Review:

Definitions

System

Internal , External

Coordinates

The coordinates of a physical system are the numbers (possibly dimensional) that describe the system at a given fixed time

Configuration

The configuration $\{q(t)\}$ of a mechanical system is a number (or vectors) consisting of values of a set of <u>independent</u> coordinate that <u>completely</u> describe the system.

Independenty ⇒ the coordinates in the configuration are independent in the sense that each can be changed independently, and each different set of va i describes something physically different

Given a set of values of the coordinates at some particular time ⇒ can figure out what the system looks like at that time.

↓

a configuration is just a mathematical snapshot of the system at a given time problem of classical mechanics

↓

want to understand how the configuration of the system evolves with time

↓

want to put the snapshots together into a mathematical movie to describe how the system moves

Degree of freedom

The number of independent components of the configuration $\{q\}$ is called the number of degree of freedom of the mechanical system

Examples

system	coordinate	# of DOFs
point mass on a track	ℓ (distance along the track)	1
point mass on a flat surface	(x, y)	2
point mass in 3-d	$\vec{r} = (x, y, z)$	3
rigid body in 3-d	$\vec{r}$ of center + 3 angles	6
2 masses + massless spring	$\vec{r}_1$ and $\vec{r}_2$	6
2 masses + massive spring	$\vec{r}_1, \vec{r}_2$ and spring	"∞"

For a system of N point mass in 3 dimension the degree of freedom is $3N$.

A continuous massive spring formally has an infinite number of degrees of freedom, because to specify its configuration we would have to give a continuous function describing how much every point on the spring is stretched. Really, of course, a physical spring has a finite but very large number of degrees of freedom, because it is not actually continuous, but is made up of atoms. But the difference between ∞ and Avogadro's number is often not very important.

This brings up an important philosophical point. What the heck is a "point" mass? What is a "rigid" body? What is a "massless" spring? Most of you have probably been dealing with physics problems for so long that you are used to these phrases. But it is important to remember that these are mathematical idealizations. Real physical systems are complicated, and in fact, what we choose for q may depend on what kind of physical questions we want to ask and what level of accuracy we need in the answer. So for example, for a hockey puck sliding on the ice at the Boston Garden, we might decide that the configuration is specified by giving the x and y coordinates that determine the puck's position in the plane of the ice. Then q would stand for the two dimensional vector, (x, y). But if we do this, we have ignored many details. For example, for a shot that comes off the ice, we would need to include the z coordinate to describe the motion of the puck. For some purposes, we would also need to include descriptions of the puck itself. For example, we have not included an angular variable that would allow us to specify how the puck is turned about its vertical axis. This is probably good enough for most problems. But sometimes, more information is required to give a good description of the physics. For example, if we wanted to understand how a rapidly rotating puck moves, we might need this more detailed information. We could also go on and describe how the puck might deform when hit by the stick, and so on. We could include more and more information until we got down to the level where we begin to see the molecular structure of the rubber of the puck. At this point, we begin to see quantum mechanical effects, and classical mechanics is no longer enough to give an accurate description.

The Art of Theoretical Physics

This is a good lesson. The coordinates that we use to describe the system may depend on what kind of information we want to get out of our mechanical model of the system, and how accurately we want our model to reproduce reality. We usually will not go over these niceties each time we discuss a system, but they are important to remember. There is really a very important point here. In physics courses, we frequently discuss "toy" systems which are obviously oversimplified, in which we have clearly left out features that are important in the "real world." This is not something to apologize for. This is precisely the art of theoretical physics. We work hard to abstract the essential physics of a system, without including things that don't matter at the level of description that we are interested in. This down-to-earth ability to focus on the crucial parameters is far more important than fancy mathematical gymnastics.

In fact, I believe, getting better at this art is one of the most useful things you can get out of this course. It is generally useful far beyond this or future physics courses. The ability to build mathematical models of phenomena is crucial to many fields. But models can be as misleading as they can be useful unless they focus on the right parameters, and unless the modeler is aware of the model's limitations. Physics is the paradigm for this kind of thinking. This is one of the reasons why, over the years, trained physicists have been so much in demand in very different fields.

System of Particles

Discrete

$\vec{r}_i(t)$

Degree of freedom $= 3N$

Continuous System

Density as a function of points at a given time

$\rho = \dfrac{\Delta M}{\Delta V}$ = volume density

$\sigma = \dfrac{\Delta M}{\Delta A}$ = surface density

$\lambda = \dfrac{\Delta M}{\Delta L}$ → linear density

For many purpose a discrete system having a very large but finit number of particles can be considered as a continuous system.

Conversely a continuous system can be considered as a discrete system consisting of a large but finite number of particles.

Center of Mass

Discrete

$$\vec{r}_{c.m.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

$$= \frac{1}{M}\sum m_i\vec{r}_i$$

Continuous

$$\vec{r}_{c.m} = \frac{\int \rho\,\vec{r}\,dV}{\int \rho\,dV}$$

In practicle it is fairly simple to go from discrete to continuous systems by merely replacing summation by integration.

$\Downarrow$

All theorems will be presented for discrete system

Momentum of a System of Particle

$$\underbrace{\vec{P}}_{\text{total momentum of the system}} \equiv \sum_i m_i \dot{\vec{r}}_i = \sum_i m_i \vec{v}_i = M\dot{\vec{r}}_{c.m} = M\vec{V}_{c.m}$$

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{\,ext}$$

(Newton's third law has been used)

Conservation of Momentum

If the resultant external force acting on a system of particles is zero then the total momentum remains constant $\Rightarrow$ total momentum is conserved.

$$\underbrace{\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{\,ext}}_{\text{Equation of Motion (Translational)}}$$

Angular Momentum of a System of Particles

This has been discussed in P. 9-1 to 9-8

Only the notation has been rewritten.

$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i = \sum \vec{L}_i$$

$$\vec{r}_i \equiv \vec{r}_{c.m.} + \vec{\rho}_i \quad \text{(definition)}$$

$$m_i \vec{r}_i = \vec{r}_{c.m} \sum m_i + \sum m_i \vec{\rho}_i = M\vec{r}_{c.m} + \sum m_i \vec{\rho}_i \Downarrow \sum m_i \rho_i = 0 \Rightarrow \sum m_i \dot{\rho}_i = 0$$

$$= \sum (\vec{r}_i - \vec{r}_{c.m}) \times m_i \vec{v}_i + \sum \vec{r}_{c.m} \times \underbrace{m_i \vec{v}_i}_{\vec{p}_i}$$

$$= \sum \vec{\rho}_i \times m_i \vec{v}_i + \vec{r}_{c.m} \times \underbrace{\sum \vec{p}_i}_{\vec{P}}$$

$$= \vec{L}_s + \vec{r}_{c.m} \times \vec{P}_{total} \text{ momentum.}$$

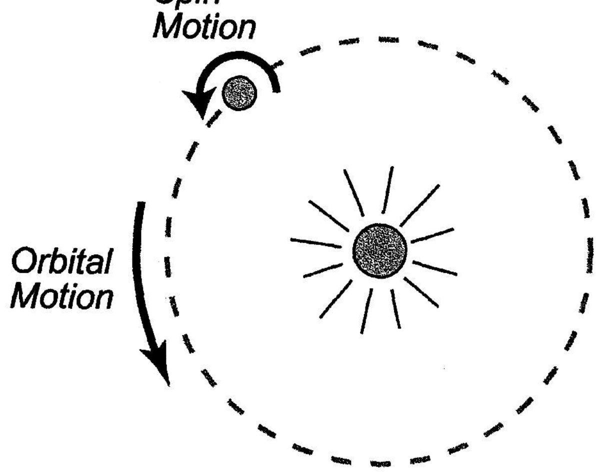
$\vec{L}_s$: spin angular momentum

$\vec{r}_{c.m} \times \vec{P}$: orbital angular momentum.

$$\sum \rho_i \times m_i \underset{(\vec{v}_{c.m.} + \dot{\vec{\rho}}_i)}{\vec{v}_i}$$

$$= (\underbrace{\sum m_i \rho_i}_{0}) \times \vec{v}_{c.m} + \sum \rho_i m_i \times \dot{\vec{\rho}}_i$$

Earth - Sun.



Equation of Motion

$$\frac{dL}{dt} = \frac{d\vec{L}_s}{dt} + \frac{d}{dt}(\vec{r}_{c.m.} \times \vec{p})$$

$$\underset{\downarrow}{\vec{\tau}_{tot}} = \sum \vec{r}_i \times \vec{F}_i^{ext} = \sum_i (\vec{r}_{cm} + \vec{\rho}_i) \times \vec{F}_i^{ext}$$

total torque

$$= \sum \vec{r}_{c.m} \times \vec{F}_i^{ext} + \sum \vec{\rho}^i \times \vec{F}_i^{ext}$$

$$\parallel$$

$$\vec{r}_{c.m} \times \vec{F}_{tot}^{ext} + \sum \vec{\rho}^i \times \vec{F}_i^{ext}$$

$$\frac{dL}{dt} = \frac{d\vec{L}_s}{dt} + \frac{d}{dt}(\vec{r}_{c.m} \times \vec{p})$$

$$= \vec{r}_{cm} \times \vec{F}_{tot}^{ext} + \sum \vec{\rho}^i \times \vec{F}_i^{ext}$$

$$\frac{d\vec{L}_s}{dt} = \sum \vec{\rho}^i \times F_i = \qquad (i)$$

$$\frac{d}{dt}(\vec{r}_{c.m} \times \vec{p}) = \vec{r}_{c.m} \times \vec{F}_{tot} \qquad (ii)$$

Total Kietic Energy of a System of Particles

$$K.E. = \sum \frac{1}{2} m_i \vec{v}_i^{\,2}$$

$$= \sum \frac{1}{2} m_i (\vec{v}_{c.m} + \vec{\rho}_i)^2$$

$$= \sum \frac{1}{2} m_i v_{c.m}^2 + \sum 2\times\frac{1}{2} m_i \vec{v}_{c.m} \cdot \vec{\rho}_i + \sum \frac{1}{2} m_i \vec{\rho}_i^{\,2}$$

$$= \frac{1}{2} M \vec{v}_{c.m}^{\,2} + \frac{1}{2} \sum m_i \vec{\rho}_i^{\,2}$$

Furthermore

The total external torque about the center of mass equals the time rate of change in angular momentum about the center of mass, i.e., $\frac{d\vec{L}_s}{dt} = \vec{\tau}_{c.m.}^{\,ext}$ holds not only for internal coordinate systems but also for system moving with the center of mass.

If motion is described relative to points other than the center of mass, the results in the above theorem become more complicated.

Impulse

If $\vec{F}$ is the total external force acting on a system of particles, then $\int_{t_1}^{t_2} F\,dt$ is the total linear impulse. As in the case of one particle, we can prove $\int_1^2 \vec{F}\,dt = \underbrace{\vec{P}_2 - \vec{P}_1}_{\text{change in linear momentum}}$

If $\vec{\tau}$ is the total external torque $\underbrace{\text{applied to}}_{\text{about origin O}}$ a system of particles the

$$\underbrace{\int_1^2 \vec{\tau}\,dt}_{\text{total angular impulse}} = \underbrace{\vec{L}_2 - \vec{L}_1}_{\text{change in angular momentum}}$$

9-7 COMBINED ROTATIONAL AND TRANSLATIONAL MOTION

Figure 9-28 shows a time-exposure photograph of a rolling wheel. This is one example of a possibly complex motion in which an object simultaneously undergoes both rotational and translational displacements.

In general, the translational and rotational motions are completely independent. For example, consider a puck sliding across a horizontal surface (perhaps a sheet of ice). You can start the puck in translational motion only (no rotation), or you can spin it in one place so that it has only rotational and no translational motion. Alternatively, you can simultaneously push the puck (with any linear velocity) and rotate it (with any angular velocity), so it moves across the ice with both translational and rotational motion. The center of mass moves in a straight line (even in the presence of an external force such as friction), but the motion of any other point of the puck may be a complicated combination of the rotational and translational motions, like the point on the rim of the wheel in Fig. 9-28.

As represented by the sliding puck or the rolling wheel, we restrict our discussion of this combined motion to cases satisfying two conditions: (1) the axis of rotation passes through the center of mass (which serves as the reference point for calculating torque and angular momentum), and (2) the axis always has the same direction in space (that is, the axis at one instant is parallel to the axis at any other instant). If these two conditions are valid, we may apply Eq. 9-11 ($\Sigma\, \tau_z = I\alpha_z$, using only *external* torques) to the rotational motion. Independent of the rotational motion, we may apply Eq. 7-16 ($\Sigma\, \vec{\mathbf{F}} = M\vec{\mathbf{a}}_{\mathrm{cm}}$, using only *external* forces) to the translational motion.

There is one special case of this type of motion that we often observe; this case is illustrated by the rolling wheel of Fig. 9-28. Note that where the illuminated point on the rim

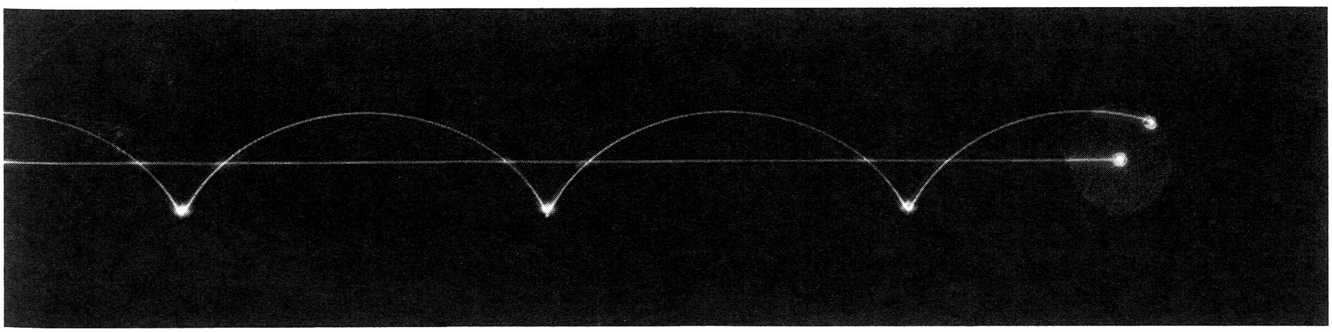


FIGURE 9-28. A time-exposure photo of a rolling wheel. Small lights have been attached to the wheel, one at its center and another at its edge. The latter traces out a curve called a *cycloid.*

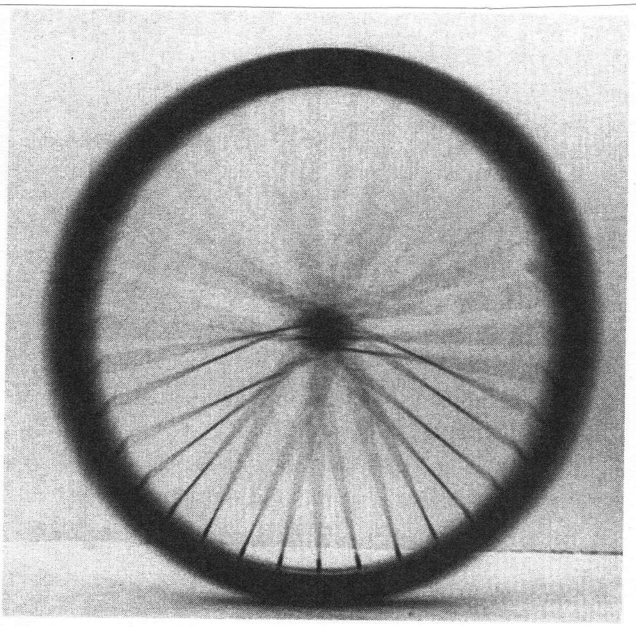


FIGURE 9-29. A photo of a rolling bicycle wheel. Note that the spokes near the top of the wheel are more blurred than those near the bottom. This is because the top has a greater linear velocity.

contacts the surface, the light seems especially bright, corresponding to a long exposure of the film. At these instants, that point is moving very slowly relative to the surface, or may perhaps be instantaneously at rest. This special case, in which an object rolls across a surface in such a way that there is no relative motion between the object and the surface at the instantaneous point of contact, is called *rolling without slipping.*

Figure 9-29 shows another example of rolling without slipping. Note that the spokes of the bicycle wheel near the bottom are in sharper focus than the spokes at the top, which appear blurred. The top of the wheel is clearly moving faster than the bottom! In rolling without slipping, the frictional force between the wheel and the surface is responsible for preventing the relative motion at the point of contact. Even though the wheel is moving, it is the force of *static* friction that applies.

Not all cases of rolling on a frictional surface result in rolling without slipping. For example, imagine a car trying to start on an icy street. At first, perhaps the wheels spin in place, so we have pure rotation with no translation. If sand is placed on the ice, the wheels still spin rapidly, but the car begins to inch forward. There is still some slipping between the tires and the ice, but we now have some translational motion. Eventually the tires stop slipping on the ice, so there is no relative motion between them; this is the condition of rolling without slipping.

Figure 9-30 shows one way to view rolling without slipping as a combination of rotational and translational motions. In pure translational motion (Fig. 9-30*a*), the center of mass C (along with every point on the wheel) moves with velocity v_{cm} to the right. In pure rotational motion (Fig. 9-30*b*) at angular speed ω, every point on the rim has tangential speed ωR. When the two motions are combined, the resulting velocity of point B (at the bottom of the wheel) is $v_{cm} - \omega R$. For rolling without slipping, the point where the wheel contacts the surface must be at rest; thus $v_{cm} - \omega R = 0$, or

$$v_{cm} = \omega R. \tag{9-36}$$

Superimposing the resulting translational and rotational motions, we obtain Fig. 9-30*c*. Note that the linear speed at the top of the wheel (point T) is exactly twice that at the center.

Equation 9-36 applies *only* in the case of rolling without slipping; in the general case of combined rotational and translational motion, v_{cm} does not equal ωR.

There is yet another instructive way to analyze rolling without slipping: we consider the point of contact B to be an instantaneous axis of rotation, as illustrated in Fig. 9-31. At each instant there is a new point of contact B and therefore a new axis of rotation, but instantaneously the motion consists of a pure rotation about B. The angular velocity of this rotation about B is the same as the angular velocity ω of the rotation about the center of mass. Since the distance from B to T is twice the distance from B to C, once again we conclude that the linear speed at T is twice that at C.

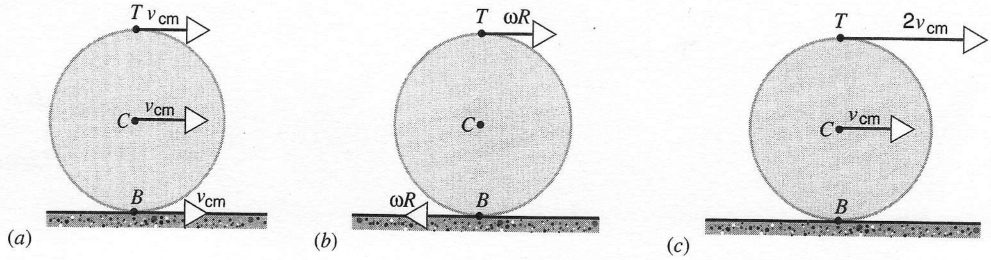


FIGURE 9-30. Rolling can be viewed as a superposition of pure translation and rotation about the center of mass. (*a*) The translational motion, in which all points move with the same linear velocity. (*b*) The rotational motion, in which all points move with the same angular velocity about the central axis. (*c*) The superposition of (*a*) and (*b*), in which the velocities at T, C, and B have been obtained by vector addition of the translational and rotational components.

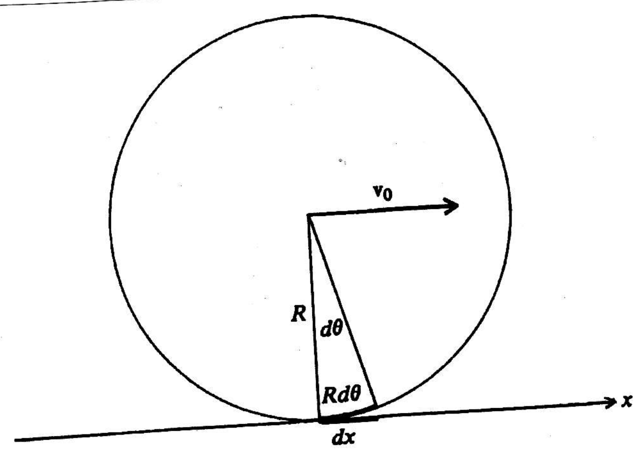


FIGURE 6-4. Wheel rolling without slipping on a level surface.

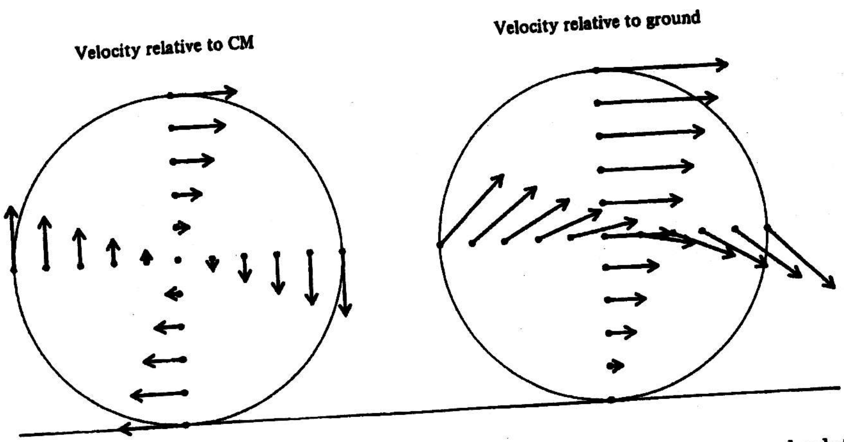


FIGURE 6-5. Velocity of points on a wheel relative to the center of mass and relative to the ground.

6.10 Impulses and Billiard Shots

For forces that act only during a very short time, it is convenient to use an integrated form of the laws of motion. The translational motion of the center-of-mass point is determined by

$$\dot{\mathbf{P}} = \mathbf{F} \tag{6.134}$$

If we multiply both sides of this equation by dt and integrate over the short time interval $\Delta t = t_1 - t_0$, during which the force acts, we obtain

$$\Delta\mathbf{P} = \mathbf{P}^1 - \mathbf{P}^0 = \int_{t_0}^{t_1} \mathbf{F}dt \tag{6.135}$$

The time integral of the force on the right is called the *impulse*. For angular motion the integrated form of the equation of motion in (6.48) is

$$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \int_{t_0}^{t_1} \mathbf{N}dt \tag{6.136}$$

The time integral of the torque is called the *angular impulse*. For rigid-body rotations about a fixed z axis, we can use (6.119) to rewrite the angular-impulse equation (6.136) as

$$\Delta L_z = I_{zz}\Delta\omega_z = I_{zz}(\omega_z^1 - \omega_z^0) = \int_{t_0}^{t_1} N_z dt \tag{6.137}$$

As an example of the usefulness of the impulse formulation of the equations of motion, we discuss the dynamics of billiard shots. For simplicity we consider only shots in which the cue hits the ball in its vertical median plane in a horizontal direction. In billiard jargon these are shots without "English".

The cue imparts an impulse to the stationary ball at a vertical distance h above the table, as illustrated in Fig. 6-16. The linear impulse from (6.135) is

$$M\Delta V_x = MV_x^1 = \int_{t_0}^{t_1} F_x dt \tag{6.138}$$

where V_x^1 is the velocity of the CM just after impact. The angular impulse of (6.137) about the z axis in Fig. 6-17 which passes through the CM of

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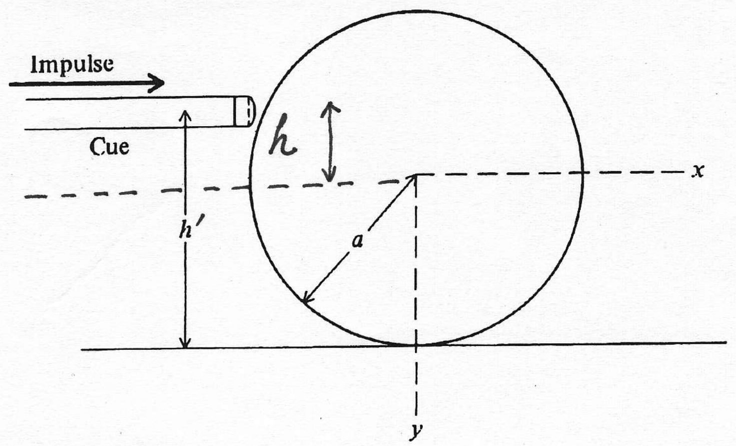


FIGURE 6-16. Impulse imparted to a billiard ball by the cue stick.

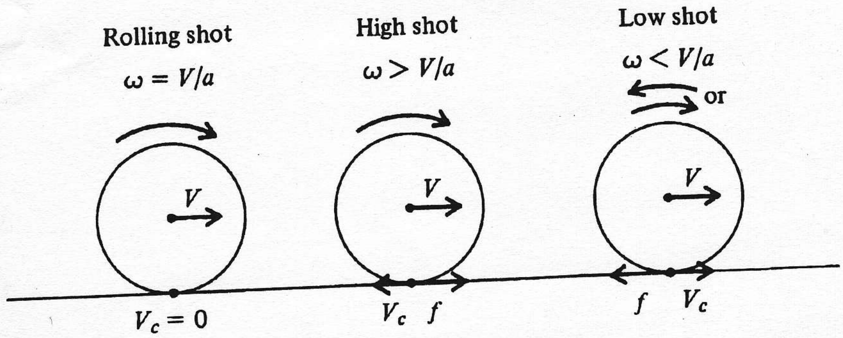
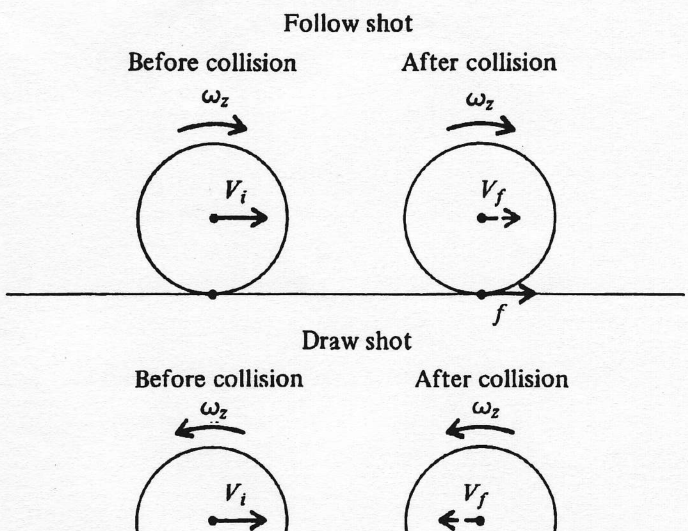
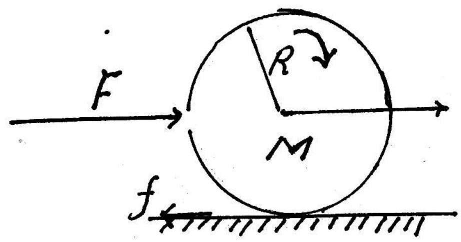


FIGURE 6-17. Rolling, high, and low shots in billiards.





Impulse through the center of mass

$\Rightarrow$ initial velocity

$$V_{cm} = V_0 \ , \quad \omega = 0$$

Due to friction force

$$\tau = fR = \mu MgR = I\alpha$$

$$I = \frac{2}{5}MR^2$$

$$\Rightarrow \quad \alpha = \frac{5}{2}\mu\frac{g}{R}$$

$$\omega = \frac{5}{2}\mu\frac{g}{R}t$$

$$Ma_c = -f$$

$$\Rightarrow \quad a_c = -\frac{\mu Mg}{M} = -\mu g$$

$$V_{c.m.} = V_0 - \mu gt = \omega R = \frac{5}{2}\mu\frac{g}{R}tR$$

Condition for rolling

$$V_{c.m.} = \omega R$$

$$t = \frac{2}{7}\frac{v_0}{\mu g}$$

time elapsed before rolling

$$S = v_0 t - \frac{1}{2}\mu g t^2$$

$$= \frac{12}{49} \cdot \frac{v_0^2}{\mu g}$$

$F \Delta t = M V_0$ *Impulse*

$F h \Delta t = I \omega$ *Angular Impulse*

$$\frac{M V_0}{\Delta t} h \Delta t = I\omega$$

$$\Rightarrow \boxed{M V_0 h} = I\omega$$

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h Impulse approximation

$$\omega = \frac{M v_0 h}{I} = \frac{5}{2} \frac{v_0 h}{R^2}$$

If $\omega R = V_0$, then $h = \frac{2}{5} R$, immediately start rolling

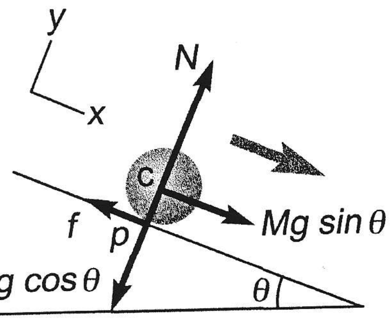
$h < \frac{2}{5} R$ $\omega < \frac{V_0}{R}$, take some time before rolling

$h > \frac{2}{5} R$ $\omega > \frac{V_0}{R}$ friction is moving forward

At the contact point, the velocity is going backward.

Example : Rolling without slipping

Any round object of radius R rolls about its CM as it translates down plane of angle θ.



Free-body diagram for a solid sphere rolling down an incline.

Mass = M

Inertia = $I = \beta M R^2$

① $\sum \tau = I\alpha$ (aboot cm)

√ ② $\tau_f + \tau_{mg} + \tau_N = Rf + 0 + 0 = I\alpha$

√ ③ $\sum f_x = Mg\sin\theta - f = Ma_{cm}$

If motion is rolling without slipping

$v_{cm} = R\omega$ and $a_{cm} = R\alpha$ → condition

$$Mg\sin\theta - \frac{I}{R}\alpha = Mg\sin\theta - \frac{\beta MR^2}{R}\frac{a_{cm}}{R}$$

$$= Mg\sin\theta - \beta M a_{cm} = Ma_{cm}$$

$$a_{cm} = \frac{g\sin\theta}{1+\beta}$$

Friction is static friction

$\therefore f_s \leq \mu_s N$

$$f_s = \frac{I\alpha}{R} = \frac{\beta M R^2}{R} \cdot \frac{1}{R} \frac{g \sin\theta}{1+\beta} \leq \mu_s M g \cos\theta$$

$$\therefore \tan\theta \leq \mu_s \frac{1+\beta}{\beta}$$

condition for angle above which object will slide as it rolls down the plane.

If object slides: $\left.\begin{array}{l} \omega R \neq v \\ \alpha R \neq a \end{array}\right\}$!!!

Hoop	$\beta = 1$
Cylinder	$\beta = 1/2$
Sphere	$\beta = 2/5$

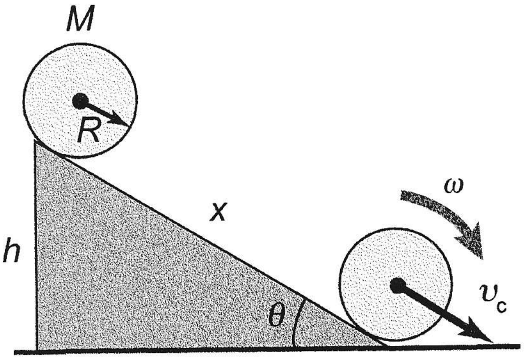
$$(a_{cm})_{sphere} = \frac{5}{7} g \sin\theta$$

$$(a_{cm})_{cyl} = \frac{2}{3} g \sin\theta$$

$$(a_{cm})_{Hoop} = \frac{1}{2} g \sin\theta$$

Example: Rolling down Incline

- Release from rest at top.
- No slipping.
- Rolling possible only if friction present to produce torque about cm.
- No energy lost since contact point does not move relative to surface.



A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

$$v_c = R\omega$$

$$K = \frac{1}{2} I_c \left(\frac{v_c}{R}\right)^2 + \frac{1}{2} M v_c^2$$

$$= \frac{1}{2}\left[\frac{I_c}{R^2} + M\right] v_c^2$$

Potential energy lost if object drops a height h:

$$\Delta U = Mgh.$$

$$\Delta K = \Delta U$$

$$\frac{1}{2}\left(\frac{I_c}{R^2} + M\right) v_c^2 = Mgh$$

$$v_c = \sqrt{\frac{2gh}{1 + I_c/MR^2}}$$

Example: Sphere down Plane

$$I_c = \frac{2}{5} MR^2$$

$$v_c = \sqrt{\frac{2gh}{1+\frac{2}{5}\frac{MR^2}{MR^2}}} = \sqrt{\frac{10}{7} gh}$$

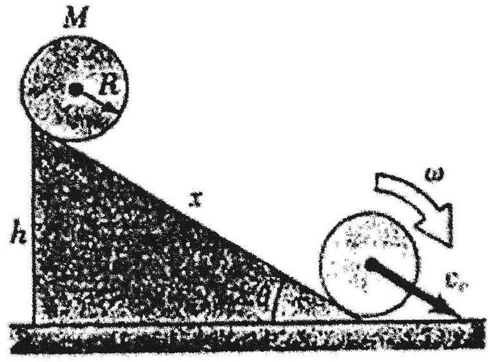


Figure A round object rolling down an incline. Mechanical energy is conserved if no slipping occurs.

x = distance along incline

$h = x \sin\theta$

$$v_c^2 = \frac{10}{7} g x \sin\theta$$

$$v_c^2 = 2 a_c x \qquad \text{[constant acceleration]}$$

$$a_c = \frac{5}{7} g \sin\theta$$

Note:

Velocity and acceleration are independent of mass and radius of sphere. All homogeneous solid spheres would have the same velocity and acceleration on a given incline.

Hollow spheres, cylinders + hoops would give similar results. Constants in expressions for v_c and a_c would be different.

Acceleration is less than for an object which does not roll.

Rigid Body

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\,ext}$$

i) In a coordinate system where the reference point is fixed or moving with constant velocity $\ddot{\vec{r}}_p = 0$

(ii) The reference point is the center of mass

(iii) $\vec{r}_p - \vec{R}$ is // to $\ddot{\vec{r}}_p$

See P. 134 - P. 135

Choose a system that rotate with the system.

↓

In this system the moments and products of inertia are time-independent

When a rigid body moves with one point stationary about the point is the total angular momentum

$$\vec{L} = \sum_i m_i (\vec{r}_i \times \vec{V}_i)$$

$\vec{r}_i$ radius vector relative to the given point

$\vec{V}_i$ velocity vector

$$\vec{V}_i = \vec{\omega} \times \vec{r}_i$$

$$\vec{L} = \sum_i m_i (\vec{\omega} r_i^2 - \vec{r}_i(\vec{r}_i \cdot \vec{\omega}))$$

$$\parallel$$

$$\sum m_i (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))$$

x, y, z coordinates $L_x = \omega_x \sum_i m_i (\vec{r}_i^{\,2} - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum m_i x_i z_i$

↓

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$I_{ij} = I_{ji}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{fixed} = \left(\frac{d\vec{A}}{dt}\right)_{body\ fixed} + \vec{\omega} \times \vec{A}$$

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{body\ fixed} + (\vec{\omega} \times \vec{L})$$

Instead of the (x, y, z) coordinates it can be shown that there is $(1,2,3)$ mutually orthogonal system in which

$$I_{ij} = 0 \quad \text{for } i \neq j$$

principal axis coordinates

$$L_1 = I_{11}\,\omega_1 = I_1\,\omega_1$$

$$L_2 = I_{22}\,\omega_2 = I_2\,\omega$$

$$L_3 = I_{33}\,\omega_3 = I_3\,\omega$$

principal axes along which $\vec{L} \parallel \vec{\omega}$

- there are at least three orthogonal principal axes. They can be found by diagonalization or by symmetry consideration.

- $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$

$\vec{\omega}$ rotation vector.

$(\omega_1, \omega_2, \omega_3)$ describe the rotation

$$\tau_1 = \frac{dL_1}{dt} + \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left|\begin{array}{ccc} \omega_1 & \omega_2 & \omega_3 \\ L_1 & L_2 & L_3 \end{array}\right|_1 \end{array}$$

$$= I_1\,\dot{\omega}_1 + \omega_2\,\omega_3\,(I_2 - I_3)$$

Similarly, we can obtain the other two equations

$$\tau_2 = I_2\,\dot{\omega}_2 - \omega_2\,\omega_3\,(I_3 - I_1)$$

$$\tau_3 = I_3\,\dot{\omega}_3 - \omega_1\,\omega_2\,(I_1 - I_2)$$

These are the famous Euler equation.

τ_1, τ_2, τ_3 + initial conditions

$\vec{\tau} \Rightarrow \vec{\omega}$

$\vec{\omega}$ and $\vec{N} = \vec{\tau} \Rightarrow$ vectors of the inertial system projected onto the principal body axes.

The Earth as a Free Symmetric Top

Nearly sphere in shape

Gravitational Torque exterted on earth by the Sun and the Moon are negligible

Nearly axially symmetric

$$I_1 = I_2 = I$$

$$\dot{\omega}_1 + \frac{I_3 - I}{I} \omega_3 \omega_2 = 0$$

$$\dot{\omega}_2 - \frac{I_3 - I}{I} \omega_1 \omega_2 = 0$$

$$\dot{\omega}_3 = 0$$

$$\omega_3(t) = \omega_3(0) = \omega_3 = \text{constant}$$

$$\dot{\omega}_1 + \underbrace{\frac{I_3 - I_1}{I} \omega_3}_{\Omega} \omega_2 = 0$$

$$\dot{\omega}_1 + \Omega \omega_2 = 0$$

$$\dot{\omega}_2 - \Omega \omega_1 = 0$$

$$\ddot{\omega}_1 + \Omega \dot{\omega}_2 = 0$$

$$\Rightarrow \quad \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

$$\omega_1 = A \cos(\Omega t + \alpha)$$

$$= A \cos \Omega t \cos \alpha - A \sin \Omega t \sin \alpha$$

$$= c_1 \cos \Omega t + c_2 \sin \Omega t$$

$$\dot{\omega}_1 + \Omega \omega_2 = 0$$

$$-A \Omega \sin(\Omega t + \alpha) + \Omega \omega_2 = 0$$

$$\omega_2 = A \sin(\Omega t + \alpha)$$

A, α are to be determined by the initial condition

$$T = \frac{2\pi}{\Omega} = \underbrace{\left(\frac{I}{I_3 - I}\right)}_{\simeq 300} \frac{2\pi}{\omega_3}$$

The Earth as a Free Symmetric Top

Since the earth is nearly spherical in shape, the gravitational torques exerted on the earth by the sun and the moon are quite small. To a good approximation the rotational motion can therefore be described by Euler's equations with no external torques. Since the earth is nearly axially symmetric, the principal moments of inertia for the two axes in the equatorial plane are equal.

$$I_1 = I_2 = I \tag{7.116}$$

The third principal axis with moment of inertia I_3 is along the polar symmetry axis. From (7.88) the differential equations for the earth's motion in an earth-based coordinate frame are

$$\begin{aligned} \dot{\omega}_1 + \frac{I_3 - I}{I}\omega_3\omega_2 &= 0 \\ \dot{\omega}_2 - \frac{I_3 - I}{I}\omega_1\omega_3 &= 0 \\ \dot{\omega}_3 &= 0 \end{aligned} \tag{7.117}$$

Any rigid body which obeys this set of torque-free equations is called a *free axially symmetric top*. The exact solution to this coupled set of equations is easily obtained. The last equation above implies that ω_3 is constant.

$$\omega_3(t) = \omega_3(0) = \omega_3 \tag{7.118}$$

The equations (7.117) can be solved using the method of (7.97)–(7.101). The solution is

$$\begin{aligned} \omega_1(t) &= a\cos(\Omega t + \alpha) \\ \omega_2(t) &= a\sin(\Omega t + \alpha) \end{aligned} \tag{7.119}$$

where

$$\Omega = \frac{I_3 - I}{I}$$

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The magnitude of the angular-velocity vector ω is

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^3} = \sqrt{a^2 + \omega_3^2} \tag{7.121}$$

Since the components ω_1 and ω_2 in (7.119) trace out a circle of radius a while ω_3 and ω remain constant, an observer on the earth sees the angular-velocity vector precesses uniformly about the symmetry axis with angular velocity Ω, as shown in Fig. 7-16.

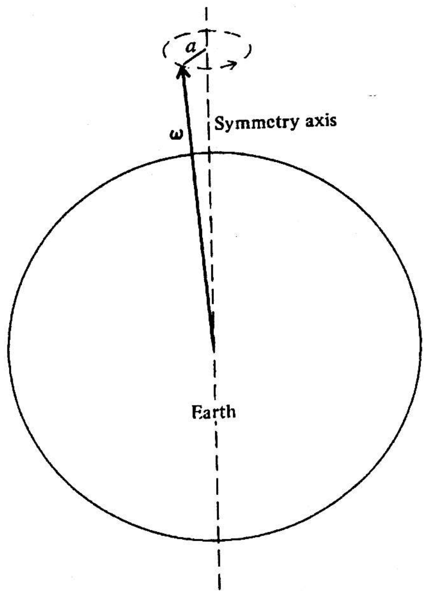

FIGURE 7-16. Precession of the earth's spin about the symmetry axis.

The period of precession of ω about the earth's symmetry axis is

$$\tau = \frac{2\pi}{\Omega} = \left(\frac{I}{I_3 - I}\right)\frac{2\pi}{\omega_3} \tag{7.122}$$

For the earth, since $2\pi/\omega_3 = 1$ day, the period of precession in days is determined by the moment-of-inertia ratio. For an earth of uniform density and oblate spheroidal shape, the value of this ratio, calculated from the measured radii of the earth, is

$$\frac{I}{I_3 - I} \approx 300 \tag{7.123}$$

Although the earth becomes more dense toward its center, the moment-of-inertia ratio is not appreciably changed from the uniform-density result.

Thus the expected precessional period is about 300 days. The precession of ω about the symmetry axis of the earth is known as the *Chandler wobble.*

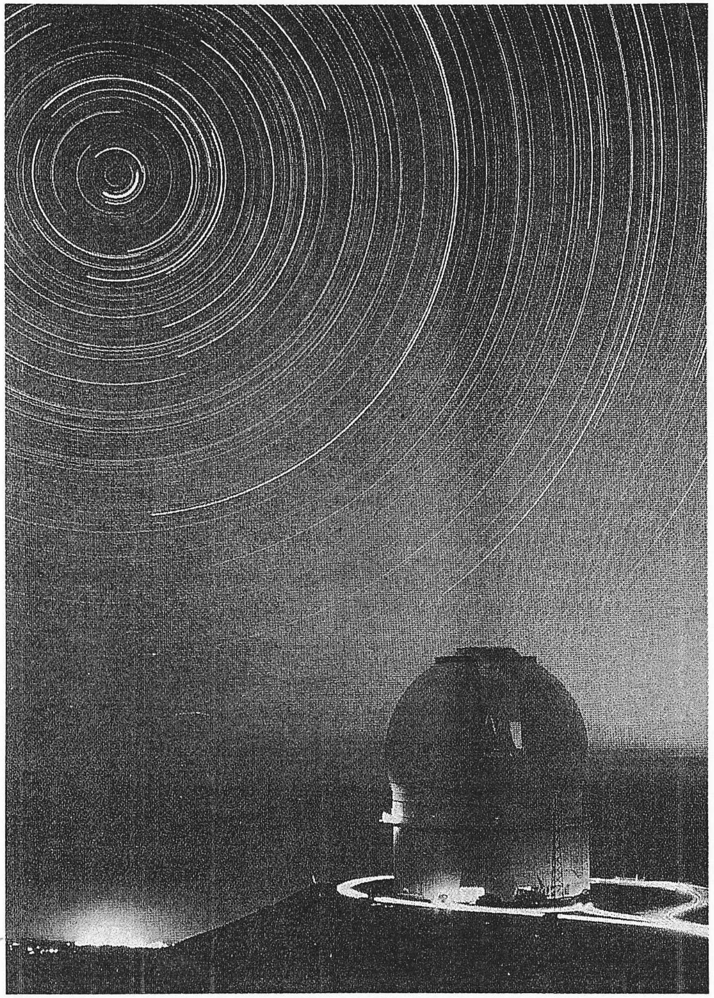

FIGURE 7-17. Star trails in the night sky above the Mauna Kea Observatory in Hawaii, photographed with a nine hour exposure camera. The stars appear as circular arcs due to the earth's rotation. The smallest bright arc is that of Polaris.

The direction of the earth's axis of rotation (*i.e.*, the direction of $\boldsymbol{\omega}$) can be experimentally determined by location of the point in the night sky which appears to remain stationary as the earth rotates, as illustrated in Fig. 7-17. The direction of the earth's rotational axis is observed to precess about the symmetry axis with a period of about 440 days. The angle between $\boldsymbol{\omega}$ and the symmetry axis is quite small. In fact, at the north pole, $\boldsymbol{\omega}$ never moves more than about 10 m from the symmetry axis. The actual motion of $\boldsymbol{\omega}$ is rather irregular, being strongly affected by earthquakes and seasonal changes. In fact, it is only due to these effects that the motion has a nonvanishing amplitude. On a quiet earth, viscous effects would damp out such a motion, and $\boldsymbol{\omega}$ would soon lie along the symmetry axis (this minimizes energy for fixed $\mathbf{L}$). The discrepancy between the expected period of 300 days and the observed value of about 440 days is primarily due to the nonrigidity of the earth.

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Appendix I

7.7 The Tennis Racket Theorem

The solution of Euler's equations for a rigid body with unequal principal moments of inertia can be beautifully illustrated with a tennis or badminton racket. The three principal axes of a tennis racket are readily identified to be (1) along the handle, (2) perpendicular to the handle in the plane of the strings, and (3) perpendicular to the handle and strings. When a tennis racket is tossed into the air with a spin about one of the principal axes, a curious phenomenon is observed. If the initial spin is about either axis (1) or axis (3), the racket continues to spin uniformly about the initial axis and can easily be recaught. On the other hand, if the initial spin is about axis (2), the motion quickly becomes irregular, with spin developing about all three principal axes, which makes it difficult to catch the falling racket. The explanation of the observed behavior follows from Euler's equations. To apply Euler's equations to the tennis racket, we choose the origin of the principal-axes coordinate system at the CM of the racket, as illustrated in Fig. 7-13.

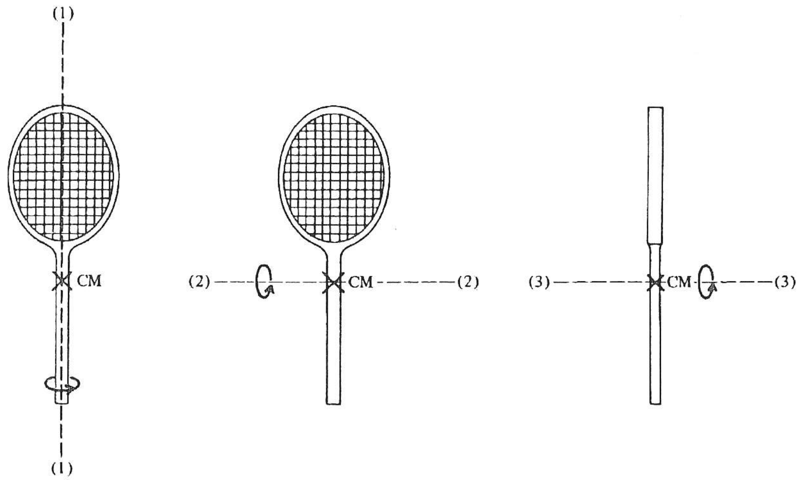

FIGURE 7-13. Principal axes of the tennis racket.

Since gravity is a uniform force in the vicinity of the earth's surface, there are no gravitational torques about the CM of the racket. If we neglect torques due to wind resistance, Euler's equations (7.75) simplify to

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0
\end{aligned}
\tag{7.88}
$$

These equations can be written as

$$\dot{\omega}_1 + r_1\omega_2\omega_3 = 0; \quad r_1 = \frac{I_3 - I_2}{I_1} \tag{7.89}$$

$$\dot{\omega}_2 - r_2\omega_3\omega_1 = 0; \quad r_2 = \frac{I_3 - I_1}{I_2} \tag{7.90}$$

$$\dot{\omega}_3 + r_3\omega_1\omega_2 = 0; \quad r_3 = \frac{I_2 - I_1}{I_3} \tag{7.91}$$

If we assume the ordering

$$I_1 \le I_2 \le I_3 \tag{7.92}$$

we consequently have all the r_i positive. We note the sign asymmetry in the three Euler equations (7.89)–(7.91).

The tennis racket theorem concerns stability of spin about the principal axes. We assume that the racket is initially spun nearly about one of the principal axes and we will determine if this spin state is stable. Assume first that the spin is initially nearly along the intermediate axis (2) or

$$\boldsymbol{\omega} \simeq \omega_2\hat{\mathbf{y}}; \quad \omega_1, \omega_3 \ll \omega_2 \tag{7.93}$$

If ω_1 and ω_3 are small as hypothesized their product is negligible and (7.90) implies that

$$\omega_2 \simeq \text{constant} \tag{7.94}$$

Equations (7.89) and (7.91) then comprise a set of coupled linear equations in ω_1 and ω_3

$$
\begin{aligned}
\dot{\omega}_1 + (r_1\omega_2)\omega_3 &= 0 \\
\dot{\omega}_3 + (r_3\omega_2)\omega_1 &= 0
\end{aligned}
\tag{7.95}
$$

For the trial exponential solution

$$
\begin{aligned}
\omega_1 &= a_1 e^{\lambda t} \\
\omega_3 &= a_3 e^{\lambda t}
\end{aligned}
\tag{7.96}
$$

two algebraic equations must be satisfied

$$
\begin{aligned}
a_1\lambda + r_1\omega_2 a_3 &= 0 \\
a_3\lambda + r_3\omega_2 a_1 &= 0
\end{aligned}
\tag{7.97}
$$

Solving both equations for the ratio a_3/a_1, we obtain

$$\frac{a_3}{a_1} = -\frac{\lambda}{r_1\omega_2} = -\frac{r_3\omega_2}{\lambda} \tag{7.98}$$

The second equality determines λ to be

$$\lambda = \pm\omega_2\sqrt{r_1 r_3} \tag{7.99}$$

Then from (7.98) the ratio of amplitudes

$$a_3 = \mp a_1\sqrt{\frac{r_3}{r_1}} \tag{7.100}$$

The general solution is a superposition of these two solutions

$$\begin{aligned} \omega_1(t) &= a e^{\omega_2\sqrt{r_1 r_3}t} + b e^{-\omega_2\sqrt{r_1 r_3}t} \\ \omega_3(t) &= \sqrt{\frac{r_3}{r_1}}\left[-a e^{\omega_2\sqrt{r_1 r_3}t} + b e^{-\omega_2\sqrt{r_1 r_3}t}\right] \end{aligned} \tag{7.101}$$

where a and b are constants determined by the initial conditions. Since r_1 and r_3 are positive the solution is a superposition of increasing and decreasing exponentials in time. The increasing exponential term will make ω_1 and ω_3 large even if they started small and hence *rotation about the intermediate axis (2) is unstable.* Our solution for axis (2) is of course strictly valid only at times for which the product $\omega_1\omega_3$ is small.

Next take the initial angular velocity nearly along one of the extreme principal moment axes, say axis (1)

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{x}}\,, \qquad \omega_2, \omega_3 \ll \omega_1 \tag{7.102}$$

By (7.89) we obtain

$$\omega_1 \simeq \text{constant} \tag{7.103}$$

and ω_2 and ω_3 satisfy

$$\begin{aligned} \dot\omega_2 - r_2\omega_1\omega_3 &= 0 \\ \dot\omega_3 + r_3\omega_1\omega_2 &= 0 \end{aligned} \tag{7.104}$$

Proceeding as before, we know the exponential solutions analogous to (7.96) must satisfy (7.104) except that in this case

$$\begin{aligned} \lambda &= \pm\omega_1\sqrt{-r_2 r_3} = \pm i\omega_1\sqrt{r_2 r_3} \\ a_3 &= \pm a_2\sqrt{r_3/r_2} \end{aligned} \tag{7.105}$$

The solutions now are oscillatory and can be written in the form

$$\begin{aligned} \omega_1(t) &= a\sin\left(\omega_1\sqrt{r_2 r_3}t + \alpha\right) \\ \omega_3(t) &= \sqrt{\frac{r_3}{r_2}}a\cos\left(\omega_1\sqrt{r_2 r_3} + \alpha\right) \end{aligned} \tag{7.106}$$

where a and α are constants determined by the initial conditions. Thus if ω_2 and ω_3 are initially small, they will remain small. *Rotation about axis (1) is thus stable.*

For an initial spin about axis (3) the solution to Euler's equations is similar to the case of axis (1). The exponential factor λ is again purely imaginary and the solutions are oscillatory. We conclude that *rotations along extreme axes are stable while the intermediate axis is unstable.*

For spin about axis (1) the angular velocity vector $\boldsymbol{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$ precesses in a small cone about principal axis (1) as shown in Fig. 7-14. A similar picture applies to the largest principal axis (3). For rotation along principal axis (2) the angular velocities about axes (1) and (3) grow rapidly with time and the racket tumbles.

To calculate the principal moments of inertia for our specific case of the tennis racket we use a grossly simplified model for the racket. We represent the mass distribution of the racket by a circular hoop of radius a and mass m_a connected to a thin rod of length ℓ and mass m_ℓ. The total mass of the racket is $M = m_a + m_\ell$. The CM of the racket is located on principal axis (1) at a distance R from the center of the hoop, where

$$MR = m_a(0) + m_\ell\left(a + \frac{\ell}{2}\right) \tag{7.107}$$

or

$$R = \frac{m_\ell}{M}\left(a + \frac{\ell}{2}\right) \tag{7.108}$$

The moment of inertia of the racket about principal axis (1) comes entirely from the hoop. We use the perpendicular-axis rule in (6.128) to obtain

$$I_1 = \tfrac{1}{2}m_a a^2 \tag{7.109}$$

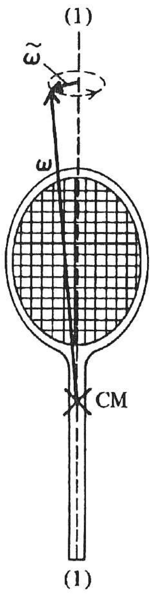

FIGURE 7-14. Stable precession of the angular velocity $\boldsymbol{\omega}$ about principal axis (1) of the tennis racket.

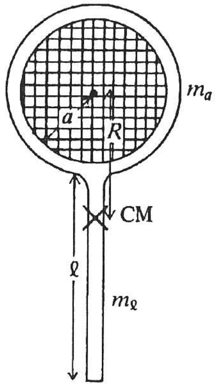

FIGURE 7-15. Dimensions of tennis racket model.

To compute the moment of inertia about principal axis (2), we will make use of the parallel-axis rule of (6.123). The moment of inertia of the hoop about an axis through its CM and parallel to principal axis (2) is $\frac{1}{2}m_a a^2$. By (6.123), the hoop makes a contribution to I_2 of

$$I_2^{hoop} = \tfrac{1}{2}m_a a^2 + m_a R^2 \tag{7.110}$$

since R is the perpendicular distance between the two parallel axes. The moment of inertia of the handle about an axis parallel to principal axis (2) passing through the CM of the handle is $\frac{1}{12}m_\ell \ell^2$. Again using (6.123), we find that the contribution of the handle to I_2 is

$$I_2^{handle} = \tfrac{1}{12}m_\ell \ell^2 + m_\ell \left(a + \frac{\ell}{2} - R\right)^2 \tag{7.111}$$

where $a + \ell/2 - R$ is the distance between these parallel axes. Combining (7.110) and (7.111) and substituting for R from (7.108), we obtain

$$I_2 = \tfrac{1}{2}m_a a^2 + m_a \left(\frac{m_\ell}{M}\right)^2 \left(a + \frac{\ell}{2}\right)^2 + \tfrac{1}{12}m_\ell \ell^2 + m_\ell \left(\frac{m_a}{M}\right)^2 \left(a + \frac{\ell}{2}\right)^2$$

This can be further simplified to

$$I_2 = \tfrac{1}{2}m_a a^2 + \tfrac{1}{12}m_\ell \ell^2 + \frac{m_a m_\ell}{M}\left(a + \frac{\ell}{2}\right)^2 \tag{7.112}$$

Finally, for principal axis (3), the racket lies in a plane perpendicular to the axis, and we can use the perpendicular-axis rule of (6.128) to obtain

$$I_3 = I_1 + I_2 \tag{7.113}$$

By comparison of (7.109), (7.112), and (7.113), we see that the principal moments of inertia are ordered as $I_1 < I_2 < I_3$. Characteristic parameters for our tennis racket model are

$$\begin{aligned} a &= 0.13\,\mathrm{m} & \ell &= 0.38\,\mathrm{m} \\ R &= 0.18\,\mathrm{m} & M &= 0.33\,\mathrm{kg} \\ m_\ell &= 0.18\,\mathrm{kg} & m_a &= 0.15\,\mathrm{kg} \end{aligned} \tag{7.114}$$

The principal moments of inertia from (7.109), (7.112), (7.113), and (7.114) are

$$\begin{aligned} I_1 &= 0.13 \times 10^{-2}\mathrm{kg}\cdot\mathrm{m}^2 \\ I_2 &= 1.24 \times 10^{-2}\mathrm{kg}\cdot\mathrm{m}^2 \\ I_3 &= 1.37 \times 10^{-2}\mathrm{kg}\cdot\mathrm{m}^2 \end{aligned} \tag{7.115}$$

The condition of stability of the motion about a principal axis which has either the largest or the smallest moment of inertia and the instability