniform rod of mass M is supported by two vertical strings, as shown. Find the tension in the remaining string immediately after one of the strings is severed.



6-23. A physical pendulum consists of a solid cylinder which is free to rotate about a transverse axis displaced by a distance ℓ along the symmetry axis from the center of mass, as illustrated. Find the value of ℓ for which the period is a minimum. Express the result in terms of the mass M and moment of inertia I about a transverse axis through the CM.



6-24. A pendulum consists of two masses connected by a very light rigid rod, as shown. The pendulum is free to oscillate in the vertical plane about a horizontal axis located a distance a from m_a at a distance b from m_b.



a) Calculate the moment of inertia of the system about 0. Find the location of the center of mass.

b) Set up the equation of motion for the system and derive the potential-energy function.

c) Take $b > a$ and determine the frequency of oscillation for small angles of displacement from the vertical.

d) Derive an exact expression for the period of the pendulum ($|\theta_{max}| < \pi$).

e) Find the minimum angular velocity which must be given to the system (starting at equilibrium) if it is to continue in rotation instead of oscillating.

6-25. A yo-yo of mass M is composed of two disks of radius R separated by a distance t by a shaft of radius r. A massless string is wound on the shaft, and the loose end is held in the hand. Upon release the yo-yo descends until the string is unwound. The string then begins to rewind, and the yo-yo climbs. Find the string tension and acceleration of the yo-yo in descent and in ascent. Neglect the mass of the shaft and assume the shaft radius is sufficiently small so that the string is essentially vertical.

6-26. Find the inertia tensor components about the origin in terms of the inertial tensor components about the CM. The position of the CM point is $\mathbf{R} = X\hat{\mathbf{x}} + Y\hat{\mathbf{y}} + Z\hat{\mathbf{z}}$.

6-27. A two-dimensional object lies in the x, y plane and is described by the moments of inertia I_{xx}, I_{yy} and the product of inertia I_{xy} for rotations in the x, y plane. In a coordinate system rotated by an angle ϕ the new coordinate components are related to x and y by

$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$

a) The tensor of inertia elements in the rotated system are $I_{x'x'}$, $I_{y'y'}$ and $I_{x'y'}$. Find the smallest angle ϕ_0 for which $I_{x'y'}$ vanishes. Show that the other solutions are $\phi_0 + n\frac{\pi}{2}$ and are equivalent. The x' and y' axes are known as *principal axes* since for rotations along these axes $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$.

b) If for two rotated systems (rotation angle $\neq \pi/2$) $I_{xy} = I_{x'y'} = 0$, prove that the principal moments are equal. As an example consider an equilateral triangle.

c) If the principal moments are equal and $I_{xy} = 0$ in some coordinate frame show that the moments are equal in any rotated coordinate system. As an example consider a uniform square.

d) Compute the principal moments in terms of a given set I_{xx}, I_{yy} and I_{xy} of inertia elements.

the point on the wheel is given by

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_{\text{rot}} = \mathbf{v}_0 + \frac{v_0}{R}\hat{\omega} \times \mathbf{r} \tag{6.72}$$

We note that $\mathbf{v} = 0$ at the point of contact. No slipping means that there is no relative motion of the wheel and surface at the contact point.

In general, the angular velocity ω is not the time derivative of any angular coordinate ϕ. As a consequence, angular displacements are quite different in nature from translational displacements. Only in the special case of fixed axis of rotation is it possible to express ω as the time derivative of a coordinate. We can show that it is not possible to write $\omega = \dot{\phi}$ in general by the following simple example. If such a representation were possible, we could compute the coordinates ϕ_x, ϕ_y, ϕ_z describing the orientation of the rigid body by integrating over the components of ω. For example,

$$\phi_x(t) - \phi_x(0) = \int_0^t \omega_x(t')dt' \tag{6.73}$$

This would say that the change in orientation $\phi(t) - \phi(0)$ resulting from a motion ω depends only on the three numbers $\int_0^t \omega_i(t')dt'$. That this is not true can be seen from the following demonstration. Take a book and choose fixed axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$. First rotate the book by 90° around the $\hat{\mathbf{x}}$ axis and then by 90° around the $\hat{\mathbf{y}}$ axis. Then start again from the original orientation and make the same rotations in the opposite order. In the two cases the resulting orientations of the book are different, but the integral $\int \omega dt$ is the same, namely,

$$\int \omega_x dt = \frac{\pi}{2}, \qquad \int \omega_y dt = \frac{\pi}{2}, \qquad \int \omega_z dt = 0 \tag{6.74}$$

6.5 Gyroscope Effect

As an interesting application of the rotational equation of motion (6.48) we will discuss the gyroscope effect experienced by a wheel spinning in a vertical plane, as illustrated in Fig. 6-6. With a counterclockwise spin, the angular-momentum vector points along the positive x axis. When a torque which tends to turn the wheel in a counterclockwise sense about the positive y axis is applied, the wheel is observed to precess about the z axis. We can predict this precession from (6.48) and derive an expression for the precession frequency. According to (6.48), the change in angular momentum in an infinitesimal time interval dt is

$$d\mathbf{L} = \mathbf{N}dt \tag{6.75}$$

The increment $d\mathbf{L}$ is parallel to $\mathbf{N}$ and perpendicular to $\mathbf{L}$, as shown in Fig. 6-6. Since $\mathbf{L}$ and $\mathbf{N}$ are perpendicular, the length of $\mathbf{L}$ is unchanged

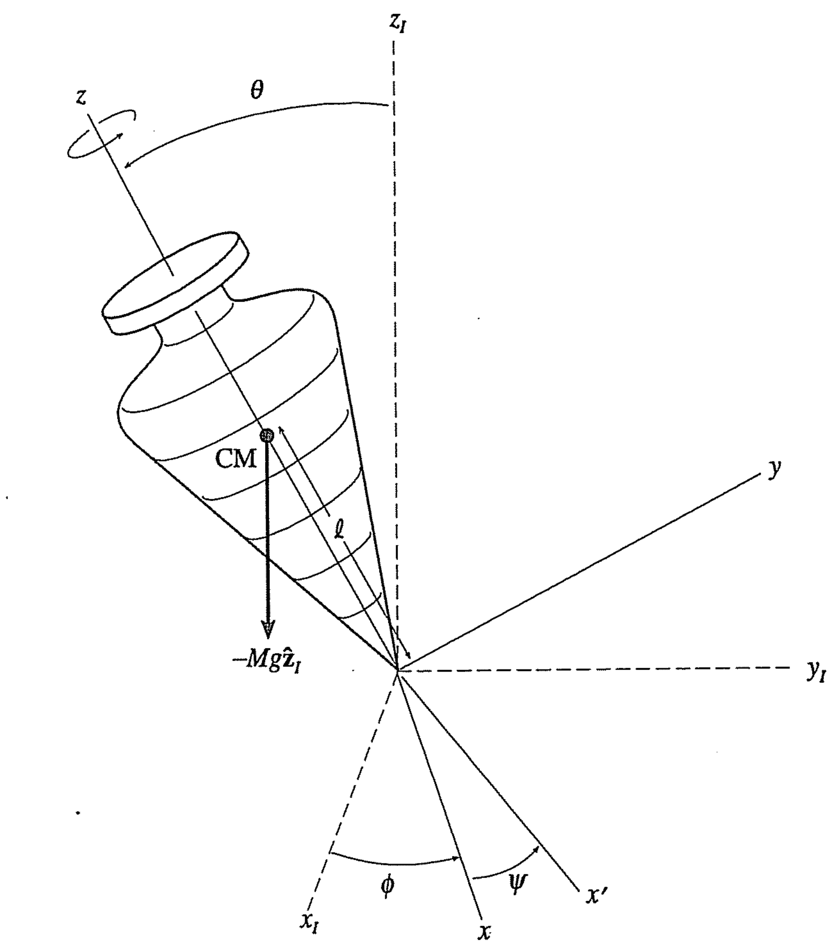


FIGURE 7-21. Spinning heavy top coordinates.

angular momentum about the origin is

$$\mathbf{L} = I\dot{\theta}\hat{\mathbf{x}} + I\dot{\phi}\sin\theta\hat{\mathbf{y}} + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{z}} \tag{7.136}$$

where $I_{zz} \equiv I_3$. Using these expressions for $\boldsymbol{\omega}$ and $\mathbf{L}$ and (6.113), the kinetic energy of the top is

$$K = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{7.137}$$

The gravitational potential energy is

$$V = Mg\ell\cos\theta \tag{7.138}$$

where ℓ is the distance from the point of contact to the CM of the top.

The Lagrangian $L = K - V$ for the top is a function of θ, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$. Since there is no dependence on ϕ or ψ there are two conserved general momenta. The first is

$$\begin{aligned} p_\psi = \frac{\partial L}{\partial\dot{\psi}} &= I_3(\dot{\psi} + \dot{\phi}\cos\theta) \\ &\equiv I_3\omega_3 \end{aligned} \tag{7.139}$$

Thus the angular velocity ω_3 along the symmetry (z) axis is constant. The second conserved momentum is

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = I\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta \tag{7.140}$$

The ϕ equation of motion is then $\dot{p}_\phi = 0$ which gives

$$I\ddot{\phi}\sin\theta = \dot{\theta}(I_3\omega_3 - 2I\dot{\phi}) \tag{7.141}$$

The Lagrange equation $\dot{p}_\theta = \frac{\partial L}{\partial\theta}$ gives the other equation of motion

$$I\ddot{\theta} = (Mg\ell - I_3\omega_3\dot{\phi} + I\dot{\phi}^2\cos\theta)\sin\theta \tag{7.142}$$

Motion in ϕ corresponds to precession and variations in θ are known as nutation.

We now use this last equation of motion to investigate the conditions under which the motion is pure precession. For pure precession the angle θ is constant. For constant $\dot{\phi}$ θ the right-hand side of (7.141) must vanish and we can solve for $\dot{\phi}$ in terms of θ.

$$\dot{\phi} = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4Mg\ell I\cos\theta}{I_3^2\omega_3^2}}\right) \tag{7.143}$$

For physical solutions, the quantity under the radical sign must not be negative. Since $\cos\theta > 0$ for a top on a table, the spin ω_3 must satisfy

$$\omega_3 \geq \sqrt{\frac{4Mg\ell I\cos\theta}{I_3^2}} \tag{7.144}$$

Only if the top has at least this minimum value of spin is pure precession possible. For a spin which is much greater than this minimum value, we can approximate the square root in (7.143) by the first two terms in a binomial expansion. We then find two possible approximate solutions for the precessional rate $\dot{\phi}$.

Slow precession:

$$\dot{\phi} = \frac{Mg\ell}{I_3\omega_3} \equiv \omega_p \tag{7.145}$$

Fast precession:

$$\dot{\phi} = \frac{I_3\omega_3}{I\cos\theta} \tag{7.146}$$

For the first solution, $\dot{\phi} \ll \omega_3$, and the angular momentum vector **L** lies nearly along the $\hat{\mathbf{z}}$ axis. This solution corresponds to slow gyroscopic precession, as discussed in § 6.5. For the second solution, $\dot{\phi} \approx \omega_3$, and **L** lies nearly along the z axis. In this case, $L\cos\theta \approx I_3\omega_3$ and $\dot{\phi} \approx L/I$, which is just the angular frequency ω_L in the force-free-top limit of (7.126). This solution with rapid precession about the vertical direction is independent of gravity in the limit $\omega_3 \gg (\omega_3)_{\text{min}}$.

For a rapidly spinning top, slow precession and small nutation are frequently observed. To find an approximate solution for the motion with this condition, the quadratic terms in $\dot{\phi}$ and $\dot{\theta}$ in the differential equations (7.141) and (7.142) can be neglected. We then obtain

$$\begin{aligned} \ddot{\phi}\sin\theta &= \frac{I_3\omega_3}{I}\dot{\theta} \\ \ddot{\theta} &= \left(\frac{Mg\ell}{I} - \frac{I_3\omega_3}{I}\dot{\phi}\right)\sin\theta \end{aligned} \tag{7.147}$$

In terms of ω_p from (7.145) and ω_L defined as

$$\omega_L \equiv \frac{I_3\omega_3}{I} \tag{7.148}$$

these equations can be written

$$\ddot{\phi}\sin\theta = \omega_L\dot{\theta} \tag{7.149}$$

$$\ddot{\theta} = \omega_L(\omega_p - \dot{\phi})\sin\theta \tag{7.150}$$

If we time-differentiate (7.149) and substitute (7.150), we find

$$\frac{d^2\dot{\phi}}{dt^2} + \omega_L^2\dot{\phi} = \omega_L^2\omega_p \tag{7.151}$$

where again we have dropped a quadratic term (of order $\ddot{\phi}\dot{\theta}$). We can immediately write down the solution to this equation for $\dot{\phi}$.

$$\dot{\phi}(t) = \omega_p + a\cos(\omega_L t + \alpha) \tag{7.152}$$

For the initial conditions $\dot{\phi} = \omega_0$, $\dot{\theta} = 0$, $\theta = \theta_0$ at $t = 0$, we find $\ddot{\phi}_0 = 0$ from (7.149) and

$$\dot{\phi}(t) = \omega_p - (\omega_p - \omega_0)\cos\omega_L t \tag{7.153}$$

The solution for $\phi(t)$ follows by integration.

$$\phi(t) = \omega_p t - \left(\frac{\omega_p - \omega_0}{\omega_L}\right)\sin\omega_L t \tag{7.154}$$

To solve for θ, we plug (7.153) into (7.149). This gives

$$\dot{\theta} = (\omega_p - \omega_0)\sin\omega_L t\sin\theta \tag{7.155}$$

Since ω_p and ω_0 are small quantities, we can make the approximation $\sin\theta \approx \sin\theta_0$ on the right-hand side of this equation. The solution for θ is then found by integration.

$$\theta(t) = \theta_0 + \left(\frac{\omega_p - \omega_0}{\omega_L}\right)\sin\theta_0(1 - \cos\omega_L t) \tag{7.156}$$

This completes the formal solution of the equations of motion in the approximation of slow precession and small nutation.

The solution for $\theta(t)$ in (7.150) exhibits nutation of the top between the angular limits θ_0 and $\theta_0 + 2[(\omega_p - \omega_0)/\omega_L]\sin\theta_0$. The sign of $(\omega_p - \omega_0)$ determines which is the upper and which is the lower bound on θ. The precession $\phi(t)$ in (7.154) has a sinusoidal motion associated with the nutation which is superimposed on the steady precession. When the initial precession ω_0 equals ω_p, the top undergoes steady precessional motion with no nutation. In Fig. 7-22 the curves traced out by the symmetry axis of the top are shown for various initial values ω_0.

The nutation frequency of the top from (7.156) is ω_L. We see from (7.145) and (7.148) that as the spin ω_3 of the top increases, the nutation frequency ω_L increases, while the precession frequency ω_p decreases. Furthermore, the nutation amplitude is inversely proportional to ω_L, so that nutation of a fast top is not so visible. When a fast top is spun on a hollow surface, however, a buzzing tone can often be heard with a frequency corresponding to the nutation frequency.

here. The friction force will also cause a torque

$$|\mathbf{N}| \approx \mu M g \ell$$

about the CM of the top which is roughly perpendicular to the peg for a thin peg, as illustrated in Fig. 7-23. For a rapidly spinning top, **L** lies nearly along the symmetry axis. Since the frictional torque is perpendicular to the symmetry axis, **L** precesses toward the vertical. The angular velocity of this precession is

$$\dot{\theta} = -\frac{|\mathbf{N}|}{|\mathbf{L}|} \approx -\frac{\mu M g \ell}{I_3 \omega_3}$$

or

$$\dot{\theta} \approx -\mu \omega_p \tag{7.161}$$

by (7.158). The angular velocity of the rising motion is just the product of the precessional angular velocity and the coefficient of skidding friction.

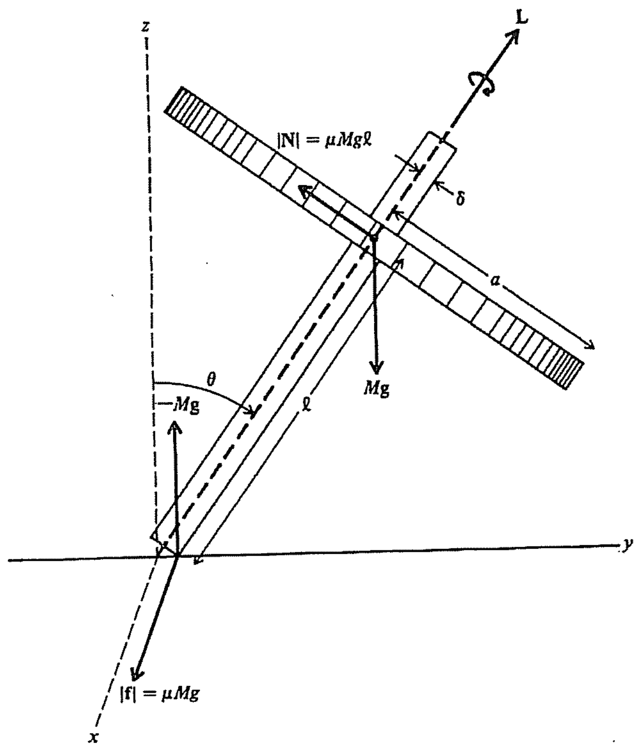


FIGURE 7-23. Forces on a rising top.

As the top rises, kinetic energy is converted into potential energy and the spin of the top decreases. In addition, some of the energy is dissipated by friction. The rate of frictional dissipation of energy,

$$\frac{dE}{dt} = -fv \tag{7.162}$$

where f is the frictional force and v is the velocity at the point of contact, can be quite small for a thin peg. Nevertheless, the effects of friction in causing the top to rise are dramatic.

Once the top has risen to a vertical position, the point of contact is the symmetry axis, and the frictional force is much smaller. From (7.142), the equation of motion for very small θ is then approximately

$$I\ddot{\theta} - (Mg\ell - I_3\omega_3\dot{\phi} + I\dot{\phi}^2)\theta = 0 \tag{7.163}$$

provided that the dissipation of energy by friction is neglected. In terms of the quantities ω_p and ω_L defined in (7.145) and (7.148), this equation can be written as

$$\ddot{\theta} + \omega_L\left(-\omega_p + \dot{\phi} - \frac{1}{\omega_L}\dot{\phi}^2\right)\theta = 0 \tag{7.164}$$

For a given $\dot{\phi}$, the motion in θ will be stable about $\theta = 0$ if

$$-\omega_p + \dot{\phi} - \frac{1}{\omega_L}\dot{\phi}^2 > 0 \tag{7.165}$$

The corresponding requirement on $\dot{\phi}$ is

$$\frac{\omega_L}{2}\left(1 - \sqrt{1 - \frac{4\omega_p}{\omega_L}}\right) < \dot{\phi} < \frac{\omega_L}{2}\left(1 + \sqrt{1 - \frac{4\omega_p}{\omega_L}}\right) \tag{7.166}$$

For a high value of the spin ω_3,

$$\frac{\omega_p}{\omega_L} = \frac{Mg\ell/I_3\omega_3}{I_3\omega_3/I} \ll 1 \tag{7.167}$$

and the condition in (7.166) is satisfied. The spinning top *sleeps* in the vertical position until friction slows down the spin to the value

$$\omega_3 = \sqrt{\frac{4Mg\ell I}{I_3^2}} \tag{7.168}$$

for which

$$\omega_L = 4\omega_p$$

and (7.166) becomes unphysical (*i.e.*, complex). At this point, the θ motion of the top becomes unstable and the top wobbles and goes down as θ increases from zero.

To develop a feeling for the motion of a typical top, we consider as an example a top made of a thin disk of radius a and mass M which is supported by a narrow peg of length $\ell = a/2$ and negligible mass, as illustrated in Fig. 7-23. The moments of inertia about the point of contact of the peg with the table are

$$\begin{aligned} I_3 &= \tfrac{1}{2}Ma^2 \\ I &= \tfrac{1}{4}Ma^2 + M\ell^2 = \tfrac{1}{2}Ma^2 \end{aligned} \tag{7.169}$$

If the top has a radius $a = 3$ cm and is set down with an initial spin of $\omega_3 = 300$ rad/s (about 50 r/s), the angular velocity of precession from (7.145) is

$$\omega_p = \frac{Mg(a/2)}{\frac{1}{2}Ma^2\omega_3} = \frac{g}{a\omega_3} = \frac{980}{3(300)} \simeq 1 \text{ rad/s} \tag{7.170}$$

For a coefficient of friction $\mu = 1/10$, the angular velocity from (7.161) of the top's rise toward the vertical is

$$\dot{\theta} = -\mu\omega_p = -0.1 \text{ rad/s} \tag{7.171}$$

If the top is started at its maximum angle of inclination,

$$\theta = \arctan\frac{a/2}{a} = 0.46 \text{ rad} \tag{7.172}$$

the time to rise to the vertical is

$$t = \frac{\theta}{|\dot{\theta}|} = \frac{0.46}{0.1} \simeq 5 \text{ s} \tag{7.173}$$

In this length of time the axis of the top has made

$$\frac{\omega_p t}{2\pi} = \frac{1(5)}{6.28} = 0.8 \text{ revolution} \tag{7.174}$$

about the vertical and

$$\frac{\omega_3 t}{2\pi} = \frac{300(5)}{6.28} = 240 \text{ revolutions} \tag{7.175}$$

about the symmetry axis. From (7.168), the condition for the motion at $\theta = 0$ to be stable is

$$\omega_3 > \sqrt{\frac{4Mg\ell I}{I_3^2}} = \sqrt{\frac{4Mg(a/2)(\frac{1}{2}Ma^2)}{(\frac{1}{2}Ma^2)^2}} = \sqrt{\frac{4g}{a}} \tag{7.176}$$

From the parameters of our top, we find

$$\sqrt{\frac{4g}{a}} = \sqrt{\frac{4(980)}{3}} = 36 \text{ rad/s} \tag{7.177}$$

Since the inequality $\omega_3 > \sqrt{4g/a}$ is satisfied for the initial spin $\omega_3 = 300$ rad/s, the top will *sleep* in its vertical position.

7.12 The Tippie-Top

When a tippie-top is spun on a smooth table, it turns itself upside-down, as pictured in Fig. 7-24. The usual high school and college rings likewise flip over to spin with their heavy ends upward. This fascinating behavior is due to a small frictional force at the point of contact with the table.

The frictional force is parallel to the table, opposing the velocity of slipping, as illustrated in Fig. 7-25. Since the horizontal direction of this force rotates rapidly with the angular frequency ω of the top, the time average of the force is zero, resulting in little effect on the motion of the CM.

The CM of a tippie-top is located close to the center of curvature, as indicated in Fig. 7-25. The gravitational torque can therefore be neglected. Furthermore, the frictional torque is nearly horizontal. This horizontal torque also rotates with angular frequency ω and time averages to zero. As a result, $d\mathbf{L}/dt \approx 0$, on the average, and the angular momentum of the top is nearly conserved. If the tippie-top is initially spun with the spin upward, as in Fig. 7-24, the approximately fixed direction of $\mathbf{L}$ is vertical with respect to the table.

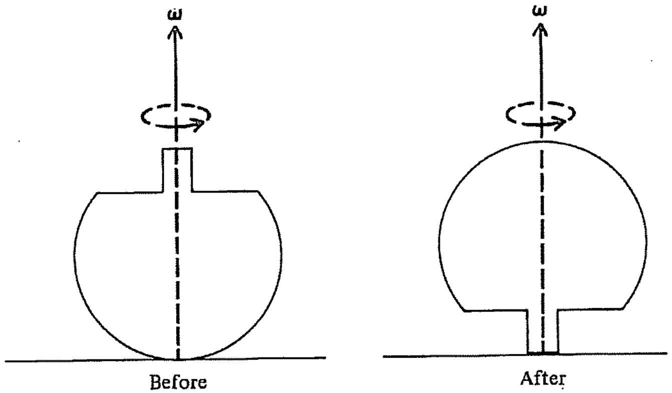


FIGURE 7-24. Flipping of a tippie-top.

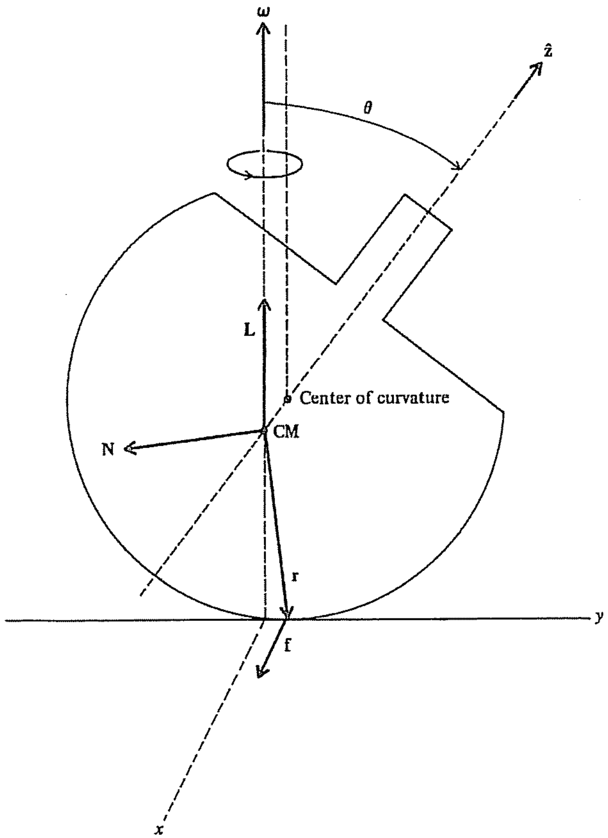


FIGURE 7-25. Frictional force and torque on a tippie-top.

The tipping motion is readily analyzed in a coordinate system which rotates with the top. In this reference frame the time average of the torque N is nonzero. The equation of motion (7.72) is

$$\mathrm{N} = \frac{\delta \mathbf{L}}{\delta t} + \omega \times \mathbf{L} \tag{7.178}$$

As a simplifying approximation, we take the three principal moments of inertia about the CM as equal (the shape of the tippie-top is nearly spherical). Then

$$\mathbf{L} \approx I\omega \tag{7.179}$$

and the $\omega \times \mathbf{L}$ term in (7.178) vanishes. $\mathbf{L}$ remains vertical and N horizontal throughout the motion. Since $\mathbf{L}$ and N are perpendicular, the torque causes the angular momentum to precess uniformly in the body frame, according to (7.178). Taking the component of (7.178) along the symmetry axis $\hat{\mathbf{z}}$, we find

$$\hat{\mathbf{z}} \cdot \mathrm{N} = \hat{\mathbf{z}} \cdot \frac{\delta \mathbf{L}}{\delta t} = \frac{\delta(\hat{\mathbf{z}} \cdot \mathbf{L})}{\delta t} \tag{7.180}$$

where θ is the angle between $\mathbf{L}$ and $\hat{\mathbf{z}}$. From (7.179) and (7.180), we obtain

$$\dot{\theta} = \frac{N}{L} \approx \frac{\mu M g R}{I\omega} \tag{7.181}$$

where R is the radius of the top. Thus θ increases with time, and the tippie-top flips over. Once the stem scrapes the table, the subsequent rise to the vertical is almost the same as an ordinary rising top, as treated in § 7.11. We can estimate the time required for the tippie-top to flip over by use of (7.181). A spin of $\omega = 300$ rad/s is easily imparted to a tippie-top of radius $R = 1.5$ cm and moment of inertia $I \approx \frac{2}{3}MR^2$ (hollow sphere). For a coefficient of friction $\mu = 1/10$, we obtain

$$\dot{\theta} = \frac{3\mu g}{2R\omega} = \frac{3(0.1)(980)}{2(1.5)(300)} = 0.3 \text{ rad/s} \tag{7.182}$$

The flip time is, roughly,

$$t = \frac{\theta}{\dot{\theta}} \approx \frac{\pi}{0.3} \approx 10 \text{ s} \tag{7.183}$$

PROBLEMS

7.2 Fictitious Forces

7-1. A particle of mass m moves in a smooth straight horizontal tube which rotates with constant velocity ω about a vertical axis which intersects the tube. Set up the equations of motion in polar coordinates and derive an expression for the distance of the particle from the rotation axis. If the particle is at $r = r_0$ at $t = 0$, what velocity must it have along the tube in order that it will be very close to the rotation axis after a very long time?

7-2. The WIYN telescope mirror blank was cast on a spinning platform in an oven. By this spin-casting technique the desired parabolic shape can be achieved without having to remove much glass by grinding. Light rays parallel to the symmetry axis of the mirror are focused to a point on the axis. This focal distance f is one-half the radius of curvature of the mirror.

a) If the mirror spins with angular velocity ω show that the focal distance is $f = g/2\omega^2$.

b) The mirror is to have a f/diameter ratio of 1.75 and a diameter of 3.5 m. Find the required spin in revolutions/min.

7-3. A bug of mass 1 g crawls out along a radius of a phonograph record turning at $33\frac{1}{3}$ r/min. If the bug is 6 cm from the center and traveling at the rate of 1 cm/s, what are the forces on the bug? What added torque must the motor supply because of the bug?

7-4. When ice skaters spin in place while pulling in their arms and legs, the striking increase in angular velocity is a consequence of angular momentum conservation. The fictitious forces which act to spin the skater are the Coriolis and azimuthal forces. For simplicity assume the spinning skater holds dumbbells initially at arm's length and by internal body forces draws them toward the rotation axis. Neglect any mass other than that of the dumbbells. Analyze the situation from the point of view of a rotating coordinate system in which the dumbbells are at rest except for radial motion. Show that the resultant forces imply that angular momentum is conserved.

7-5. For problem 7-1 use the Lagrangian method to

a) Find the radial equation of motion of the mass m.

b) Find the general constraint force Q'_θ.

c) Interpret this constraint force in terms of the Coriolis force.

7-6. A bead of mass m is constrained to move frictionlessly on a hoop of radius R. The hoop rotates with constant angular velocity ω about a vertical axis which coincides with a diameter of the hoop.

a) Obtain the equation of motion by applying Newton's law in the rest frame of the hoop.

b) Find the critical angular velocity Ω below which the bottom of the hoop provides a stable-equilibrium position for the bead.

c) Find the stable-equilibrium position for $\omega > \Omega$.

7.3 Motion on the Earth

7-7. A spherical planet of radius R rotates with a constant angular velocity ω. The effective gravitational acceleration g_{eff} is some constant, g, at the poles and $0.8\,g$ at the equator. Find g_{eff} as a function of the polar angle θ and g. With what velocity must a rocket be fired vertically upward from the equator to escape completely from the planet?

7-8. A particle of mass m is constrained to move in a vertical plane which rotates with constant angular velocity ω. Find the equations of motion of the particle, including the force of gravity.

7-9. A particle moves with velocity v on a smooth horizontal plane. Show that the particle will move in a circle due to the rotation of the earth; find the radius of the circle.

7-10. A ball is thrown vertically upward with velocity v_0 on the earth's surface. If air resistance is neglected, show that the ball lands a distance $(4\omega \sin\theta v_0^3/3g^2)$ to the west, where ω is the angular velocity of the earth's rotation and θ is the colatitude angle.

7.4 Foucault's Pendulum

7-11. Show using (7.45) that the Foucault pendulum equations of motion in cartesian coordinates are

$$\ddot{x} + \omega_0^2 x - 2\omega\cos\theta\,\dot{y} = 0$$
$$\ddot{y} + \omega_0^2 y + 2\omega\cos\theta\,\dot{x} = 0$$

where ω is the earth's angular velocity and θ is the colatitude angle. Solve this system of coupled equations retaining only the leading order in ω/ω_0 using the trial solutions $x = c_x e^{i\Omega t}$ and $y = c_y e^{i\Omega t}$.

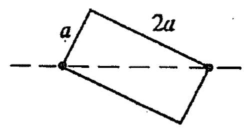
Show that the two allowed angular frequencies are $\Omega_\pm = \omega_0 \pm \omega \cos\theta$ and that $(c_y/c_x)_\pm = \pm i$. Impose the initial values $x(0) = a$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = 0$. Use the trigonometric identity in (2.168) to find the Foucault rotation in the x, y plane and determine the period of the rotation.

7.6 Principal Axes and Euler's Equations

7-12. Prove that if $I_1 \neq I_2 \neq I_3$ the principal axes are orthogonal. *Hint: start with (7.85) for $\boldsymbol{\omega}^{(a)}$ and I_a and dot with $\boldsymbol{\omega}^{(b)}$. Then do the same with $\boldsymbol{\omega}^{(b)}$ and I_b and dot with $\boldsymbol{\omega}^{(a)}$. Subtract the two equations and use the fact that $\mathbb{I}$ is a symmetric tensor.*

7-13. Show that the principal moments of inertia are real. *Hint: starting with (7.85) dot with $\boldsymbol{\omega}^*$ (the complex conjugate) and subtract the resulting equation from its complex conjugate. Then use the fact that $\mathbb{I}$ is real and symmetric.*

7-14. A flat rectangular plate of mass M and sides a and $2a$ rotates with angular velocity ω about an axle through two diagonal corners, as shown. The bearings supporting the plate are mounted just at the corners. Find the force on each bearing.



7-15. A point particle of mass $m = 1\,\text{kg}$ is located at the point $(x_0,\ y_0,\ 0)$

a) Calculate the tensor of inertia.

b) Find the principal axes and interpret the result.

7-16. For a prism mass distribution show that if any two axes perpendicular to the axis of the prism have equal moments of inertia, then all the axes in this plane are principal axes. *Hint: consider the expression for kinetic energy K in the ω_1, ω_2 plane where ω_1 and ω_2 lie in the plane perpendicular to the prism axes. The curve of constant K is an ellipse. The result follows from the geometry of the ellipse.*

7.7 The Tennis Racket Theorem

7-17. Show that the tennis racket in § 7.7 is properly designed so that a hard stroke to the ball at the center of the racket head does not jar the player's hand by causing impulsive torques.

7-18. A tennis racket is swung underhand and released so that it rises vertically with an initial spin about the unstable principal axis. At the instant of release the end of the handle is at rest. The racket subsequently rises to a height of 5 m.

a) Determine the time of rise to maximum height.

b) Find the initial angular velocity $\omega_2(0)$ about the CM.

c) For an initial spin axis $\omega_3(0)$ that is 1 percent of $\omega_2(0)$, compute the time at which the racket begins to tumble.

7-19. Write $2K$ and L^2 for a general rigid body, in terms of the principal-axis components of $\boldsymbol{\omega}$. From this, demonstrate the *tennis racket theorem* for a free rigid body using conservation of K and L^2.

7.9 The Free Symmetric Top: External Observer

7-20. A coin in a horizontal plane is tossed into the air with angular velocity components ω_1 about a diameter through the coin and ω_3 about the principal axis perpendicular to the coin. If ω_3 were equal to zero, the coin would simply spin around its diameter. For ω_3 nonzero, the coin will precess. What is the minimum value of ω_3/ω_1 for which the wobble is such that the same face of the coin is always exposed to an observer looking from above? With a little practice, this is a clever way to arrange the outcome of a coin flip!

7-21. A satellite with three distinct principal axes tumbles as it orbits the earth. A flexible antenna is deployed which slowly dissipates energy as the spacecraft tumbles. Eventually the spacecraft stabilizes without further action. Find the final regular rotational state of the satellite. *Hint: consider what happens to the energy and angular momentum in the principal axes coordinate system.*

7.10 The Heavy Symmetric Top

7-22. Why is it difficult to spin a pencil on its point? Illustrate quantitatively with a pencil of length 10 cm and diameter 0.5 cm.

7-23. Give an expression, in terms of rotations about the CM, for the kinetic energy of a heavy top whose point is fixed but pivots freely. When the kinetic energy of the CM motion is included show explicitly that the total kinetic energy is the same as when calculated relative to the pivot point.

7-24. A spinning heavy top is placed on a smooth horizontal table. The top point will now trace out a complex curve and the CM will repeatedly rise and fall. Find the Lagrangian and discuss how it differs from the fixed-point case.

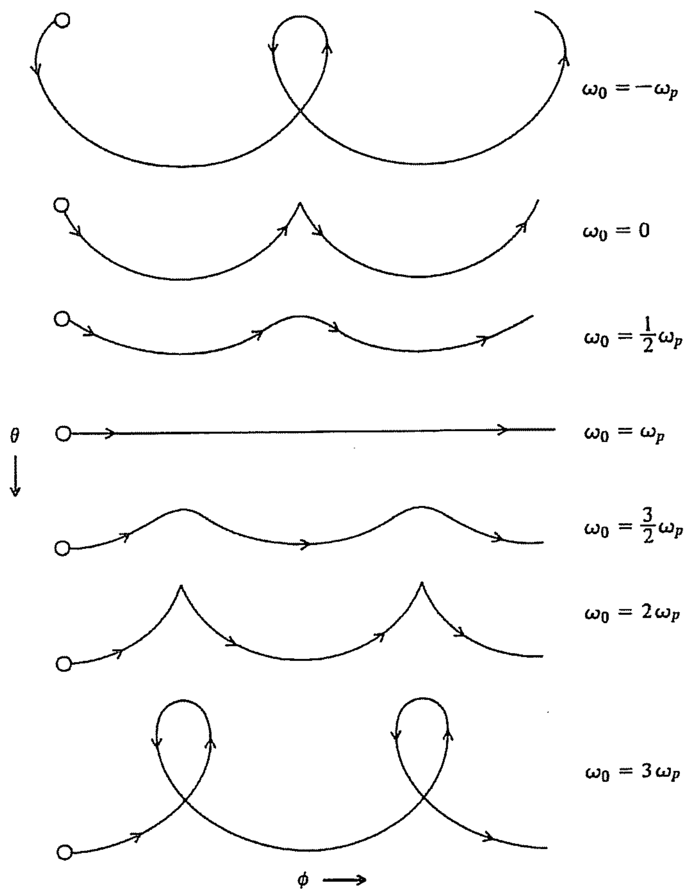


FIGURE 7-22. Nutation curves traced out by the symmetry axis of the top for various initial conditions. The top is started at the same value of θ in each case and the resulting curves are trochoids.

The phenomenon of nutation exhibited by our formal solution above can be understood from a more elementary viewpoint. For a top which is spinning rapidly, the angular momentum $\mathbf{L}$ is nearly along the symmetry axis $\hat{\mathbf{z}}$ of the top. The gravitational torque is obtained from (7.138)

$$\mathrm{N} = -\frac{dV}{d\theta}\hat{\boldsymbol{\theta}} = Mg\ell\sin\theta\hat{\mathbf{x}} \tag{7.157}$$

N is perpendicular to the z_I axis and causes the precession about the vertical direction. The angular velocity of precession $\boldsymbol{\omega}_p = \omega_p\hat{\mathbf{z}}_I$ can be found by equating

$$\frac{d\mathbf{L}}{dt} = \mathrm{N} = -Mg\ell\sin\theta\hat{\mathbf{x}}$$

to

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega}_p\times\mathbf{L} = \omega_p L(\hat{\mathbf{z}}_I\times\hat{\mathbf{z}})$$

Since $\hat{\mathbf{z}}_I\times\hat{\mathbf{z}} = -\hat{\mathbf{x}}\sin\theta$ and $L \approx I_3\omega_3$, we obtain the expected result

$$\omega_p \approx \frac{Mg\ell}{I_3\omega_3} \tag{7.158}$$

for the angular frequency of steady precession about the vertical z_I axis. For ω_3 very large, the precession rate ω_p is quite slow and the symmetry axis is nearly stationary. If the top is now given a slight push, it instantaneously acquires a small angular-momentum component $\Delta\mathbf{L}$ perpendicular to $\hat{\mathbf{z}}$. The resulting total angular momentum $\bar{\mathbf{L}} \equiv \mathbf{L} + \Delta\mathbf{L}$ points in a direction slightly different from the symmetry axis $\hat{\mathbf{z}}$. The ensuing motion is like that of a free top with the symmetry axis precessing around $\bar{\mathbf{L}}$ in a small circle. The angular frequency of this circular motion is found from (7.136) to be

$$\omega_L = \frac{\bar{\mathbf{L}}}{I} \approx \frac{I_3\omega_3}{I} \tag{7.159}$$

The complete motion of the symmetry axis is a superposition of this rapid free-top circular motion about the direction $\bar{\mathbf{L}}$ on the slow precession of $\hat{\mathbf{L}}$ about the vertical direction.

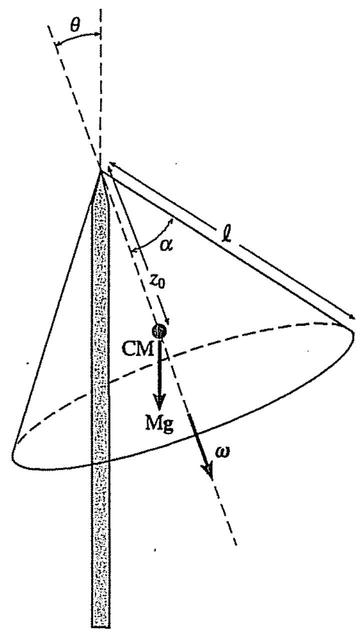
7.11 Slipping Tops: Rising and Sleeping

When a spinning top similar to that in Fig. 7-21 is set down on a rough surface, the top usually slips initially. A frictional force directed opposite to the instantaneous skidding velocity acts to accelerate the CM of the top until the velocity of slipping is reduced to zero and pure rolling motion sets in. If the top is spinning rapidly when it is set down, it tends to maintain a fixed angle θ with the vertical, since the nutation is small. The normal force at the point of contact is then essentially the entire weight of the top, and so the frictional force is

$$|\mathbf{f}| \approx \mu Mg \tag{7.160}$$

where μ is the coefficient of friction. This friction force will cause the top to move in the direction of the force but that effect will not be considered

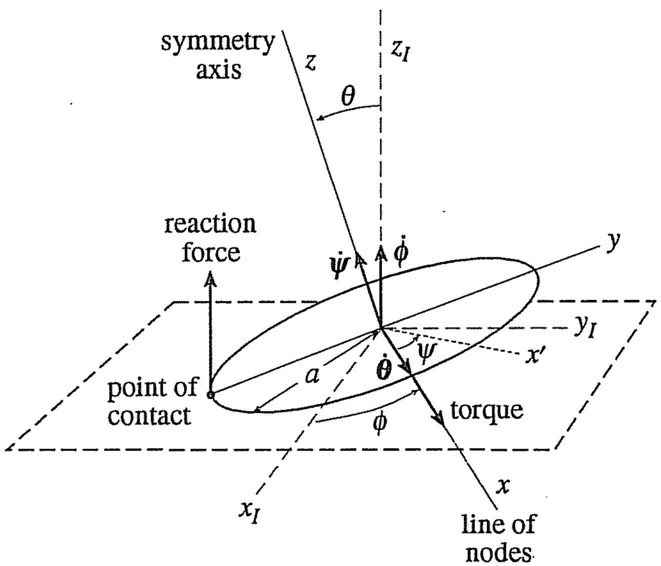
7-25. A hollow conical segment of half angle α, mass M and side length ℓ spins on a sharp pivot at its apex as shown. The cone is made from a uniform thin sheet and has an open base.



a) The cone is initially rapidly spun clockwise as viewed from above with angular velocity ω_3 about the symmetry axis. Find the direction and rate of the slow precession in terms of M, g, I_3 and the distance z_0 of the CM from the apex.

b) Calculate the principal moments, the location of the CM and the rate of slow precession.

7-26. A disk is spun about a vertical diameter. As it looses energy through friction it begins to wobble with slowly decreasing angle θ. Assume that the only motion of the CM is to fall slowly and that the disk rolls without slipping.



a) Show that the rolling condition is $\omega_3 = 0$.

b) Show that the precession of the line of nodes $\dot{\phi}$ is given by $\dot{\phi} = \omega_0/\sqrt{\sin\theta}$, where $\omega_0 = 2\sqrt{g/a}$. The wobble rate thus increases as the disk lies down. *Hint: Use part a) to establish that* **L** *lies along the* $\hat{\mathbf{y}}$ *axis. The torque lies along the* $\hat{\mathbf{x}}$ *axis. Use* $\mathbf{N} = \dot{\mathbf{L}} = \dot{\phi} \times \mathbf{L}$.

c) Show that the component of spin along the vertical x-axis is given by $\omega_{\text{vert}} = \omega_0(\sin\theta)^{3/2}$. Thus even though the wobble rate increases without limit as the disk lies down, the spin as seen from above actually comes to a stop. This effect can be impressively demonstrated using a heavy disk, with a mark on top, spinning on a cement floor.

7-27. Apply Euler's equations to obtain the equations of motion for the symmetric heavy top. Use the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ coordinate system of Fig. 7-20, where $\hat{\mathbf{x}}$ is the line of nodes. Because of the symmetry of the top these coordinates form a principal system even though the top is not at rest. Show that the equations of motion are equivalent to those of § 7.10. *Hint: The* $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ *system has angular velocity* $\boldsymbol{\omega} - \dot{\psi}\hat{\mathbf{z}}$ *relative to the inertial system.*

7.11 Slipping Tops: Rising and Sleeping

7-28. For a top with a spherical peg end, show that the effective peg radius is $\delta = \delta_0 \sin\theta$, where θ is the inclination angle. Would you expect this top to rise higher than a similar top with a cut-off peg?

7.12 The Tippie-Top

7-29. Analyze the motion of a tippie-top using an inertial frame on the table. *Hint: Find the implications for* $\boldsymbol{\omega}$ *of the precession of* **L** *about the vertical direction.*

9-6 NONEQUILIBRIUM APPLICATIONS OF NEWTON'S LAWS FOR ROTATION

In this section we remove the restriction from the previous section in which the angular acceleration was zero because the net torque was zero. Here we consider cases in which a nonzero net torque acts on a body and imparts an angular acceleration to it.

In the case of linear motion in one dimension, we solve similar problems using Newton's second law, $\Sigma F_x = ma_x$, where one component of the net force produces a component of the acceleration along the same coordinate axis. To maintain the analogy with Newton's laws for linear motion, we continue the restriction that the body rotate about a single fixed axis. We use the rotational form of Newton's second law (Eq. 9-11), $\Sigma \tau_z = I\alpha_z$, where (as in the previous section) we have for convenience dropped the "ext" subscript with the understanding that we are considering *only* external torques in our analysis.

In this section we will analyze problems involving angular accelerations produced by a torque applied to an object with a fixed axis of rotation. In the next section we will broaden the discussion somewhat to include cases in which the object rotates and also moves linearly (but keeps the axis of rotation in a fixed direction). In Chapter 10 we consider rotations in which the axis is not fixed in direction.

SAMPLE PROBLEM 9-9. A playground merry-go-round is pushed by a parent who exerts a force $\vec{F}$ of magnitude 115 N at a point P on the rim a distance of $r = 1.50$ m from the axis of rotation (Fig. 9-25). The force is exerted in a direction at an angle 32° below the horizontal, and the horizontal component of the force is in a direction 15° inward from the tangent at P. (*a*) Find the magnitude of the component of the torque that accelerates the merry-go-round. (*b*) Assuming that the merry-go-round can be represented as a steel disk 1.5 m in radius and 0.40 cm thick and that the child riding on it can be represented as a 25-kg "particle" 1.0 m from the axis of rotation, find the resulting angular acceleration of the system including the merry-go-round and child.

Solution (*a*) Only the horizontal component of $\vec{F}$ produces a vertical torque. Let us find $F_\perp$, the component of $\vec{F}$ along the horizontal line perpendicular to $\vec{r}$. The horizontal component of $\vec{F}$ is

$$F_h = F \cos 32° = 97.5 \text{ N}.$$

The component of F_h perpendicular to $\vec{r}$ is

$$F_\perp = F_h \cos 15° = 94.2 \text{ N}.$$

The (vertical) torque along the axis of rotation is thus

$$\tau = rF_\perp = (1.50 \text{ m})(94.2 \text{ N}) = 141 \text{ N}\cdot\text{m}.$$

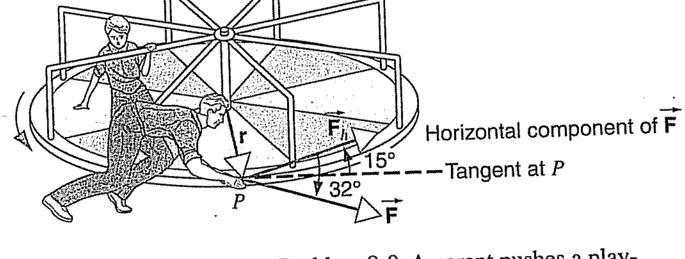


FIGURE 9-25. Sample Problem 9-9. A parent pushes a playground merry-go-round. The parent is leaning down, so the force has a downward component. Furthermore, because the parent is outside the rim, the force is directed slightly inward. The horizontal component of the force, F_h, is in the plane of the rotating platform and makes an angle of 15° with the tangent at P, the point at which the force is applied.

The component of F_h parallel to $r\,(= F_h \sin 15°)$ produces no torque at all about the axis of rotation, and the vertical component of $F\,(= F \sin 32°)$ produces a torque perpendicular to the axis that would tend to tip the rotating platform out of the horizontal plane (because the parent is pushing *down* on the platform) if that torque were not opposed by an equal and opposite torque from the bearings.

(*b*) The merry-go-round is a circular disk of radius $R = 1.5$ m and thickness $d = 0.40$ cm. Its volume is $\pi R^2 d = 2.83 \times 10^4$ cm³. The density of steel is 7.9 g/cm³, so the mass of the merry-go-round is $(2.8 \times 10^4 \text{ cm}^3)(7.9 \text{ g/cm}^3) = 2.23 \times 10^5 \text{ g} = 223$ kg. From Figure 9-15*c* we obtain the rotational inertia of a disk rotated about an axis perpendicular to its center:

$$I_m = \tfrac{1}{2}MR^2 = \tfrac{1}{2}(223 \text{ kg})(1.5 \text{ m})^2 = 251 \text{ kg}\cdot\text{m}^2.$$

The rotational inertia of the child, whom we treat as a particle of mass $m = 25$ kg at a distance of $r = 1.0$ m from the axis of rotation, is

$$I_c = mr^2 = (25 \text{ kg})(1.0 \text{ m})^2 = 25 \text{ kg}\cdot\text{m}^2.$$

The total rotational inertia is $I_t = I_m + I_c = 251 \text{ kg}\cdot\text{m}^2 + 25 \text{ kg}\cdot\text{m}^2 = 276 \text{ kg}\cdot\text{m}^2$. The angular acceleration can now be found from Eq. 9-11:

$$\alpha_z = \frac{\tau_z}{I_t} = \frac{141 \text{ N}\cdot\text{m}}{276 \text{ kg}\cdot\text{m}^2} = 0.51 \text{ rad/s}^2.$$

Based on the direction of the force shown in Fig. 9-25, the right-hand rule indicates that both τ_z and α_z point vertically upward from the plane of the merry-go-round.

SAMPLE PROBLEM 9-10. Figure 9-26*a* shows a pulley, which can be considered as a uniform disk of mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed (frictionless) horizontal axle. A block of mass $m = 1.2$ kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the

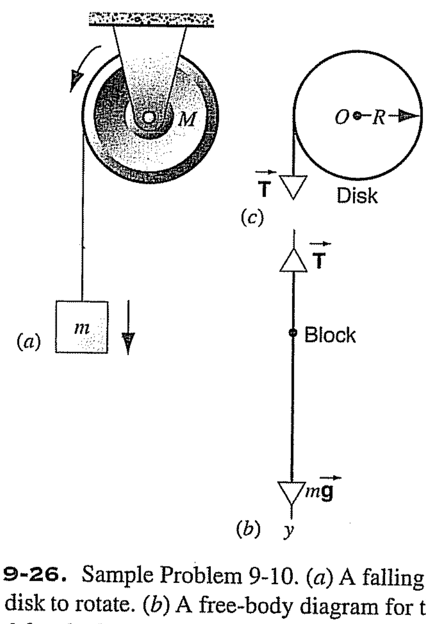


FIGURE 9-26. Sample Problem 9-10. (*a*) A falling block causes the disk to rotate. (*b*) A free-body diagram for the block. (*c*) A partial free-body diagram for the disk. The directions taken as positive are shown by the arrows in (*a*). The positive z axis is out of the page.

7.9 The Free Symmetric Top: External Observer

The description of the earth's rotational motion as a free symmetric top in § 7.8 was appropriate for an observer at rest in the rotating reference frame. In this section we concentrate on the motion of a free symmetric top as viewed by an external observer in an inertial frame. Since any object tossed into the air is basically a free top, the inertial description has a wide range of applications.

For a symmetric top the angular momentum and angular velocity projected onto the principal axes $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ in the top are

$$\begin{aligned} \mathbf{L} &= I(\omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}}) + I_3\omega_3\hat{\mathbf{z}} \\ \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}}) + \omega_3\hat{\mathbf{z}} \end{aligned} \tag{7.124}$$

where $\hat{\mathbf{z}}$ is in the direction of the symmetry axis. By eliminating $(\omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}})$ in these equations, the angular-velocity vector $\boldsymbol{\omega}$ can be expressed in terms of $\hat{\mathbf{L}}$ and $\hat{\mathbf{z}}$ as

$$\boldsymbol{\omega} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{z}} \tag{7.125}$$

where

$$\Omega \equiv \left(\frac{I_3 - I}{I}\right)\omega_3$$

as before, in (7.120). Since (7.125) is a linear relation among $\boldsymbol{\omega}$, $\mathbf{L}$, and $\hat{\mathbf{z}}$, these three vectors must lie in a plane. The absence of torques on the top implies that $\mathbf{L}$ is constant in the inertial system. Thus the $\boldsymbol{\omega}, \hat{\mathbf{z}}$ plane rotates (precesses) around the direction of $\mathbf{L}$. According to (7.125), the motion of the top as viewed from the inertial frame can be resolved into non-orthogonal components ω_L along $\hat{\mathbf{L}}$ and ω_3 along $\hat{\mathbf{z}}$ as

$$\begin{aligned}\omega_L &= \frac{L}{I} \\ \omega_3 &= -\Omega\end{aligned} \tag{7.126}$$

Since $\hat{\mathbf{z}}$ is a vector fixed in the body (*i.e.*, it rotates with the body), we have from (7.6) and (7.125)

$$\dot{\hat{\mathbf{z}}} = \boldsymbol{\omega}\times\hat{\mathbf{z}} = (\omega_L\hat{\mathbf{L}})\times\hat{\mathbf{z}} \tag{7.127}$$

Hence the symmetry axis $\hat{\mathbf{z}}$ rotates (or precesses) with fixed angular velocity $\omega_L\hat{\mathbf{L}}$ about the fixed inertial axis $\hat{\mathbf{L}}$. If we were riding on the top what would we experience? The angular velocity $\boldsymbol{\omega}^*$ of the top as observed from the *precessing frame* which rotates with angular velocity $\omega_L\hat{\mathbf{L}}$ is

$$\boldsymbol{\omega}^* = \boldsymbol{\omega} - \omega_L\hat{\mathbf{L}} = -\Omega\hat{\mathbf{z}}$$

The motion of the top as seen from this body-fixed frame is a rotation about the symmetry axis $\hat{\mathbf{z}}$ at the angular rate $-\Omega$. Since this is the rate that the top rotates with respect to $\boldsymbol{\omega}$ (which is a fixed vector in the precessing frame), we conclude that $+\Omega$ is the rate that $\boldsymbol{\omega}$ rotates with respect to the body, in agreement with the result (7.120) found from Euler's equations.

In the motion of the top, the angles that the symmetry axis $\hat{\mathbf{z}}$ makes with the vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ remain constant, as can be shown from (7.125) and (7.127) or from energy- and angular-momentum conservation. Since there are no torques, both the angular momentum $\mathbf{L}$ and the rotational kinetic energy K are constant. From (7.124) we can write $\mathbf{L}$ as

$$\mathbf{L} = I\omega_n\hat{\mathbf{n}} + I_3\omega_3\hat{\mathbf{z}} \tag{7.128}$$

where

$$\omega_n\hat{\mathbf{n}} \equiv \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} \tag{7.129}$$

is orthogonal to $\hat{\mathbf{z}}$. In terms of the components ω_n and ω_3, we have

$$\begin{aligned}L^2 &= I^2\omega_n^2 + I_3^2\omega_3^2 \\ 2K &= \mathbf{L}\cdot\boldsymbol{\omega} = I\omega_n^2 + I_3\omega_3^2\end{aligned} \tag{7.130}$$

where we used the expression (6.113) for the rotational kinetic energy. The constancy of L^2 and K in (7.130) requires in turn that ω_n and ω_3 be constant. The magnitude of $\boldsymbol{\omega}$ in (7.124),

$$\omega = \sqrt{\omega_n^2 + \omega_3^3}$$

is then also constant. From the geometry of Fig. 7-18, the angles of interest are determined by

$$\begin{aligned}\tan\alpha &= \frac{\omega_n}{\omega_3} \\ \tan\theta &= \frac{L_n}{L_3} = \frac{I\omega_n}{I_3\omega_3}\end{aligned} \tag{7.131}$$

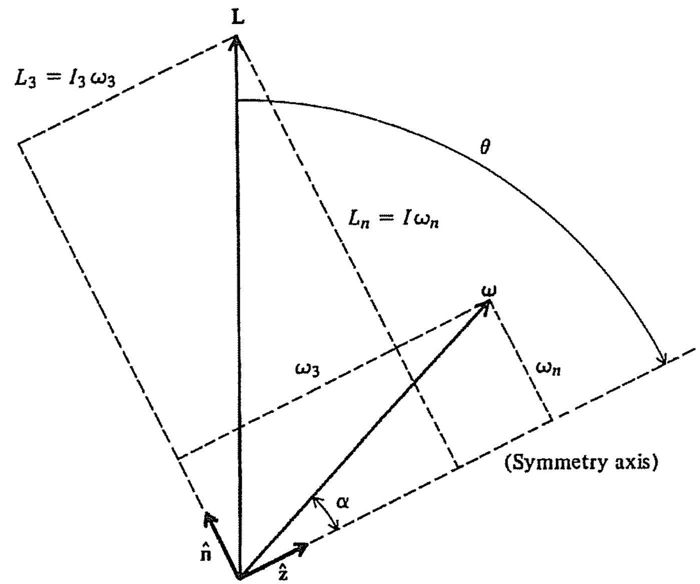


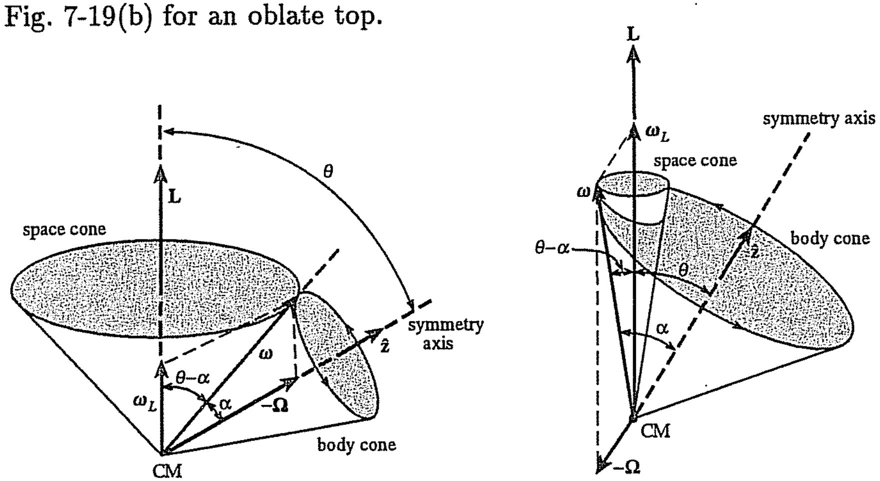
FIGURE 7-18. Components of angular velocity and angular momentum along the symmetry axis $\hat{\mathbf{z}}$ of the top and an axis $\hat{\mathbf{n}}$ perpendicular to the symmetry axis in the plane of $\boldsymbol{\omega}$ and $\mathbf{L}$.

The fixed relative orientation of $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{z}}$ follows immediately from these results. If we eliminate ω_n/ω_3 in (7.131), we obtain

$$\tan\alpha = \frac{I_3}{I}\tan\theta \tag{7.132}$$

For an oblate top (pancake or coinlike), $I_3 > I$, and the angle α is larger than θ. For a prolate top (football or cigar-shape), $\alpha < \theta$, which is the case illustrated in Fig. 7-18.

A simple geometric construction can be made to illustrate symmetrical free-top motion in an inertial reference frame. This construction is based on the constancy of the angles θ and α. As the plane containing the vectors $\boldsymbol{\omega}$ and $\hat{\mathbf{z}}$ precesses about $\mathbf{L}$, the vector $\boldsymbol{\omega}$ sweeps out a cone (the space cone) of half-angle $(\theta - \alpha)$ about the fixed direction $\mathbf{L}$. In the coordinate system fixed in the top, the vector $\boldsymbol{\omega}$ sweeps out a cone (the body cone) of half-angle α. Since $\boldsymbol{\omega}$ sweeps out both the space and body cones, the line of contact between the two cones is simply the vector $\boldsymbol{\omega}$. The points on the body cone which lie on the vector $\boldsymbol{\omega}$ are instantaneously at rest with respect to the fixed-space cone because $\boldsymbol{\omega}$ is the instantaneous axis of rotation of the top. As a consequence, the body cone must roll on the fixed-space cone without slipping. Thus we have a qualitative picture of the top's motion as the body cone rolling on the space cone. This is illustrated in Fig. 7-19(a) for a prolate top, and in Fig. 7-19(b) for an oblate top.



(a) Prolate top (b) Oblate top

FIGURE 7-19. Space and body cones for (a) prolate top, (b) oblate top.

7.10 The Heavy Symmetric Top

Untold generations of children have been fascinated by the precessing, rising, sleeping, and dying of spinning tops. The theory of spinning tops plays an important role in a wide variety of disciplines ranging from astronomy to applied mechanics to nuclear physics. In this section we discuss the motion of a symmetric top in a gravity field for a special case in which the point of contact of the top with supporting surface, the pivot, is fixed.

To analyze the motion of the top it is convenient to introduce the *Euler angle coordinates* ϕ, θ, ψ shown in Fig. 7-20. In this figure the origin of the inertial coordinate system x_I, y_I, z_I is at the fixed point of contact of the top. A coordinate system x, y, z is obtained by the rotation through an angle ϕ about $\hat{\mathbf{z}}_I$, followed by a rotation through θ about $\hat{\mathbf{x}}$. The $\hat{\mathbf{x}}$ axis is called the *line of nodes*. Then a coordinate system x', y', z' is obtained by a rotation through an angle ψ about the $\hat{\mathbf{z}}$ axis. The angles ϕ, θ, ψ uniquely specify the x', y', z' system relative to x_I, y_I, z_I and are useful in describing the orientation of a symmetric top as illustrated in Fig. 7-21, where x', y', z' are taken to be the body-fixed axes of the top.

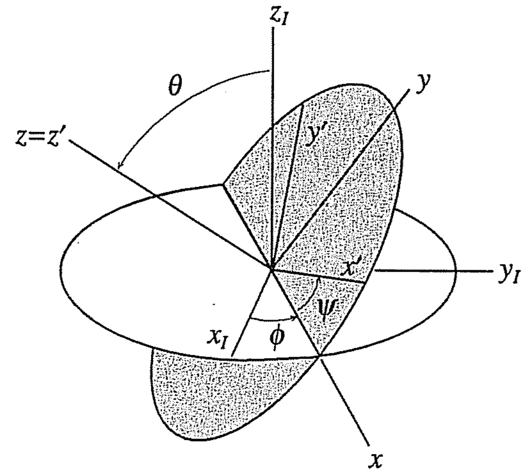


FIGURE 7-20. Euler angle coordinates describing rotations.

The angular velocity of the top is given in terms of the Euler angles by

$$\omega = \dot{\phi}\hat{\mathbf{z}}_I + \dot{\theta}\hat{\mathbf{x}} + \dot{\psi}\hat{\mathbf{z}} \tag{7.133}$$

Using the geometry of Fig. 7-20

$$\hat{\mathbf{z}}_I = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{y}} \tag{7.134}$$

and thus

$$\omega = \dot{\theta}\hat{\mathbf{x}} + \dot{\phi}\sin\theta\hat{\mathbf{y}} + (\dot{\psi}\dot{\phi}\cos\theta)\hat{\mathbf{z}} \tag{7.135}$$

Because the top is symmetric, the moments of inertia $I_{xx} = I_{yy} \equiv I$ are the same about any set of orthogonal axes in the $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ plane. Thus the

Symmetrical Top

$(I_1 = I_2 = I \neq I_3)$

$I\dot{\omega}_1 = \omega_2 \omega_3 (I - I_3)$

$I\dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I)$

$I\dot{\omega}_3 = 0 \Rightarrow \omega_3$ is constant

$$\begin{cases} \dot{\omega}_1 = \left(\frac{I - I_3)\omega_3}{I}\right. \omega_2 = -\Omega\omega_2 \\ \dot{\omega}_2 = \frac{(I_3 - I)\omega_3}{I}\omega_1 = \Omega\omega_1 \end{cases}$$

$$\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\Omega^2\dot{\omega}_1$$

$\Omega = \omega_3$

$\omega_1 = \omega_\perp \cos(\Omega t + \delta)$

(ω) $\omega_2 = \omega_\perp \sin(\Omega t + \delta)$

$\omega_1^2 + \omega_3^2 = \omega_\perp^2$

$\omega_3 =$ constant

$$L = \frac{(I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2}{2I}$$

Feynman's Wobbing Plate

$\Omega_W \sin\theta = \omega \sin\alpha$

$$\frac{I\omega_\perp}{L} = \frac{I\omega_\perp}{L}$$

justify

$$\Omega = \frac{(I_3 - I)\omega_3}{I} = \omega_3$$

asymmetrical Top
↓
can be solved analytically using elliptic function.

邵錦昌

第　　章

非慣性座標上的力學

一、概述：

我們已經知道了，在慣性座標中的觀察者所看的物體運動是遵守簡單的牛頓第二定律：

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(3)

(4)

故

$$a'=A+a \qquad (5)$$

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邵錦昌

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20–3 The gyroscope

Let us now return to the law of conservation of angular momentum. This law may be demonstrated with a rapidly spinning wheel, or gyroscope, as follows (see Fig. 20–1). If we sit on a swivel chair and hold the spinning wheel with its axis horizontal, the wheel has an angular momentum about the horizontal axis.

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* That this is true can be derived by compounding the displacements of the particles of the body during an infinitesimal time Δt. It is not self-evident, and is left to those who are interested to try to figure it out.

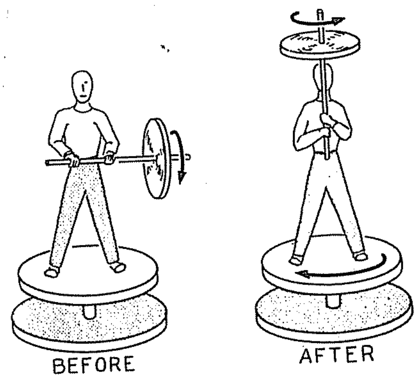


Fig. 20–1. *Before:* axis is horizontal; moment about vertical axis = 0. *After:* axis is vertical; momentum about vertical axis is still zero; man and chair spin in direction opposite to spin of the wheel.

Angular momentum around a *vertical* axis cannot change because of the (frictionless) pivot of the chair, so if we turn the axis of the wheel into the vertical, then the wheel would have angular momentum about the vertical axis, because it is now spinning about this axis. But the *system* (wheel, ourself, and chair) *cannot* have a vertical component, so we and the chair have to turn in the direction opposite to the spin of the wheel, to balance it.

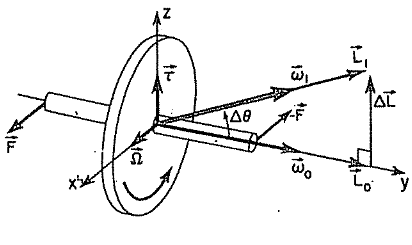


Fig. 20–2. A gyroscope.

First let us analyze in more detail the thing we have just described. What is surprising, and what we must understand, is the origin of the forces which turn us and the chair around as we turn the axis of the gyroscope toward the vertical. Figure 20–2 shows the wheel spinning rapidly about the y-axis. Therefore its angular velocity is about that axis and, it turns out, its angular momentum is likewise in that direction. Now suppose that we wish to rotate the wheel about the x-axis at a small angular velocity Ω; what forces are required? After a short time Δt, the axis has turned to a new position, tilted at an angle $\Delta\theta$ with the horizontal. Since the major part of the angular momentum is due to the spin on the axis (very little is contributed by the slow turning), we see that the angular momentum vector has changed. What is the change in angular momentum? The angular momentum does not change in *magnitude*, but it does change in *direction* by an amount $\Delta\theta$. The magnitude of the vector ΔL is thus $\Delta L = L_0\,\Delta\theta$, so that the torque, which is the time rate of change of the angular momentum, is $\tau = \Delta L/\Delta t = L_0\,\Delta\theta/\Delta t = L_0\Omega$. Taking the directions of the various quantities into account, we see that

$$\boldsymbol{\tau} = \boldsymbol{\Omega} \times \mathbf{L}_0. \tag{20.15}$$

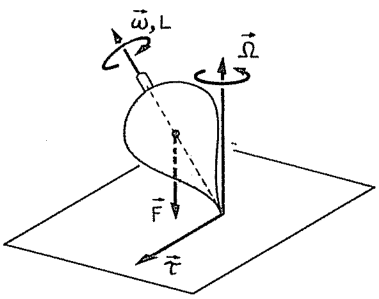


Fig. 20–3. A rapidly spinning top. Note that the direction of the torque vector is the direction of the precession.

Thus, if $\boldsymbol{\Omega}$ and $\mathbf{L}_0$ are both horizontal, as shown in the figure, $\boldsymbol{\tau}$ is *vertical*. To produce such a torque, horizontal forces $\mathbf{F}$ and $-\mathbf{F}$ must be applied at the ends of the axle. How are these forces applied? By our hands, as we try to rotate the axis of the wheel into the vertical direction. But Newton's Third Law demands that equal and opposite forces (and equal and opposite *torques*) act on *us*. This causes us to rotate in the opposite sense about the vertical axis z.

This result can be generalized for a rapidly spinning top. In the familiar case of a spinning top, gravity acting on its center of mass furnishes a torque about the point of contact with the floor (see Fig. 20–3). This torque is in the horizontal direction, and causes the top to precess with its axis moving in a circular cone about the vertical. If $\boldsymbol{\Omega}$ is the (vertical) angular velocity of precession, we again find that

$$\boldsymbol{\tau} = d\mathbf{L}/dt = \boldsymbol{\Omega} \times \mathbf{L}_0.$$

Thus, when we apply a torque to a rapidly spinning top, the direction of the precessional motion is in the direction of the torque, or at right angles to the forces producing the torque.

We may now claim to understand the precession of gyroscopes, and indeed we do, mathematically. However, this is a mathematical thing which, in a sense, appears as a "miracle." It will turn out, as we go to more and more advanced physics, that many simple things can be deduced mathematically more rapidly than they can be really understood in a fundamental or simple sense. This is a strange characteristic, and as we get into more and more advanced work there are circumstances in which mathematics will produce results which *no one* has really been able to understand in any direct fashion. An example is the Dirac equation, which appears in a very simple and beautiful form, but whose consequences are hard to understand. In our particular case, the precession of a top looks like some kind of a miracle involving right angles and circles, and twists and right-hand screws. What we should try to do is to understand it in a more physical way.

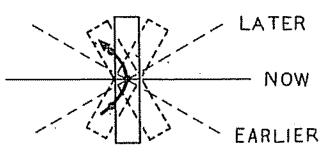


Fig. 20–4. The motion of particles in the spinning wheel of Fig. 20–2, whose axis is turning, is in curved lines.

How can we explain the torque in terms of the real forces and the accelerations? We note that when the wheel is precessing, the particles that are going around the wheel are not really moving in a plane because the wheel is precessing (see Fig. 20–4). As we explained previously (Fig. 19–4), the particles which are crossing through the precession axis are moving in *curved paths*, and this requires application of a lateral force. This is supplied by our pushing on the axle, which then communicates the force to the rim through the spokes. "Wait," someone says, "what about the particles that are going back on the other side?" It does not take long to decide that there must be a force in the *opposite direction* on that side. The net force that we have to apply is therefore zero. The *forces* balance out, but one of them must be applied at one side of the wheel, and the other must be applied at the other side of the wheel. We could apply these forces directly, but because the wheel is solid we are allowed to do it by pushing on the axle, since forces can be carried up through the spokes.

What we have so far proved is that if the wheel is precessing, it can balance the torque due to gravity or some other applied torque. But all we have shown is that this is *a* solution of an equation. That is, if the torque is given, and *if we get the spinning started right*, then the wheel will precess smoothly and uniformly. But we have not proved (and it is not true) that a uniform precession is the *most general* motion a spinning body can undergo as the result of a given torque. The general motion involves also a "wobbling" about the mean precession. This "wobbling" is called *nutation.*

Some people like to say that when one exerts a torque on a gyroscope, it turns and it precesses, and that the torque *produces* the precession. It is very strange that when one suddenly lets go of a gyroscope, it does not *fall* under the action of gravity, but moves sidewise instead! Why is it that the *downward* force of the gravity, which we *know* and *feel*, makes it go *sidewise*? All the formulas in the world like (20.15) are not going to tell us, because (20.15) is a special equation, valid only after the gyroscope is precessing nicely. What really happens, in detail, is the following. If we were to hold the axis absolutely fixed, so that it cannot precess in any manner (but the top is spinning) then there is no torque acting, not even a torque from gravity, because it is balanced by our fingers. But if we suddenly let go, then there will instantaneously be a torque from gravity. Anyone in his right mind would think that the top would fall, and that is what it starts to do, as can be seen if the top is not spinning too fast.

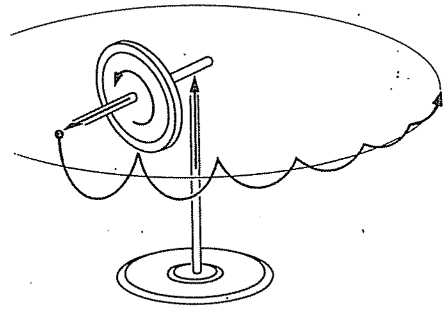


Fig. 20–5. Actual motion of tip of axis of gyroscope under gravity just after releasing axis previously held fixed.

The gyro actually does fall, as we would expect. But as soon as it falls, it is then turning, and if this turning were to continue, a torque would be required. In the absence of a torque in this direction, the gyro begins to "fall" in the direction opposite that of the missing force. This gives the gyro a component of motion around the vertical axis, as it would have in steady precession. But the actual motion "overshoots" the steady precessional velocity, and the axis actually rises again to the level from which it started. The path followed by the end of the axle is a cycloid (the path followed by a pebble that is stuck in the tread of an automobile tire). Ordinarily, this motion is too quick for the eye to follow, and it damps out quickly because of the friction in the gimbal bearings, leaving only the steady precessional drift (Fig. 20–5). The slower the wheel spins, the more obvious the nutation is.

When the motion settles down, the axis of the gyro is a little bit lower than it was at the start. Why? (These are the more complicated details, but we bring them in because we do not want the reader to get the idea that the gyroscope is an absolute miracle. It *is* a wonderful thing, but it is not a miracle.) If we were holding the axis absolutely horizontally, and suddenly let go, then the simple precession equation would tell us that it precesses, that it goes around in a horizontal plane. But that is impossible! Although we neglected it before, it is true that the wheel has *some* moment of inertia about the precession axis, and if it is moving about that axis, even slowly, it has a weak angular momentum about the axis. Where did it come from? If the pivots are perfect, there is no torque about the vertical axis. How then *does* it get to precess if there is no change in the angular momentum? The answer is that the cycloidal motion of the end of the axis damps down to the average, steady motion of the center of the equivalent rolling circle. That is, it settles down a little bit low. Because it is low, the spin angular momentum now has a small vertical component, which is exactly what is needed for the precession. So you see it has to go down a little, in order to go around. It has to yield a little bit to the gravity; by turning its axis down a little bit, it maintains the rotation about the vertical axis. That, then, is the way a gyroscope works.

20–4 Angular momentum of a solid body

Before we leave the subject of rotations in three dimensions, we shall discuss, at least qualitatively, a few effects that occur in three-dimensional rotations that are not self-evident. The main effect is that, in general, the angular momentum of a rigid body is *not necessarily* in the same direction as the angular velocity. Consider a wheel that is fastened onto a shaft in a lopsided fashion, but with the axis through the center of gravity, to be sure (Fig. 20–6). When we spin the wheel around the axis, anybody knows that there will be shaking at the bearings because of the lopsided way we have it mounted. Qualitatively, we know that in the rotating system there is centrifugal force acting on the wheel, trying to throw its mass as far as possible from the axis. This tends to line up the plane of the wheel so that it is perpendicular to the axis. To resist this tendency, a torque is exerted by the bearings. If there is a torque exerted by the bearings, there must be a rate of change of angular momentum. How can there be a rate of change of angular momentum when we are simply turning the wheel about the axis? Suppose we break the angular velocity ω into components ω_1 and ω_2 perpendicular and parallel to the plane of the wheel. What is the angular momentum? The moments of inertia about these two axes are *different*, so the angular momentum components, which (in these particular, special axes only) are equal to the moments of inertia times the corresponding angular velocity components, are in a *different ratio* than are the angular velocity components. Therefore the angular momentum vector is in a direction in space *not* along the axis. When we turn the object, we have to turn the angular momentum vector in space, so we must exert torques on the shaft.

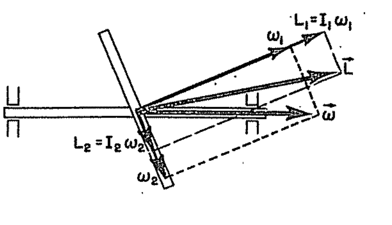


Fig. 20–6. The angular momentum of a rotating body is not necessarily parallel to the angular velocity.

Although it is much too complicated to prove here, there is a very important and interesting property of the moment of inertia which is easy to describe and to use, and which is the basis of our above analysis. This property is the following: Any rigid body, even an irregular one like a potato, possesses three mutually perpendicular axes through the CM, such that the moment of inertia about one of these axes has the greatest possible value for any axis through the CM, the moment of inertia about another of the axes has the *minimum* possible value, and the moment of inertia about the third axis is intermediate between these two (or equal to one of them). These axes are called the *principal axes* of the body, and they have the important property that if the body is rotating about one of them, its angular momentum is in the same direction as the angular velocity. For a body having axes of symmetry, the principal axes are along the symmetry axes.

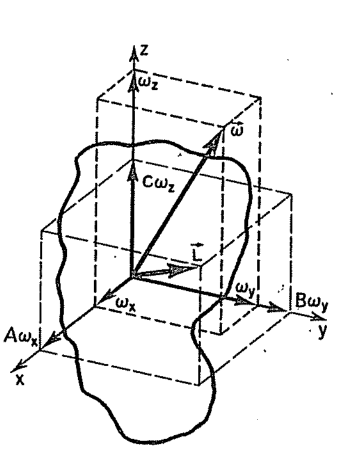


Fig. 20–7. The angular velocity and angular momentum of a rigid body ($A > B > C$).

If we take the x-, y-, and z-axes along the principal axes, and call the corresponding principal moments of inertia A, B, and C, we may easily evaluate the angular momentum and the kinetic energy of rotation of the body for any angular velocity ω. If we resolve ω into components ω_x, ω_y, and ω_z along the x-, y-, z-axes, and use unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, also along x, y, z, we may write the angular momentum as

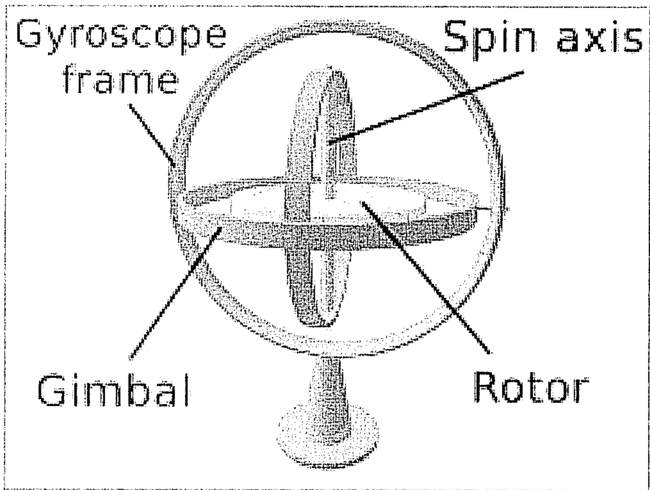
$$\mathbf{L} = A\omega_x\mathbf{i} + B\omega_y\mathbf{j} + C\omega_z\mathbf{k}. \tag{20.16}$$

The kinetic energy of rotation is

$$\begin{aligned}\text{KE} &= \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2) \\ &= \tfrac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega}.\end{aligned} \tag{20.17}$$

Gyroscope

From Wikipedia, the free encyclopedia



A gyroscope

A **gyroscope** is a device for measuring or maintaining orientation, based on the principles of conservation of angular momentum.[1] In essence, a mechanical gyroscope is a spinning wheel or disk whose axle is free to take any orientation. Although this orientation does not remain fixed, it changes in response to an external torque much less and in a different direction than it would without the large angular momentum associated with the disk's high rate of spin and moment of inertia. Since external torque is minimized by mounting the device in gimbals, its orientation remains nearly fixed, regardless of any motion of the platform on which it is mounted.

Gyroscopes based on other operating principles also exist, such as the electronic, microchip-packaged MEMS gyroscope devices found in consumer electronic devices, solid-state ring lasers, fibre optic gyroscopes, and the extremely sensitive quantum gyroscope.

Applications of gyroscopes include inertial navigation systems where magnetic compasses would not work (as in the Hubble telescope) or would not be precise enough (as in ICBMs), or for the stabilization of flying vehicles like radio-controlled helicopters or unmanned aerial vehicles. Due to their precision, gyroscopes are also used to maintain direction in tunnel mining.[2]

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Description and diagram

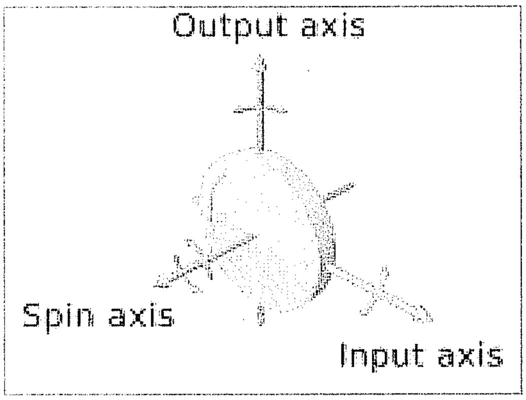


Diagram of a gyro wheel. Reaction arrows about the output axis (blue) correspond to forces applied about the input axis (green), and vice versa.

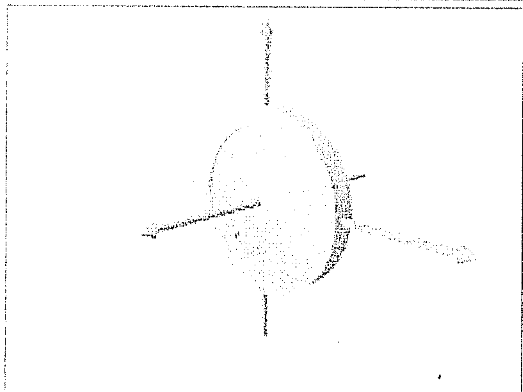
Within mechanical systems or devices, a conventional *gyroscope* is a mechanism comprising a rotor journaled to spin about one axis, the journals of the rotor being mounted in an inner gimbal or ring; the inner gimbal is journaled for oscillation in an outer gimbal for a total of two gimbals.

The **outer gimbal** or ring, which is the gyroscope frame, is mounted so as to pivot about an axis in its own plane determined by the support. This outer gimbal possesses one degree of rotational freedom and its axis possesses none. The next **inner gimbal** is mounted in the gyroscope frame (outer gimbal) so as to pivot about an axis in its own plane that is always perpendicular to the pivotal axis of the gyroscope frame (outer gimbal). This inner gimbal has two degrees of rotational freedom.

to spin about an axis, which is always perpendicular to the axis of the inner gimbal. So the rotor possesses three degrees of rotational freedom and its axis possesses two. The wheel responds to a force applied about the input axis by a reaction force about the output axis.

The behaviour of a gyroscope can be most easily appreciated by consideration of the front wheel of a bicycle. If the wheel is leaned away from the vertical so that the top of the wheel moves to the left, the forward rim of the wheel also turns to the left. In other words, rotation on one axis of the turning wheel produces rotation of the third axis.

A **gyroscope flywheel** will roll or resist about the output axis depending upon whether the output gimbals are of a free- or fixed- configuration. Examples of some free-output-gimbal devices would be the attitude reference gyroscopes used to sense or measure the pitch, roll and yaw attitude angles in a spacecraft or aircraft.

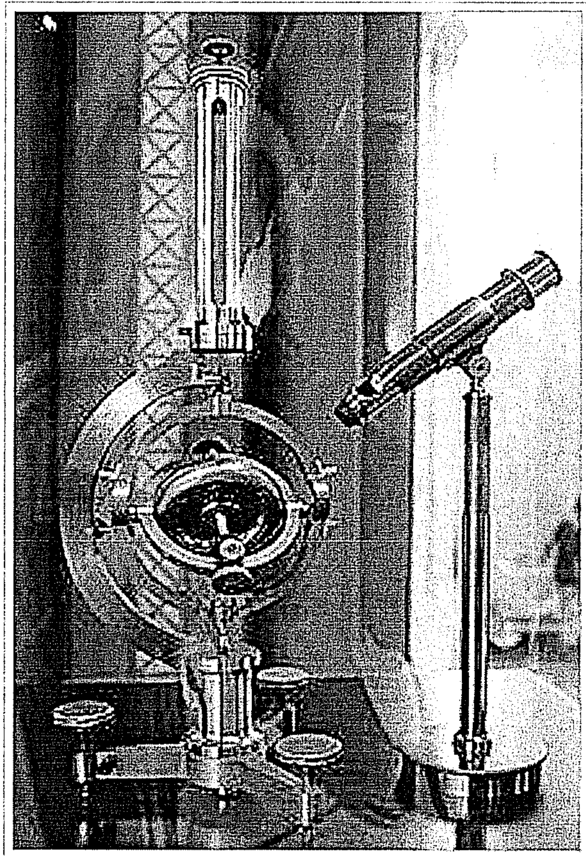


Animation of a gyro wheel in action

The centre of gravity of the rotor can be in a fixed position. The rotor simultaneously spins about one axis and is capable of oscillating about the two other axes, and, thus, except for its inherent resistance due to rotor spin, it is free to turn in any direction about the fixed point. Some gyroscopes have mechanical equivalents substituted for one or more of the elements. For example, the spinning rotor may be suspended in a fluid, instead of being pivotally mounted in gimbals. A control moment gyroscope (CMG) is an example of a fixed-output-gimbal device that is used on spacecraft to hold or maintain a desired attitude angle or pointing direction using the gyroscopic resistance force.

In some special cases, the outer gimbal (or its equivalent) may be omitted so that the rotor has only two degrees of freedom. In other cases, the centre of gravity of the rotor may be offset from the axis of oscillation, and, thus, the centre of gravity of the rotor and the centre of suspension of the rotor may not coincide.

The earliest known gyroscope-like instrument was made by German Johann Bohnenberger, who first wrote about it in 1817. At first he called it the "Machine".[3][4] Bohnenberger's machine was based on a rotating massive sphere.[5] In 1832, American Walter R. Johnson developed a similar device that was based on a rotating disk.[6][7] The French mathematician Pierre-Simon Laplace, working at the École Polytechnique in Paris, recommended the machine for use as a teaching aid, and thus it came to the attention of Léon Foucault.[8] In 1852, Foucault used it in an experiment involving the rotation of the Earth.[9][10] It was Foucault who gave the device its modern name, in an experiment to see (Greek *skopeein*, to see) the Earth's rotation (Greek *gyros*, circle or rotation),[11] which was visible in the 8 to 10 minutes before friction slowed the spinning rotor.



Gyroscope invented by Léon Foucault in 1852. Replica built by Dumoulin-Froment for the Exposition universelle in 1867. National Conservatory of Arts and Crafts museum, Paris.

In the 1860s, the advent of electric motors made it possible for a gyroscope to spin indefinitely; this led to the first prototype gyrocompasses. The first functional marine gyrocompass was patented in 1904 by German inventor Hermann Anschütz-Kaempfe.[12] The American Elmer Sperry followed with his own design later that year, and other nations soon realized the military importance of the invention—in an age in which naval prowess was the most significant measure of military power—and created their own gyroscope industries. The Sperry Gyroscope Company quickly expanded to provide aircraft and naval stabilizers as well, and other gyroscope developers followed suit.[13]

In 1917, the Chandler Company of Indianapolis, created the "Chandler
" t ith ll t i d d t l Ch dl

inc. in 1982. The chandler toy is still produced by TEDCO today.[14]

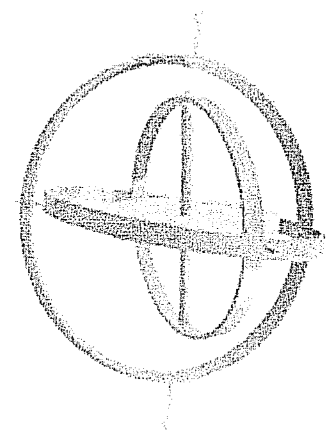
In the first several decades of the 20th century, other inventors attempted (unsuccessfully) to use gyroscopes as the basis for early black box navigational systems by creating a stable platform from which accurate acceleration measurements could be performed (in order to bypass the need for star sightings to calculate position). Similar principles were later employed in the development of inertial guidance systems for ballistic missiles.[15]

During World War II, the gyroscope became the prime component for aircraft and anti-aircraft gun sights.[16]

3-axis MEMS-based gyroscopes are also being used in portable electronic devices such as Apple's current generation of iPad and iPhone.[17] This adds to the 3-axis acceleration sensing ability available on previous generations of devices. Together these sensors provide 6 component motion sensing; acceleration for X,Y, and Z movement, and gyroscopes for measuring the extent and rate of rotation in space (roll, pitch and yaw).

Properties

A gyroscope exhibits a number of behaviours including precession and nutation. Gyroscopes can be used to construct gyrocompasses, which complement or replace magnetic compasses (in ships, aircraft and spacecraft, vehicles in general), to assist in stability (Hubble Space Telescope, bicycles, motorcycles, and ships) or be used as part of an inertial guidance system. Gyroscopic effects are used in tops, boomerangs, yo-yos, and Powerballs. Many other rotating devices, such as flywheels, behave in the manner of a g roscope altho gh the g roscopic



A gyroscope in operation with freedom in all three axes. The rotor will maintain its spin axis

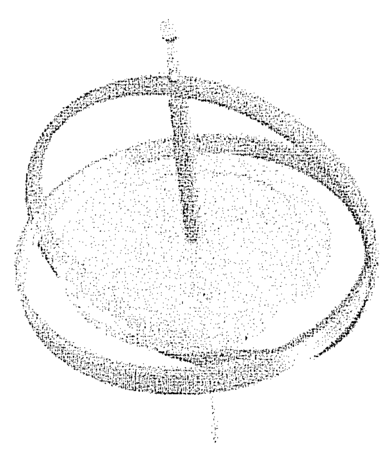
The fundamental equation describing the behavior of the gyroscope is:

$$\tau = \frac{d\mathbf{L}}{dt} = \frac{d(I\omega)}{dt} = I\alpha$$

where the pseudovectors $\boldsymbol{\tau}$ and $\mathbf{L}$ are, respectively, the torque on the gyroscope and its angular momentum, the scalar I is its moment of inertia, the vector $\boldsymbol{\omega}$ is its angular velocity, and the vector $\boldsymbol{\alpha}$ is its angular acceleration.

It follows from this that a torque $\boldsymbol{\tau}$ applied perpendicular to the axis of rotation, and therefore perpendicular to $\mathbf{L}$, results in a rotation about an axis perpendicular to both $\boldsymbol{\tau}$ and $\mathbf{L}$. This motion is called *precession*. The angular velocity of precession $\boldsymbol{\Omega}_{\mathrm{P}}$ is given by the cross product:

$$\tau = \boldsymbol{\Omega}_{\mathrm{P}} \times \mathbf{L}.$$



Precession on a gyroscope

Precession can be demonstrated by placing a spinning gyroscope with its axis horizontal and supported loosely (frictionless toward precession) at one end. Instead of falling, as might be expected, the gyroscope appears to defy gravity by remaining with its axis horizontal, when the other end of the axis is left unsupported and the free end of the axis slowly describes a circle in a horizontal plane, the resulting precession turning. This effect is explained by the above equations. The torque on the gyroscope is supplied by a couple of forces: gravity acting downward on the device's centre of mass, and an equal force acting upward to support one end of the device. The rotation resulting from this torque is not downward, as might be intuitively expected, causing the device to fall, but perpendicular to both the gravitational torque (horizontal and perpendicular to the axis of rotation) and the axis of rotation (horizontal and outwards from the point of support), i.e., about a vertical

Under a constant torque of magnitude τ, the gyroscope's speed of precession Ω_P is inversely proportional to L, the magnitude of its angular momentum:

$$\tau = \Omega_\mathrm{P} L \sin\theta,$$

where θ is the angle between the vectors $\mathbf{\Omega}_\mathrm{P}$ and $\mathbf{L}$. Thus, if the gyroscope's spin slows down (for example, due to friction), its angular momentum decreases and so the rate of precession increases. This continues until the device is unable to rotate fast enough to support its own weight, when it stops precessing and falls off its support, mostly because friction against precession cause another precession that goes to cause the fall.

By convention, these three vectors - torque, spin, and precession - are all oriented with respect to each other according to the right-hand rule.

To easily ascertain the direction of gyro effect, simply remember that a rolling wheel tends, when it leans to the side, to turn in the direction of the lean.

Variations

Gyrostat

A **gyrostat** is a variant of the gyroscope. It consists of a massive flywheel concealed in a solid casing. Its behaviour on a table, or with various modes of suspension or support, serves to illustrate the curious reversal of the ordinary laws of static equilibrium due to the gyrostatic behaviour of the interior invisible flywheel when rotated rapidly. The first gyrostat was designed by Lord Kelvin to illustrate the more complicated state of motion of a spinning body when free to wander about on a horizontal plane, like a top spun on the pavement, or a hoop or bicycle on the road.

MEMS

A MEMS gyroscope takes the idea of the Foucault pendulum and uses a vibrating element, known as a MEMS (Micro Electro-Mechanical System). The MEMS-based gyro was initially made practical and producible by gyroscopes.

FOG

A fiber optic gyroscope (FOG) is a gyroscope that uses the interference of light to detect mechanical rotation. The sensor is a coil of as much as 5 km of optical fiber. The development of low-loss single-mode optical fiber in the early 1970s for the telecommunications industry enabled the development of Sagnac effect fiber optic gyros.

VSG or CVG

A vibrating structure gyroscope (VSG), also called a Coriolis Vibratory Gyroscope (CVG),[18] uses a resonator made of different metallic alloys. It takes a position between the low-accuracy, low-cost MEMS gyroscope and the higher-accuracy and higher-cost FOG. Accuracy parameters are increased by using low-intrinsic damping materials, resonator vacuumization, and digital electronics to reduce temperature dependent drift and instability of control signals.[19]

High-Q Wine-Glass Resonators for precise sensors like HRG [20] or CRG [21] are based on Bryan's "wave inertia effect". They are made from high-purity quartz glass or from single-crystalline sapphire.

DTG

A dynamically tuned gyroscope (DTG) is a rotor suspended by a universal joint with flexure pivots.[22] The flexure spring stiffness is independent of spin rate. However, the dynamic inertia (from the gyroscopic reaction effect) from the gimbal provides negative spring stiffness proportional to the square of the spin speed (Howe and Savet, 1964; Lawrence, 1998). Therefore, at a particular speed, called the tuning speed, the two moments cancel each other, freeing the rotor from torque, a necessary condition for an ideal gyroscope.

London moment

A London moment gyroscope relies on the quantum-mechanical phenomenon, whereby a spinning superconductor generates a magnetic field whose axis lines up exactly with the spin axis of the gyroscopic rotor. A magnetometer determines the orientation of the generated field, which is interpolated to determine the axis of rotation. Gyroscopes of this type can be extremely accurate and stable. For example, those used in the Gravity Probe B experiment measured changes in gyroscope spin axis orientation to better than 0.5 milliarcseconds (1.4×10^{-7} degrees) over a one-year period.[23] This is equivalent to an angular separation the width of a human hair viewed from 32 kilometers (20 mi) away.[24]

The GP-B gyro consists of a nearly-perfect spherical rotating mass made of fused quartz, which provides a dielectric support for a thin layer of niobium superconducting material. To eliminate friction found in conventional bearings, the rotor assembly is centered by the electric field from six electrodes. After the initial spin-up by a jet of helium which brings the rotor to 4,000 RPM, the polished gyroscope housing is evacuated to an ultra-high vacuum to further reduce drag on the rotor. Provided the suspension electronics remain powered, the extreme rotational symmetry, lack of friction, and low drag will allow the angular momentum of the rotor to keep it spinning for about 15,000 years.[25]

A sensitive DC SQUID magnetometer able to discriminate changes as small as one quantum, or about 2×10^{-15} Wb, is used to monitor the gyroscope. A precesses, or tilt, in the orientation of the rotor causes the London moment magnetic field to shift relative to the housing. The moving field passes through a superconducting pickup loop fixed to the housing, inducing a small electric current. The current produces a voltage across a shunt resistance, which is resolved to spherical coordinates by a microprocessor. The system is designed to minimize Lorentz torque on the rotor.[26]

Modern uses

In addition to being used in compasses, aircraft, computer pointing devices, etc., gyroscopes have been introduced into consumer electronics. Since the gyroscope allows the calculation of orientation and rotation, designers have than the previous lone accelerometer within a number of smartphones. Scott Steinberg, known for his critiques on newly released technology, says that the new addition of the gyroscope in the iPhone 4 may "completely redefine the way we interact with downloadable apps".[27]

Nintendo has integrated a gyroscope into the Wii console's Wii Remote controller by an additional piece of hardware called "Wii MotionPlus".[28] It is also included in the 3DS, which detects movement when turning.

See also

- Aerotrim
- Anti-rolling gyro — Ship gyroscopic roll stabilizers.
- Attitude indicator
- Balancing machine
- Countersteering
- Euler angles
- Eric Laithwaite
- Gyro monorail
- Gyrocar
- Gyroscopic exercise tool
- Heading indicator
- Reaction wheel
- Rifling
- Top
- Turn and bank indicator
- Turn coordinator
- LN-3 Inertial Navigation System
- Stabilizer (ship)

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11 ^ L Foucault (1852) *Comptes rendus* vol 35 page 427

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External links

- The Royal Institution's 1974–75 Christmas Lecture (http://www.gyroscopes.org/1974lecture.asp) Professor Eric Laithwaite
- One-Wheeled Robot-Gyrostat (http://demonstrations.wolfram.com/OneWheeledRobotGyrostat/) by Olga Kapustina and Yuri Martynenko Wolfram Demonstrations Project
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Categories: Gyroscopes | 1852 introductions

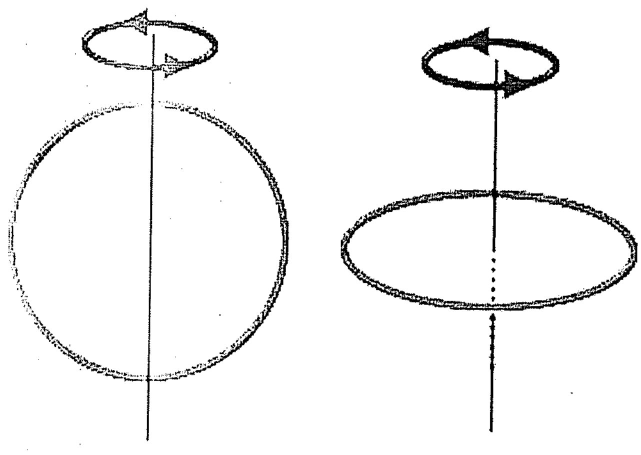
The Motion of a Spinning Top

What drives the motion of a Spinning Top?

There are many forms and shapes of spinning tops, and they are put into motion in an interesting variety of ways. Some are spun by snap-twisting a center stem with your fingers and releasing, while the top remains on the ground. Others are held by a support at the top while a cord wound around the top is pulled to spin it. The spinning top many of us know is launched from about waist level to the floor by snapping your wrist as you release it, while maintaining a grip on the cord wound around its body. However they are spun, each type behaves in a similar fashion.

The physics of rotation.

The body of a top has at least one axis about which it will spin steadily and smoothly. This rotation axis is a symmetry axis of the top, known as a *principal axis*. For example, the red hoop in the figure below has two unique symmetry axes indicated, for rotations of the type specified by the blue arrows.



For each unique symmetry axis, the object has a *moment of inertia* value that determines how it will spin when a *torque* is applied. The way this all works through is described by **Newton's Laws of Rotation**. While this can get pretty complicated in detail, there are some circumstances where the object will spin in a very simple manner. *The object's spin about the rotation axis gives it an angular momentum,* ***which will remain constant*** *until some outside torque works on it.*

The ideal top.

Suppose a top is so perfectly fashioned that its principal rotation axis (spin axis) goes through its center of mass. (The center of mass, also known as the center of gravity, is the balance point of the object.) If we spin this top carefully, so that it remains perfectly upright while spinning (and gravity can't exert a torque on it about its point), it will spin at a steady angular velocity almost indefinitely. Sliding friction between its tip and the floor does slow it gradually. But if the point is very sharp, sliding friction there exerts very little torque on the top about its rotational axis. Because it's unable to exert a torque on the ground, the top can't exchange angular momentum with the earth. It spins on until it slowly gets rid of its angular momentum through sliding friction and air resistance.

(5)

Name: ______________________

Problem 3: Rotational Collision (15 points)

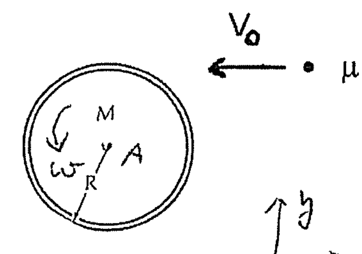
A uniform cylindrical shell (hoop) sits on one of its flat sides on a frictionless surface. The hoop has mass M, radius R and height H. A bullet of mass μ moving horizontally with velocity V_0 strikes the hoop with impact parameter R at mid-height (H/2 from the surface). After the collision the bullet continues with velocity $V_0/2$ in its original direction. Ignore any hole the bullet creates. Give all your answers in terms of M, R, H, V_0, and μ.

a) What was the angular momentum, $\vec{L}$, of the system about the center of the hoop before the collision?

b) What is the linear velocity, $\vec{V}$, of the center of the hoop after the collision?

c) What is the angular velocity, $\vec{\omega}$, of the hoop after the collision?

a) $\boxed{L = \mu V_0 R}$ pointing up (out of paper) relative to A.



Top View

b) $M\vec{V} + \mu \frac{\vec{V_0}}{2} = \vec{V_0}\mu$

$\Rightarrow$ $\boxed{V_x = -\frac{\mu/2}{M} V_0 \qquad V_y = 0}$

c) $I\omega + \mu \frac{V_0}{2} R = \mu V_0 R$

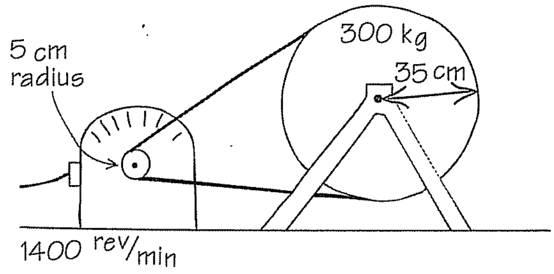
$\omega = \frac{\mu V_0 R}{2I} = \boxed{\frac{\mu V_0}{2MR}}$

60. (II) Electric power is used to speed up a centrifuge whose rotational inertia is 1.2 kg · m^2; 1.6 kW of power were used to make the centrifuge accelerate at a steady rate from rest to 17,000 rev/min. If the electricity use was 100 percent efficient, how much time is required to speed up the centrifuge?

61. (II) A point mass $m = 0.2$ kg is attached to a string, which passes through a hole in a table and rotates in a circle of radius $r = 0.8$ m with an angular velocity of 40 rad/s. What mass M must be attached to the end of the string under the table to maintain this motion? Suppose that mass M is slowly increased by an amount that makes it descend a distance 0.1 m. What is the amount of the increase of M? What will the new angular velocity of the point mass be? [*Hint*: Use angular momentum conservation.]

62. (II) Show that $(\vec{r} \times \vec{p}) \cdot (\vec{r} \times \vec{p}) = r^2p^2 - (\vec{r} \cdot \vec{p})^2$. [*Hint*: It is convenient, without any loss of generality, to assume that both $\vec{r}$ and $\vec{p}$ lie in the xy-plane.] Use this result to express the kinetic energy of a particle in terms of the momentum in the radial direction and of the square of the angular momentum.

63. (II) A cylindrical shaft of radius 5 cm is connected by a band to a solid cylindrical flywheel of mass 300 kg and of radius 0.35 m (Fig. 10–45). A motor brings the shaft up to a rotational rate of 1400 rev/min. Calculate the amount of work done by the motor, neglecting the rotational inertia of the shaft.



▲ **FIGURE 10–45** Problem 63.

64. (II) A thin rod of mass M, length ℓ, and constant density is standing on end on a rough table that forms the xy-plane. The rod begins to fall, with its top moving in the $+x$-direction, but as it falls, its point of contact does not move. As the rod hits the table, what are its (a) angular velocity, (b) angular momentum, and (c) kinetic energy?

65. (II) An object of mass M moves in a circular planar orbit about a center of gravitational attraction. The force of attraction has magnitude $F = K/r^2$, where r is the radius of the circle, and it is directed toward the center. Calculate (a) the velo
dius, (c) the period, T, and (d) the acceleration o
in terms of the angular momentum L, M, and K.

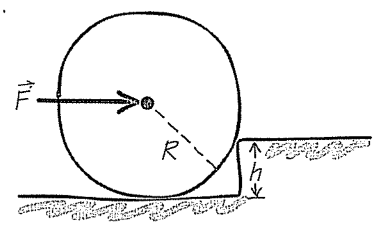
66. (II) An object of mass m moves in a plane wi
scribed by the radius vector $\vec{r} = \hat{i}A\cos\omega_1 t + \hat{j}$
culate the angular momentum about the o
direction will the angular momentum point? U
cumstances will the angular momentum be cons

67. (II) A hurricane is a vast swirl of Earth's atmosph
knowledge of the size of such storms, the dept
phere, the speed of the winds, the density of air, a
timate the kinetic energy contained as well
momentum. Compare your estimate of the angu
with that of Earth itself (see Section 9–5).

68. (III) A putty ball of mass $m = M/5$ is throw
$\vec{v} = v\hat{i}$ and hits the top of the thin rod in Proble
stands vertically. If the putty ball makes a com
collision, and if again the point of contact betw
the table does not move, what are the angular v
momentum, and kinetic energy of the system as i

69. (III) A particular top can be approximated as a s
mass 100 g and radius 2 cm. A string of negligible
1 m is wound around the top, which is started by
tally on the string with a constant force of magni
top starts from rest at point O, and the string is pull
friction between the top and the table on which it r
is the final velocity of the center of mass of the top?
gular velocity of the top about its center of mass?

70. (III) A uniform solid cylinder of radius R and
against a vertical curb of height h, where $h <$
The cylinder is mounted through its axis on a fr
zontal axle. You exert a horizontal force of mag
axle, pushing the cylinder against the curb. What
value of F that will cause the cylinder to roll up



▲ **FIGURE 10–46** Problem 70.

must be constant. In this time interval, $v_{fx} = v_{cm}$ and $v_{ix} = 0$. The acceleration is then

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{v_{fx} - v_{ix}}{t} = \frac{v_{cm} - 0}{t} = \frac{v_{cm}}{t}.$$

The x component of Newton's second law then gives

$$f = Ma_x = \frac{Mv_{cm}}{t}$$

Only the frictional force gives a torque about the center of mass, so the net torque is $\Sigma\, \tau_z = fR$. With $\omega_i = -\omega_0$ and $\omega_f = -v_{cm}/R$ at the instant when rolling without slipping begins (the minus signs indicating that the cylinder is spinning clockwise), the angular acceleration is

$$\alpha_z = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t} = \frac{-v_{cm}/R + \omega_0}{t}.$$

Newton's second law for rotation gives $fR = I_{cm}\alpha_z$. Substituting for f and α_z from the above two equations, we obtain

$$\left(\frac{Mv_{cm}}{t}\right)R = \frac{\frac{1}{2}MR^2(-v_{cm}/R + \omega_0)}{t}$$

using $I_{cm} = \frac{1}{2}MR^2$ from Fig. 9-15. After eliminating common factors we can solve for v_{cm} to find

$$v_{cm} = \tfrac{1}{3}\omega_0 R = \tfrac{1}{3}(15\text{ rev/s})(2\pi\text{ rad/rev})(0.12\text{ m}) = 3.8\text{ m/s}.$$

Note that v_{cm} does not depend on the values of M, g, or μ_k. What, however, would occur if any of these quantities were zero?

(*b*) With $f = Mv_{cm}/t$ and also $f = \mu_k N = \mu_k Mg$, we can eliminate f and solve for t:

$$t = \frac{v_{cm}}{\mu_k g} = \frac{3.8\text{ m/s}}{(0.21)(9.80\text{ m/s}^2)} = 1.8\text{ s}.$$

SAMPLE PROBLEM 9-13. A toy yo-yo* of total mass $M = 0.24$ kg consists of two disks of radius $R = 2.8$ cm connected by a thin shaft of radius $R_0 = 0.25$ cm (Fig. 9-34*a*). A string of length $L = 1.2$ m is wrapped around the shaft. If the yo-yo is thrown downward with an initial velocity of $v_0 = 1.4$ m/s, what is its rotational velocity when it reaches the end of the string?

Solution The free-body diagram for the yo-yo is shown in Fig. 9-34*b*. The net force is $\Sigma\, F_y = Mg - T$ (taking the downward direction to be positive), and the net torque about the center of mass is $\Sigma\, \tau_z = TR_0$ (taking counterclockwise torques to be positive). The translational and rotational forms of Newton's second law then give

$$Mg - T = Ma_y \qquad \text{and} \qquad TR_0 = I\alpha_z.$$

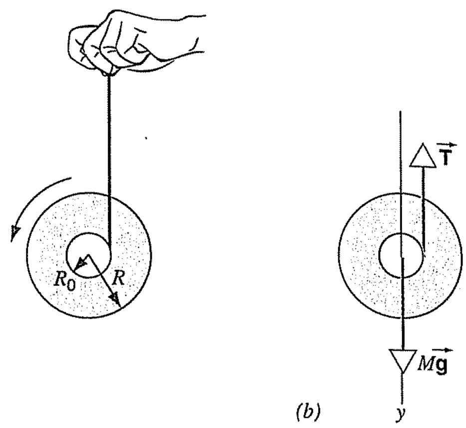


FIGURE 9-34. Sample Problem 9-13. (*a*) A yo-yo falls as the string unwinds from the axle. (*b*) The force diagram.

We consider the string to be of negligible thickness and assume that it does not slip as it is unwinding. The point where the string contacts the shaft is instantaneously at rest, just like the point B in Figs. 9-30 and 9-31. With $v_{cm} = \omega R_0$, it follows that (in magnitudes only) $a_{cm} = \alpha R_0$. In our notation for this problem, $a_{cm} = a_y$ (a positive quantity) and $\alpha = \alpha_z$ (also a positive quantity). Thus, taking $a_y = \alpha_z R_0$ and combining the force and torque equations to eliminate the tension, we solve for the angular acceleration:

$$\alpha_z = \frac{g}{R_0}\frac{1}{1 + I/MR_0^2}.$$

To complete the solution, we need the rotational inertia, which is not given. Let us assume that the thin shaft makes a negligible contribution to I (the mass and radius of the shaft are both small compared to the disks). Then the rotational inertia is $I = \frac{1}{2}MR^2$ and

$$\alpha_z = \frac{g}{R_0}\frac{1}{1 + R^2/2R_0^2}$$

$$= \frac{980\text{ cm/s}^2}{0.25\text{ cm} + (2.8\text{ cm})^2/2(0.25\text{ cm})} = 61.5\text{ rad/s}^2.$$

To find the final angular velocity from this acceleration, we can use Eq. 8-6, $\omega_z = \omega_{0z} + \alpha_z t$, if we know the time t for the yo-yo to unwind. This time can be found from Eq. 8-7, $\phi = \phi_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$. The angle through which the yo-yo rotates as the string unwinds is $\phi - \phi_0 = L/R_0 = 480$ rad, and the initial angular velocity is $\omega_{0z} = v_0/R_0 = (1.4\text{ m/s})(0.0025\text{ m}) = 560$ rad/s. With these substitutions, Eq. 8-7 then gives

$$(30.75\text{ rad/s}^2)t^2 + (560\text{ rad/s})t - 480\text{ rad} = 0.$$

Solving this quadratic equation, we find $t = 0.82$ s or -19 s. The positive value is the physically meaningful one, and so

$$\omega_z = \omega_{0z} + \alpha_z t = 560\text{ rad/s} + (61.5\text{ rad/s}^2)(0.82\text{ s}) = 610\text{ rad/s}.$$

* See "The Yo-Yo: A Toy Flywheel," by Wolfgang Burger, *American Scientist*, March–April 1984, p. 137.

falling block, the tension in the cord, and the angular acceleration of the disk.

Solution Figure 9-26*b* shows a free-body diagram for the block. Note that, in drawing the free-body diagram for the analysis of rotations, it is necessary to show the forces *and* their points of application, so that we may determine the line of action for each force in calculating the corresponding torque. We choose the y axis to be positive downward, so that the net force is $\Sigma F_y = mg - T$, which is a positive quantity if the block accelerates downward. Using the y component of Newton's second law ($\Sigma F_y = ma_y$), we have

$$mg - T = ma_y.$$

Figure 9-26*c* shows a partial free-body diagram for the disk. Choosing the positive z axis to be out of the plane of the figure, the z component of the net torque about O is $\Sigma \tau_z = TR$ (neither the weight of the disk nor the upward force exerted at its point of support contribute to the torque about O, because both their lines of action pass through O). Applying the rotational form of Newton's second law (Eq. 9-11) gives $TR = I\alpha_z$, where α_z is positive for the counterclockwise rotation. With $I = \frac{1}{2}MR^2$ and $\alpha_z = a_T/R$, we obtain $TR = (\frac{1}{2}MR^2)(a_T/R)$ or

$$T = \tfrac{1}{2}Ma_T.$$

Because the cord does not slip or stretch, the acceleration a_y of the block must equal the tangential acceleration a_T of a point on the rim of the disk. With $a_y = a_T = a$, we can combine the equations for the block and the disk to obtain

$$a = g\frac{2m}{M + 2m} = (9.8 \text{ m/s}^2)\frac{(2)(1.2 \text{ kg})}{2.5 \text{ kg} + (2)(1.2 \text{ kg})} = 4.8 \text{ m/s}^2,$$

and

$$T = mg\frac{M}{M + 2m} = (1.2 \text{ kg})(9.8 \text{ m/s}^2)\frac{2.5 \text{ kg}}{2.5 \text{ kg} + (2)(1.2 \text{ kg})}$$
$$= 6.0 \text{ N}.$$

As expected, the acceleration of the falling block is less than g, and the tension in the cord (= 6.0 N) is less than the weight of the hanging block ($= mg = 11.8$ N). We see also that the acceleration of the block and the tension depend on the mass of the disk but not on its radius. As a check, we note that the formulas derived above predict $a = g$ and $T = 0$ for the case of a massless disk ($M = 0$). This is what we expect; the block simply falls as a free body, trailing the cord behind it.

The angular acceleration of the disk follows from

$$\alpha_z = \frac{a}{R} = \frac{4.8 \text{ m/s}^2}{0.20 \text{ m}} = 24 \text{ rad/s}^2 = 3.8 \text{ rev/s}^2$$

and it is positive, corresponding to a rotation in the direction of the arrow in Fig. 9-26*a*.

For rotations about a fixed axis, the angular velocity and acceleration have only one component, and therefore only that same component of the torque enters into Newton's laws. However, we may apply a force to a rigid body in any direction, and there will in general be two or three components to the torque, only one of which actually produces rotations. What happens to the other components?

Consider the bicycle wheel shown in Fig. 9-27. The axle of the wheel is fixed in direction by the two bearings, so that the rotation axis corresponds to the z axis. A force $\vec{F}$ is applied to the wheel in an arbitrary direction, and in general the associated torque may have x, y, and z components, as shown in Fig. 9-27. Each component of the torque tends to produce rotation about its corresponding axis. However, we have assumed that the body is fixed in such a way that rotation about only the z axis is possible. The x and y components of the torque produce no motion. In this case, the bearings serve to constrain the system to rotate about only the z axis, and they must therefore provide torques that cancel the x and y components of the torque from the applied force. This indicates what is meant by a body constrained to move about a fixed axis: only torque components parallel to that axis are effective in rotating the body, and torque components perpendicular to the axis are assumed to be balanced by other parts of the system. The bearings *must* provide torques with x and y components to keep the direction of the axis of rotation fixed; the bearings *may* also provide a torque in the z direction, such as in the case of non-ideal bearings that exert frictional forces on the axle of the wheel. Since the center of mass of the wheel does not move, the forces exerted by the bearings must add to the external forces to give a net force of zero.

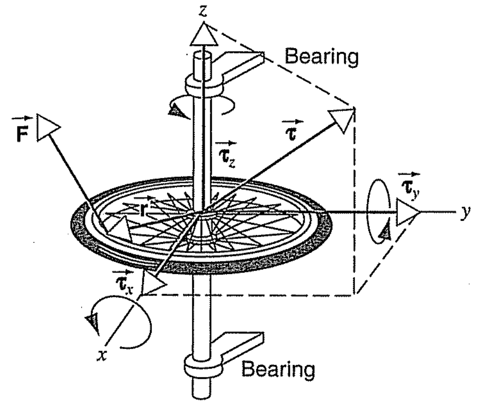


FIGURE 9-27. A rigid body, in this case a wheel, is free to rotate about the z axis. An arbitrary force $\vec{F}$, shown acting at a point on the rim, can produce torque components along the three coordinate axes. Only the z component is successful in rotating the wheel. The x and y components of the torque would tend to tip the axis of rotation away from the z axis. This tendency must be opposed by equal and opposite torques (not shown) exerted by the bearings, which hold the axis in a fixed direction.

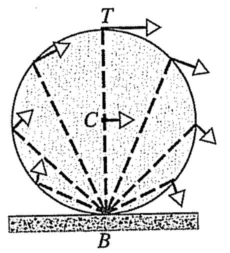


FIGURE 9-31. A rolling body can be considered to be rotating about an instantaneous axis at the point of contact B. The vectors show the instantaneous linear velocities of selected points.

SAMPLE PROBLEM 9-11. A solid cylinder of mass M and radius R starts from rest and rolls without slipping down an inclined plane of length L and height h (Fig. 9-32). Find the speed of its center of mass when the cylinder reaches the bottom.

Solution The free-body diagram of Fig. 9-32b shows the forces acting on the cylinder: the weight $M\vec{g}$, the normal force $\vec{N}$, and the frictional force $\vec{f}$. Based on the choice of x and y axes shown in the figure, the components of the net force on the cylinder are $\Sigma F_x = Mg \sin\theta - f$ and $\Sigma F_y = N - Mg\cos\theta$. If we apply Newton's second law with $a_x = a_{cm}$ and $a_y = 0$, we obtain for the x and y equations

$$Mg\sin\theta - f = Ma_{cm} \quad \text{and} \quad N - Mg\cos\theta = 0.$$

To find the net torque about the center of mass, we note that the lines of action of both $\vec{N}$ and $M\vec{g}$ pass through the center of mass and so their moment arms are zero. Only the frictional force contributes to the torque, and so $\Sigma \tau_z = -fR$. Newton's second law for rotation then gives

$$-fR = I_{cm}\alpha_z.$$

In Fig. 9-32, the z axis is out of the page and so α_z is indeed negative. The condition for rolling without slipping is $v_{cm} = \omega R$; differentiating this expression gives $a_{cm} = \alpha R$, which relates the magnitudes of a_{cm} and α. Substituting $\alpha_z = -a_{cm}/R$ and $I_{cm} = \frac{1}{2}MR^2$ (for a cylinder), we find

$$f = -\frac{I_{cm}\alpha_z}{R} = -\frac{(\frac{1}{2}MR^2)(-a_{cm}/R)}{R} = \frac{1}{2}Ma_{cm}.$$

Substituting this into the first translational equation, we find

$$a_{cm} = \tfrac{2}{3}g\sin\theta.$$

That is, the acceleration of the center of mass for the rolling cylinder ($\frac{2}{3}g\sin\theta$) is less than its acceleration would be if the cylinder were sliding down the incline ($g\sin\theta$). This result holds at any instant, regardless of the position of the cylinder along the incline.

Because the acceleration is constant, we can use the equations of Chapter 2 to find the velocity. With $v_{0x} = 0$ and taking $x - x_0 = L$ (where the x axis lies along the plane), Eqs. 2-26 and 2-28 respectively become $v_{cm} = a_{cm}t$ and $L = \frac{1}{2}a_{cm}t^2$. Solving the second equation for the time t, we find $t = \sqrt{2L/a_{cm}}$. With this result the first equation gives

$$v_{cm} = a_{cm}t$$
$$= a_{cm}\sqrt{\frac{2L}{a_{cm}}} = \sqrt{2La_{cm}} = \sqrt{2L(\tfrac{2}{3}g\sin\theta)} = \sqrt{\tfrac{4}{3}Lg\sin\theta}$$

This method also determines the force of static friction needed for rolling:

$$f = \tfrac{1}{2}Ma_{cm} = (\tfrac{1}{2}M)(\tfrac{2}{3}g\sin\theta) = \tfrac{1}{3}Mg\sin\theta.$$

What would happen if the force of static friction between the surfaces were less than this value?

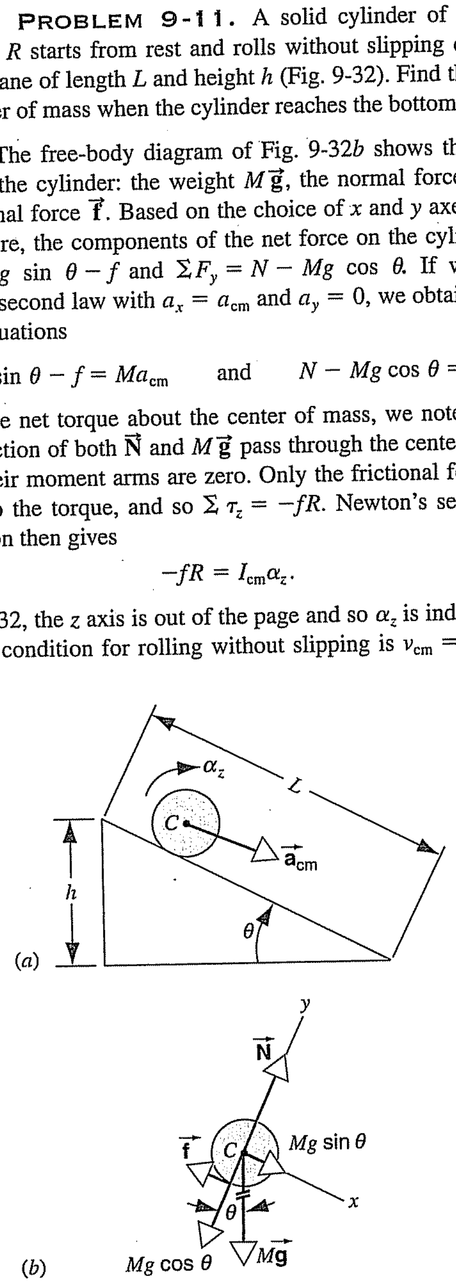


FIGURE 9-32. Sample Problem 9-11. (a) A cylinder rolls without slipping down the incline. (b) The free-body diagram of the cylinder.

SAMPLE PROBLEM 9-12. A uniform solid cylinder of radius R ($=12$ cm) and mass M ($=3.2$ km) is given an initial (clockwise) angular velocity ω_0 of 15 rev/s and then lowered on to a uniform horizontal surface (Fig. 9-33). The coefficient of kinetic friction between the surface and the cylinder is $\mu_k = 0.21$. Initially, the cylinder slips as it moves along the surface, but after a time t, pure rolling without slipping begins. (a) What is the velocity v_{cm} of the center of mass at the time t? (b) What is the value of t?

Solution (a) Figure 9-33b shows the forces that act on the cylinder. The x and y components of the net force are $\Sigma F_x = f$ and $\Sigma F_y = N - Mg$. During the interval from time 0 to time t while slipping occurs, the forces are constant and so the acceleration

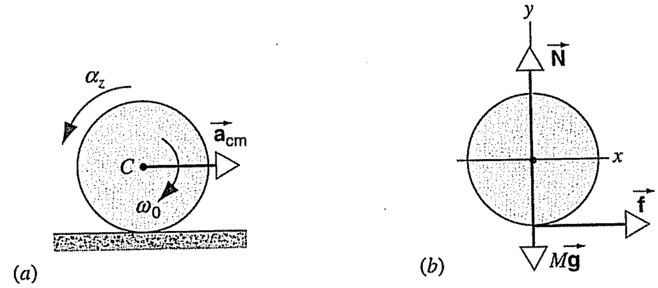


FIGURE 9-33. Sample Problem 9-12. (a) The rotating cylinder initially slips as it rolls. (b) The free-body diagram of the cylinder.

Body Rolling Down an Inclined Plane

As an illustration of laminar motion, we shall study the motion of a round object (cylinder, ball, and so on) rolling down an inclined plane. As shown in Figure 8.14, there are three forces acting on the body. These are (1) the downward force of gravity, (2) the

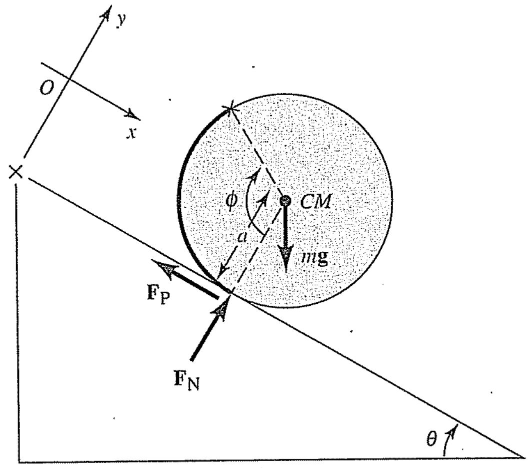


Figure 8.14 Body rolling down an inclined plane.

normal reaction of the plane: $\mathbf{F}_N$, and (3) the frictional force parallel to the plane: $\mathbf{F}_P$. Choosing axes as shown, the component equations of the translation of the center of mass are

$$m\ddot{x}_{cm} = mg \sin \theta - F_P \tag{8.60}$$

$$m\ddot{y}_{cm} = -mg \cos \theta + F_N \tag{8.61}$$

where θ is the inclination of the plane to the horizontal. Since the body remains in contact with the plane, we have

$$y_{cm} = \text{constant}$$

Hence,

$$\ddot{y}_{cm} = 0$$

Therefore, from Equation 8.61,

$$F_N = mg \cos \theta \tag{8.62}$$

The only force that exerts a moment about the center of mass is the frictional force $\mathbf{F}_P$. The magnitude of this moment is $F_P a$ where a is the radius of the body. Hence, the rotational equation (Equation 8.59) becomes

$$I_{cm}\dot{\omega} = F_P a \tag{8.63}$$

To discuss the problem further, we need to make some assumptions regarding the contact between the plane and the body. We shall solve the equations of motion for two cases.

Motion with No Slipping

If the contact is very rough so that no slipping can occur, that is, if $F_P \leq \mu_s F_N$ where μ_s is the coefficient of *static* friction, we have the following relations:

$$\dot{x}_{cm} = a\dot{\phi} = a\omega \tag{8.64}$$

$$\ddot{x}_{cm} = a\ddot{\phi} = a\dot{\omega}$$

where ϕ is the angle of rotation. Equation 8.63 can then be written

$$\frac{I_{cm}}{a^2}\ddot{x}_{cm} = F_P \tag{8.65}$$

Substituting the above value for F_P into Equation 8.60 yields

$$m\ddot{x}_{cm} = mg \sin \theta - \frac{I_{cm}}{a^2}\ddot{x}_{cm}$$

Solving for $\ddot{x}_{cm}$, we find

$$\ddot{x}_{cm} = \frac{mg \sin \theta}{m + (I_{cm}/a^2)} = \frac{g \sin \theta}{1 + (k_{cm}^2/a^2)} \tag{8.66}$$

10-5 THE SPINNING TOP*

A spinning top provides us with what is perhaps the most familiar example of the phenomenon shown in Fig. 10-4*b*, in which a lateral torque changes the direction but not the magnitude of an angular momentum. Figure 10-18*a* shows a top spinning about its axis. The bottom point of the top is assumed to be fixed at the origin O of our inertial reference frame. We know from experience that the axis of this rapidly spinning top will move slowly about the vertical axis. This motion is called *precession,* and it arises from the configuration illustrated in Fig. 10-4*b*, with gravity supplying the external torque.

Figure 10-18*b* shows a simplified diagram, with the top replaced by a particle of mass M located at the top's center of mass. The gravitational force Mg gives a torque about O of magnitude.

$$\tau = Mgr \sin \theta. \tag{10-18}$$

The torque, which is perpendicular to the axis of the top and therefore perpendicular to $\vec{\mathbf{L}}$ (Fig. 10-18*c*), can change the direction of $\vec{\mathbf{L}}$ but not its magnitude. The change in $\vec{\mathbf{L}}$ in a small increment of time dt is given by

$$d\vec{\mathbf{L}} = \vec{\tau}\, dt \tag{10-19}$$

and is in the same direction as $\vec{\tau}$—that is, perpendicular to $\vec{\mathbf{L}}$. The effect of $\vec{\tau}$ is therefore to change $\vec{\mathbf{L}}$ to $\vec{\mathbf{L}} + d\vec{\mathbf{L}}$, a vector of the same length as $\vec{\mathbf{L}}$ but pointing in a slightly different direction.

If the top has axial symmetry, and if it rotates about its axis at high speed, then the angular momentum will be along the axis of rotation of the top. As $\vec{\mathbf{L}}$ changes direc-

* See "The Amateur Scientist: The Physics of Spinning Tops, Including Some Far-Out Ones," by Jearl Walker, *Scientific American,* March 1981, p. 185.

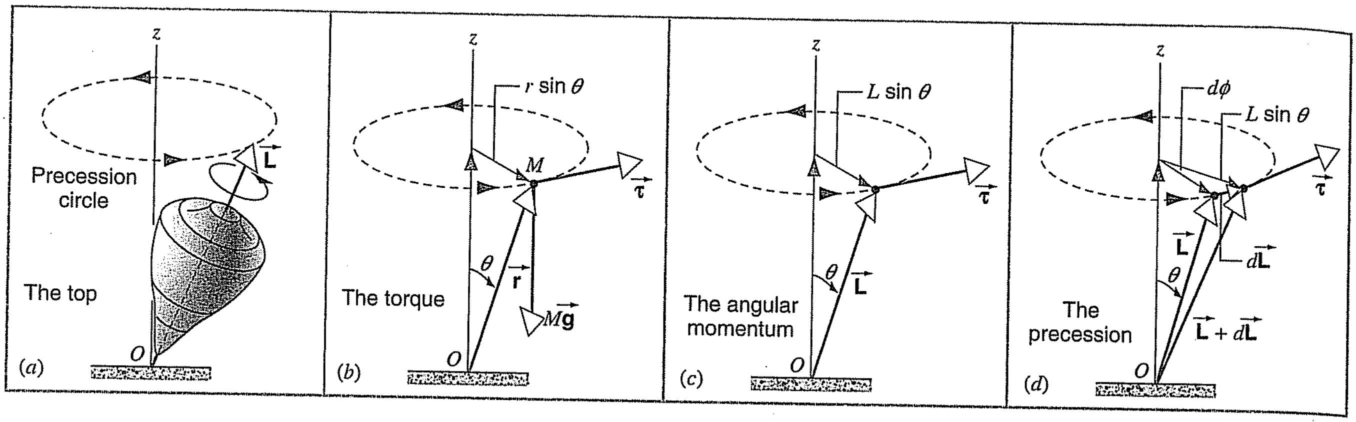


FIGURE 10-18. (*a*) A spinning top precesses about a vertical axis. (*b*) The weight of the top exerts a torque about the point of contact with the floor. (*c*) The torque is perpendicular to the angular momentum vector. (*d*) The torque changes the direction of the angular momentum vector, causing precession.

tion, the axis changes direction too. The tip of the $\vec{\mathbf{L}}$ vector and the axis of the top trace out a circle about the z axis, as shown in Fig. 10-18*a*. This motion is the precession of the top.

In a time dt, the axis rotates through an angle $d\phi$ (see Fig. 10-18*d*), and thus the angular speed of precession ω_P is

$$\omega_P = \frac{d\phi}{dt}. \qquad (10\text{-}20)$$

From Fig. 10-18*d*, we see that

$$d\phi = \frac{dL}{L \sin\theta} = \frac{\tau\, dt}{L \sin\theta}. \qquad (10\text{-}21)$$

Thus

$$\omega_P = \frac{d\phi}{dt} = \frac{\tau}{L \sin\theta} = \frac{Mgr \sin\theta}{L \sin\theta} = \frac{Mgr}{L}. \qquad (10\text{-}22)$$

The precessional speed is inversely proportional to the angular momentum and thus to the rotational angular speed; the faster the top is spinning, the slower it will precess. Conversely, as friction slows down the rotational angular speed, the precessional angular speed increases.

Equation 10-22 gives the relationship between the magnitudes of $\vec{\omega}_P$, $\vec{\mathbf{L}}$, and $\vec{\tau}$. These quantities are vectors, and the vector relationship among them is

$$\vec{\tau} = \vec{\omega}_P \times \vec{\mathbf{L}}. \qquad (10\text{-}23)$$

You should be able to show that this relationship is consistent with the relationship between the magnitudes (Eq. 10-22) and also with the directions of the vectors given in Fig. 10-18. Note that for precessional motion about the z axis, the vector $\vec{\omega}_P$ is in the z direction.

Precession is commonly observed for spinning tops and gyroscopes. Even the Earth can be considered to be a spinning top, and the gravitational pull of the Sun and Moon on the tidal bulges near the equator causes a precession (called in astronomy the "precession of the equinoxes") in which the Earth's rotational axis traces out the surface of a cone (as in Fig. 10-18) with half-angle $\theta = 23.5°$, taking about 26,000 years to complete a full cycle.

There are two components to the angular momentum of the top: its rotational angular momentum about its symmetry axis and the precessional angular momentum. The total angular momentum is the sum of these two vectors, which in general does not lie along the symmetry axis of the top. Therefore our assumption that the symmetry axis of the top follows the direction of the angular momentum vector is not quite correct. If, however, the precessional angular momentum is much smaller than the rotational angular momentum of the top, there is only a very small deviation between the direction of the symmetry axis and the direction of the angular momentum. This small deviation causes a slight oscillation, called a *nutation,* of the axis of the top back and forth about the precessional circle.

Appendix I

7.7 The Tennis Racket Theorem

The solution of Euler's equations for a rigid body with unequal principal moments of inertia can be beautifully illustrated with a tennis or badminton racket. The three principal axes of a tennis racket are readily identified to be (1) along the handle, (2) perpendicular to the handle in the plane of the strings, and (3) perpendicular to the handle and strings. When a tennis racket is tossed into the air with a spin about one of the principal axes, a curious phenomenon is observed. If the initial spin is about either axis (1) or axis (3), the racket continues to spin uniformly about the initial axis and can easily be recaught. On the other hand, if the initial spin is about axis (2), the motion quickly becomes irregular, with spin developing about all three principal axes, which makes it difficult to catch the falling racket. The explanation of the observed behavior follows from Euler's equations. To apply Euler's equations to the tennis racket, we choose the origin of the principal-axes coordinate system at the CM of the racket, as illustrated in Fig. 7-13.

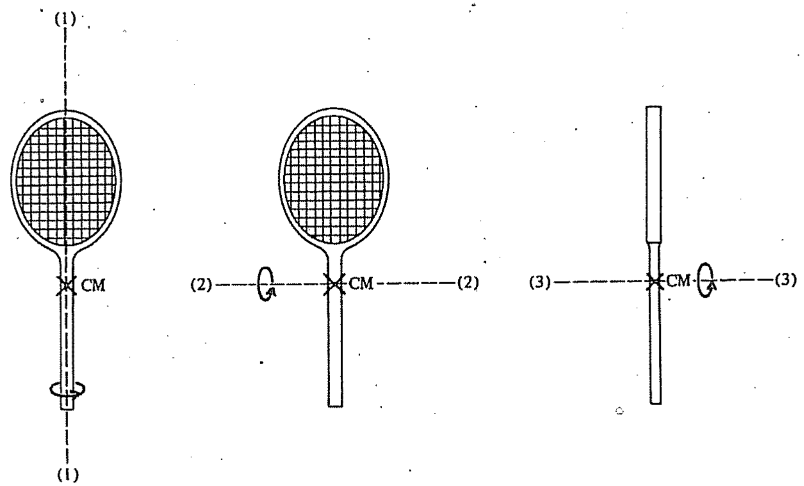


FIGURE 7-13. Principal axes of the tennis racket.

Since gravity is a uniform force in the vicinity of the earth's surface, there are no gravitational torques about the CM of the racket. If we neglect torques due to wind resistance, Euler's equations (7.75) simplify to

$$\begin{aligned} I_1\dot\omega_1 + (I_3 - I_2)\omega_3\omega_2 &= 0 \\ I_2\dot\omega_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot\omega_3 + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned} \tag{7.88}$$

These equations can be written as

$$\dot\omega_1 + r_1\omega_2\omega_3 = 0; \quad r_1 = \frac{I_3 - I_2}{I_1} \tag{7.89}$$

$$\dot\omega_2 - r_2\omega_3\omega_1 = 0; \quad r_2 = \frac{I_3 - I_1}{I_2} \tag{7.90}$$

$$\dot\omega_3 + r_3\omega_1\omega_2 = 0; \quad r_3 = \frac{I_2 - I_1}{I_3} \tag{7.91}$$

If we assume the ordering

$$I_1 \le I_2 \le I_3 \tag{7.92}$$

we consequently have all the r_i positive. We note the sign asymmetry in the three Euler equations (7.89)–(7.91).

The tennis racket theorem concerns stability of spin about the principal axes. We assume that the racket is initially spun nearly about one of the principal axes and we will determine if this spin state is stable. Assume first that the spin is initially nearly along the intermediate axis (2) or

$$\boldsymbol{\omega} \simeq \omega_2\hat{\mathbf{y}}; \quad \omega_1, \omega_3 \ll \omega_2 \tag{7.93}$$

If ω_1 and ω_3 are small as hypothesized their product is negligible and (7.90) implies that

$$\omega_2 \simeq \text{constant} \tag{7.94}$$

Equations (7.89) and (7.91) then comprise a set of coupled linear equations in ω_1 and ω_3

$$\begin{aligned} \dot\omega_1 + (r_1\omega_2)\omega_3 &= 0 \\ \dot\omega_3 + (r_3\omega_2)\omega_1 &= 0 \end{aligned} \tag{7.95}$$

For the trial exponential solution

$$\begin{aligned} \omega_1 &= a_1 e^{\lambda t} \\ \omega_3 &= a_3 e^{\lambda t} \end{aligned} \tag{7.96}$$

two algebraic equations must be satisfied

$$\begin{aligned} a_1\lambda + r_1\omega_2 a_3 &= 0 \\ a_3\lambda + r_3\omega_2 a_1 &= 0 \end{aligned} \tag{7.97}$$

Solving both equations for the ratio a_3/a_1, we obtain

$$\frac{a_3}{a_1} = -\frac{\lambda}{r_1\omega_2} = -\frac{r_3\omega_2}{\lambda} \tag{7.98}$$

The second equality determines λ to be

$$\lambda = \pm\omega_2\sqrt{r_1 r_3} \tag{7.99}$$

Then from (7.98) the ratio of amplitudes

$$a_3 = \mp a_1\sqrt{\frac{r_3}{r_1}} \tag{7.100}$$

The general solution is a superposition of these two solutions

$$\begin{aligned} \omega_1(t) &= ae^{\omega_2\sqrt{r_1r_3}t} + be^{-\omega_2\sqrt{r_1r_3}t} \\ \omega_3(t) &= \sqrt{\frac{r_3}{r_1}}\left[-ae^{\omega_2\sqrt{r_1r_3}t} + be^{-\omega_2\sqrt{r_1r_3}t}\right] \end{aligned} \tag{7.101}$$

where a and b are constants determined by the initial conditions. Since r_1 and r_3 are positive the solution is a superposition of increasing and decreasing exponentials in time. The increasing exponential term will make ω_1 and ω_3 large even if they started small and hence *rotation about the intermediate axis (2) is unstable.* Our solution for axis (2) is of course strictly valid only at times for which the product $\omega_1\omega_3$ is small.

Next take the initial angular velocity nearly along one of the extreme principal moment axes, say axis (1)

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{x}}\,, \qquad \omega_2, \omega_3 \ll \omega_1 \tag{7.102}$$

By (7.89) we obtain

$$\omega_1 \simeq \text{constant} \tag{7.103}$$

and ω_2 and ω_3 satisfy

$$\begin{aligned} \dot{\omega}_2 - r_2\omega_1\omega_3 &= 0 \\ \dot{\omega}_3 + r_3\omega_1\omega_2 &= 0 \end{aligned} \tag{7.104}$$

Proceeding as before, we know the exponential solutions analogous to (7.96) must satisfy (7.104) except that in this case

$$\begin{aligned} \lambda &= \pm\omega_1\sqrt{-r_2r_3} = \pm i\omega_1\sqrt{r_2r_3} \\ a_3 &= \pm a_2\sqrt{r_3/r_2} \end{aligned} \tag{7.105}$$

The solutions now are oscillatory and can be written in the form

$$\begin{aligned} \omega_1(t) &= a\sin\left(\omega_1\sqrt{r_2r_3}t + \alpha\right) \\ \omega_3(t) &= \sqrt{\frac{r_3}{r_2}}a\cos\left(\omega_1\sqrt{r_2r_3} + \alpha\right) \end{aligned} \tag{7.106}$$

where a and α are constants determined by the initial conditions. Thus if ω_2 and ω_3 are initially small, they will remain small. *Rotation about axis (1) is thus stable.*

For an initial spin about axis (3) the solution to Euler's equations is similar to the case of axis (1). The exponential factor λ is again purely imaginary and the solutions are oscillatory. We conclude that *rotations along extreme axes are stable while the intermediate axis is unstable.*

For spin about axis (1) the angular velocity vector $\boldsymbol{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$ precesses in a small cone about principal axis (1) as shown in Fig. 7-14. A similar picture applies to the largest principal axis (3). For rotation along principal axis (2) the angular velocities about axes (1) and (3) grow rapidly with time and the racket tumbles.

To calculate the principal moments of inertia for our specific case of the tennis racket we use a grossly simplified model for the racket. We represent the mass distribution of the racket by a circular hoop of radius a and mass m_a connected to a thin rod of length ℓ and mass m_ℓ. The total mass of the racket is $M = m_a + m_\ell$. The CM of the racket is located on principal axis (1) at a distance R from the center of the hoop, where

$$MR = m_a(0) + m_\ell\left(a + \frac{\ell}{2}\right) \tag{7.107}$$

or

$$R = \frac{m_\ell}{M}\left(a + \frac{\ell}{2}\right) \tag{7.108}$$

The moment of inertia of the racket about principal axis (1) comes entirely from the hoop. We use the perpendicular-axis rule in (6.128) to obtain

$$I_1 = \tfrac{1}{2}m_a a^2 \tag{7.109}$$

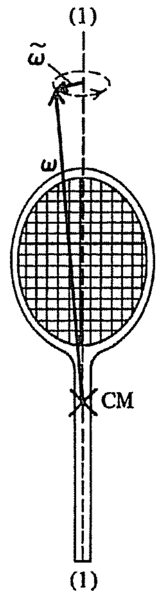


FIGURE 7-14. Stable precession of the angular velocity ω about principal axis (1) of the tennis racket.

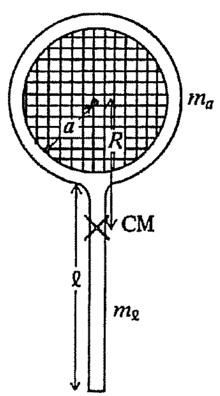


FIGURE 7-15. Dimensions of tennis racket model.

To compute the moment of inertia about principal axis (2), we will make use of the parallel-axis rule of (6.123). The moment of inertia of the hoop about an axis through its CM and parallel to principal axis (2) is $\frac{1}{2}m_a a^2$. By (6.123), the hoop makes a contribution to I_2 of

$$I_2^{hoop} = \tfrac{1}{2}m_a a^2 + m_a R^2 \tag{7.110}$$

since R is the perpendicular distance between the two parallel axes. The moment of inertia of the handle about an axis parallel to principal axis (2) passing through the CM of the handle is $\frac{1}{12}m_\ell \ell^2$. Again using (6.123), we find that the contribution of the handle to I_2 is

$$I_2^{handle} = \tfrac{1}{12}m_\ell \ell^2 + m_\ell\left(a + \frac{\ell}{2} - R\right)^2 \tag{7.111}$$

where $a + \ell/2 - R$ is the distance between these parallel axes. Combining (7.110) and (7.111) and substituting for R from (7.108), we obtain

$$I_2 = \tfrac{1}{2}m_a a^2 + m_a\left(\frac{m_\ell}{M}\right)^2\left(a + \frac{\ell}{2}\right)^2 + \tfrac{1}{12}m_\ell \ell^2 + m_\ell\left(\frac{m_a}{M}\right)^2\left(a + \frac{\ell}{2}\right)^2$$

This can be further simplified to

$$I_2 = \tfrac{1}{2}m_a a^2 + \tfrac{1}{12}m_\ell \ell^2 + \frac{m_a m_\ell}{M}\left(a + \frac{\ell}{2}\right)^2 \tag{7.112}$$

Finally, for principal axis (3), the racket lies in a plane perpendicular to the axis, and we can use the perpendicular-axis rule of (6.128) to obtain

$$I_3 = I_1 + I_2 \tag{7.113}$$

By comparison of (7.109), (7.112), and (7.113), we see that the principal moments of inertia are ordered as $I_1 < I_2 < I_3$. Characteristic parameters for our tennis racket model are

$$\begin{aligned} a &= 0.13\,\text{m} & \ell &= 0.38\,\text{m} \\ R &= 0.18\,\text{m} & M &= 0.33\,\text{kg} \\ m_\ell &= 0.18\,\text{kg} & m_a &= 0.15\,\text{kg} \end{aligned} \tag{7.114}$$

The principal moments of inertia from (7.109), (7.112), (7.113), and (7.114) are

$$\begin{aligned} I_1 &= 0.13 \times 10^{-2}\text{kg}\cdot\text{m}^2 \\ I_2 &= 1.24 \times 10^{-2}\text{kg}\cdot\text{m}^2 \\ I_3 &= 1.37 \times 10^{-2}\text{kg}\cdot\text{m}^2 \end{aligned} \tag{7.115}$$

The condition of stability of the motion about a principal axis which has either the largest or the smallest moment of inertia and the instability