Angular Momentum

1. Single Particle

$\vec{L}_o = \vec{r} \times \vec{p}$ definition

$\downarrow$

respect to a point

$\frac{d\vec{L}_o}{dt} = \vec{r} \times \vec{F} = \vec{\tau}_o = \vec{N}_o$ Equation of motion

2. A System of Particles

$\vec{L}_{tot} = \sum m_i \vec{r}_i \times \vec{v}_i = \sum_i \vec{r}_i \times \vec{P}_i$ definition

$$\frac{d\vec{L}_{tot}}{dt} = \sum \vec{r}_i \times \vec{F}_i^{\,ext}$$

3. Seperation of the c.m. part

$$\vec{L}_o = \sum \vec{r}_i \times m_i \vec{v}_i$$

Define $\vec{\rho}_i = \vec{r}_i - \vec{R}_{c.m}$

$$\begin{aligned}
\vec{L}_o &= \sum_i \vec{r}_i \times m_i \vec{v}_i \\
&= \sum_i (\vec{\rho}_i + \vec{R}_{c.m}) \times m_i \vec{v}_i \\
&= \underbrace{\sum_i \vec{\rho}_i \times m_i \vec{v}_i}_{\vec{L}_{c.m.}} + \underbrace{\sum \vec{R}_{c.m} \times m_i \vec{v}_i}_{= \vec{R}_{c.m.} \times \vec{P}}
\end{aligned}$$

$\Uparrow$ definition

$$\frac{d\vec{L}_o}{dt} = \frac{d}{dt}\vec{L}_{c.m.} + \frac{d}{dt}(\vec{R}_{cm.} \times \vec{P})$$

$$\frac{d}{dt}(\vec{R}_{cm.} \times \vec{P}) = \frac{d\vec{R}_{c.m.}}{dt} \times \vec{P} + \vec{R}_{c.m} \times \frac{d\vec{P}}{dt}$$

$$\frac{d\vec{R}_{c.m.}}{dt} \times \vec{P} = \frac{1}{M}\vec{P} \times \vec{P} = 0$$

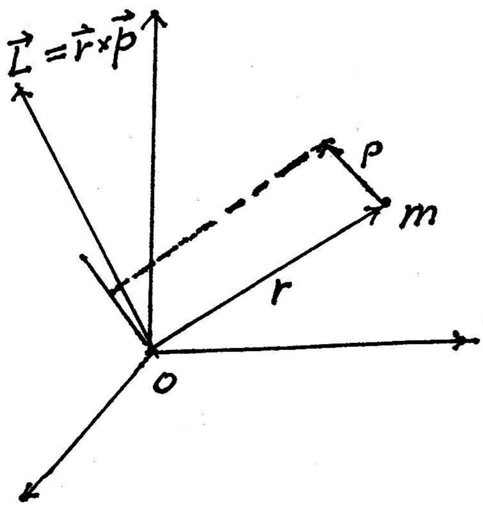
$$\vec{R}_{c.m} \times \frac{d\vec{P}}{dt} = \vec{R}_{c.m} \times \vec{F}_{tot}^{\,ext} = \vec{\tau}_{tot}^{\,ext} - \vec{\tau}_{c.m}^{\,ext}$$

$$\begin{aligned}
\vec{\tau}_{tot} &= \sum \vec{r}_i \times \vec{F}_i^{\,ext} \\
&= \sum_i (\vec{R}_{c.m} + \vec{\rho}_i) \times \vec{F}_i^{\,ext} \\
&= (\vec{R}_{cm} \times \vec{F}_{tot} + \sum_i \vec{\rho}_i \times \vec{F}_i^{\,ext})
\end{aligned}$$

$$\frac{d\vec{L}_o}{dt} = \tau_{tot}^{\,ext} \quad ; \quad \frac{d\vec{L}_{c.m}}{dt} = \tau_{c.m}^{\,ext}$$

Angular Momentum for a Particle

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$



Since $\vec{L}$ depends on the reference point O

↓ should say

the angular momentum with respect to O

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

↓ torque on the particle with respect to the point O

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{\tau} \, (= \vec{N})$$

Angular momentum theorem

$$\frac{d\vec{L}}{dt} = \vec{N} = \vec{\tau}$$

The rate of change of the angular momentum of a particle around some point O equals the torque on the particle with respect to O.

Conservation of Angular Momentum

If no torque acts on a particle, the angular momentum of that particle with respect to O is constant in time.

Example

(i) Central force problem

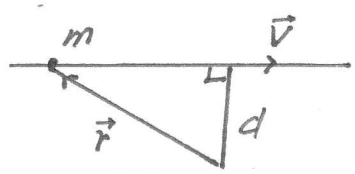
$\downarrow$

the orbits of the planets around the Sun are planar

and

Kepler's second law

(ii) Angular momentum is not solely associated with closed orbits.



A free particle of mass m moves along a straight line with velocity $\vec{v}$ in an inertial frame

$$\vec{L}_o = \vec{r} \times m\vec{v}$$

$\downarrow$

$\perp$ to the plane pointing into the plane of the paper

$$L_o = |\vec{L}_o| = m v d$$

$\uparrow$

is the $\perp$ distance from the line to O

We have used the result when we discussed the Rutherford Scattering.

Torque and Angular Momentum Around an Axis

Torque around an axis is simply the projection of the torque vector on the line.

$$\vec{\tau} = \vec{r} \times F$$
$$\parallel$$
$$\vec{N}$$

In Cartesian coordinates

$$N_x = yF_z - zF_y$$
$$N_y = zF_x - xF_z$$
$$N_z = xF_y - yF_x$$

$\downarrow$

projection of $\vec{N}$ on z axis

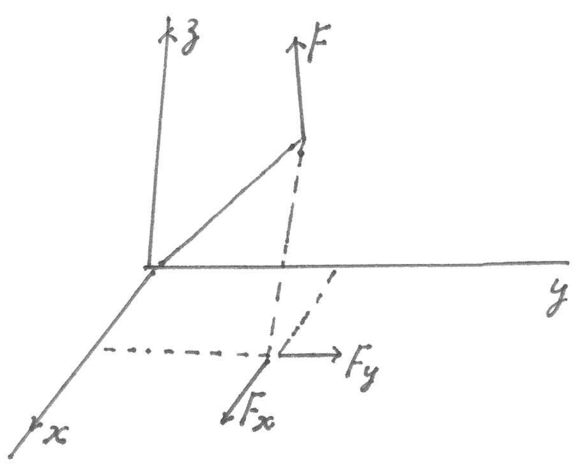
$\Downarrow$

torque around the z axis

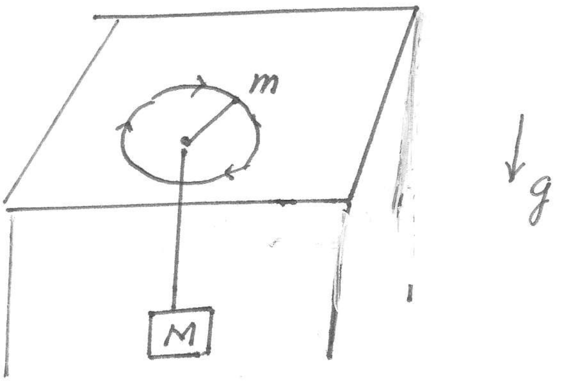
F_y acts over the arm x $\Longrightarrow$ "create" a rotation in the positive direction around the z-axis

F_x acts over the arm y $\Rightarrow$ "create" a rotation in the negative direction around the z-axis

In angular momentum around some axis, this is the projection of $\vec{L}$ onto the axis



Example



Particle of mass in circular motion on a smooth horizontal table

↓

uniform circular motion with velocity v_0 radius R_0

$R_0 \to R_1$

(a) Calculate v_1

(b) Calculate increase in K.E. of the particle

(c) Calculate the work done by the string.

(a) string force has no torque around the center of the circle $\Rightarrow$ angular momentum is conserved

$$m v_0 R_0 = m v_1 R_1 = m v r$$

(b) $$\Delta K.E. = \frac{1}{2} m (v_1^2 - v_0^2) = \frac{1}{2} m v_0^2 \left(\frac{R_0^2}{R_1^2} - 1 \right)$$

(c) $$W = \int \vec{F} \cdot d\vec{r} = -\int_{R_0}^{R_1} m \frac{v^2}{r} dr$$

$$= -\int_{R_0}^{R_1} m \frac{v_0^2 R_0^2}{r^2 r} dr = \frac{1}{2} m v_0^2 \left[\frac{R_0^2}{R_1^2} - 1 \right]$$

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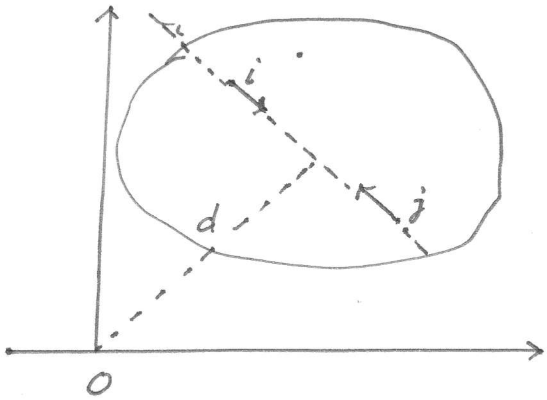
The Angular Momentum Theorem for a System of Particles



The contribution of the internal force to the torque vanishes

$$\vec{N}_{ij} = \vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}$$
$$= (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \qquad (\vec{F}_{ij} = -\vec{F}_{ij})$$
$$= 0$$

since $(\vec{r}_i - \vec{r}_j)$ and $\vec{F}_{ij}$ are parallel.

$$\Rightarrow \quad \frac{d}{dt} \vec{L}^{tot} = \sum_i N_i^{ext}$$

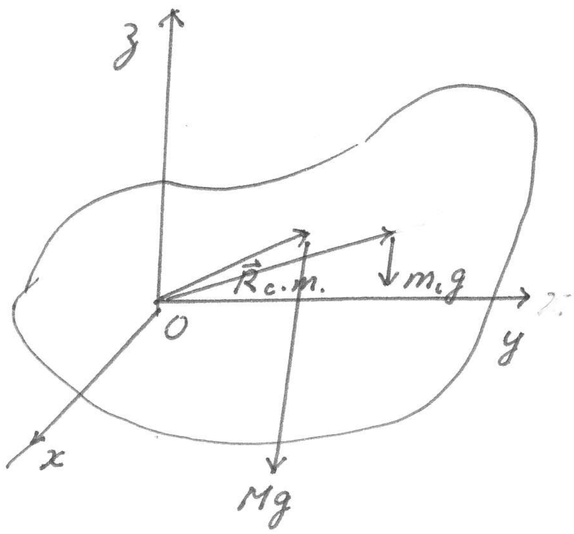
The time derivative of total angular momentum
for a particle system relative to a point O
fixed in an inertial frame, equals the sum
of the torques of the external forces, around O

When the sum of external torque around a point O
(fixed in an inertial frame) vanishes, the total
angular momentum of the particle system relative
to the point O is a constant of motion.

No external forces act on a system of particles
$\Rightarrow$ a closed system.

For a closed system the total angular momentum
is conserved.

Center of Gravity



$$\vec{N}_0 = \sum m_i \vec{r}_i \times \vec{g}$$
$$= M\vec{R}_{c.m.} \times \vec{g}$$
$$= \vec{R}_{c.m.} \times Mg$$

The total torque of gravity on a body can be calculated by assuming the entire mass to be concentrated at the center of mass.

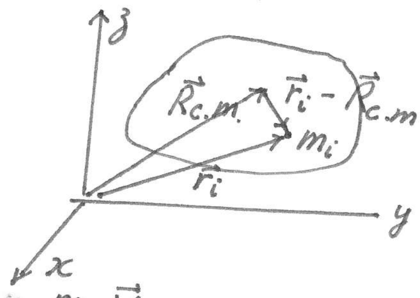
$\Rightarrow$ center of mass $\longleftrightarrow$ center of gravity

$\Rightarrow$ torque of gravity about the C.M is zero

$\downarrow$

a body in a homogeneous gravitational field and supported at the C.M. is in equilibrium.

Angular Momentum Around the Center of Mass



$$\vec L_o = \sum \vec r_i \times m_i \vec v_i$$

$$= \sum \overbrace{(\vec r_i - \vec R_{c.m})}^{\rho_i} \times m_i \vec v_i + \sum \vec R_{c.m.} \times m_i \vec v_i$$

$$= \vec L_{c.m.} + \vec R_{cm} \times \vec P$$

independent of the choice of origin. — depend on choice of the origin O

$$= \vec R_{c.m} \times \vec P + \sum \rho_i \times \vec P_i$$

$$\frac{dL_o}{dt} = \frac{d\vec L_{c.m.}}{dt} + \frac{d}{dt}(\vec R_{c.m} \times \vec P)$$

$$\vec \tau_{tot} = \sum \vec r_i \times \vec F_i = \sum_i (\vec R_{c.m} + \rho_i) \times \vec F_i$$

$$= \sum_i \vec R_{c.m} \times \vec F_i + \sum (\rho_i \times \vec F_i)$$

$$= \vec R_{c.m.} \times \vec F_{tot} + \sum_i \underbrace{\rho_i \times \vec F_i}_{\vec \tau_{cm}}$$

$$\frac{d\vec L_o}{dt} = \frac{d\vec L_{c.m}}{dt} + \frac{d}{dt}(\vec R_{c.m.} \times \vec P)$$

$$\parallel$$

$$\frac{d\vec R_{c.m}}{dt} \times \vec P + \vec R_{c.m} \times \frac{d\vec P}{dt}$$

$$\frac{1}{M} \vec P \times \vec P + \vec R_{c.m} \times \vec F_{tot}$$

$$\Rightarrow \quad \vec R \times \vec F_{tot} = \frac{d}{dt}(\vec R \times \vec P)$$

and

$$\frac{d\vec L_{c.m}}{dt} = \frac{d}{dt} \vec L_{c.m} = \vec \tau^{\,ext}_{c.m}$$

Rigid Bodies

A system of particles in which the distance between any two particles do not change regardless of the forces acting is called a rigid body.

Translations and Rotations

A displacement of a rigid body is a change from one position to another

If during a displacement all points of the body on some line remain fixed, the displacement is called the rotation

If during a displacement all points of the rigid body move in a line parallel to each other

Chasle's Theorem

The general motion of a rigid body can be considered as a translation plus a rotation about a suitable point which is often taken to be the center of mass.

A rigid body in the form of a triangle ABC [Fig. 9-7] is moved in a plane to position DEF, i.e. the vertices A, B and C are carried to D, E and F respectively. Show that the motion can be considered as a translation plus a rotation about a suitable point.

Choose a point G on triangle ABC which corresponds to the point H on triangle DEF. Perform the translation in the direction GH so that triangle ABC is carried to $A'B'C'$. Using H as center of rotation perform the rotation of triangle $A'B'C'$ through the angle θ as indicated so that $A'B'C'$ is carried to DEF. Thus the motion has been accomplished by a translation plus a rotation.

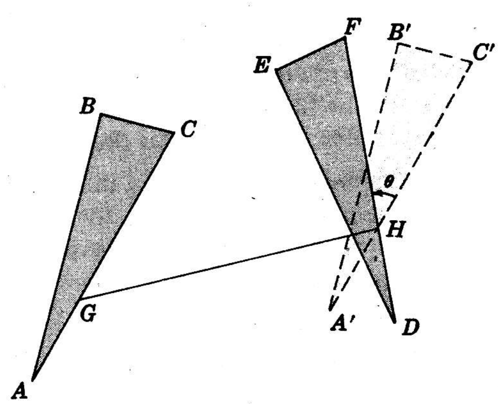


Fig. 9-7

Euler's theorem

A rotation of rigid point of the body is equivalent to a rotation about a line which passes through the point. The line is referred to as the instanteous axis of rotation.

Plane Motion of a Rigid Body

is simplified considerably when all points move parallel to a fixed plane.

In such case two type of motion, i.e., plane motion are possible

1. Rotation about a fixed axis

The rigid body rotate about a fixed axis ⊥ to the fixed

The system has only one degree of freedom and only one coordinate is required for describing the motion.

2. Translation parallel to the given plane.

Rotation can be finite or infinitesimal

Finite rotation cannot be represented by vectors since the commutative law fails

However, infinitesimal rotations can be represented by vectors

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Give an example to show that finite rotations cannot be represented by vectors.

Let A_x represent a rotation of a body [such as the rectangular parallelepiped of Fig. 9-8(a)] about the x axis while A_y represents a rotation about the y axis. We assume that such rotations take place in a positive or counterclockwise sense according to the right hand rule.

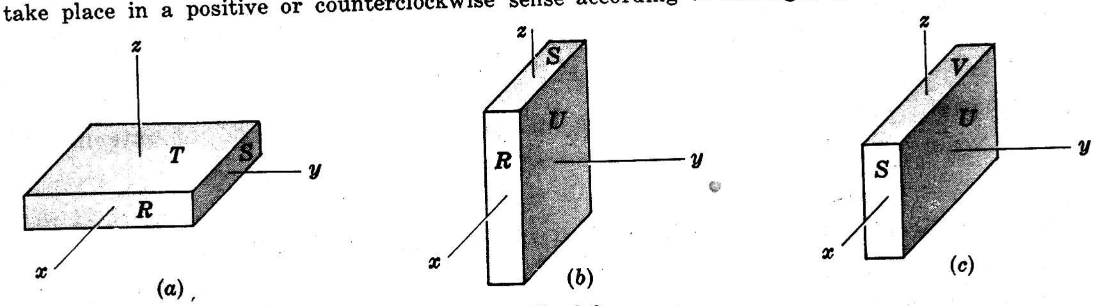


Fig. 9-8

9-11

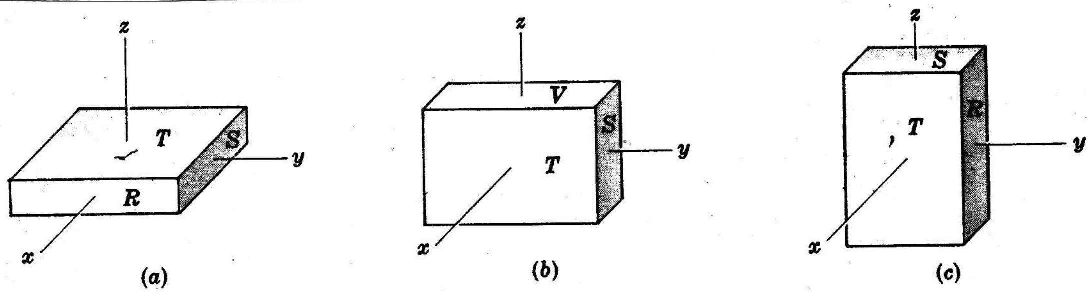


Fig. 9-9

In Fig. 9-8(a) we start with the parallelepiped in the indicated position and perform the rotation A_x about the x axis as indicated in Fig. 9-8(b) and then the rotation about the y axis as indicated in Fig. 9-8(c). Thus Fig. 9-8(c) is the result of the rotation $A_x + A_y$ on Fig. 9-8(a).

In Fig. 9-9(a) we start with the parallelepiped in the same position as in Fig. 9-8(a), but this time we first perform the rotation A_y about the y axis as indicated in Fig. 9-9(b) and then the rotation A_x about the x axis as indicated in Fig. 9-9(c). Thus Fig. 9-9(c) is the result of the rotation $A_y + A_x$ on Fig. 9-9(a).

Since the position of the parallelepiped of Fig. 9-8(c) is not the same as that of Fig. 9-9(c), we conclude that the operation $A_x + A_y$ is not the same as $A_y + A_x$. Thus the commutative law is not satisfied, so that A_x and A_y cannot possibly be represented by vectors.

Rigid Body

1. Definitions
   - Rigid Body
   - Translation
   - Rotation
2. Rotation about a fixed axis
   - $\vec{\omega}$
   - Relation between Angular and Linear velocity and acceleration
   - Rotation kinetic energy
3. Moments of Inertia

Rigid Body Kinematics

Objects in the real world are not point-like particles that we have been dealing with up to now. A real object has a mass distribution associated with its size and shape.

The motion of a real object involves both:

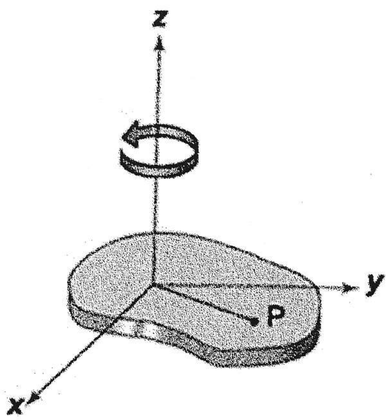
- translational motion of the cm.
- rotational motion about an axis (usually take to be an axis through the cm or some other fixed axis).

We will restrict our discussions to that of rigid bodies. A rigid body is one in which the relative coordinates connecting all the constituent particles remain constant. This is of course an idealized situation.

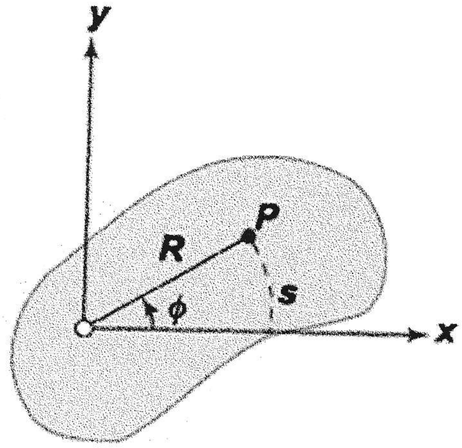
Rotations about a Fixed Axis

We will initially study the motion of a rigid body rotating about an axis that is fixed in an inertial frame.

Consider motion around the z-axis. Reference point P (which is not on the axis) represents the rotational motion of the body and of its angular position.



Rotation of a rigid body about a fixed axis (z axis).



z out of the paper

Given a reference point P, its angular position is measured by the angle ϕ, between position vector $\vec{r}$ and the x-axis.

As the particle moves in a circle from the positive x-axis ($\varphi = 0$) to the point P, it moves through an arc length

$$s = R\phi$$

$$\phi(\text{rad}) = \frac{\pi}{180}\,\phi(\text{deg})$$

ϕ = positive counterclockwise

$\phi = 0 \Rightarrow x$-axis

$\phi = 2\pi \Rightarrow x$-axis again.

ϕ: is not a vector [rotations do not commute]

$\vec{d\phi} = d\varphi\,\hat{k}$ [infinitesimal rotation is a vector]

The rotational motion of a body is described by the rate of change of ϕ. In general the position angle is a function of time:

$$\phi = \phi(t)$$

Suppose the particle moves from P to Q. The reference line OP makes an angle ϕ_1 at the time t_1, and an angle ϕ_2 at the time t_2. Define the average angular velocity of the body, $\bar{\omega}$, in the time interval $\Delta t = t_2 - t_1$

as the ratio of angular displacement $\Delta\phi = \phi_2 - \phi_1$ to Δt.

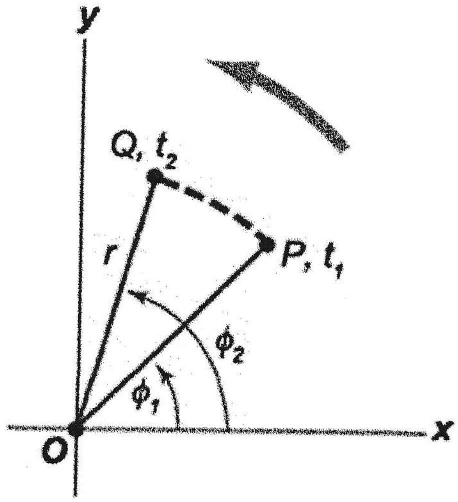
$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t} \quad \text{rad/s or } s^{-1}$$

$\hat{k}$ = unit vector along axis of rotation (z-axis)

$\vec{\omega}$ = points along axis of rotation.
[RH Rule for sign convention]

Analagous to linear velocity, the instantaneous angular velocity, is defined as the limit of this ratio as $\Delta t \to 0$. Becomes time rate-of-change of $\phi(t)$.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt}\hat{k} \quad (s^{-1})$$



A particle on a rotating rigid body moves from *P* to *Q* along the arc of a circle. In the time interval $\Delta t = t_2 - t_1$, the radius vector sweeps out an angle $\Delta\phi = \phi_2 - \phi_1$.

<u>If</u> the angular velocity, ω, is a constant
$\omega = \omega_0$,
the rate of rotation is often given in terms of the frequency, or number of revolutions per unit time.

1 revolution = $\Delta\phi = 2\pi$ radians

Time per revolution, or period $T = \frac{2\pi}{\omega_0}$ (s)

Frequency of revolution is $\cancel{\nu} = \frac{1}{T} = \frac{\omega_0}{2\pi}$ (Hz)

If the angular velocity of the body is changing with time (i.e. ω is not constant), then there is an angular acceleration.
If the angular velocities are ω_1 and ω_2 at the times t_1 and t_2, the average angular acceleration is

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is the limit of this ratio as $\Delta t \to 0$.

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\vec{\omega}}{dt}. \qquad (s^{-2})$$

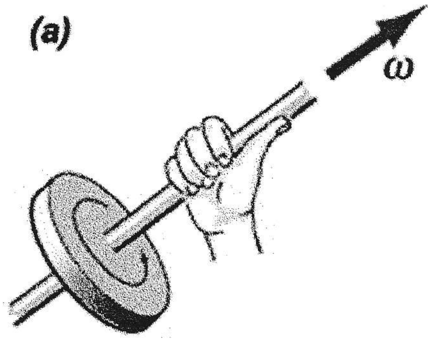
Since $\omega = \frac{d\phi}{dt}$, we also have

$$\vec{\alpha} = \frac{d^2\vec{\phi}}{dt^2}$$

For <u>rotation</u> about a <u>fixed axis</u>, every particle on the rigid body has the <u>same angular velocity</u> and the <u>same angular acceleration</u>.
The direction of $\vec{\alpha}$ is along the same axis as $\vec{\omega}$. If the axis of rotation is changing then $\vec{\alpha}$ is not in the same direction as $\vec{\omega}$.

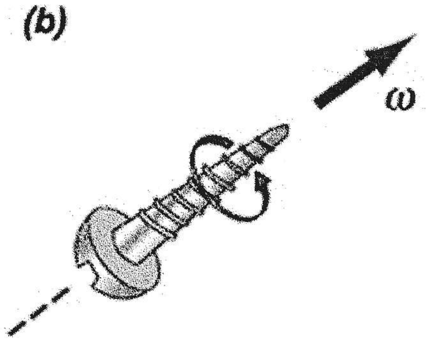
Direction - Right Hand Rule.

(a)



The right-hand rule for determining the direction of the angular velocity.

(b)



The direction of ω is in the direction of advance of a right-handed screw.

Fingers of right hand are wrapped along direction of rotation. Then $\vec{\omega}$ points along thumb.

Direction of $\vec{\alpha}$ is related to $\frac{d\vec{\omega}}{dt}$.

$\frac{d\vec{\omega}}{dt} > 0$ $\quad$ $\vec{\alpha}$ same as $\vec{\omega}$

$\frac{d\vec{\omega}}{dt} < 0$ $\quad$ $\vec{\alpha}$ opposite to $\vec{\omega}$

[Fixed-Axis Rotation]

Rotational Motion with Constant Angular Acceleration

- Assume motion along a fixed axis.
- Ignore vector notation (sign designates direction)
- Results also hold for axis in linear translation

$$\frac{d\omega}{dt} = \alpha \qquad (\alpha = \text{constant})$$

$$\int d\omega = \int \alpha \, dt$$

$$\omega = \alpha t + C$$

If $\omega = \omega_0$ at $t = 0$, $\Rightarrow C = \omega_0$.

and

$$\boxed{\omega = \omega_0 + \alpha t} \qquad ①$$

$$\frac{d\phi}{dt} = \omega = \omega_0 + \alpha t$$

$$\int d\phi = \int \omega_0 dt + \alpha \int t \, dt$$

$$\phi = \omega_0 t + \frac{1}{2}\alpha t^2 + C$$

If $\phi = \phi_0$ at $t = 0$, $\Rightarrow C = \phi_0$.

and

$$\boxed{\phi = \phi_0 + \omega_0 t + \frac{\alpha t^2}{2}} \qquad ②$$

Solve Eq. ① for t and substitute in Eq. ②

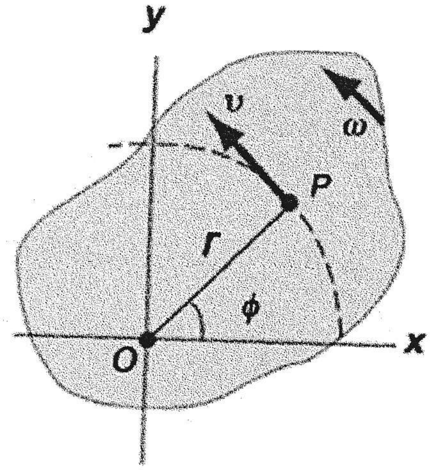
$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)} \qquad ③$$

Motion with constant linear acceleration	Motion with constant angular acceleration
$a =$ constant	$\alpha =$ constant
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Relation between Angular and Linear Velocity and Acceleration

As a rigid body rotates about a fixed axis, every particle in the body moves in a circle the center of which is on the axis of rotation.

Consider the point P. P moves in a circle, the linear velocity vector is thus tangent to this circle. Magnitude is ds/dt, where s is distance travelled along the circular path.



As a rigid body rotates around the fixed axis through O, the point P has a linear velocity v, which is always tangent to the circular path of radius r.

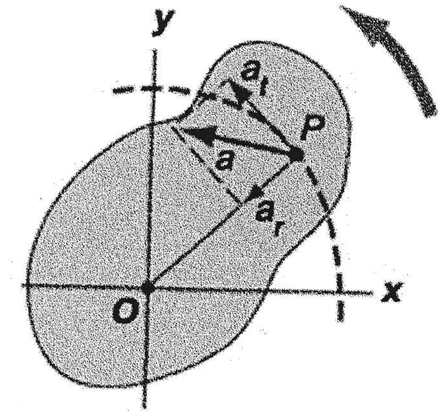
$$s = r\phi \qquad [\phi \text{ in radians}]$$

$$v = \frac{ds}{dt} = r\frac{d\phi}{dt}$$

$$v = r\omega$$

Speed of the particle is directly proportional to its distance from the axis of rotation. The further from the axis the higher its velocity.

To relate the linear acceleration of the point P to the angular acceleration of the rigid body about a fixed axis, we take the time derivative of v:



As a rigid body rotates about a fixed axis through O, the point P experiences a tangential component of acceleration, a_t, and a centripetal component of acceleration, a_r. The total acceleration of this point is $a = a_t + a_r$.

$$a_t = a_{\parallel} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

This is the tangential (parallel) component of the linear acceleration of a point at a distance r from the axis of rotation. It is related to the change in speed of the particle.

Since the particle moves in a circle, we have seen that it also has a radial or centripetal acceleration due to the changing direction of its velocity.

$$a_r = a_{\perp} = \frac{v^2}{r} = r\omega^2$$

Total linear acceleration of the particle is $\vec{a}$:

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

$$a = \sqrt{a_t^2 + a_r^2} = r\sqrt{\alpha^2 + \omega^4} \quad m/s^2$$

Note:

All points in a rotating rigid body have the same value of ω and the same value of α. Points that are different distances from the axis have different values of v and different values of a_t and a_c.

Example - Rotating Turntable

Record player rotates at 33 rev/min and takes 20s to come to rest.

a) What is angular acceleration, assuming it is uniform?

$$\omega_0 = \left(33 \frac{\text{rev}}{\text{min}}\right)\left(2\pi \frac{\text{rad}}{\text{rev}}\right) \times \left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 3.46 \text{ rad/s}.$$

$$\omega = \omega_0 + \alpha t$$

$$\omega = 0 \text{ at } t = 20\text{s}$$

$$\therefore \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 3.46}{20} = -0.173 \text{ rad/s}^2$$

(<0, decellerating)

b) How many rotations before it comes to rest?

$$\Delta\phi = \phi - \phi_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 3.46\,(20) - \frac{1}{2}(0.173)\,20^2$$

$$= 34.6 \text{ rad}$$

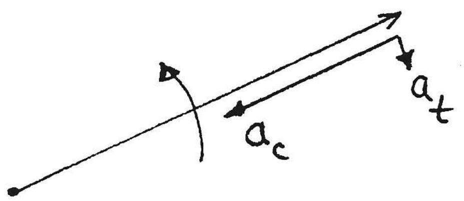
$$= 34.6/2\pi = 5.51 \text{ rev}.$$

c) If rim is at radius $r = 14$cm, what is the acceleration of a point on the rim at $t = 0$.

$$a_t = r\alpha = 14\text{ cm}\,(0.173 \text{ rad/s}^2) = 2.42 \text{ cm/s}^2$$

$$a_c = r\omega_0^2 = 14\text{cm}\,(3.46 \text{ rad/s})^2 = 168 \text{ cm/s}^2 \quad (t=0)$$

$$a = \sqrt{2.42^2 + 168^2} = 168.0 \text{ cm/s}^2$$



Velocity at rim ($t=0$):

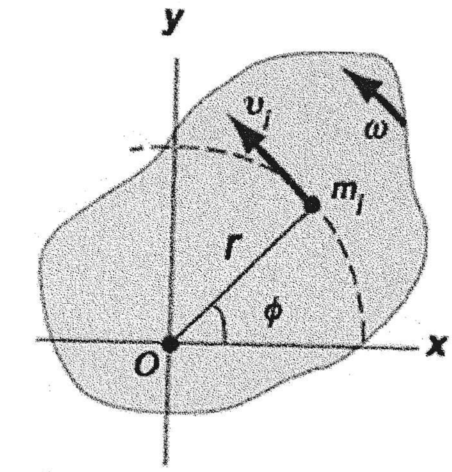
$$v = r\omega_0 = 14\text{cm} \times 3.46 \text{ rad/s}$$

$$= 48.4 \text{ cm/s}$$

Rotational Kinetic Energy

Consider a rigid body as a collection of small particles. The KE of a rotating rigid body is the sum of the individual KE's of all the particles.

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$



A rigid body rotating about the z axis with angular velocity ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_i v_i^2$. The total kinetic energy of the body is $\frac{1}{2} I\omega^2$.

Suppose the rigid body is rotating about a fixed z-axis with an angular velocity ω. All particles execute circular motion with same angular speed.

$$v_i = r_i \omega$$

r_i = particle distance from axis of rot.

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

where $I = \sum_i m_i r_i^2$ [Moment-of-Inertia]

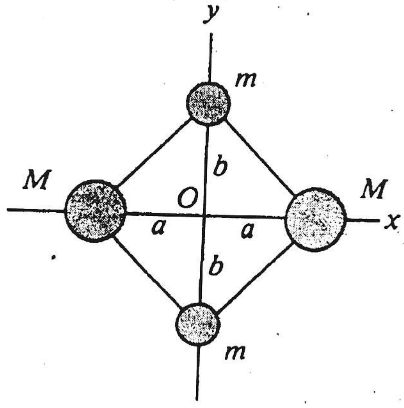
↑ Particles at large r_i have higher speed and contribute more to KE

$[I] = kg \cdot m^2$ (SI)
$slug \cdot ft^2$ (Br)

ω, I ← Resistance to rotational motion
v, M ← Resistance to linear motion
} Inertial quant.

Example : Four Rotating Particles

- Four point masses fastened to a very light frame. lying in xy-plane.



a) Rotation about y-axis with ang. velocity ω.

- masses m do not contribute since $r_i = 0$ for them. and they have no motion about y. !

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

$$K = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2$$

b) Rotation about z-axis, $\perp$ to xy-plane

r_i, in each case is the $\perp$ distance to axis of rot.

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$$

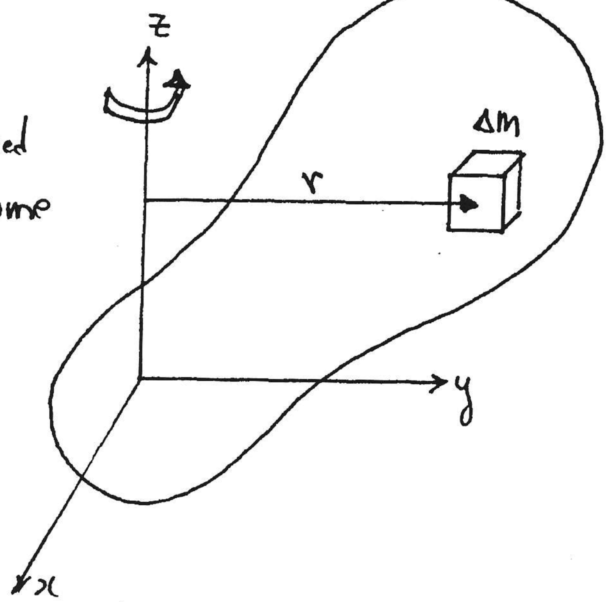
$$= 2Ma^2 + 2mb^2$$

$$K = \frac{1}{2} I_z \omega^2 = (Ma^2 + mb^2)\omega^2$$

Summary:

- Moment-of-Inertia depends on axis of rotation.
- It will take more work, for this example, to set the system into rotation about z-axis than about y-axis. Depends on the distribution of mass.

Moments of Inertia for Rigid Bodies



we evaluate I_1 for a rigid body rotating about a fixed axis by dividing it up into volume elements of mass Δm.
Use $I = \Sigma r^2 \Delta m$ and take the limit of this sum as $\Delta m \to 0$ we have an integral over the volume.
r: ⊥ distance from rotation axis to Δm.

$$I = \lim_{\Delta m \to 0} \Sigma r^2 \Delta m = \int_V r^2 dm$$

dm: must be expressed in terms of its coordinates.
- For a 3-dimensional object it is convenient to do this in terms of the local volume density, i.e mass per unit volume

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$\therefore dm = \rho dV$$

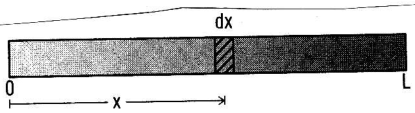
and $\boxed{I = \int \rho r^2 dV}$

2'ND moment of mass distribution.

$$I = \overset{\rho_H}{\frac{M}{V}} \int r^2 dV$$, for homogeneous bodies

Non-uniform rod.

$\rho(x) = bx$



A nonuniform rod of length L has a density that increases linearly from the end at $x = 0$ to the end $x = L$. The density (mass per unit length) is given by $\rho = bx$, where b is a constant and x is the distance from the end. The density at the $x = 0$ end is zero.

a. At a distance x from the $x = 0$ end, what is the mass of an increment of length dx? [G]

b. Sum up the contributions to the total mass from all of the elements of length dx to find the total mass of the rod (in terms of b and L). [O]

c. Where is the center of mass of this rod? If you can't recall the definition of this, refer to a text. [Q]

d. What is the moment of inertia, relative to an axis through $x = 0$, of the element of mass located in thickness dx at distance x from the $x = 0$ end? [X]

e. What is the total moment of inertia of this rod relative to an axis through the $x = 0$ end? [L] What is the radius of gyration of this rod relative to the $x = 0$ end? [Y]

f. What is the moment of inertia relative to an axis through the center of mass? (Use the parallel axis theorem.) [I]

g. What is the moment of inertia relative to an axis through the $x = L$ end? (Get the answer using the parallel axis theorem and by direct integration.) [W]

(a) $\rho(x)\,dx = bx\,dx$

(b) $M = \int_0^L \rho(x)\,dx = \frac{1}{2} bL^2$

(c) $M x_{c.m.} = \int_0^L x\,\rho(x)\,dx$

$= \int_0^L bx^2\,dx = \frac{1}{3} bL^3$

$\frac{1}{2} bL^2\, x_{c.m.} = \frac{1}{3} bL^3$

$\Rightarrow x_{c.m.} = \frac{2}{3} L$

(d) $dI = x^2 \rho(x)\,dx$

$= bx^3\,dx$

(e) $I = \int_0^L bx^3\,dx = \frac{1}{4} bL^4$

$K^2 = \frac{I}{M} = \frac{\frac{1}{4} bL^4}{\frac{1}{2} bL^2}$

$= \frac{1}{2} L^2$

$K = 0.707\,L$

(f) $I = I_c + Md^2$

$\frac{1}{2} ML^2 = I_c + M\left(\frac{2}{3} L\right)^2$

$\frac{1}{2} ML^2 = I_c + \frac{4}{9} ML^2 \Rightarrow I_c = \frac{1}{18} M L^2$

(g) $I = \frac{1}{18} ML^2 + M\left(\frac{1}{3} L\right)^2$

$= \frac{1}{18} ML^2 (1 + 2) = \frac{3}{18} ML^2$

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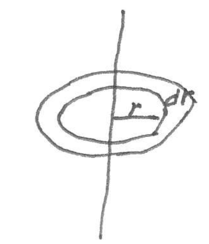
Moment of Inertia for a solid sphere

First we calculate for a disk



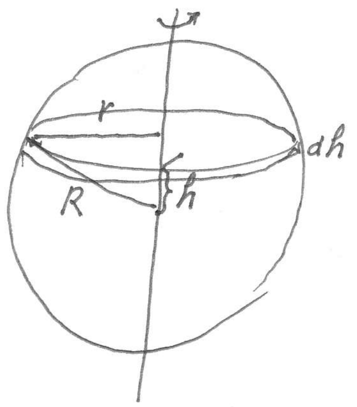
$$dm = \overset{\sigma(r)}{\overset{\parallel}{\rho(r)}}\, 2\pi r\, dr$$

$$I = \int r^2\, dm$$

$$= \int_0^R r^2\, 2\pi r\, \overset{\sigma}{\overset{\downarrow}{\rho}}\, dr$$

$$= 2\pi\, \frac{1}{4} R^4 \underset{\sigma}{\rho}\, \cancel{dr} = \frac{1}{2} M R^2$$

$$M = \underset{\sigma}{\rho} \cdot \pi R^2 \qquad \underset{\sigma}{\rho} = \frac{M}{\pi R^2}$$



$$dI = \frac{1}{2}\, dm\, r^2$$

$$= \frac{1}{2} \rho \pi r^2 dh\, r^2$$

$$r^2 = R^2 - h^2$$

$$dI = \frac{1}{2} \rho \pi (R^2 - h^2)^2 dh$$

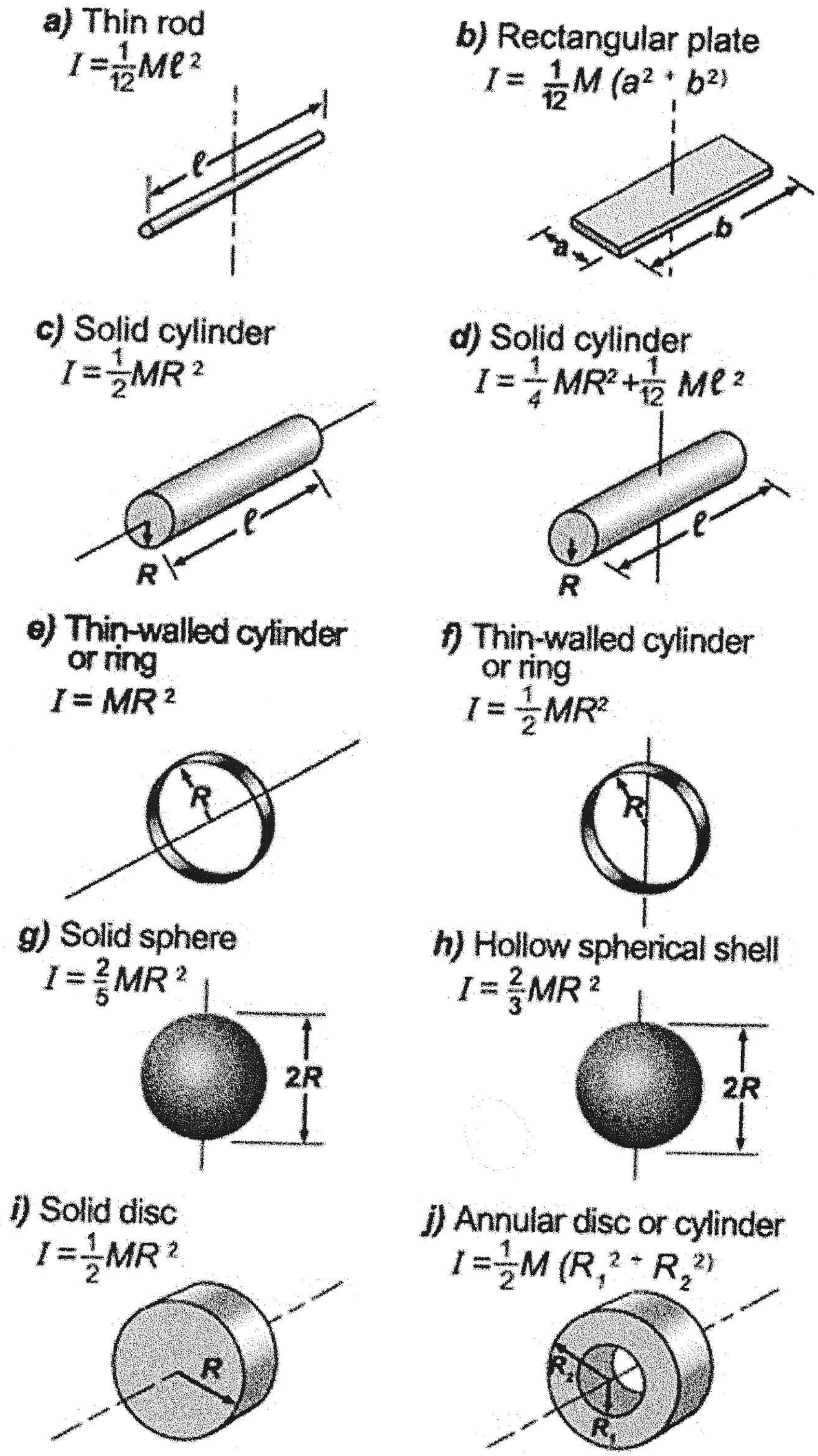
$$I = \int_{-R}^{R} \frac{1}{2} \pi \rho (R^4 - 2R^2 h^2 + h^4) dh$$

$$= \frac{1}{2} \pi \rho \left(R^4 h - \frac{2}{3} R^2 h^3 + \frac{1}{5} h^5 \right|_{-R}^{R}$$

$$= \frac{8}{15} \pi \rho R^5$$

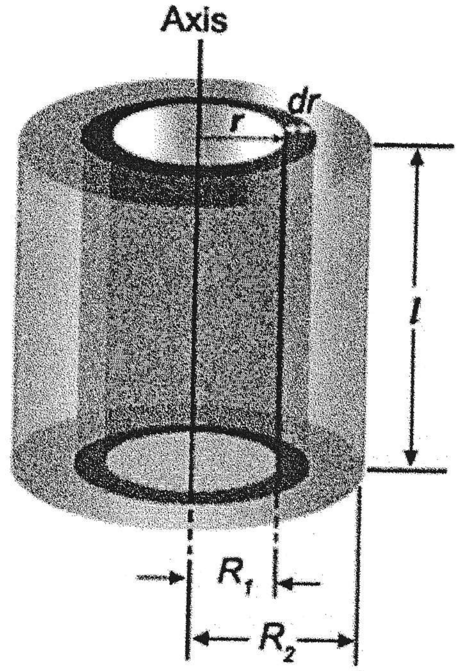
$$\rho = \frac{M}{\frac{4}{3} \pi R^3}$$

$$\Rightarrow \quad I = \frac{8}{15} \pi R^5 \frac{M}{\frac{4}{3} \pi R^3} = \frac{2}{5} M R^2$$



Rational Inertia values for various objects for the indicated axes

Example: I – Uniform Hollow Cylinder



Moment of inertia of a hollow cylinder. The mass element is a cylindrical shell of a radius *r* and thickness *dr*.

- Always need to choose mass elements which are a fixed radius from axis of rotation.

We choose a thin cylindrical shell of radius r, thickness dr, and length ℓ.
The volume of such a shell is that of a flat sheet of length ℓ, thickness dr and width $2\pi r$.

$$dV = 2\pi \ell r\, dr$$

$$dm = \rho\, dV = 2\pi \rho \ell r\, dr$$

$$I = \rho \int r^2 dV = 2\pi\rho\ell \int_{R_1}^{R_2} r^3 dr$$

$$= \frac{\pi\rho\ell}{2}\left(R_2^4 - R_1^4\right)$$

$$= \frac{\pi\rho\ell}{2}\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)$$

Mass of Cylinder

$$M = \rho V = \pi \ell \rho \left(R_2^2 - R_1^2\right)$$

$$I = \frac{M}{2}\left(R_2^2 + R_1^2\right)$$

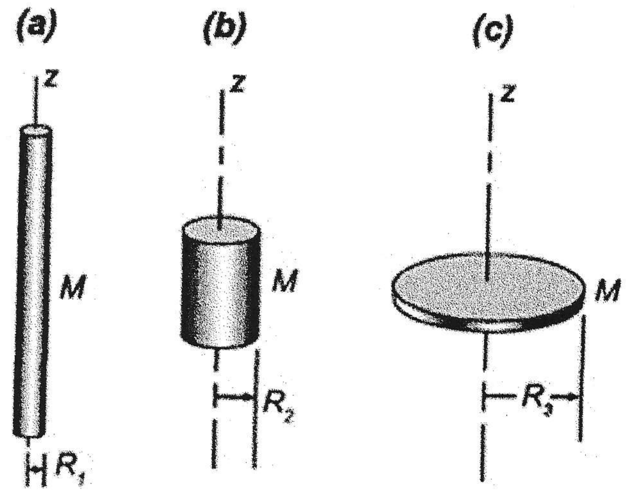
If cylinder is solid, $R_1 = 0$.

$$I = \frac{1}{2} M R^2$$

If cylinder is a very thin shell, $R_1 \sim R_2 = R$

$$I = M R^2$$

I in all cases does not depend on l. The distribution along the axis does not matter. Moment-of-inertia depends on radial distribution.



Three different distributions about the z axis of the same mass M of material. The rotational inertia values are $I_z(R_1) < I_z(R_2) < I_z(R_3)$.

When using I, it is often convenient to do so in terms of a "Radius of Gyration", k

$$I = M k^2$$

It is defined such that if all of the mass of an object were located a distance k from the axis, it would have the same moment-of-inertia as the actual object.

Parallel-Axis Theorem

The moment-of-inertia depends on the location of the axis of rotation.

KE for a body about a fixed axis is

$$K = \frac{1}{2} I_z \omega^2$$

We had before that the KE is the sum of the translational energy of motion of the cm and the internal energy of motion relative to the cm.

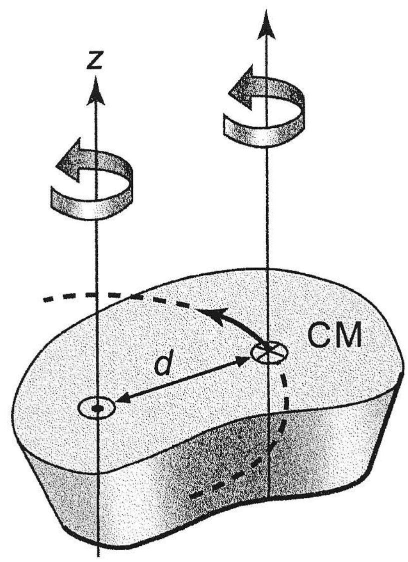
$$K = \frac{1}{2} M v_{cm}^2 + K_{Int}.$$

Consider rotation about z-axis not through cm.

cm moves in a circle of radius d around this axis.

$$\therefore v_{cm} = d\omega$$

$$\therefore \frac{1}{2} M v_{cm}^2 = \frac{1}{2} M d^2 \omega^2$$



Two alternative parallel axes of rotation of a rigid body. The z axis is fixed. The center-of-mass axis moves along a circle of radius *d* around the z axis. The body is in rotational motion relative to each of these axes.

The rotation of the body with angular velocity ω about a fixed z-axis is also a rotation about a parallel axis through the cm with the same angular velocity ω. One turn around z corresponds to one turn around axis through cm.

The KE associated with rotational motion around axis through cm is

$$K_{Int} = \frac{1}{2} I_{cm} \omega^2$$

$$\therefore K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M d^2 \omega^2$$

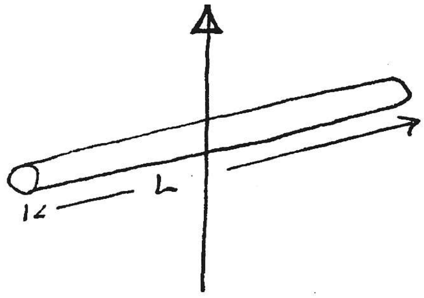
$$\frac{1}{2} I_z \omega^2 = \frac{1}{2} \left[I_{cm} + M d^2 \right] \omega^2$$

Comparing, we must have

$$I_z = I_{cm} + M d^2$$

[Parallel Axis Theorem]

Example: Rod



- Thin rod axis through its midpoint

$$I_{cm} = \frac{1}{12} ML^2$$

- what is I about an axis through its end?

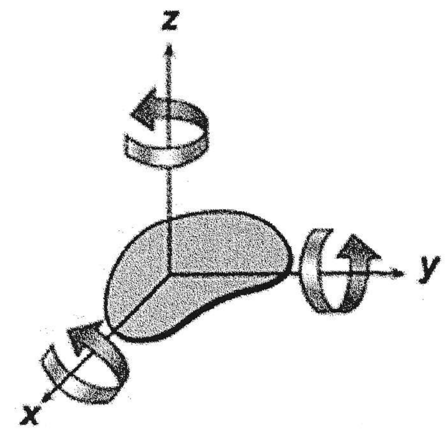
$$d = L/2$$

$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$I = \frac{ML^2}{3}$$

Perpindicular - Axis Theorem.

Relates moments-of-inertia of a thin flat plate about three mutually perpindicular axes.



A thin, flat plate. The plate may rotate about either the the *x* axis, the *y* axis, or the *z* axis.

Consider a thin plate which can rotate about any of three $\perp$ axes

$$\left.\begin{array}{l} I_x \\ I_y \\ I_z \end{array}\right\} \text{corresponding moments-of-inertia}$$

Let plate be in xy-plane. The distance from z-axis to reference point P is

$$R = \sqrt{x^2 + y^2}$$

$$I_z = \int \rho R^2 dV = \int \rho (x^2+y^2)\, dV$$

$$I_x = \int \rho y^2 dV$$

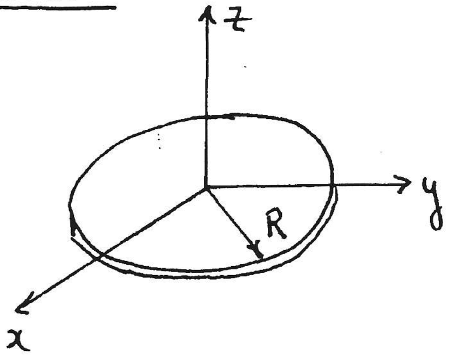
$$I_y = \int \rho x^2 dV$$

$$\therefore \boxed{I_z = I_x + I_y}$$

[Perpindicular-Axis Thm]

Example

i) Disk



• Disk in xy-plane

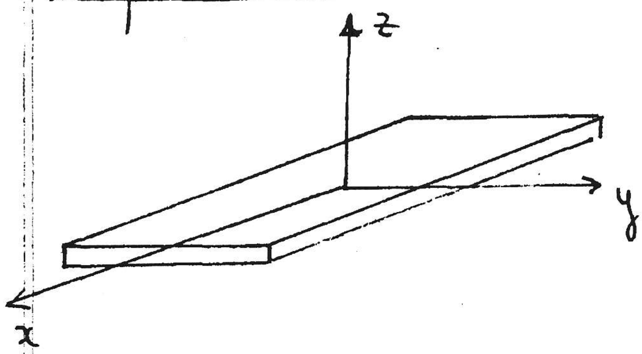
$$I_z = \frac{1}{2} M R^2$$

By symmetry
$I_x = I_y$

$$\therefore \quad I_z = 2 I_x$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{4}$$

ii) Square Plate.



Side = a

By symmetry

$I_x = I_y.$

$$2 I_x = I_z = \frac{1}{6} M a^2$$

Measure the moment of Inertia

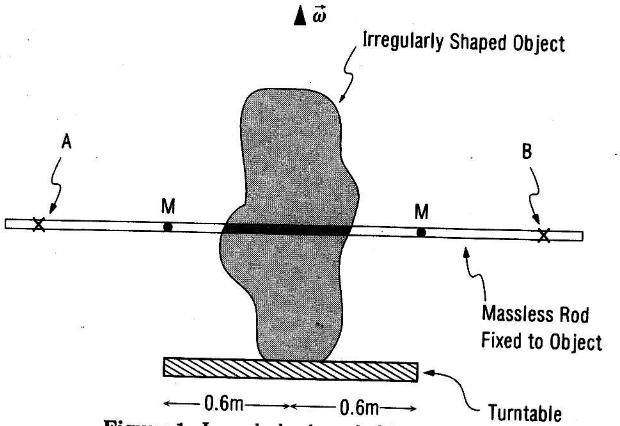


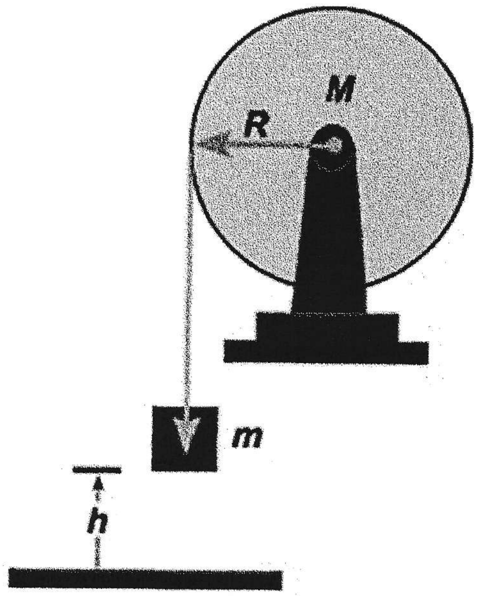
Figure 1. Irregularly shaped object rotating about vertical axis on a turntable which rotates with very little frictional loss. Two masses M are located on a massless rod held horizontally fixed to the object.

$$(I + 2 \times 4 \times 0.6^2)\,30 = (I + 2 \times 4\,(1.2)^2) \cdot 10$$

$$\Rightarrow \quad I = 1.44 \ \text{kg m.}$$

Example

A light rope is wrapped around a solid cylinder of mass M and radius R.
Mass m tied to rope and released a height h above the floor. Assuming frictionless motion, what is speed of m and angular velocity of cylinder when m strikes floor?



As the cylinder rotates, the rope unwinds and mass *m* drops.

Ans: System initially has no kinetic energy but has PE.
Finally both m and M have KE and the PE of m is decreased.

$$E_1 = K_1 + U_1$$
$$= 0 + mgh$$

$$E_2 = K_2 + U_2$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$v = R\omega$ [Related by geometry - see Kinematics]
$I = \frac{1}{2}MR^2$ [solid cylinder]

$E_1 = E_2$ conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\tfrac{1}{2}MR^2\right)(v/R)^2 = \tfrac{1}{2}(m + M/2)v^2$$

$$v = \sqrt{2gh/(1 + M/2m)}$$

Review

System of Particle

$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i$$

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i^{\,ext}$$

valid in inertial frame of reference

Center of mass

$$\vec{r}_{c.m} \equiv \vec{R}$$

$$\vec{r}_i = \vec{R} + \vec{\rho}_i$$

$$\vec{v}_i = \dot{\vec{R}} + \dot{\vec{\rho}}_i$$

$$\sum m_i \vec{r}_i = M\vec{R}$$

$$\sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$$

$$\sum m_i (\underbrace{\vec{R}}_{=} + \vec{\rho}_i) = M\vec{R}$$

$$\sum m_i \rho_i = 0$$

$$\sum m_i \dot{\vec{\rho}}_i = 0$$

$$\vec{L} = \sum_i (\vec{R} + \vec{\rho}_i) \times m_i (\dot{\vec{R}} + \dot{\vec{\rho}}_i)$$

$$= \sum_i \vec{R} \times m_i \dot{\vec{R}} + \sum_i m_i (\vec{R} \times \dot{\vec{\rho}}_i) + \sum_i \vec{\rho}_i \times m_i \dot{\vec{R}} + \sum_i (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i)$$

$$= \underbrace{\vec{R} \times M\dot{\vec{R}}}_{\vec{R} \times \vec{P}} + \vec{R} \times \underbrace{\sum m_i \dot{\vec{\rho}}_i}_{0} + \underbrace{\sum m_i \rho_i}_{0} \times \dot{\vec{R}} + \underbrace{\sum \rho_i \times m_i \dot{\vec{\rho}}_i}_{\vec{L}_{c.m.}}$$

motion of the center of mass

about the center of mass

$$\vec{\tau}_{tot} = \vec{N} = \sum \vec{r}_i \times \vec{F}_i^{\,ext}$$

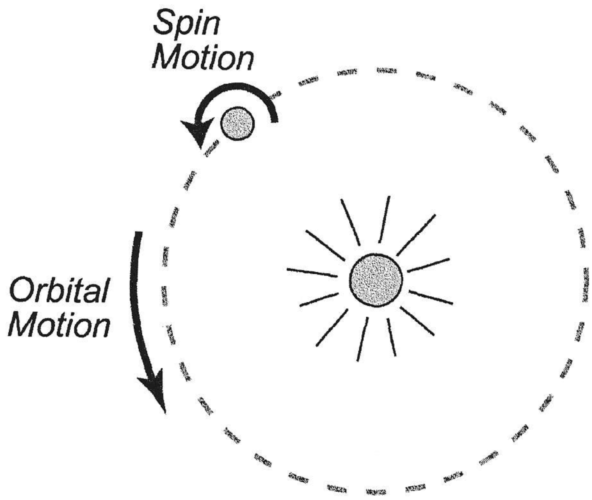
$$= \sum_i (\vec{R} + \rho_i) \times F_i^{\,ext}$$

$$= \underbrace{\sum_i \vec{R} \times \vec{F}_i^{\,tot}}_{\vec{R} \times \vec{F}_{tot}^{\,ext}} + \underbrace{\sum_i \vec{\rho}_i \times \vec{F}_i^{\,ext}}_{\vec{\tau}_{c.m}^{\,ext}}$$

$$\frac{d\vec{L}}{dt} = \underbrace{\frac{d}{dt}(\vec{R} \times \vec{P})}_{\equiv} + \frac{dL_{c.m.}}{dt}$$

$$= \vec{R} \times \vec{F}_{tot}^{\,ext} + \underbrace{\sum_i \vec{\rho}_i \times \vec{F}_i^{\,ext}}_{\tau_{c.m}}$$

Earth - Sun



Example

Rod of mass M and length ℓ free to swing in a vertical plane about a fixed pivot

$$\vec{L} = \vec{R} \times \vec{P} + \vec{L}_{c.m.}$$

$$L_z = M \cdot \frac{\ell}{2} \cdot \frac{\ell}{2} \omega = \frac{1}{4} M \ell^2 \omega$$

$$2 r v \, dm = 2r(r\omega)\lambda \, dr \Rightarrow 2\lambda\omega \int_0^{\ell/2} r^2 dr = \left(\frac{1}{12} \ell^2\right)\omega$$

$dm = \lambda \, dr$

$\lambda = \frac{M}{\ell}$

$$= \frac{4}{12} M \ell^2 \omega = \left(\frac{1}{3} M \ell^2\right) \omega$$

Related to the parallel axis theorem

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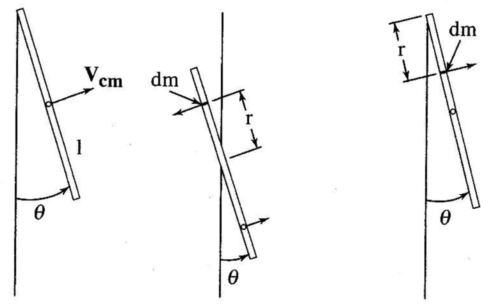
$$L_{cm.} = \frac{l}{2}|\vec{P}_{cm}| = m\frac{l}{2} v_{c.m} = m\frac{l}{2}(\frac{l}{2}\omega) = \frac{l}{4} ml^2\omega$$



(a) (b) (c)

Figure 7.4 Rod of mass m and length l free to swing in a vertical plane about a fixed pivot.

Figure 7.4(b) depicts the motion of the rod as seen from the perspective of its center of mass. The angular momentum dL_{rel} of two small mass elements, each of size dm symmetrically disposed about the center of mass of the rod, is given by

$$dL_{rel} = 2rdp = 2rv\ dm = 2r(r\omega)\lambda dr$$

where λ is the mass per unit length of the rod. The total relative angular momentum is obtained by integrating this expression from $r = 0$ to $r = l/2$.

$$L_{rel} = 2\lambda\omega\int_0^{l/2} r^2 dr = \frac{1}{12}(\lambda l)l^2\omega = \left(\frac{1}{12}ml^2\right)\omega$$

We can see in the equation above that the angular momentum of the rod about its center of mass is directly proportional to the angular velocity ω of the rod. The constant of proportionality $ml^2/12$ is called the *moment of inertia* I_{cm} of the rod about its center of mass. Moment of inertia plays a role in rotational motion similar to that of inertial mass in translational motion as we shall see in the next chapter.

Finally, the total angular momentum of the rod is

$$L_{tot} = L_{cm} + L_{rel} = \frac{1}{3}ml^2\omega$$

Again, note that the total angular momentum of the rod is directly proportional to the angular velocity of the rod. Here, though, the constant of proportionality is the moment of inertia of the rod about the pivot point at the end of the rod. This moment of inertia is larger than that about the center of mass. The reason is that more of the mass of the rod is distributed farther away from its end than from its center, thus making it more difficult to rotate a rod about an end.

The total angular momentum can also be obtained by integrating down the rod, starting from the pivot point, to obtain the contribution from each mass element dm, as shown in Figure 7.4(c)

$$dL_{tot} = rdp = r(vdm) = r(r\omega)\lambda dr$$

$$L_{tot} = \lambda\omega\int_0^l r^2 dr = \frac{1}{3}ml^2\omega$$

And, indeed, the two methods yield the same result.

Kinetic Energy of the System.

$$K.E. = \sum \frac{1}{2} m_i \vec{V}_i^{\,2}$$

$$= \sum_i \frac{1}{2} m_i (\vec{V}_i \cdot \vec{V}_i)$$

$$= \sum_i \frac{1}{2} m_i (\dot{\vec{R}} + \dot{\vec{\rho}}_i) \cdot (\dot{\vec{R}} + \dot{\vec{\rho}}_i)$$

$$= \sum_i \frac{1}{2} m_i \dot{\vec{R}}^2 + \sum m_i (\dot{\vec{R}} \cdot \dot{\vec{\rho}}_i) + \sum_i \frac{1}{2} m_i \dot{\vec{\rho}}_i^{\,2}$$

$$\sum m_i (\dot{\vec{R}} \cdot \dot{\vec{\rho}}_i) = \dot{\vec{R}} \cdot \underbrace{\sum m_i \dot{\vec{\rho}}_i}_{=0}$$

$$= \frac{1}{2} M \vec{V}_{c.m.}^{\,2} + \underbrace{\sum_i \frac{1}{2} m_i \dot{\vec{\rho}}_i^{\,2}}_{(I)}$$

Do the same problem as before

$$\frac{1}{2} M V_{c.m} = \frac{1}{2} M \left(\frac{l}{2} \omega\right)^2 = \frac{1}{8} M l^2 \omega^2$$

$$dT = \frac{1}{2} (2\, \underset{\lambda dr}{dm}) (r\omega)^2 \qquad \lambda = \frac{M}{l}$$

$$(I) = \lambda \omega^2 \int_0^{l/2} r^2 dr$$

$$= \lambda \omega^2 \frac{1}{3} \left(\frac{l}{2}\right)^3$$

$$= \frac{l^3}{24} \frac{M}{l} \omega^2$$

$$= \frac{1}{2} \underbrace{\left(\frac{1}{12} m l^2\right)}_{I_{c.m.}} \omega^2$$

$$\text{Total} = \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \omega^2 = \frac{3+1}{24} M l^2 \omega^2 = \frac{1}{6} M l^2 \omega^2$$

Direct calculation

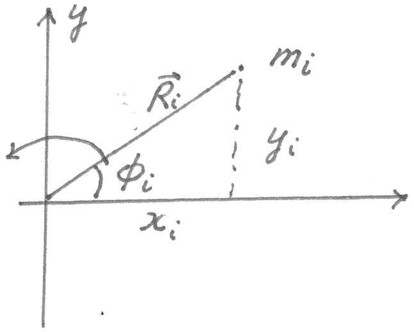
$$T = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} \left(\frac{1}{3} M l^2\right) \omega^2$$

Rotation of a Rigid Body About a Fixed Axis

The simplest type of rigid body, other than pure translation, is that in which the body is constrained to rotate about a fixed axis.

$z \rightarrow$ axis of rotation



$\vec{\omega}$

direction and magnitude

$$\vec{V}_i = \vec{\omega} \times \vec{r}_i$$

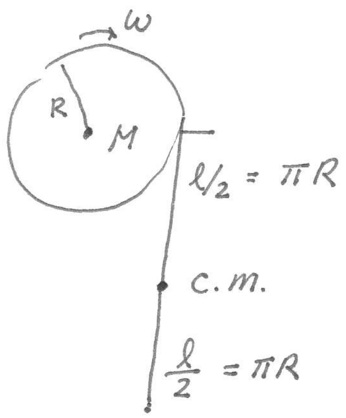
$$L_z = I_z \, \omega$$

$$\underset{\substack{\parallel \\ \tau^{z}_{tot}}}{\frac{dL_z}{dt}} = \frac{d(I_z \omega)}{dt} = I_z \frac{d\omega}{dt}$$

$\downarrow$

the component

Example



center of mass of the chain

$l = 2\pi R$ mass $m = \frac{M}{2}$

Energy conservation

M is unchanged.

$$\underset{\text{initial}}{\frac{M}{2}\, g \,\frac{l}{2}} = \overset{\text{K.E. of the disk}}{\frac{1}{2} I \omega^2} + \boxed{\frac{1}{2} m v^2}$$

$$\frac{1}{2} I\omega^2 = \underbrace{\frac{1}{2}\,\frac{1}{2} M R^2}_{\text{final.}} \omega^2 \qquad \frac{1}{2} m v^2 = \frac{1}{2} m R^2 \omega^2$$

$$\omega^2 = \pi \frac{g}{R}$$

Shown in Figure 8.7 is a uniform chain of length $l = 2\pi R$ and mass $m = M/2$ that was initially wrapped around a uniform, thin disc of radius R and mass M. One tiny piece of chain initially hung free, perpendicular to the horizontal axis. When the disc was released, the chain fell and unwrapped. The disc began to rotate faster and faster about its fixed z-axis, without friction. (a) Find the angular speed of the disc at the moment the chain completely unwrapped itself. (b) Solve for the case of a chain wrapped around a wheel whose mass is the same as that of the disc, but concentrated in a thin rim.

Solution:

(a) Figure 8.7 shows the disc and chain at the moment the chain unwrapped. The final angular speed of the disc is ω. Energy was conserved as the chain unwrapped.

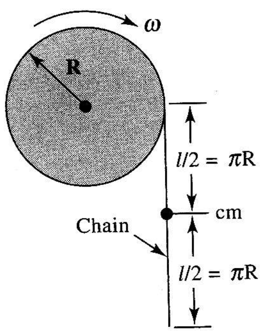


Figure 8.7 Falling chain attached to disc, free to rotate about a fixed z-axis.

Since the center of mass of the chain originally coincided with that of the disc, it fell a distance $l/2 = \pi R$, and we have

$$mg\frac{l}{2} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$\frac{l}{2} = \pi R \qquad v = \omega R \qquad I = \frac{1}{2}MR^2$$

Solving for ω^2 gives

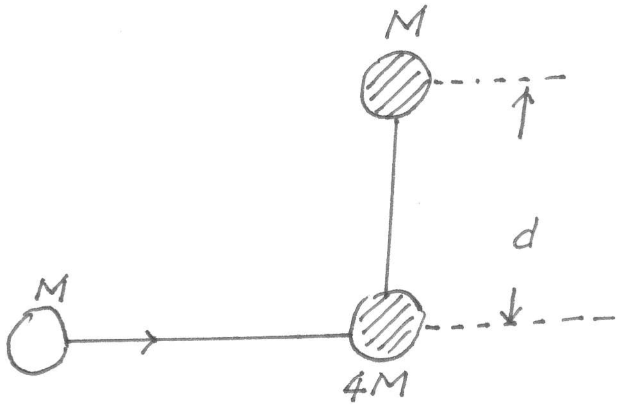
$$\omega^2 = \frac{mg\frac{l}{2}}{\left[\frac{1}{2}\left(\frac{M}{2}\right) + \frac{1}{2}m\right]R^2} = \frac{mg\pi R}{\left(\frac{1}{2}m + \frac{1}{2}m\right)R^2}$$

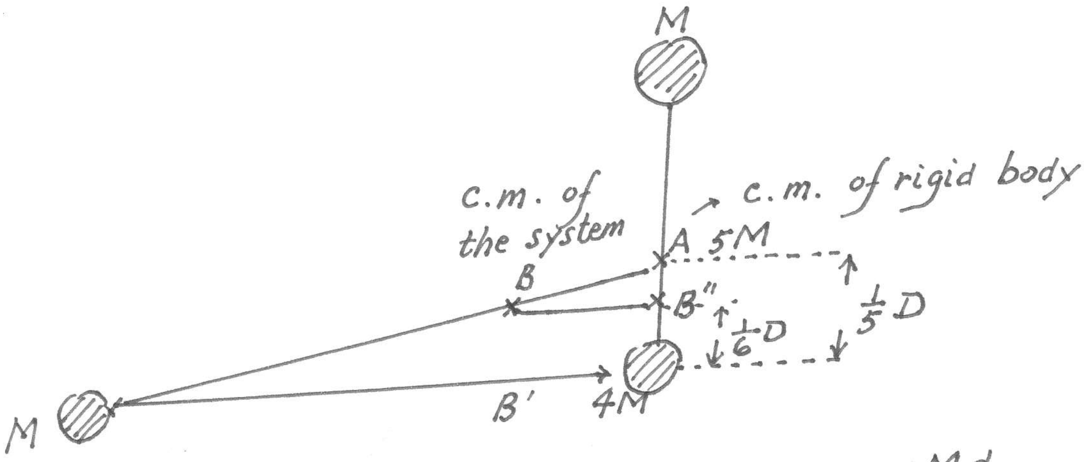
$$= \pi\frac{g}{R}$$

(b) The moment of inertia of a wheel is $I = MR^2$. Substituting this into the above equation yields

$$\omega^2 = \pi\frac{2g}{3R}$$

Even though the mass of the wheel is the same as that of the disc, its moment of inertia is larger, since all its mass is concentrated along the rim. Thus, its angular acceleration and final angular velocity are less than that of the disc. ■

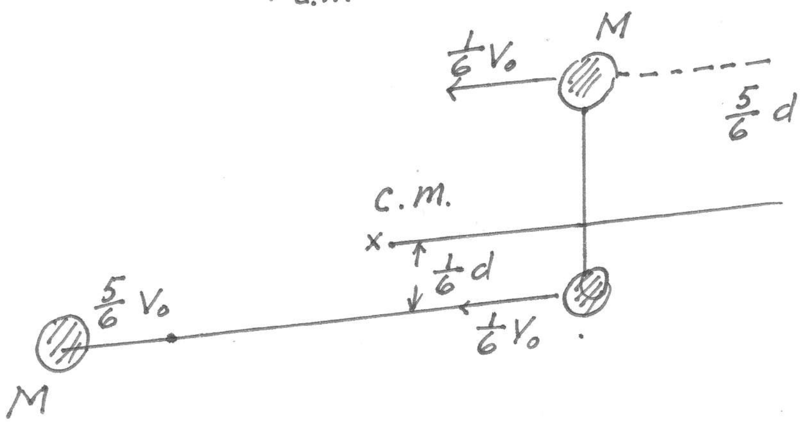




$$A \Rightarrow \frac{Md}{5M} \qquad B'' \Rightarrow \frac{Md}{6M}$$

$$\vec{V}_{c.m.} = \frac{1}{6}\vec{V}_0$$

$$\vec{P}_{c.m} = \text{constant vector}$$



$\vec{r} \times \vec{V}$

$M \quad \frac{5}{6}V_0 \cdot \frac{1}{6}d = \frac{5}{36}MV_0$ up, out of the page

$[-x\hat{i} + (-\frac{1}{6}d)\hat{j}] \times V\frac{5}{6}\hat{i} = + \hat{k}$

$4M \quad \frac{1}{6}V_0 \cdot \frac{1}{6}d = \frac{4}{36}MV_0$ down into the page

$-\frac{1}{6}d\,\hat{j} \times (-\hat{i}) = -\hat{k}$

$M \quad \frac{1}{6}V_0 \quad \frac{5}{6}d = \frac{5}{36}MV_0 d$

$(+\hat{j}) \times (-\hat{i}) = \hat{k}$

$$\frac{10-4}{36} = \frac{1}{6}MV_0 d$$ up, out of page

$$I = M\left(\frac{5}{6}d\right)^2 + 5M\left(\frac{1}{6}d\right)^2 = \frac{5}{6}Md^2$$

$$\frac{1}{6}MV_0 d = \frac{5}{6}Md^2\omega$$

$$\omega = \frac{1}{5}\frac{V_0}{d}$$

Before collision

$$\frac{1}{2}MV_0^2$$

After the collision

$$E_k = \frac{1}{2}6M\left(\frac{1}{6}V_0\right)^2 + \frac{1}{2}I\omega^2$$

$$\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{5}{6}M\cancel{d^2} \cdot \frac{1}{25}\frac{V_0^2}{\cancel{d^2}}$$

$$= \frac{1}{2}\frac{1}{6}MV_0^2 + \frac{1}{60}MV_0^2$$

$$= \frac{5+1}{60}MV_0^2 = \frac{1}{10}MV_0^2$$

$$Q = \frac{1}{10}MV_0^2 - \frac{1}{2}MV_0^2 = \frac{1-5}{10}MV_0^2$$

$$= -\frac{4}{5}\left(\frac{1}{2}MV_0^2\right)$$

Q → change in K.E.

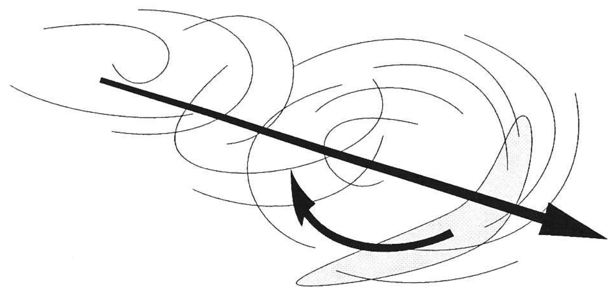
→ 80% of the original K.E. is lost.

Appendix I

PHYSNET

MISN-0-43

TRANSLATIONAL & ROTATIONAL MOTION OF A RIGID BODY



Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

TRANSLATIONAL & ROTATIONAL MOTION OF A RIGID BODY

by

J. S. Kovacs

Title: **Translational & Rotational Motion of a Rigid Body**

Author: J. S. Kovacs, Michigan State University

Version: 2/1/2000 Evaluation: Stage 4

Length: 1 hr; 20 pages

Input Skills:

1. Calculate the kinetic energy loss in an inelastic collision (MISN-0-21).
2. Calculate the kinetic energy of a rigid rotating object (MISN-0-36).

Output Skills (Problem Solving):

S1. Apply the following theorems to determine completely the motion of a rigid body in the case when the torque on the body is along one of the principal axes: (a) Relative to an inertial reference system the time rate of change of the center of mass momentum of a rigid body is equal to the net external force acting on the rigid body; (b) Relative to any point in an inertial reference system, the time derivative of the angular momentum vector of a rigid body is equal to the net external torque acting on the rigid body; and (c) The total kinetic energy of a rigid body relative to an inertial reference system is the kinetic energy of the center of mass of that body relative to the inertial system, plus the kinetic energy of the body relative to the center of mass.

Post-Options:

1. "Ideal Collisions Between a Frictionful Sphere and a Flat Surface: The Superball" (MISN-0-53).
2. "Euler's Equations: The Tennis Racket Theorem" (MISN-0-57).

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New authors, reviewers and field testers are welcome.

TRANSLATIONAL & ROTATIONAL MOTION OF A RIGID BODY

by

J. S. Kovacs

1. Introduction

1a. General Description of the Motion of a System of Particles. The general description of the motion of a system of particles consists of specifying the motion of the center of mass (CM) of the system (as if the whole mass of the system were located at the center of mass) plus the description of the motion of the particles of the system relative to the center of mass. For a rigid body the particles of the system are constrained to move such that the relative separation of all pairs of particles remains unchanged. Theorems about the motion of rigid bodies will be demonstrated as they apply to a simple rigid system.

1b. Theorems on the Motion of Rigid Bodies. When a set of forces acting upon a rigid body are such that they combine to produce an external torque which is directed along one of the principal axes of the body (see Fig. 1), the following theorems may be applied to determine completely the motion of the rigid body:

a. Relative to an inertial reference system the time rate of change of the center of mass momentum of a rigid body is equal to the net external force acting on the rigid body:

$$\frac{d\vec{P}_{CM}}{dt} = \vec{F}_{ext}.$$

a′. If there is no net external force on a rigid body, the center of mass momentum is constant, and hence the center of mass velocity is

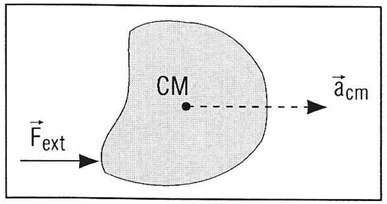


Figure 1. Torque-producing forces acting on a body.

constant in magnitude and direction: $\vec{P}_{CM}$ = constant vector, $\vec{V}_{CM}$ = constant vector.

b. Relative to any point in an inertial reference system, the time derivative of the angular momentum vector of a rigid body is equal to the net external torque acting on that body relative to the same point:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}.$$

b′. With no external torque on the body the angular momentum vector relative to any point in an inertial reference system is constant.

c. The total kinetic energy of a rigid body relative to an inertial reference system is the kinetic energy of the center of mass of that body relative to the inertial system, plus the kinetic energy of the body relative to the center of mass. (For a rigid body this latter is the kinetic energy of the rotation of the body about the center of mass.

2. Application of Rigid Body Theorems to a Simple System

2a. Description of the Motion of a Rigid Body. The complete description of the motion of a rigid body consists of the description of the translational motion of its center of mass as if it were a point mass, plus the description of its rotational motion about an axis through its center of mass.[1] The net external force on the system will determine the former, while the net external torque will determine the latter.

2b. Example: Collision of a Point Mass with a Dumbbell. As an illustration of the theorems listed in Sect. 1b, consider the following example:

In far outer space, far from the influence of any appreciable gravitational force, a rigid body consisting of two point masses, of mass M and $4M$, at the ends of a massless rod of length d is at rest relative to an inertial reference frame.[2] Another mass, M, makes a collision approach

[1] The axis is not necessarily fixed in its direction in space. In cases when it is not, the motion of the direction of the axis is necessary to complete the description. We will not consider such cases here. See, however, "Torque and Angular Momentum in Circular Motion" (MISN-0-34) and "Euler's Equations: The Tennis Racket Theorem" (MISN-O-57).

[2] An inertial reference frame is a frame in which Newton's First Law of Motion is true.

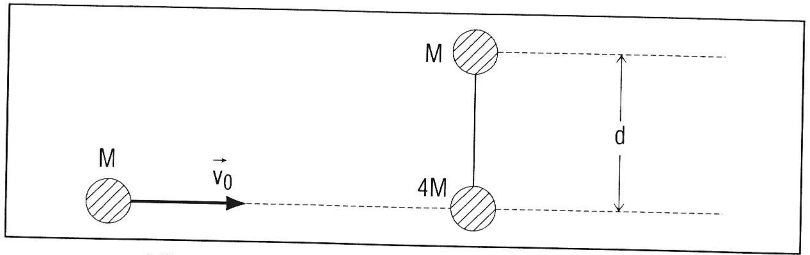


Figure 2. A mass M with initial speed $\vec{V}_0$ approaches to make a head on collision with a mass $4M$, initially at rest. The mass $4M$ is attached via a rigid massless rod to another mass M with the orientation of the rod such that it is perpendicular to the line of flight of the moving mass.

toward this rigid body, moving with a velocity $\vec{V}_o$ toward a head on collision with the $4M$ mass directed perpendicularly to the rod connecting the rigid body masses (See Fig. 2).

2c. Mass and Dumbbell: Translational Motion. *Problem*: To demonstrate that the center of mass of the three mass system moves with the same velocity before and after the collision.

Answer: First, the center of mass must be located, then its velocity before the collision occurs must be determined. The center of mass of the rigid body is located one-fifth of the distance from the $4M$ mass along the line connecting the two masses (at point A in Fig. 3). The center of mass of the combination of this mass ($5M$) and the incident mass M is located along the line joining these masses and one-sixth of the distance from the $5M$ mass to the incident mass (at point B). Figure 2 shows this location at the instant when the incident mass is distance D from the mass $4M$.

Problem: How high above the line joining the incident mass with the struck mass is the center of mass?

Answer: From the similar triangles OPA and $OB'B$ we see that BB' is five-sixths of AP so that the length of BB' is $(d/6)$.

Problem: Where will the center of mass be when the incident mass strikes the $4M$ mass?

Answer: It will be at B'', a distance $(d/6)$ above point P, the location of the mass $4M$. With B and B'' both the same distance above the line OP, it is clear that the line BB'' is parallel to the line OP. Thus, while

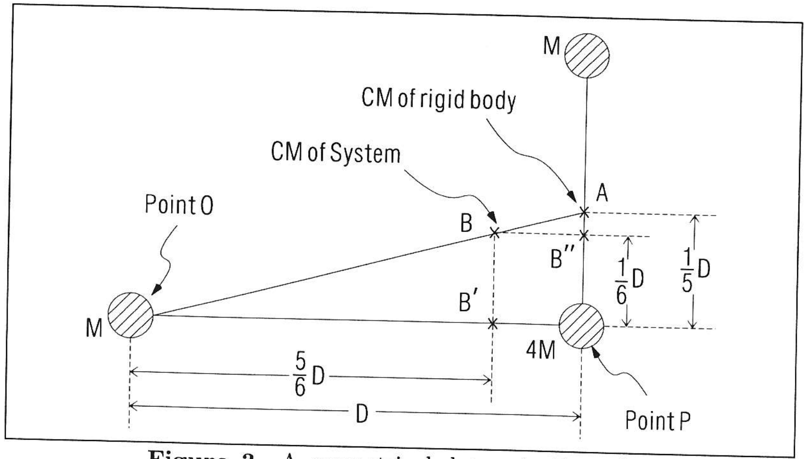


Figure 3. A geometrical determination of the location of the center of mass of the system described in Fig. 2.

the incident mass moved distance D, the center of mass moved from B to B'' along a line parallel to line OP. Line BB'' is of length $(D/6)$, hence the center of mass velocity must be one-sixth of the incident velocity and parallel to the incident velocity:

$$\vec{V}_{CM} = \frac{1}{6}\vec{V}_0. \qquad (1)$$

Problem: What is the CM velocity after the collision?

Answer: It must be the same because there are no external forces on the three mass system. (The incident mass and the struck mass exert forces on each other but these are internal forces.) Because :

$$\frac{d\vec{P}_{CM}}{dt} = \vec{F}_{ext} = 0, \qquad (2)$$

$$\vec{P}_{CM} = \text{ constant vector.} \qquad (3)$$

To demonstrate this, consider the case where the incident mass sticks to the mass $4M$ upon colliding. Before the collision, as observed in an inertial frame at rest with respect to the rigid body, the only momentum is $M\vec{V}_0$. After the collision the momentum is that of the new rigid body ($5M$ at one end, M at the other). The momentum is thus $6M\vec{V}_{CM}$ where

$\vec{V}_{CM}$ is the velocity of the center of mass after the collision. Equating these two (because there are no external forces) we get:

$$\vec{V}_{CM} = \frac{1}{6}\vec{V}_0, \tag{4}$$

exactly what it was before the collision.

2d. Mass and Dumbbell: Rotational Motion. *Problem*: Is this the only motion of the combined system?

Answer: Obviously not. The system spins around as it moves. (If the rigid rod were struck at the center of mass of the system, no spinning would occur.) To describe completely the motion we must also include a description of the rotation of the system. Rotational motion is determined by external torques on the system and how they affect the angular momentum of the system:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}, \tag{5}$$

where $\vec{L}$ and $\vec{\tau}_{ext}$ are both defined with respect to the same point in some inertial reference frame. With our system (consisting of the three masses) there is no net external torque. Therefore the angular momentum, evaluated relative to any point in an inertial frame, is constant:

$$\vec{L} = \text{ constant vector.} \tag{6}$$

It is convenient to take the point relative to which the torque is evaluated as the center of mass of the system.

Problem: First, is this a point in an inertial reference frame?

Answer: If the reference frame in which the rigid rod (before collision) is at rest is an inertial reference frame, then so is a frame which is attached to the center of mass and moving with it. That's because the center of mass reference frame is moving with constant velocity with respect to the reference frame which is at rest with respect to the rigid rod. (In frames of reference which don't accelerate with respect to each other, observers see the same force or effect of a force.)

Problem: Relative to this point, what is the angular momentum of the system before the collision takes place?

Answer: Relative to this point the incident mass is moving to the right with velocity $(5/6)\vec{V}_0$ (recall that the center of mass itself is moving to the right with velocity $(1/6)\vec{V}_0$). The masses on the ends of the rigid rod

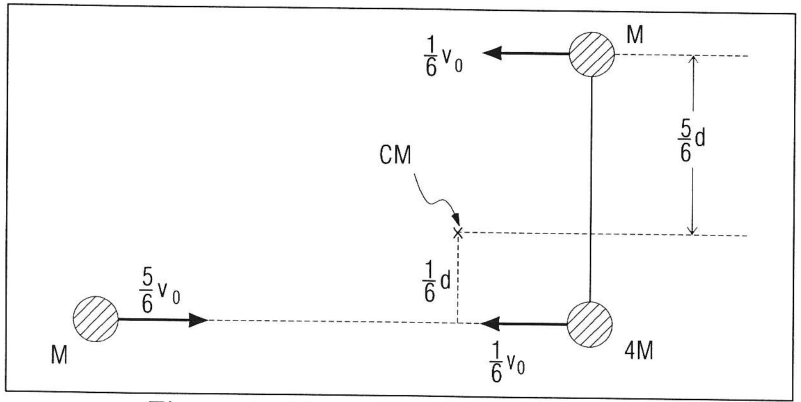


Figure 4. The motion of the system described in Fig. 2 (before collision) as observed in a reference frame in which the center of mass is at rest.

are thus both moving to the left with a speed $(1/6)\vec{V}_0$ or with velocity $-(1/6)\vec{V}_0$. Hence, viewed at rest with respect to the center of mass, the motion observed is as shown in Fig. 4.

The total angular momentum of this system as seen relative to the center of mass is the vector sum of the individual angular momenta. Each $\vec{L} = \vec{r} \times m\vec{v}$, where $\vec{r}$ is the vector from the center of mass to the instantaneous location of the mass, and $\vec{v}$ is the velocity of that mass. However, the magnitude of this cross product is the magnitude of the momentum times the "lever arm."[3]Its direction may be evaluated separately using the right hand rule. The individual angular momenta are determined as indicated in Table 1.

[3]See "Force and Torque" (MISN-0-5)

Table 1. Determination of the individual angular momenta relative to the center of mass of the system for each of the three point masses of the system shown in Fig. 2.

Object	Mass	Speed	Lever Arm	Magn. of Ang. Mom.	Direct. of Ang. Mom.
incident mass	M	$\frac{5}{6}V_0$	$\frac{1}{6}d$	$\frac{5}{36}MV_0d$	up, out of page
mass on rod	$4M$	$\frac{1}{6}V_0$	$\frac{1}{6}d$	$\frac{4}{36}MV_0d$	down, into page
mass on rod	M	$\frac{1}{6}V_0$	$\frac{5}{6}d$	$\frac{5}{36}MV_0d$	up, out of page

The resultant angular momentum of the system (before collision) is thus:

$$\frac{1}{6}MV_0d \text{ directed up, out of page.} \tag{7}$$

This is also the value of the angular momentum of the rod after the incident mass struck and attached to the $4M$ mass. Hence the newly formed rigid body (M and $5M$) at two ends of a rigid rod of length d rotates counter-clockwise about the center of mass with an angular momentum whose magnitude is $(d/6)MV_0$.

Problem: What is the angular velocity of rotation of this rigid body?

Answer: For a rigid body rotating about a principal axis $\vec{L} = I\vec{\omega}$ and $\vec{\omega}$ is in the same direction as $\vec{L}$. In this case, up out of page: the rotation is counter-clockwise. I, the moment of inertia, from its definition is the sum of the "mass times distance squared" contributions for each mass relative to the axis parallel to $\vec{L}$ through the center of mass. It is, therefore,

$$I = M\left(\frac{5}{6}d\right)^2 + 5M\left(\frac{1}{6}d\right)^2 = \frac{5}{6}Md^2. \tag{8}$$

We thus find: $\vec{\omega} = V_0/(5d)$. Consequently, the complete description of the motion of the system after impact is as follows:

i. The center of mass moves with constant velocity $(1/6)\vec{V}_0$ in a straight line parallel to the direction of the velocity of the incident mass.

ii. The rod rotates counter-clockwise about an axis through the center of mass (perpendicular to the plane of the paper) with a constant angular velocity of $V_0/(5d)$

2e. Mass and Dumbbell: Kinetic Energy. *Problem*: What is the kinetic energy of the system?

Answer: Again the answer depends upon the reference frame with respect to which this kinematical quantity is to be observed. Hence, making the question more specific:

Problem: What is the kinetic energy of the system relative to a system which is at rest with respect to the rigid rod before the collision?

Answer: The kinetic energy before the collision is that of the incident mass:

$$\frac{1}{2}MV_0^2.$$

After the collision it is the kinetic energy of the center of mass as if all the mass were concentrated there: $(1/2)6M\frac{V_0}{6}^2$ plus the kinetic energy of all the parts of the system relative to the center of mass. For a rigid body this is $(1/2)I\vec{\omega}^2$. For this object it is:

$$\frac{1}{2}I\vec{\omega}^2 = \frac{1}{60}MV_0^2. \tag{9}$$

The total kinetic energy is thus:

$$E_k(\text{total}) = \frac{1}{12}MV_0^2 + \frac{1}{60}MV_0^2 = \frac{1}{10}MV_0^2. \tag{10}$$

(5 times as much kinetic energy is in translational motion as there is in rotational motion. If the incident mass had struck the center of mass, all the kinetic energy would have been in translational motion.)

Problem: What is the "Q" of this collision?

Answer: According to its definition, Q is the change in the kinetic energy that occurs as a result of the reaction. If Q is positive, kinetic energy has been gained as a result of the reaction or collision. For this collision:

$$Q = -\frac{4}{5}\left(\frac{1}{2}MV_0^2\right). \tag{11}$$

As a result of the collision 80% of the original kinetic energy is lost. (The collision is obviously inelastic[4]).

[4]See "Potential Energy, Conservative Forces, The Law of Conservation of Energy" (MISN-0-21).

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

1. For the system shown in Fig. 1 of the textual material, determine the velocity of the center of mass of the system before the collision. Show that it is parallel to the velocity of the incident mass.
2. Determine the velocity of the center of mass of the system of Problem 1 after the collision, assuming the projectile mass sticks to the mass $4M$.
3. Relative to the center of mass of this above system, determine the velocity of each of the three component masses before the collision.
4. Determine the total angular momentum relative to the center of mass of the above mentioned system before the collision.
5. Determine the total angular momentum relative to the stationary mass M (on the rod) before the collision.
6. Assuming conservation of angular momentum, calculate the angular velocity of the system relative to the center of mass after the collision.
7. Assuming conservation of angular momentum, calculate the instantaneous angular velocity of the system after the collision relative to the location of the initially stationary mass M. (This will be the angular velocity with which that upper mass M sees the other masses begin to rotate around it.)
8. Determine the total kinetic energy, relative to a frame at rest with respect to the rod, of the system before the collision occurs.
9. Determine the total kinetic energy relative to this same frame (as in 8 above) of the system after the collision.
10. Determine the gain or loss of kinetic energy as a result of the collision.

Brief Answers:

1. Equation (1).
2. Equation (2).
3. See Fig. 3.
4. Equation (7).
5. MV_0d, directed up, out of page.
6. $\omega = V_0/(5d)$ counter-clockwise as viewed looking at the plane of the paper.
7. $\omega = V_0/(5d)$.
8. $\frac{1}{2}MV_0$.
9. Equation (11).

MODEL EXAM

1. For the same problem as was illustrated in the *text*, Sect. 2, find the kinetic energy of the system before the collision [K], the kinetic energy after the collision [C] and the Q of the collision [F], all as observed in a reference frame attached to the center of mass.
2. In far outer space free from any outside forces and observed in an inertial frame, there is at rest a uniform rod of length L and mass M. A point mass, also of mass M approaches the rod with a velocity $\vec{V}_0$ directed perpendicular to the long axis of the rod. The mass strikes the rod at a point that is a distance d away from the center of the rod, toweard the end of the rod. After the collision, the incident mass continues along the same straight line in its original direction but with a reduced velocity, $\vec{V}_0/2$.

   a. Determine the location of the center of mass of the system at an instant when the incident mass is a distance D from the rod. [L] Determine the velocity of the center of mass at this instant. [A]
   b. Determine the velocity of the center of mass of the rod after the collision, as observed in the same frame where the rod was initially at rest. [I]
   c. Determine the velocity of the center of mass of the rod relative to the center of mass of the system after the collision. [J]
   d. Determine the angular momentum of the system, before the collision, relative to the center of mass of the rod. [B]
   e. Determine the angular velocity of the rod about its own center of mass after the collision. [E]
   f. Determine the kinetic energy of the system before the collision, as observed in the original inertial frame. [H]
   g. Determine the kinetic energy of the system after the collision, as observed in this same frame. [M]
   h. Determine the Q of this reaction. [G]
   i. Assuming an elastic collision, determine the value of d in terms of L. [D]

Brief Answers:

A. $\vec{V}_0/2$

B. (MV_0d), directed up, out of page, as seen in a sketch where d is below center.

C. $MV_0^2/60$

D. $d = L/\sqrt{6}$

E. $6V_0d/L^2$

F. $-(4/10)MV_0^2$

G. $(1/2)MV_0^2\left(\frac{3d^2}{L^2} - \frac{1}{2}\right)$

H. $MV_0^2/2$

I. $(\vec{V}_0/2)$, same direction as original velocity of point mass.

J. Zero

K. $(5/12)MV_0^2$

L. Along a line joining the center of the rod to the incident mass and half-way between the two.

M. $(1/2)MV_0^2\left(\frac{1}{2} + \frac{3d^2}{L^2}\right)$

Center of Percussion: The "Baseball Bat Theorem"

Appendix II

As an example of the above theory, let us discuss the collision of a ball of mass m, treated as a particle, with a rigid body (bat) of mass M. For simplicity we shall assume that the body is initially at rest on a smooth horizontal surface and is free to move in laminar-type motion. Let $\hat{\mathbf{P}}$ denote the impulse delivered to the body by the ball. Then the equations for translation are

$$\hat{\mathbf{P}} = M\mathbf{v}_{cm} \tag{8.80}$$

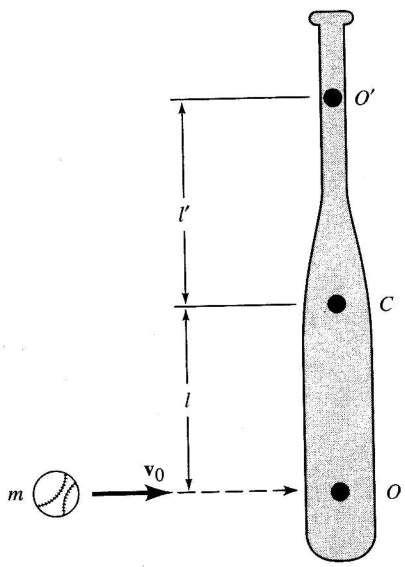
$$-\hat{\mathbf{P}} = m\mathbf{v}_1 - m\mathbf{v}_0 \tag{8.81}$$

where $\mathbf{v}_0$ and $\mathbf{v}_1$ are, respectively, the initial and final velocities of the ball and $\mathbf{v}_{cm}$ is the velocity of the mass center of the body after the impact. Clearly, the above two equations imply conservation of linear momentum.

Since the body is initially at rest, the rotation about the center of mass, as a result of the impact, is given by

$$\omega = \frac{\hat{P}l}{I_{cm}} \tag{8.82}$$

in which l is the distance OC from the center of mass C to the line of action of $\hat{\mathbf{P}}$, as shown in Figure 8.16. Let us now consider a point O' located a distance l' from the



Baseball colliding with a bat.

center of mass such that the line CO' is the extension of OC, as shown. The (scalar) velocity of O' is obtained by combining the translational and rotational parts, namely,

$$v_{O'} = v_{cm} - \omega l' = \frac{\hat{P}}{M} - \frac{\hat{P}l}{I_{cm}}l' = \hat{P}\left(\frac{1}{M} - \frac{ll'}{I_{cm}}\right) \tag{8.83}$$

In particular, the velocity of O' will be zero if the quantity in parentheses vanishes, that is, if

$$ll' = \frac{I_{cm}}{M} = k_{cm}^2 \tag{8.84}$$

where k_{cm} is the radius of gyration of the body about its center of mass. In this case the point O' is the instantaneous center of rotation of the body just after impact. O is called the *center of percussion* about O'. The two points are related in the same way as the centers of oscillation, defined previously in our analysis of the physical pendulum (Equation 8.43).

Anyone who has played baseball knows that if the ball hits the bat in just the right spot there will be no "sting" upon impact. This "right spot" is just the center of percussion about the point at which the bat is held.