Chapter 11

Statics

Rigid Body
Approximately Rigid Body } motionless → statics

Static Conditions for Rigid Bodies

Equilibrium

No center of mass velocity
and
no angular velocity about any point

Equilibrium conditions

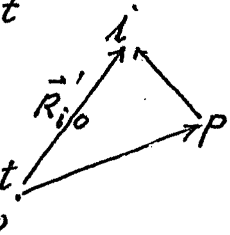
$$\sum \vec{F}_i = 0 \qquad \longrightarrow \text{"external"}$$

$$\sum_i \vec{R}_i \times \vec{F}_i = 0$$

The equilibrium condition for the torque is true for any choice of the axis about which the torques are calculated

The Condition of No Torque is Independent of the Choice of Reference Point

$$\vec{R}'_{Oi} = \vec{R}_{OP} + \underbrace{\vec{D}_{PO}}_{\text{independent of } i}$$



$$\vec{R}_i' = \vec{R}_i - \vec{D}$$

$$\underset{\substack{\parallel \\ 0}}{\sum (\vec{R}_i \times \vec{F}_i)} = \sum [(\vec{R}_i' + \vec{D}) \times \vec{F}_i] = \underset{\substack{\parallel \\ 0}}{\sum (\vec{R}_i' \times \vec{F}_i)} + \underset{\substack{\parallel \\ \vec{D} \times \sum_i F_i \\ \parallel \\ 0}}{\sum_i (\vec{D} \times \vec{F}_i)}$$

We can use any reference point to calculate the torques

Gravity and Rigid Body

Near the earth surface

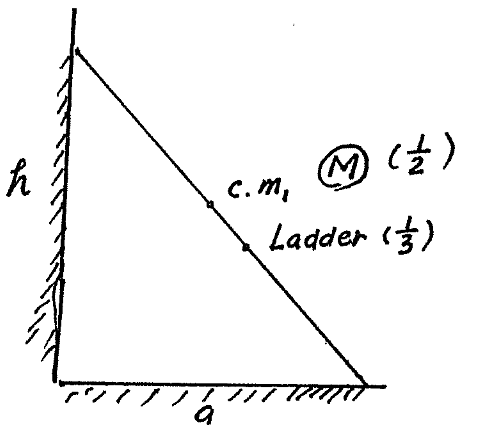
The torque due to gravity on axtended object of total mass M may be represented by the torque due to gravity acting on a particle of mass M located at the object's center of mass.

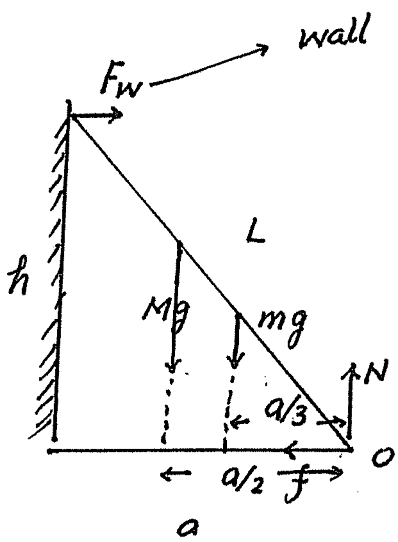
Example

Bridge

Method of determing the center of gravity (mass)

Example





$a = \sqrt{L^2 - h^2}$

calculate the torque through O is simpler

f, N will not contribute

$\Sigma F_x = 0 \qquad F_W - f = 0$

$\Sigma F_y = 0 \qquad N - Mg - mg = 0 \Rightarrow N$ is known

$\Sigma \tau = 0 \qquad -F_W h + \frac{Mga}{2} + \frac{mga}{3} = 0$ taking O as the reference point

2 unknown, 2 equation

$L = 12 m \qquad M = 72 kg$

$m = 45 Kg$

$h = 9.3 m$

$\Rightarrow \quad F_W = f = 410 N$

$N = 1150 N$

Example: Raising a Cylinder

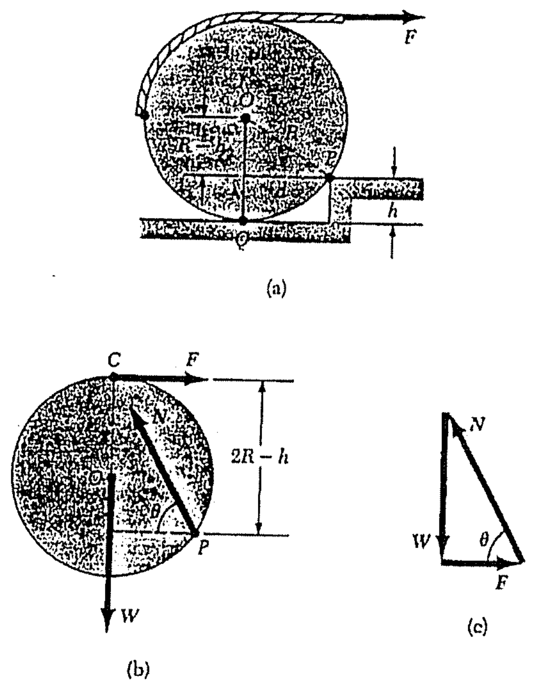


Figure (a) A cylinder of weight W being pulled by a force F over a step. (b) The free-body diagram for the cylinder when it is just ready to be raised. (c) The *vector* sum of the three external forces is zero.

Cylinder of weight W and radius R is to be raised onto a step of height h. A rope is wrapped around cylinder and pulled horizontally. No slipping.

What is minimum F and reaction force at P?

When cylinder is ready to be raised, reaction force at Q goes to zero. ∴ Three forces on cylinder.

$$d = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2} \qquad \text{(moment arm of } \vec{W} \text{ about } P)$$

Torques about P: $\quad Wd - F(2R-h) = 0$

$$\therefore F = \frac{W\sqrt{2Rh - h^2}}{2R-h}$$

$$\Sigma F_x = 0 \qquad F - N\cos\theta = 0$$

$$\Sigma F_y = 0 \qquad N\sin\theta - W = 0$$

$$\tan\theta = \frac{W}{F} \quad \text{and} \quad N = \sqrt{W^2 + F^2}$$

$W = 500$ N, $h = 0.3$ m, $R = 0.8$ m.

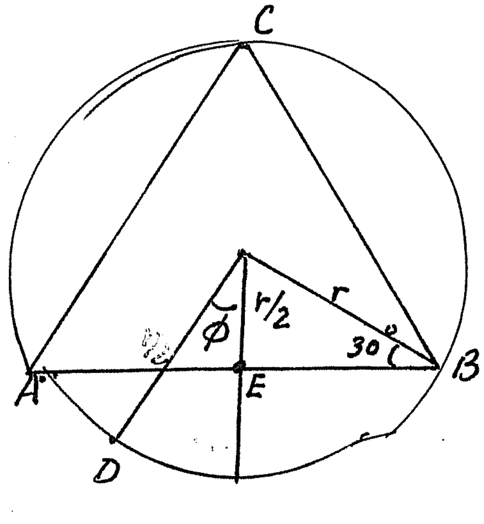
Solving: $F = 385$ N, $\theta = 52.4°$ and $N = 631$ N.

3 legs. table

When reaching D

$F_c = 0$

$\Rightarrow$ the table start to flip



$r = 2m$

Table $= \underset{\substack{\uparrow \\ 50\,kg}}{M}\,g = W$

Man $= \underset{\substack{\downarrow \\ 75\,kg}}{m}\,g = W'$

$F_o = 50\ kg\ g$ 向下

$F_D = 75\ kg\ g$

$\vec{F}_A,\ \vec{F}_B,\ \underset{\substack{\| \\ 0}}{\vec{F}_c}$ 向上

$$\boxed{F_o + F_D = F_A + F_B}$$ ✓

$\Sigma \tau_x = 0$ $\quad \vec{EB} \times \underset{\text{direction}}{\ }\ \vec{F}_B$ 向上

$F_A,\ F_B$ gives no contribution

$F_o \quad \hat{y} \times \hat{z} = \hat{x}$

$$\boxed{F_o \frac{r}{2} = F_D\,(r\cos\phi - r/2)}$$

$\Rightarrow \cos\phi = \frac{5}{6}$

y

$$\boxed{F_A \frac{\sqrt{3}}{2}\,r - F_D\, r \sin\phi - F_B \frac{\sqrt{3}}{2}\,r = 0}$$

2 equation solve for F_A and F_B

undertmined System.

$\downarrow$

see the textbook

分類：	
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第六章 力，力矩，質量中心及靜力平衡

第一節 力，力矩及其和

簡介：我們在這一節中將討論如何來組合力及力距，這些物理量均為向量。在此一節中我們將直覺地，(用日常經驗)來介紹力這個觀念(力的確切之義已在第五章中討論動力時仔細研討過)力距是兩個物理向量之向量積。在此節中我們將只討論作用於質點及剛体之力。

基本觀念

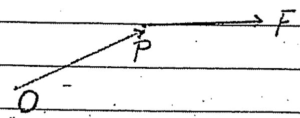
共點力：若是一組力作用於同一點，則這些力稱為共點力。

若是 $\vec{F}_1, \vec{F}_2, \cdots \vec{F}_N$ 是一組共點力則其合力 $\vec{R}$ 即為此組共點力之向量和

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \quad (1)$$

力距：若一力 $\vec{F}$ 作用於 P 點則此力對於 O 點之力距為[(1),(2),(3),(4)]

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (2)$$



圖一

此處 $\vec{r}$ 是由 O 至 P 點之向量

若 $\vec{F}_1, \vec{F}_2, \vec{F}_3, \cdots \vec{F}_N$ 作用於同一點 P 則此組力對 O 點之力距分別為 $\vec{r}\times\vec{F}_1$, $\vec{r}\times\vec{F}_2$, $\vec{r}\times\vec{F}_3$ … $\vec{r}\times\vec{F}_N$，因此這組力對 O 點之力距和

$$\vec{\tau} = \vec{r}\times\vec{F}_1 + \vec{r}\times\vec{F}_2 + \vec{r}\times\vec{F}_3 + \cdots \vec{r}\times\vec{F}_N$$

$$= \vec{r} \times \sum_{i=1}^{N} \vec{F}_i = \vec{r} \times \vec{R}. \quad (3)$$

也就是說要求一組同點力 $\vec{F}_1, \vec{F}_2, \vec{F}_3, \cdots \vec{F}_N$ 對 O 點之力距和，我們先將這些同點力先將此組同點力相加得 $\vec{R}$，然後求其對 O 點之力距，其結果與此組力對 O 點力距之和相等[(5)]，由此可知若一組力作用於一點，若其合力為零則此組外力對任何一點之力距和亦為零。

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(6)

若 $\vec{F}_1, \vec{F}_2, \vec{F}_3 \cdots \vec{F}_N$ 分別作用於 $P_1, P_2, \cdots P_N$ 點則

(1) 其合力為 $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_N$

(2) 其對於 O 點之合力距 $\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \cdots \vec{r}_N \times \vec{F}_N$ (4)

此處 $\vec{r}_1, \vec{r}_2, \vec{r}_3 \cdots \vec{r}_N$ 分別為由 O 至 P_1，由 O 至 P_2，由 O 至 P_3，⋯ 由 O 至 P_N 之向量

剛体 為一物体其中任意兩點之間之距離永遠保持不變者 (7)

討論

(1) 要決定一個力我們必需知道其大小、方向及其作用之位置

(2) $\vec{\tau} = \vec{r} \times \vec{F}$ 之幾何意義為一垂直於 $\vec{r}$ 及 $\vec{F}$ 所組成平面之向量 其指向由 $\vec{r}$, $\vec{F}$ 之右手定則所決定 其大小為 $|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta$ 此處 θ 為 $\vec{r}$ 及 $\vec{F}$ 之夾角 $\vec{\tau}$ 為一向量 當 $|\vec{r}| = 0$ 時 也即是當 $\vec{F}$ 作用於 O 點，其對 O 點之力距顯然地為零 同時若 $\vec{r} \parallel \vec{F}$ 時其對於 O 點之力距亦為零

若我們將 $\vec{r}$ 及 $\vec{F}$ 以下列直角坐標表出

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

則

$$\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

(3) 注意力距之定義與 O 點有關，因為 $\vec{r}$ 是 O 點至 P 點(力之作用點)之向量 因此當我們討論力距時必需首先決定對那一點來計算

(4) 以後我們將仔細討論其物理意義 但此處我們可以提出 $\vec{\tau}$ 的效果是使此物体對 O 點轉動

(5) 因此一組同點力與其合力 $\vec{R}$ 之效果完全相同

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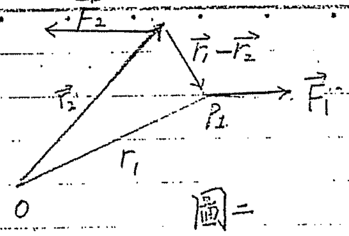
特殊情況 $N=2$, $\vec{F_1}=-\vec{F_2}$ 則

(1) 其合力 $\vec{R}=\vec{F_1}+\vec{F_2}=0$

(2) 其對 O 點之力距和為 $\vec{\tau}=\vec{\tau_1}+\vec{\tau_2}$

$=\vec{r_1}\times\vec{F_1}+\vec{r_2}\times\vec{F_2}=(\vec{r_1}-\vec{r_2})\times\vec{F}$



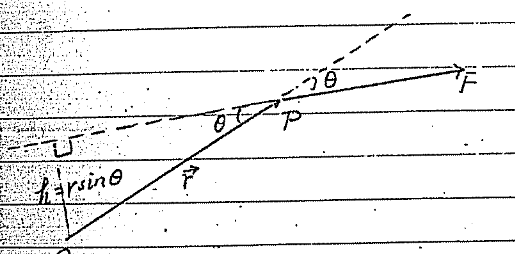
圖二

這樣一組力稱為力偶 由此式中可看出其力距与 O 點無關

(3) 由於在剛体中任意兩質點間之距離不變，因此此一系統之質点只可能同時移動或對一點 O 同時轉動 此一移動是由此剛体所受之合力 R 決定 而其對一點 O 之轉動則由其對 O 點之力距和 $\vec{\tau}$ 決定

應用

(1)



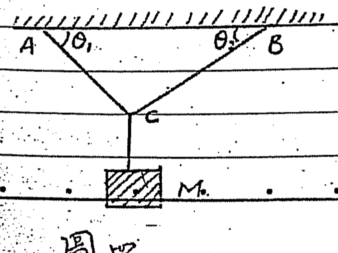
圖三

若一力作用於 P 點則其對 O 點之力距之定義為 $\vec{\tau}=\vec{r}\times\vec{F}$.

其大小為 $|\vec{\tau}|=rF\sin\theta=Fh$ 此處 h 是由 O 點力之作用線間之垂直距離 所以我們可以用 Fh 來決定 $|\vec{\tau}|$ 其方向垂直於 $\vec{r}$, $\vec{F}$ 平面

(2) 若討論力學問題時，應經常將作用於一質點或一剛体上所有之力畫出來 這樣計算合力或合力距之計算即可一目了然，我們特舉一些例子來說明

(A)



假如我們想找出在 C 點及 M 點所受之力，我們首先將 C 點及 M 點獨立起來而將所有作用於 C 及 M 點之力畫出來 這樣的

圖四

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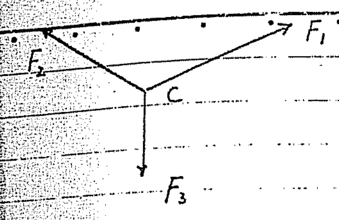
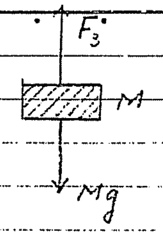
故稱為自由力圖



圖五-甲

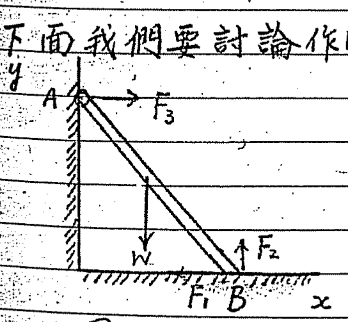


圖五-乙

當我們畫這些圖時我們如何決定倒底有那些作用力參與及作用力之方向呢?

一般講起來任何與質點相聯之物体均可加力於此質點 如C點與三個繩子相聯,因此這三個繩子可以對C點加力. 此外如C點有質量則我們就必須加入其所受的重力. 此時我們只知道其所可能所受的力. 下一步我們就必須來決定這些力之大小及方向. 首先我們應將已知之力(方向及大小均為已知者)標出. 因為力為一向量,任一未知力通常引進三個需要解的未知數. 但是假如此力是由直而且是能移動的繩子所產生則此力之方向即可決定. 唯一需要決定的是其大小. 這是因為一個直繩不能在其垂直方向受力或拖力. 同時它也不施一推力,因為它會彎曲. 因此一個繩子只能施一"拖"力. 繩子是一非常特殊之例子,即是一繩子所拖之力為未知,它的方向却完全決定. 因此在(甲)圖中,所需要求的是 $|\vec{F_1}|$, $|\vec{F_2}|$, $|\vec{F_3}|$. 在(乙)圖中只需求 $|\vec{F_3}|$. 因此要解這一問題只需找到三個包含此三未知數之三個独立聯立方程式即可.(通常這些方程式即是平衡之條件方程式.)

(B) 下面我們要討論作用於剛体之力



我們在此種情況下也是首先將此一剛体独立出來,討論其所受之力. 也即是我們要畫出對該剛体之自由力圖

圖六.

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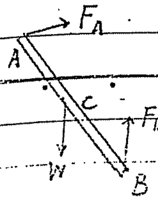
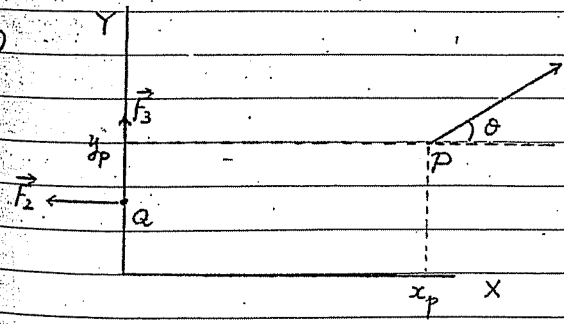


圖七

因此剛体在 A, B 點与壁接觸 因此壁可以於 A, B 點對此一剛体施力 因此在圖中我們有 $\vec{F_A}$ 与 $\vec{F_B}$ 兩力（此時我們只知其施力點而尚不知其大小及方向）。若此一剛体帶有質量而且其質量平均分佈於此桿上則地球對此一剛体在其中心點施一重力（其大小為 Mg，方向為向地心）因此如果該剛体桿為已知，則其所受之重力為一已知力。由圖中之對稱性可以很容易的看出所有之力均在 xy 平面上（也即是所有之力在 z 軸方向之分力為零）因此 $\vec{F_A}$, $\vec{F_B}$ 及 $\vec{W}$ 均為兩度空間向量。此外若一壁為平滑或其与剛体支接之處裝以滑輪而因此壁与剛体間之摩擦力可略出不計時，同時要求其支接點固定時則壁對該剛体所施力之方向也需垂直於壁之表面。如上圖中我們在 A 處壁作用於剛体之力是沿 x 軸方向 在 B 點則由於可能有摩擦力之存在 壁對剛体所施的力不一定需要垂直於壁面。但是由於任意一二度空間之向量均可分解為互相垂直的兩個向量，也即是 $\vec{F_B} = \vec{F_1} + \vec{F_2}$ 此處 $\vec{F_1}$, $\vec{F_2}$ 分別為沿 x 及 y 軸之向量 因此我們得到圖六。此中所以力之方向及作用點均為已知。所需要求的只是這些力之大小而已。

習題

(1)



z 軸是指向紙外

有三個力 $\vec{F_1}$, $\vec{F_2}$ 及 $\vec{F_3}$ 作用於此一系統上。$\vec{F_1}$ 作用於 XY 平面上一之一點 P

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外的兩個力作用於XY平面上之另一點Q。P點位於 $x_P = 12m$ 及 $y_P = 5m$ 處

點則位於Y軸上 $y_Q = 3m$ （見圖八） $\vec{F_1}$ 之大小為40牛頓其方向是如

圖所示在XY平面上與X軸成 $30°$ 之交角 $\vec{F_2}$ 則是沿負x軸之方向其大小為

0牛頓，$\vec{F_3}$ 則平行於Y軸，其大小為10牛頓

a) 將P點之坐標向量 $\vec{r_P}$ 用單位向量表出 [J]

) 將Q點之坐標向量 $\vec{r_Q}$ 用單位向量表出 [Q]

) 將 $\vec{F_1}, \vec{F_2}, \vec{F_3}$ 以單位向量表出 [C]

) 作用於此一系統之合力為何？用單位向量將其表出 [F]

) 這些力是否共點力？ [K]

) 合力之大小為何？ [O]

) 合力之方向為何？ [T]

) $\vec{F_2}$ 及 $\vec{F_3}$ 合力之大小及方向為何？ [G]

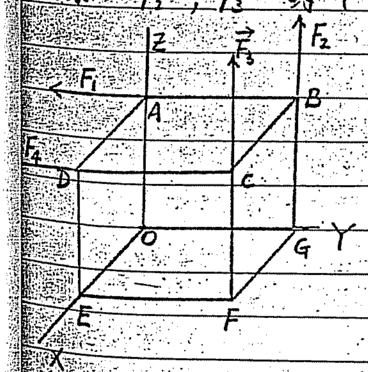
) $\vec{F_2}$ 及 $\vec{F_3}$ 是否共點力？ [M]

) $\vec{F_1}$ 對原點之力距為何？ $\vec{F_2}$ 對原點之力距為何？ $\vec{F_3}$ 對原點之力距為何？

) 求此三力所產生對原點之力距和？ [S]

) 求 $\vec{F_1}$ 對 (12, 0, 0) 點之力距 [V]

) 求 $\vec{F_2}$，$\vec{F_3}$ 對 (12, 0, 0) 點之力距 [D]



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ABCDEFGO 為一每邊長 2m 之正方体　$\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$, $\vec{F_4}$ 分別作用於 A, B, C, D 點，其中 $\vec{F_2}$, $\vec{F_3}$ 平行於 z 軸大小均為 5 牛頓　$\vec{F_1}$, $\vec{F_4}$ 均沿 −y 軸大小分別為 3 及 4 牛頓

(a) $\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$, $\vec{F_4}$ 中那些力是共點力？ [A]

(b) 此系統所受之合力為何？ [E]

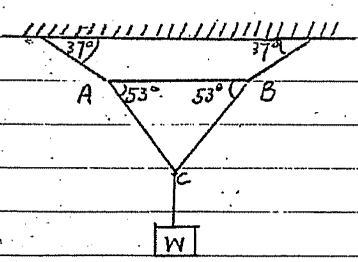
(c) 將 A, B, C, D 點之位置向量以單位向量表出. [N]

(d) 求 $\vec{F_1}$ 對原點之力距. 求 $\vec{F_2}$, $\vec{F_3}$, $\vec{F_4}$ 對原點之力距 [I]

(e) 求 $\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$, $\vec{F_4}$ 對原點之力距和 [U]

(f) 求 $\vec{F_1}$ 對 C 點之力距 求 $\vec{F_2}$, $\vec{F_3}$, 及 $\vec{F_4}$ 對 C 點之力距 [R]

(g) 求 $\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$, $\vec{F_4}$ 對 C 點之力距和 [W]



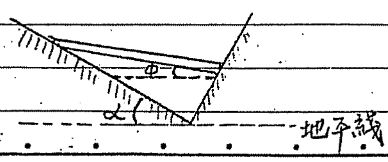
(a) 繪出在 A 點之自由力圖 [H]

(b) 繪出在 B 點之自由力圖 [L]

(c) 繪出在 C 點之自由力圖 [P]

(d) 繪出在 W 點之自由力圖 [X]

一質量為 m 長為 l 之棒. 放置在一平滑直角內如圖所示. 將作用於此桿上之力的着力點及方向標出.



[B]

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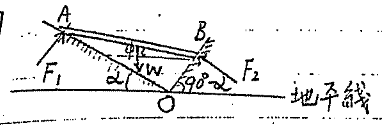
答案

[A] 均非共點力

[B]



$\vec{W}$ 是重力，作用點是桿之中點，方向是與地平線垂直
$\vec{F_1}$ 垂直於 OA 平面，因為該平面為平滑表面，無摩擦力
$\vec{F_2}$ 〃〃 〃 OB 平面 〃〃〃〃〃〃〃〃〃〃〃

所以所有力之方向均為已知，所未知者為力之大小而已
若取地平線之方向為 x 軸，垂直方向為 y 軸
$\vec{F_1} = |\vec{F_1}| \sin\alpha \, \hat{i} + |\vec{F_1}| \cos\alpha \, \hat{j}$ ，$\vec{F_2} = -|\vec{F_2}| \cos\alpha \, \hat{i} + |\vec{F_2}| \sin\alpha \, \hat{j}$，$\vec{W} = -mg \, \hat{j}$

[C] $\vec{F_1} = 34.6 \, \hat{i} + 20 \, \hat{j}$ 牛頓，$\vec{F_2} = -30 \, \hat{i}$ (牛頓)，$\vec{F_3} = +10 \, \hat{j}$ (牛頓)

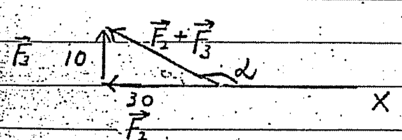
[D] $\vec{F_2}$ 對 (12, 0, 0) 點之力距為 $-90 \, \hat{k}$

$\vec{F_3}$ 對 (12, 0, 0) 點之力距為 $-120 \, \hat{k}$

[E] $\Sigma \vec{F} = -7 \, \hat{j} + 10 \, \hat{k}$ (牛頓)

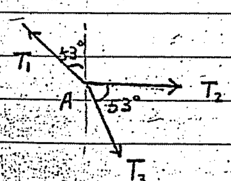
[F] $\Sigma \vec{F} = 4.6 \, \hat{i} + 30 \, \hat{j}$

[G] $\vec{F_2} + \vec{F_3} = -30 \, \hat{i} + 10 \, \hat{j}$



$\tan\alpha = -\frac{1}{3}$ (第二象限)

[H]



[I] $\vec{F_1}$ 對原點之力距為 $6 \, \hat{i}$ (牛頓米)

$\vec{F_2}$ 對原點之力距為 $10 \, \hat{i}$ (牛頓米)

$\vec{F_3}$ 對原點之力距為 $10 \hat{i} - 10 \hat{j}$ (牛頓米)

$\vec{F_4}$ 對原點之力距為 $-8 \, \hat{k}$ (牛頓米)

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第二節 質量中心

引介 在此節中我們將討論一系統在 $\vec{r}_1 \cdots \vec{r}_N$ 處受平行力之特殊情形 當然由對該系統的移動運動而言 此一組平行力之效果與 $\vec{R} = \sum \vec{F}_i$ 相同 若是此組力為平行時我們可以找到一點 C 具有下列之特性：此一組平行力對任何一點 O 之力距和與 $\vec{R}$ 作用於 C 點對 O 點之力距相同 這就是說這一組平行力與一作用於 C 點之 $\vec{R}$ 完全相同 這一點 C 被稱為力中心 因此一組平行力可簡化成單一力作用於力中心 一具体之物体可看成一組質點 在地球表面附近這一組質點均受重力，其大小與其質量成正比而方向則指向地球中心(對一般物体而言，其方向均平行) 因此質量中心即是此組平行重力之力中心 質量中心之功效在當我們討論重力對該物体產生之效果時我們可以用一單力作用於其質量中心來代替 而且在討論很多力與力距之效果時，我們可以把該物体當作一簡單之質點

基本觀念

若一組平行力 $\vec{F}_1, \vec{F}_2 \cdots \vec{F}_N$ 作用於 $\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N$ (A)

(B) 則很明顯的其合力為 $\vec{R} = \sum \vec{F}_i = \hat{u} \sum F_i$ (2)

此組對任何一點 O (其坐標為 $\vec{r}_0$) 之力距和為

$$\vec{\tau} = \sum (\vec{r}_i - \vec{r}_0) \times \vec{F}_i \quad (3)$$

(1),(2)

[因為 $\vec{F}_1, \vec{F}_2 \cdots \vec{F}_N$ 為平行之力，所以

$$\vec{F}_i = F_i \hat{u} \quad (1)$$

此處 F_i 是一純量而 $\hat{u}$ 是沿此組平行力方向之單位向量] (B)

$$\vec{\tau} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i = \sum_i (\vec{r}_i - \vec{r}_0) F_i \times \hat{u} \quad (4)$$

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$$= \left(\sum_i (\vec{r}_i - \vec{r}_o) F_o\right) \times \hat{u} = \frac{\sum_i (\vec{r}_i - \vec{r}_o) F_i}{\sum_i F_i} \times \hat{u}(\Sigma F_i)$$

$$= \frac{\sum_i (\vec{r}_i - \vec{r}_o) F_i}{\Sigma F_i} \times \vec{R} \qquad (5)$$

因此若我們定義 力中心 $\vec{r}_c$ 為

$$\vec{r}_c = \frac{\Sigma \vec{r}_i F_i}{\Sigma F_i} \qquad (6)$$

則 $\vec{\tau} = \frac{\sum_i (\vec{r}_i - \vec{r}_o) F_i}{\Sigma F_i} \times \vec{R} = (\vec{r}_c - \vec{r}_o) \times \vec{R} \qquad (7)$

這即是說此一組平行力對任何一點 0 之力距和即等於 $\vec{R}$ 作用於 $\vec{r}_c$ 對 0 點

之力距[3]

在地球表面附近一具質量為 m_i 之質點所受之重力

$$\vec{W}_i = m_i \vec{g} = m_i \hat{u} g \qquad (8)$$

代入(7)式，$\vec{r}_c = \frac{\Sigma \vec{r}_i m_i g}{\Sigma m_i g} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} \qquad (9)$

此時所得之 $\vec{r}_c$ 稱為質量中心[4],[5]

論

(1) 此處最重要的點是 $\vec{F}_1, \cdots \vec{F}_N$ 為平行力 因此 (1) 式 而且 F_i 為一純量 在導 (5) 式過程，我們一直利用到 F_i 是純量之特性

(2) 若 $\vec{F}_1, \cdots \vec{F}_N$ 不平行時 則通常其對 0 点之力距和 不能寫成 $\vec{\tau} = (\vec{r}_c - \vec{r}_o) \times \vec{R}$ 因為 $\sum_i (\vec{r}_i \times \vec{F}_i)$ 並不一定与 $\vec{R}$ 相垂直 而 $\vec{\tau}$ 若能寫成 $(\vec{r}_c - \vec{r}_o) \times \vec{R}$ 則必需与 $\vec{R}$ 垂直 当 $\vec{F}_1 \cdots \vec{F}_N$ 平行時 $\vec{R} \parallel \hat{u}$ 而因為 $\Sigma(\vec{r}_i \times \vec{F}_i)$ 均垂直於 $\hat{u}$ 所以其和也必垂直於 $\hat{u}$ 因此 $\vec{\tau} \perp \vec{R}$

(3) 若我們對 $\vec{r}_c$ 点求此組平行力之力距和 顯然的為零

(4) 第 (6) 式及第 (9) 式均為向量公式 第(9)式可寫成下列三公式

$$x_c = \frac{\Sigma m_i x_i}{\Sigma m_i}, \quad y_c = \frac{\Sigma m_i y_i}{\Sigma m_i}, \quad z_c = \frac{\Sigma m_i z_i}{\Sigma m_i} \qquad (10)$$

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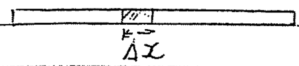
(5) 若我們有一度空間具体之物体如下圖所示



我們將此一桿分成 N 丁 Δx 大小之小段

x_i 是第 i 個小段之 x 軸坐標 $\Delta m_i = \rho(x_i)\Delta x$

是第 i 段之質量，此處 $\rho(x_i)$ 是在第 i 個小段的質量密度，若此桿並不均勻

則 ρ 可以為一 x_i 之函数

由 (10) 式中我們可得其質量中心

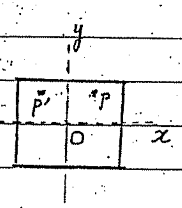
$$x_c = \frac{\sum_i m_i x_i}{\sum m_i} = \frac{\sum x_i \rho(x_i)\Delta x}{\sum \rho(x_i)\Delta x} \tag{11}$$

当我們令 $\Delta x \to 0$ 時 $N \to \infty$，上式中之和趨向於積分，(11) 式变成

$$x_c = \frac{\int x\rho(x)\,dx}{\int \rho(x)\,dx} \tag{12}$$

應用

(1) 經常我們會遭遇到均勻密度及具有對稱性之物体在此種情況之下我們可利用對稱性來求其質點中心．現在我們舉例來說明



如圖中我們有一均勻的正方平面体．直覺告訴我們其質量中心應位於此方塊之中心點．現在我們來討論為何如此．

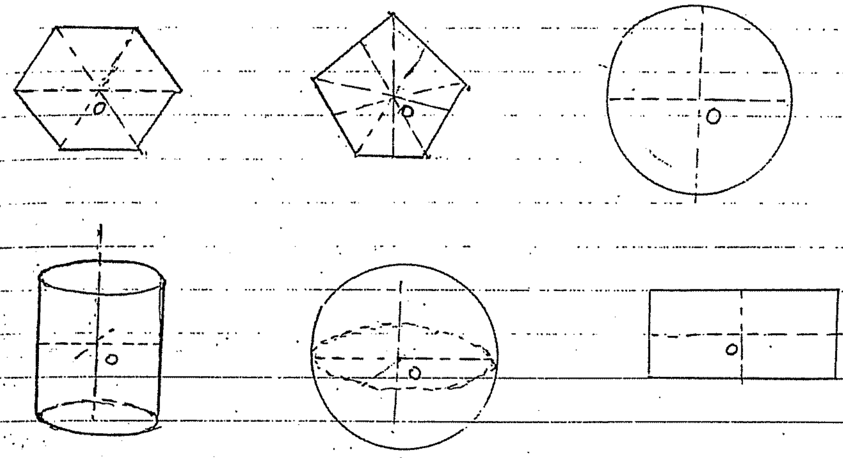
在討論(3)中我們曾提及對質量中心此組平行力之力距應為 $\vec{0}$．

在上圖中我們取 x; y 軸如圖，而取重力之方向為 $-\hat{z}$ 方向．

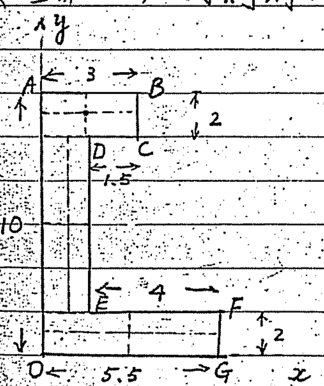
則 y 軸上之任一点其 y 方向之力距為 0．因為 P 及 P′ 對 O点之坐標為 (x,y) 及 $(-x,y)$．因為作用於 P 及 P′ 之重力（方向相同，均為 $-\hat{z}$），大小相同（因為我們有一均勻之物体）所以作用於 P，P′ 点之重力對 O 點在 y 方面之力距大小相等而方向相反．因此其和為 0．若我們計算 P，P′ 對 y 軸任何一点其結果也相同．因為對此軸，任意一点 P 均可找到其對應点 P′．所以在 y 軸上任一点其重力在

y 方向之力矩和為零 因此質量中心必須位於 y 軸上 同理質量中心也必需位於 x 軸上 所以 O 點是此正方平面体之質量中心

其他具有對稱性及其質量中心將列於下圖



2) 若一圖形不具良好之對稱性,但可以分解為具有良好對稱性的幾部分 我們可以先將各部分之質量中心及該部分之質量和求出,然後再計算該系統之質量中心, 我們將舉例來說明

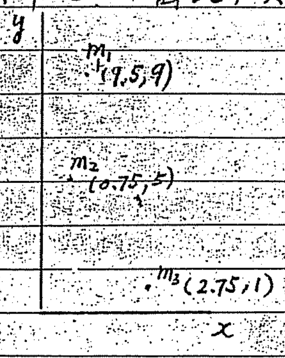


均勻之平板如圖所示求其質點中心

因為是均勻之平板,所以其質量與其面積成正比

令其比例常數為 c

此一平板可分解為三個長方形,由(1)中可知長方形之重心位於其中心 因此,以上之題目可簡化成求下圖之質量中心



$m_1 = c\,(3\times 2) = 6c$

$m_2 = c\,(1.5\times 6) = 9c$

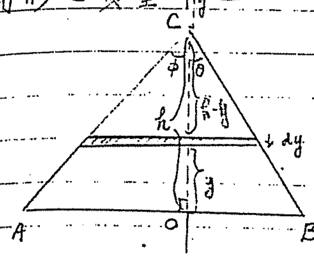
$m_3 = c\,(5.5\times 2) = 11c$

因此 $x_c = \dfrac{1.5\cdot 6c + 0.75\cdot 9c + 2.75\cdot 11c}{26c} = 1.77$

$y_c = \dfrac{6c\times 9 + 9c\times 5 + 11c\times 1}{26c} = 4.25$

注意此一結果與比例常數 c 無關

三角形之質量中心



首先我們將證明三角形之質量中心位於高 $\frac{1}{3}h$ 處

我們取 $\overrightarrow{CO} \perp \overrightarrow{AB}$, OC 之方向為 y 軸

取三角形沿 y 方向分成分多片 由圖上可看出

在 Δy 之間之質量為 $c\underbrace{(h-y)(\tan\theta+\tan\phi)}_{\text{面積}}\Delta y$, c 即為(2)中之比例常數

所以利用(12)式

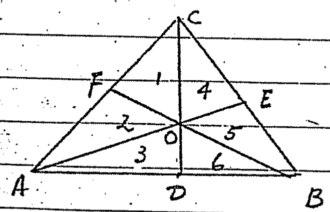
$$y_c = \frac{\int_0^h c\,(h-y)(\tan\theta+\tan\phi)\,y\,dy}{\int_0^h c\,(h-y)(\tan\theta+\tan\phi)\,dy}$$

$$= \frac{\int_0^h (hy-hy^2)\,dy}{\int_0^h (h-y)\,dy} = \frac{\frac{1}{2}h^3-\frac{1}{3}h^3}{h^2-\frac{1}{2}h^2} = \frac{1}{3}h.$$

此處 h 是此三角形對 AB 底邊之高 所以 C 位於離 AB 底邊之高度之 $\frac{h}{3}$ 處

用同樣的方法我們可證明 C 亦位於離 BC, AC 底邊之高度為 $\frac{h'}{3}$, $\frac{h''}{3}$ 處

此處 h', h'' 分別為此三角形對 BC, 及 AC 底邊之高



其次我們將證明三角形三中線之交點即滿足以上之要求

圖中 O 點為三中線之交點, 我們要證明 O 與 AB, BC, AC 之垂直距離分別為 $\frac{1}{3}h$, $\frac{1}{3}h'$ 及 $\frac{1}{3}h''$

證明: 令 Δ_i = 第 i 個三角形之面積

因為 D, E, F 分別為 AB, BC, 及 CA 之中點

$$\Delta_3 = \Delta_6, \quad \Delta_1 = \Delta_2, \quad \Delta_4 = \Delta_5$$

$$\Delta_1+\Delta_2+\Delta_3 = \Delta_2+\Delta_3+\Delta_6 = \Delta_3+\Delta_6+\Delta_5 = \Delta_6+\Delta_5+\Delta_4 = \Delta_5+\Delta_4+\Delta_1$$
$$= \Delta_4+\Delta_1+\Delta_2$$

所以 $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = \frac{1}{6}\Delta_{ABC}$

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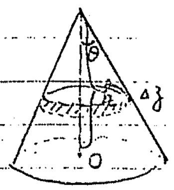
$\Delta_3 = \frac{1}{2}\cdot(AD)\cdot$(O与底邊 AB 之垂直距離) $= \frac{1}{6}\cdot\frac{1}{2}(AB)(h)$

$AD = \frac{1}{3}AB$, 因此 (O与底邊 AB 之垂直距離) $= \frac{1}{3}h$ 同理,我們可證明O与底邊 BC, CA 之垂直距離也分別為 $\frac{1}{3}h'$, 及 $\frac{1}{3}h''$. 因此三角形之三中線之交点即為該三角形之質量中心. 這也是在幾何中三角形三中線之交点被稱為重心的理由

(4) 圓錐体之質點中心



由對稱性可知其質量中心必位於圓底之中心与頂點上, 我們取此線為 z 軸. 我們將圓錐切在 Δz 厚之圓餅

由簡單之幾何可知 在 Δy 間之質量為 $c\pi[(h-z)\tan\theta]^2\Delta z$. C 即為比例常數 h 是錐之高.

所以利用(12)式可得

$$z_c = \frac{\int_0^h c\pi[(h-z)\tan\theta]^2 z\,dz}{\int_0^h c\pi[(h-z)\tan\theta]^2 dz}$$

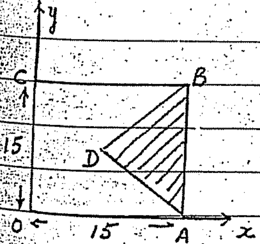
$$= \frac{\int_0^h [h^2 z - 2hz^2 + z^3]dz}{\int_0^h (h^2 - 2hz + z^2)dz} = \frac{\frac{1}{2}h^4 - \frac{2}{3}h^4 + \frac{1}{4}h^4}{h^3 - h^3 + \frac{1}{3}h^3}$$

$$= \frac{\frac{6-8+3}{12}h^4}{\frac{1}{3}h^3} = \frac{1}{4}h$$

因此我們求得均匀圓錐之質量中心位於底圓中心与頂点联线上離底圓 $\frac{1}{4}h$ 處

習題

(1)



設此平均分佈之平面之單位面積之質量為1

(a) 求 OABC 正方形之質量及其質量中心之位置 [C]

(b) 求三角形 ABD 之質量及其質量中心之位置 [G]

(c) 求正方形 OABC 切去三角形以後多邊形 OADBC 之質量及其質量中心的位置 [I]

(2) 一均匀分佈之平面物体其面積為70平方米. 它的質量中心在一直角坐標

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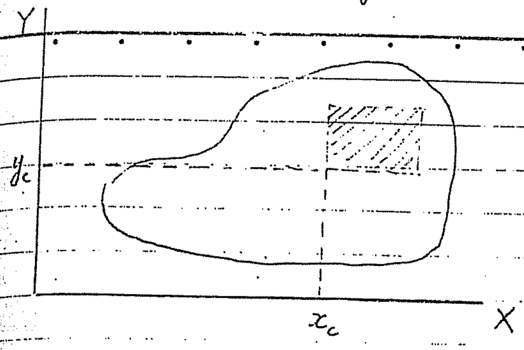
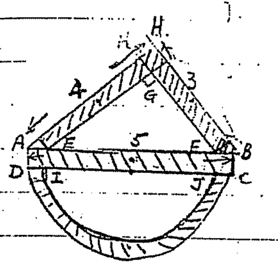
系統中位於 $x_c = 9.5$ 米，$y = 6.0$ 米處（見附圖）



現在我們將一左下角位於原來物体之質量中心之長方形（其于 x 軸平行之边長為 5 米，与平行於 y 軸之邊長為 4 米）切除

求切除此長方形以後物体之質量中心. [A]

(3)



我們有一物体如圖所示 其分佈是平均分佈 $AB = 5$ 米，$BI = 3$ 米，$AI = 4$ 米

所有木條之寬度均為 0.5 米 設單位面積之質量為 1

(a) 求 AB 平方形之質量為何？其質量中心位於何處？

取該点為原点，取 AB 之方向為 x 軸 [M]

(b) 求 完整三角形 ABH 之質量与質量中心之位置，求三角形 EFG 之質量与質量中心之位置 [E]

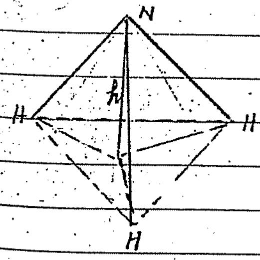
(c) 求 T字尺 AHBFGE 之質量及其質量中心之位置 [L]

(D) 求 DC 半圓之質量及其質量中心之位置，求 IJ 半圓之質量及質量中心之位置 [H]
求弧 DCJC 之質量及其質量中心之位置

(E) 求此一系統之質量及其質量中心之位置（注意此点並不一定在有物体處）[O]

(4) NH_3 為金字塔形之分子 N 原子位於頂点 N-H 間之距離為 $1.01 \times 10^{-10} m$

兩個 N-H 為邊之夾角為 108° N 原子之質量為氫分子之 14 倍

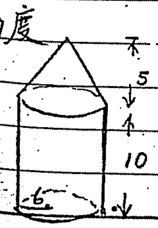


(a) 求 H-H 之距離 [B]

(b) 求 h 之長度 [J]

(c) 決定 NH_3 之質量中心離 H-H-H 所構成之三角形底之高度 [N]

(5)



圓筒体之單位体積質量為 1

錐形之單位体積質量為 2

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(a) 求圓筒之質量及質量中心之位置 [D]

(b) 求錐体之体積及質量 [P]

(c) 求錐体之質量中心之位置 [F]

(d) 求整個物体之質量及其質量中心之位置 [K]

答案

[A] $x = 8.5$米, $y = 5.2$米 [B] 1.63×10^{-10} m [C] $x = 7.5$米, $y = 7.5$

[D] 1130.4；以圓筒之圓底之中心為原点圓筒之軸為 z 軸. 圓筒之質量中心位於 (0, 0, 5) [E] 6, (0.234, 1.05) 3.01 ; (0.27, 0.817)

[F] (0, 0, 11.25) [G] 質量為 $\frac{225}{4}$, 質量中心位於 (12.5, 7.5)

[H] 9.817, (-0.1, -1.31) ; 6.283, (0, -1.099) ; 3.534, (0, -1.685)

[I] $x = 5.83$, $y = 7.5$ [J] 3.8×10^{-11} m [K] 1507.2 ; 6.63

[L] 2.99 (0.198, 1.285)

[M] 2.5 ; (0, 0) 由定義而來 [N] 距 H-H-H 底 $= 3.13 \times 10^{-11}$ m

[O] 9.024, (0.066, -0.234)

[P] 188.4 ; 376.8

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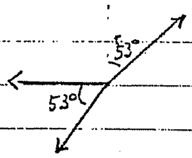
[J] $\vec{F_P} = 12\hat{i} + 5\hat{j}$

[K] $\vec{F_2}, \vec{F_3}$ 是共點力

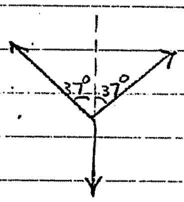
[L]



[M] 是 [N] $\vec{F_A} = 2\hat{k}$, $\vec{F_B} = 2\hat{j} + 2\hat{k}$, $\vec{F_C} = 2\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{F_D} = 2\hat{i} + 2\hat{k}$

[O] $|\Sigma \vec{F}| = \sqrt{46^2 + 30^2}$ 牛頓

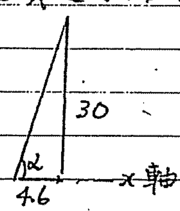
[P]



[Q] $r_Q = 3\hat{j}$ (米) [R] $\vec{F_1}$ 對 c 點之力距為 $+6\hat{k}$

$\vec{F_2}$ 對 c 點之力距為 $10\hat{j}$ $\vec{F_3}$ 對 c 點之力距為 $\vec{0}$, $\vec{F_4}$ 對 c 點之力距為 $\vec{0}$

[S] $157\hat{k}$ (牛頓米) (T)

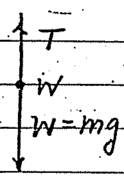


$\tan\alpha = \frac{30}{46}$ (第一象限)

[U] $26\hat{i} - 10\hat{j} - 8\hat{k}$ (牛頓米) [V] $173\hat{k}$ (牛頓米)

[W] $10\hat{j} + 6\hat{k}$ (牛頓米)

[X]



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第三節 靜力平衡

簡介　當一力作用於一質點時，會使該質點有沿受力之方向運動之趨向，若一組力作用於一質點則會使該質點有沿其合力之方向運動之趨向。但是，質點只是一個數學上之假想，真實的物体均有一定的大小，通常当一組力作用於此一具体之物体，此物体不但會有沿着這些力之合力方向運動之趨向，同時它也有轉動之趨向。在物理及工程應用裏一極重要的問題是在何種條件之下，一個系統既沒有移動也沒有轉動之趨向。我們發現其條件為該系統所受之總合力及總合力距為零。這些條件被稱為平衡條件。

基本觀念

(a) 一質點之平衡條件　此質點所受力之和為零 (1)

$$\Sigma \vec{F_i} = \vec{0} \qquad (1)$$

(b) 一剛体之平衡條件 (2)(3)(4)

(1) 此剛体所受力之和為零

$$\Sigma \vec{F_i} = \vec{0} \qquad (2)$$

(2) 此剛体對任何一點所受之力距和為零

$$\Sigma \vec{\tau_i} = \vec{0} \qquad (3)$$

討論

(1) 此一公式是一向量公式，此一公式在直角坐標可寫成下列三個公式

$$\Sigma F_{ix} = 0,\quad \Sigma F_{iy} = 0,\quad \Sigma F_{iz} = 0. \qquad (4)$$

(2) 當我們討論質點時，其所受之力為共点力，因此若其合力為零則對任何一點之力距也自然的為零，所以 (a) 是 (b) 之一特殊情況而已。注意当我們討論一剛体時其所受之力並不一定是共點力，因此滿足 (2) 並不保證 (3) 式一定也會滿足，而一剛体

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之平衡條件則要求 (2),(3) 兩式必須同時滿足方可

(3) 第三式亦為一向量公式 此一公式在直角坐標中可寫成下列三個公式

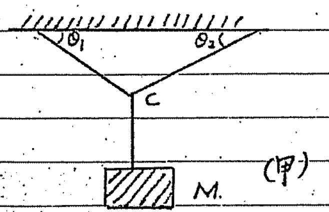
$$\Sigma \tau_{ix}=0, \quad \Sigma \tau_{iy}=0, \quad \Sigma \tau_{iz}=0 \qquad (5)$$

(4) 照理我們可對任何點來計算力距 但通常我們均盡量選擇對使我們計算簡化的點也計算力距 我們將在應用部分中討論這一點

應用

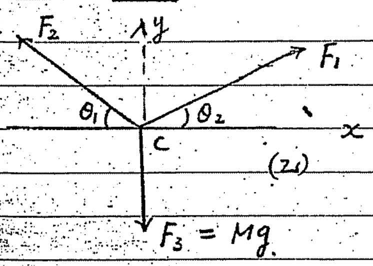
我們在討論靜力平衡中的步驟大致可分為 (一) 畫出有關質點及剛体之自由力圖 (二) 清楚的標出對這些力已知之資料, 如它的着力點, 大小, 或方向 (三) 決定未知之變數為何 (四) 將平衡條件寫成未知變數之聯立方程式 (五) 解聯立方程式來決定所要求的答案 以下, 我們將舉例來說明

(a)

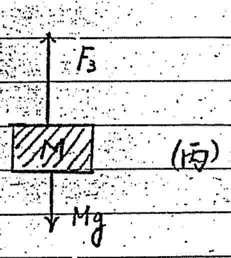


此處 M 及 θ_1, θ_2 為已知 所需求者為各繩上之張力

在上節中我們已將其自由力圖繪出



此圖各力之方向為已知 因為一繩只能施一拉力 (見上一節之討論)



此圖中向上之拉力為 F_3, 這是因為作用力反反作用力之關係 其向下之力為重力 故為 Mg

M 點平衡條件為 $F_3 = Mg$

現在我們回到 (乙) 圖, 此圖中未知者為 F_1, F_2 之大小

c 點之平衡條件為

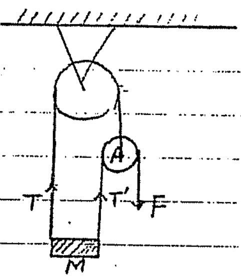
$$F_{1x} + F_{2x} = |F_1| \cos\theta_2 - |F_2| \cos\theta_1 = 0 \qquad (6)$$

$$F_{1y} + F_{2y} + F_{2z} = |F_1| \sin\theta_1 + |F_2| \sin\theta_2 - Mg = 0 \qquad (7)$$

此處我們取 xy 軸如圖所示. 因爲這些力在 z 方向之分力均爲零 所以 $\Sigma F_z = 0$ 是自然滿足的. 現在我們有兩個聯立方程式兩個未知數 $|F_1|, |F_2|$, 所以 $|F_1|, |F_2|$ 可由 (6), (7) 兩式中解出 其結果爲

$$|F_1| = \frac{Mg\cos\theta_1}{\sin(\theta_1+\theta_2)} , \quad |F_2| = \frac{Mg\cos\theta_2}{\sin(\theta_1+\theta_2)} \qquad (8)$$

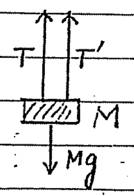
(b)



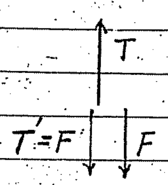
已知 M 求保持平衡時所需之力 F.

此一問題極爲簡單, 因爲所有之力均沿一方向. 因此是一度空間之問題

一平滑滑輪之特性: 它只能改變力的方向卻不更改其大小



我們首先將在 M 及 A 點之自由力圖繪出. 在 M 點之平衡條件爲 $T + T' = Mg$ (9)

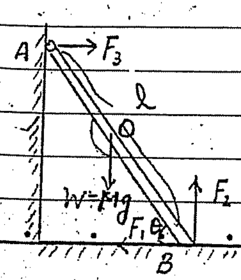


在 A 點, 由滑輪之特性我們知道 向上之拉力爲 T 向下之兩拉力均爲 F 而由作用力及反作用之關係得知 $F = T'$ 故在 A 點之平衡條件爲

$$T = 2T' \qquad (10)$$

由 (9), (10) 兩式可得 $3T' = Mg$. 因此 $T' = \frac{1}{3}Mg \quad T = \frac{2}{3}Mg$ (11)

(c)



此題中 l, θ 及 W 爲已知 求 $\vec{F}_B = \vec{F}_1 + \vec{F}_2$, 及 $\vec{F}_3 = \vec{F}_A$

在 A 處之接觸處加有滑輪 故其摩擦力可略出不計, 因此 壁作用於剛体桿之力垂直於壁面.

此處我們用到一平滑表面之特性，一平滑表面所施之力必須与其表面垂直

如同在上節中所討論，此題之三個未知數為 $|\vec{F_1}|$，$|\vec{F_2}|$，及 $|\vec{F_3}|$

此剛体得平衡條件之一為

$$|\vec{F_2}| = Mg \qquad \Sigma F_y = 0 \tag{12}$$

$$|\vec{F_1}| = |\vec{F_3}| \qquad \Sigma F_x = 0 \tag{13}$$

因此我們只剩一未知数　因為所有的力均在 xy 平面上，所以 $\Sigma F_z = 0$ 是当然滿足的，因此不能給我們新的独立公式，因此我們必須利用力距公式

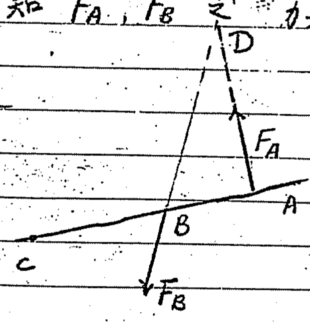
照理我們可以對任何一點寫下力距和為零之平衡條件，但在此題中取 B 點則最為方便，因為 $\vec{F_1}$，$\vec{F_2}$ 作用於 B 點所以它們對 B 點之力距為零

其力距為 $-\frac{1}{2} l\, Mg\cos\theta + l|\vec{F_3}|\sin\theta = 0$ 由此可得

$$|\vec{F_3}| = \frac{1}{2}\frac{Mg\cos\theta}{\sin\theta} \tag{14}$$

當然我們也可計算這些力對 A 点或 O 点之力距和而得到同樣的結果，但計算却較為複雜，因此一般來講我們利用力距平衡公式時通常均對未知力最多的着力点來計算力距

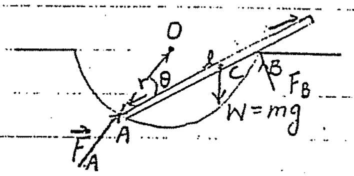
(d) 若一剛体有三力 $\vec{F_A}$，$\vec{F_B}$ 及 $\vec{F_C}$ 作用於其上而此一剛体位於平衡之狀況，如果已知 $\vec{F_A}$，$\vec{F_B}$ 之着力点及方向，同時也知道 $\vec{F_C}$ 之着力点則 $\vec{F_C}$ 之方向即可決定



如圖所示 $\vec{F_A}$，$\vec{F_B}$ 之延長線交於 D 点，則對 D 點求力距，$\vec{F_A}$，$\vec{F_B}$ 對 D 点之力距均為 0，由於 $\Sigma \vec{\tau_i} = 0$ 因此 $\vec{F_C}$ 對 D 点之力距也必須為 0，因此 $\vec{F_C}$ 若不為 $\vec{0}$ 則必須平行於 $\vec{CD}$ 向量，若 $\vec{F_A}+\vec{F_B} \neq \vec{0}$，則 $\vec{F_C} \neq 0$ 否則 $\Sigma \vec{F_i} \neq 0$，因此 $\vec{F_C}$ 必須与 $\vec{CD}$ 向量平行或与 $-\vec{CD}$ 平行

若 $\vec{F_A}$ 與 $\vec{F_B}$ 不相交 則 $\vec{F_A} \parallel \vec{F_B}$，由 $\vec{F_A}+\vec{F_B}+\vec{F_C}=0$ 可知 $\vec{F_C}$ 也必與 $\vec{F_A}$ 及 $\vec{F_B}$ 平行 因此其方向也完全決定

(e) 一質量為 m 長為 ℓ 的棒，放置在半徑為 r 完全平滑的半圓形球体中，試求棒的平衡位置，計算半圓形球体作用於棒的反作用力

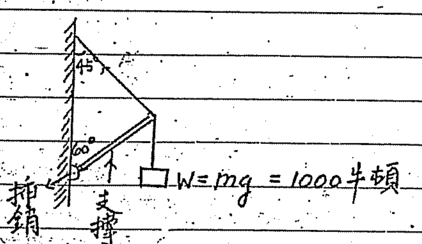


首先我們討論此桿所受之力

此桿共受三力 $\vec{F_A}$，$\vec{F_B}$ 及 $\vec{W}$ 分別着力於 A，B及C 點．$\vec{W}$ 為重力，其大小為 mg 其着力點位於此桿之中點．由於是完全平滑之半圓球形 $\vec{F_A}$ 必須垂直於球体之表面，因此 $\vec{F_A}$ 是沿 $\vec{AO}$ 向量之方向．$\vec{W}$ 之方向為向地心．由 (d) 之討論中我們可以決定 $\vec{F_B}$ 之方向 因此在此題只剩 $|\vec{F_A}|$，$|\vec{F_B}|$ 及 θ 為未知數．然後利用平衡條件公式可解此一題中之未知數

習題

(1)



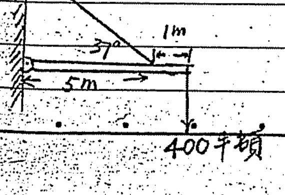
假設支持之質量可略去不計

(a) 繪出支撐上所受之自由力圖 [G]

(b) 插銷作用於支撐之力之方向為何？其大小為何？ [D]

(c) 求纜繩之張力之大小 [N]

(2) 在圖中免計支撐之重量



(a) 繪出支撐所受之自由力圖 [C]

(b) 插銷作用於支撐之力之方向為何？其大小為何？ [K]

(c) 求繩上之張力 [M]

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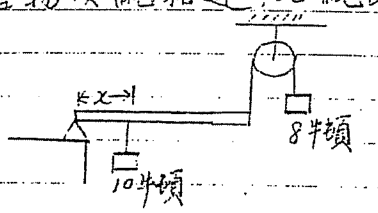
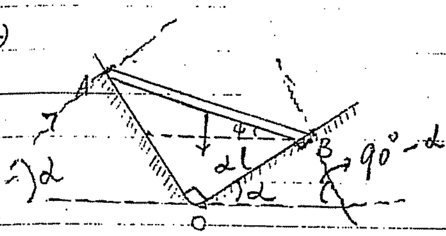
(3) 一均勻桿長 2 米重 3 牛頓，假如棒之一端支於一刀口上，另一端與 8 牛頓的重物以繩相連，此繩跨過一滑輪如圖所示



(a) 求一重 10 牛頓之方塊應懸於桿之何處可使桿平衡？[A]

(b) 作用於刀口的力方向及大小為何？ [E]

(4) 一質量為 6 kg 長 0.8 米之棒放置在一平滑的直角內如圖所示

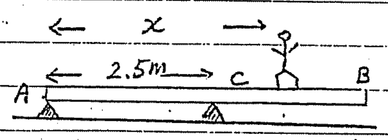


(1) 繪出作用於桿上之自由力圖 [J]

(2) 求 $\vec{F_A}$，$\vec{F_B}$ 之方向與大小 [H]

(3) 求平衡位置時 α 與 φ 之關係 [F]

(5) 在圖中之均勻桿長為 4 米重為 50 kgf 有一固定支点 C，桿圍繞此點可轉動，此桿靜止於 A 點，一人重 75 kgf，正由 A 點開始沿此桿行走



(a) 繪出作用於此桿之自由力圖 [B]

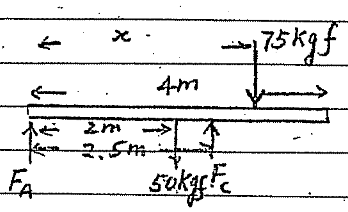
(b) 當人走至 x 点時寫出其平衡條件，將作用於 A 点及 C 点之力寫成 x 之函数 [I]

(c) 由上式中決定此人能行走距 A 端最大距離為何值時仍可使桿保持平衡？ [L]

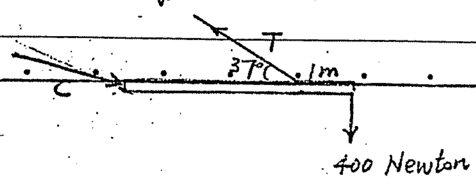
答案

[A] $x = 1.3$ 米

[B]



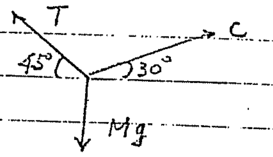
[C]



[D] 其方向力支持之方向平行，$|\vec{C}| = \frac{2000}{\sqrt{3}+1}$ 牛頓　[E] 方向向上，

大小為 5 牛頓 [F] $\tan\phi = \cot 2\alpha$

[G]



[H] $\vec{F_A}$，$\vec{F_B}$ 分別垂直於 OA 及 OB 平面

$|\vec{F_A}| = 6\sin\alpha$ kgf，$|\vec{F_B}| = 6\cos\alpha$ kgf

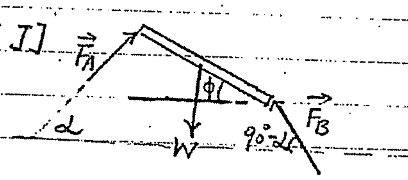
[I] $F_A + F_C = (75+50)\,kgf = 125\ kgf$

對 A 計算力距　$F_C(2.5) - 50(2) - 75x = 0$

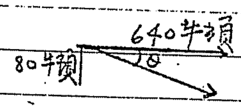
對 C 計算力距　$F_A(-2.5) + (-50)(-0.5) + (-75)(x-2.5) = 0$

$F_C = 40 + 30x$，$F_A = 85 - 30x$　[kgf 單位]

[J]



[K]



$\tan\theta = \frac{1}{8}$

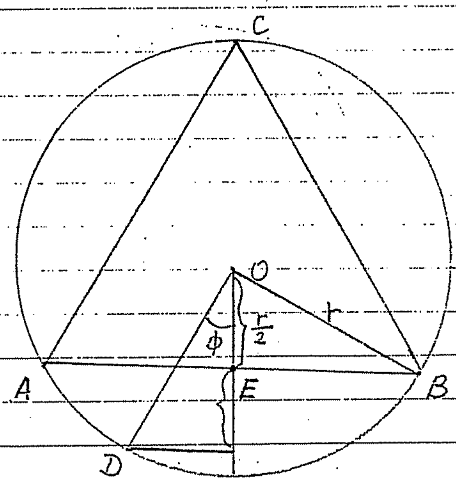
[L] $x = 2.83$ 米

[M] $T = 800$ 牛頓

[N] $T = \frac{\sqrt{2}\cdot 1000}{\sqrt{3}+1}$ 牛頓

其他問題

一均勻圓桌面其半徑為 2 米重量為 50 kgf 在 A, B, C 處有三支腿支撐，在下圖中我們繪出由上而下看的圖



一人重 75 Kgf 沿桌邊由 A 走向 B

(a) 此人走至 D 點時桌面開始倒翻，求於 D 点之 ϕ 角

(b) 求在 D 點時桌腿 A, B, C 之反作用力為何

我們首先討論此桌面所受的力 $\vec{F_o}$, $\vec{F_D}$ 為向下（重力） $\vec{F_A}$, $\vec{F_B}$, $\vec{F_C}$ 為向上（反作用力） $\vec{F_o}$, $\vec{F_D}$ 之大小亦為已知，很顯然地，当桌面開始倒翻時，$\vec{F_C}=0$

所以力平衡公式為 $F_o + F_D = F_A + F_B$ (15)

然後我們來討論力距之平衡公式，首先我們取 E 為我們之坐標原点 AB 為 x 軸 向上為 y 軸，然後我們對 E 點來計算力距，這麼做的原因是在此一坐標中 $\vec{F_A}$, $\vec{F_B}$ 對 E 點之力距均在 y 方向，其 x 方向之力距為 0，平衡條件要求 x 方向之力距合為 0，這一公式現在變得非常簡單

$$F_o \cdot \frac{r}{2} = F_D \cdot \left(r \cdot \cos\phi - \frac{r}{2}\right)$$

因為 F_o, F_D, r 均為已知 所以很快的可將 $\cos\phi$ 求出，其結果為

$$\cos\phi = \frac{5}{6} \qquad (16)$$

因此 D 點即完全確定，對 E 点之力距在 y 方面也必須為 0，所以

$$F_A \cdot \frac{\sqrt{3}}{2} r - F_D \cdot r\sin\phi - F_B \frac{\sqrt{3}}{2} r = 0 \qquad (17)$$

將 (15), (16), (17) 合併，我們可得 F_A 及 F_B

此一例子說明了 (一) $\sum \vec{\tau_i} = \vec{0}$ 為一向量公式因此 $\sum \tau_{ix}$, $\sum \tau_{iy}$, $\sum \tau_{iz}$ 均必需為 0.

(二) $\sum \vec{\tau_i} = 0$ 對任何一點來計算均需成立

習題 (i) 對 A 點來計算力距而解 (b) 部分

(ii) 對 B 點來計算力距而解 (b) 部分

9-5 EQUILIBRIUM APPLICATIONS OF NEWTON'S LAWS FOR ROTATION

It is possible for the net external force acting on a body to be zero, while the net external torque is nonzero. For example, consider two forces of equal magnitude that act on a body in opposite directions but not along the same line. This body will have an angular acceleration but no linear or translational acceleration. It is also possible for the net external torque on a body to be zero, while the net external force is not (a body falling in gravity); in this case there is a translational acceleration but no angular acceleration. For a body to be in equilibrium *both the net external force and the net external torque must be zero.* In this case the body will have *neither* an angular acceleration nor a translational acceleration. According to this definition, the body could have a linear or an angular velocity, as long as that velocity is constant. However, we will most often consider the special case in which the body is at rest.

We therefore have two conditions of equilibrium:

$$\sum \vec{\mathbf{F}}_{\text{ext}} = 0 \tag{9-22}$$

and

$$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0. \tag{9-23}$$

Each of these vector equations can be replaced with its equivalent three component (scalar) equations:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0 \tag{9-24}$$

and

$$\sum \tau_x = 0, \quad \sum \tau_y = 0, \quad \sum \tau_z = 0, \tag{9-25}$$

where for convenience we have dropped the subscript "ext" from these equations. At equilibrium, the sum of the external force components and the sum of the external torque components along each of the coordinate axes must be zero. This must be true for any choice of the directions of the coordinate axes.

The equilibrium condition for the torques is true for any choice of the axis about which the torques are calculated. To prove this statement, we consider a rigid body on which many forces act. Relative to the origin O, force $\vec{F}_1$ is applied at the point located at $\vec{r}_1$, force $\vec{F}_2$ at $\vec{r}_2$, and so on. The net torque about an axis through O is therefore

$$\begin{aligned}\vec{\tau}_O &= \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N \\ &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N. \qquad (9\text{-}26)\end{aligned}$$

Suppose a point P is located at displacement $\vec{r}_P$ with respect to O (Fig. 9-21). The point of application of $\vec{F}_1$, with respect to P, is $(\vec{r}_1 - \vec{r}_P)$. The torque about P is

$$\begin{aligned}\vec{\tau}_P &= (\vec{r}_1 - \vec{r}_P) \times \vec{F}_1 + (\vec{r}_2 - \vec{r}_P) \times \vec{F}_2 \\ &\quad + \cdots + (\vec{r}_N - \vec{r}_P) \times \vec{F}_N \\ &= [\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N] \\ &\quad - [\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots + \vec{r}_P \times \vec{F}_N].\end{aligned}$$

The first group of terms in the brackets gives $\vec{\tau}_O$ according to Eq. 9-26. We can rewrite the second group by removing the constant factor of $\vec{r}_P$:

$$\begin{aligned}\vec{\tau}_P &= \vec{\tau}_O - [\vec{r}_P \times (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N)] \\ &= \vec{\tau}_O - [\vec{r}_P \times (\textstyle\sum \vec{F}_{\text{ext}})] \\ &= \vec{\tau}_O,\end{aligned}$$

where we make the last step because $\sum \vec{F}_{\text{ext}} = 0$ for a body in translational equilibrium. Thus the torque about any two points has the same value when the body is in translational equilibrium.

Often we deal with problems in which all the forces lie in the same plane. In this case the six conditions of Eqs. 9-24 and 9-25 are reduced to three. We resolve the forces into two components:

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad (9\text{-}27)$$

and, if we calculate torques about a point that also lies in the xy plane, all torques must be in the direction perpendicular to the xy plane. In this case we have

$$\sum \tau_z = 0, \qquad (9\text{-}28)$$

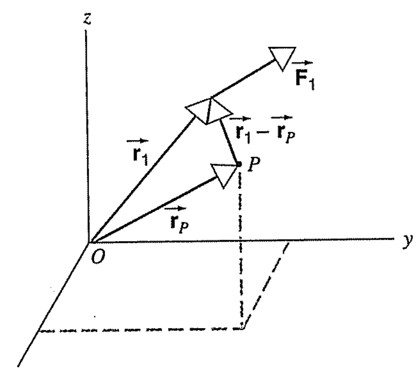


FIGURE 9-21. The force $\vec{F}_1$ is one of N external forces that act on a rigid body (not shown). The vector $\vec{r}_1$ locates the point of application of $\vec{F}_1$ relative to O and is used in calculating the torque of $\vec{F}_1$ about O. The vector $\vec{r}_1 - \vec{r}_P$ is used in calculating the torque of $\vec{F}_1$ about P.

We limit ourselves mostly to planar problems to simplify the calculations; this condition does not impose any fundamental restriction on the application of the general principles of equilibrium.

Equilibrium Analysis Procedures

Usually in equilibrium problems, we are interested in determining the values of one or more unknown forces by applying the conditions for equilibrium (zero net external force and zero net external torque). Here are the procedures you should follow:

1. *Draw a boundary around the system,* so that you can clearly separate the system you are considering from its environment.

2. *Draw a free-body diagram showing all external forces that act on the system and their points of application.* External forces are those that act through the system boundary that you drew in step 1; these often include gravity, friction, and forces exerted by wires or beams that cross the boundary. Internal forces (those that objects within the system exert on each other) should not appear in the diagram. Sometimes the direction of a force may not be obvious in advance. If you imagine making a cut through the beam or wire where it crosses the boundary, the ends of this cut will pull apart if the force acts outward from the boundary. If you are in doubt, choose the direction arbitrarily, and if you have guessed wrong your solution will result in negative values for the components of that force.

3. *Set up a coordinate system and choose the direction of the axes.* This coordinate system will be used to resolve the forces into their components.

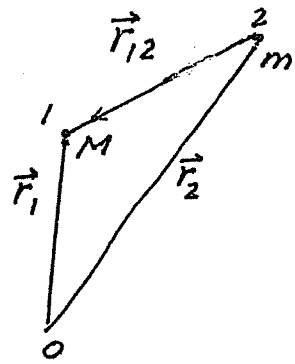
4. *Set up a coordinate system and axes for resolving the torques into their components.* In equilibrium, the net external torque must be zero about *any* axis. Often you can choose to calculate torques about a point through which several forces act, thereby eliminating those forces from the torque equation. In adding torque components, we follow the sign convention that *the torque along any axis is positive if acting alone it would produce a counterclockwise rotation about that axis.* The right-hand rule for torques can also be used to establish this convention.

Once we have carried out these steps in setting up the problem, we can carry out the solution using Eqs. 9-22 and 9-23 or 9-27 and 9-28, as the following problems illustrate.

Chapter 12

Gravitation

Newton's Law of Universal Gravitation



$$\vec{F}_{12} = -G \frac{m_1 m_2}{\vec{r}_{12}^{\,2}} \vec{r}_{12} \qquad \text{force on } \underset{m}{\underset{\downarrow}{m_2}} \text{ due to } \underset{M}{\underset{\downarrow}{m_1}}$$

$G \downarrow$ Newton's constant

- Universal
- Supperposition principle
- The force is conservative $\downarrow$

$$E_p = -G \frac{m_1 m_2}{r^{\cancel{2}}} \hat{r}$$

$E_p \downarrow$ potential energy

Chapter 12

Outline

Gravitation

Angular Momentum

Angular Momentum Conservation

Kepler - Newton's Problem

Rutherford Scattering Problem

Variable Mass Case; Rocket Problem.

Angular Momentum Conservation

$$mr^2\dot{\theta} = L_o$$

Energy Conservation

$$\frac{1}{2}m\dot{r}^2 + \frac{L_o^2}{2mr^2} + V(r) = E \qquad V(r) = -\frac{k}{r}$$

See P. 6' to P. 8 $\qquad k = GmM$

$\downarrow$ effective potential

P. 10

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L_o^2}{2mr^2} - \frac{k}{r} = E$$

$$\downarrow$$

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = \left(E + \frac{k}{r}\right) - \frac{L^2}{2mr^2}$$

$$\downarrow$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left(E + \frac{k}{r}\right) - \frac{L^2}{m^2r^2}$$

$$\downarrow$$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right) - \frac{L^2}{m^2r^2}}$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta}$$ (see P. 11)

$$\updownarrow$$

$$\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right) - \frac{L^2}{m^2r^2}}$$

$$d\theta = \frac{L}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right) - \frac{L^2}{m^2r^2}}}$$

$$= \frac{(L/r^2)\,dr}{\sqrt{2m\left(E + \frac{k}{r}\right) - \frac{L^2}{r^2}}}$$

Let $\quad u = \frac{1}{r} \qquad du = -\frac{1}{r^2}\,dr \Rightarrow$

$$d\theta = \frac{-L\,du}{\sqrt{2m(E+ku)-L^2u^2}}$$

$$L^2 \qquad -\left(u-\frac{mk}{L^2}\right)^2 + \left(\frac{2mE}{L^2}+\frac{m^2k^2}{L^4}\right)$$

$$L^2 \qquad = -u^2 + \frac{2mk}{L^2}u + \cancel{\frac{m^2k^2}{L^4}} + \frac{2mE}{L^2} + \cancel{\frac{m^2k^2}{L^4}}$$

Check

$$d\theta = \frac{-du}{\sqrt{\left(\frac{2mE}{L^2}+\frac{m^2k^2}{L^4}\right)-\left(u-\frac{mk}{L^2}\right)^2}}$$

$$= -\frac{du'}{\sqrt{a^2-u'^2}}$$

Let $\quad u' = u - \frac{mk}{L}$

$u' = a\cos y$

$du' = -a\sin y\,dy$

$$\int -\frac{du'}{\sqrt{a^2-u'^2}} = \int \frac{+a\sin y}{a\sin y}\,dy$$

$$= y = \cos^{-1}\frac{u'}{a}$$

$$= \cos^{-1}\frac{\left(u-\frac{mk}{L}\right)}{\left(\frac{2mE}{L^2}+\frac{m^2k^2}{L^4}\right)^{\frac{1}{2}}}$$

$$\theta-\theta_0 = \cos^{-1}\frac{\left(\frac{1}{r}-\frac{mk}{L}\right)}{\frac{mk}{L^2}\left[1+\frac{2EL^2}{mk^2}\right]}$$

Take $\quad \theta_0 = 0$

$$= \cos^{-1}\frac{\frac{L^2}{mk^2}\frac{1}{r}-1}{\sqrt{1+\frac{2EL^2}{mk^2}}}$$

$$\epsilon = \sqrt{1+\frac{2EL^2}{mk^2}}$$

$$\theta-\theta_0 = \cos^{-1}\frac{\frac{L^2}{mk^2}\frac{1}{r}-1}{\epsilon}$$

$$\epsilon \cos\theta = \frac{L^2}{mk}\frac{1}{r} - 1$$

$$= \frac{\alpha}{r} - 1 \qquad \alpha = \frac{L^2}{mk}$$

$$\Rightarrow \quad \frac{\alpha}{r} = 1 + \epsilon \cos\theta$$

See P. 13

$$\epsilon r \cos\theta = \alpha - r$$

$$\Rightarrow \quad \alpha = r + \epsilon \underbrace{r\cos\theta}_{x}$$

$$(\alpha - \epsilon x)^2 = r^2$$

$$\alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 = x^2 + y^2$$

$$\alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 = x^2 + y^2$$

claim

$$\frac{\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2}{\left(\frac{\alpha}{1-\epsilon}\right)^2} + \frac{y^2}{\frac{\alpha^2}{(1-\epsilon^2)}} = 1$$

ellipse

P. 24

$$a = \frac{\alpha}{1-\epsilon^2}$$

$$b = \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

$$a^2 - b^2 = \epsilon^2 a^2$$

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta$$

$$r = \frac{l}{1 - e\cos\theta} \qquad \text{in reference}$$

$$\Rightarrow 1 - e\cos\theta = \frac{l}{r} \qquad \alpha = l, \quad e = -\epsilon$$

Conic Section

Write out in Cartesian Coordinates

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta \qquad \alpha = \frac{L^2}{mk}, \quad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

$$\Downarrow$$

$$\alpha = r + \epsilon r\cos\theta = r + \epsilon x$$

$$(\alpha - \epsilon x)^2 = r^2$$

$$\alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 = x^2 + y^2$$

$$\alpha^2 - 2\epsilon\alpha x + (\epsilon^2 - 1)x^2 = y^2$$

$$\frac{y^2}{\frac{\alpha^2}{(1-\epsilon^2)}} = 1 - \frac{\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2}{\left(\frac{\alpha}{1-\epsilon^2}\right)^2}$$

$$y^2 = \frac{\alpha^2}{1-\epsilon^2} - \left(\cancel{\frac{\alpha^2}{1-\epsilon^2}}\right) \frac{\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2}{\frac{\alpha^2}{(1-\epsilon^2)^2}}$$

$$= \frac{\alpha^2}{1-\epsilon^2} - (1-\epsilon^2)\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2$$

$$= \frac{\alpha^2}{1-\epsilon^2} - (1-\epsilon^2)\left[x^2 + 2\frac{x\epsilon\alpha}{1-\epsilon^2} + \frac{\epsilon^2\alpha^2}{1-\epsilon^2}\right]$$

x^2 $\quad \epsilon^2 - 1$ $\quad$ check

x $\quad -2x\epsilon\alpha$ $\quad$ check

1 $\quad \frac{\alpha^2}{1-\epsilon^2} - \alpha^2 \frac{\epsilon^2}{1-\epsilon^2}$

$$= \frac{\alpha^2}{1-\epsilon^2}(1-\epsilon^2) = \alpha^2 \qquad \text{Check}$$

$$a = \frac{\alpha}{1-\epsilon^2}$$

$$b = \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

$$a^2 - b^2 = \epsilon^2 a^2$$

$$a^2(1-\epsilon^2) = b^2$$

$$\frac{\alpha^2}{(1-\epsilon^2)^2}(1-\epsilon^2) = \frac{\alpha}{1-\epsilon^2} \qquad \text{check}$$

$$L_0 = m v r$$

$$\frac{\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2}{\left(\frac{\alpha}{1-\epsilon^2}\right)^2} = 1 \qquad \text{along } y=0$$

$$x + \frac{\epsilon\alpha}{1-\epsilon^2} = \pm \frac{\alpha}{1-\epsilon^2}$$

$$x = \pm \frac{\alpha}{1-\epsilon^2} - \frac{\epsilon\alpha}{1-\epsilon^2}$$

$(+)$ $$x = \frac{\alpha}{1-\epsilon^2} - \frac{\epsilon\alpha}{1-\epsilon^2}$$

$$= \frac{\alpha(1-\epsilon)}{1-\epsilon^2} = \frac{\alpha}{1+\epsilon} \quad ✓$$

$-$ $$-\frac{\alpha}{1-\epsilon^2} - \frac{\epsilon\alpha}{1-\epsilon^2}$$

$$= \frac{-\alpha(1+\epsilon)}{1-\epsilon^2} = -\frac{\alpha}{1-\epsilon}$$

$$v_a \frac{\alpha}{1-\epsilon} = v_p \frac{\alpha}{1+\epsilon} = \frac{L}{m}$$

Two equations

$$\frac{1}{2m}\frac{L^2(1-\epsilon)^2}{\alpha^2} - \frac{k(1-\epsilon)}{\alpha} = E$$

$$\frac{1}{2m}\frac{L^2(1+\epsilon)^2}{\alpha^2} - \frac{k(1+\epsilon)}{\alpha} = E$$

Two equations, two unknowns

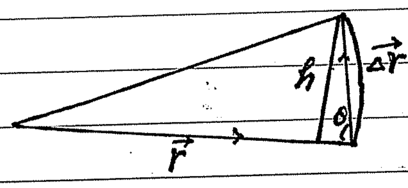
$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2}$$

$$\alpha = \frac{L^2}{mk}$$

Angular Momentum Conservation

$\downarrow$

Kepler's Second Law



$$\Delta A = \frac{1}{2} \left| \vec{r} \times \Delta\vec{r} \right|$$

Divide by Δt and letting $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} \left| \vec{r} \times \vec{v} \right| = \frac{1}{2m} \left| \vec{L} \right|$$

$\downarrow$ constant

$\downarrow$ conservation of angular momentum.

$\downarrow$

Kepler's second law

Semi - major axes

$$2a = r_{max} + r_{min} = \frac{\alpha}{1-\epsilon} + \frac{\alpha}{1+\epsilon}$$

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|}$$

$$a^2 - b^2 = \epsilon^2 a^2$$

$$b^2 = a^2 - \epsilon^2 a^2 = (1-\epsilon^2)\, a^2$$

$$\underset{\sqrt{1-\epsilon^2}\, a}{\underset{\parallel}{b}} = \frac{L}{\sqrt{2m|E|}}$$

$$\pi ab = \underset{\frac{dA}{dt}}{\underset{\parallel}{\frac{L}{2m}}}\, T$$

$$T^2 = \left(\frac{2m\pi ab}{L}\right)^2 = \frac{4m^2}{L^2}\pi^2 a^2 (1-\epsilon^2) a^2$$

$$= \frac{4m^2}{L^2}\pi^2 \underbrace{a(1-\epsilon^2)}_{\alpha} a^3$$

$$\frac{T^2}{a^3} = \frac{4m^2}{L^2}\pi^2 \underbrace{a(1-\epsilon^2)}_{\alpha = \frac{L^2}{mk},\ k = GmM} = \frac{4\pi^2}{GM}$$

independent of m!

Kepler's third law

The Sun is located at $(0,0)$

$y = 0$

$$\left(x + \frac{\epsilon\alpha}{1-\epsilon^2}\right)^2 = \left(\frac{\alpha}{1-\epsilon^2}\right)^2$$

$$x + \frac{\epsilon\alpha}{1-\epsilon^2} = \pm \frac{\alpha}{1-\epsilon^2}$$

$$x = \frac{-\epsilon\alpha}{1-\epsilon^2} \pm \frac{\alpha}{1-\epsilon^2}$$

$(+)$ $$x = -\frac{\epsilon\alpha}{1-\epsilon^2} + \frac{\alpha}{1-\epsilon^2} = \frac{\alpha(1-\epsilon)}{1-\epsilon^2} = \frac{\alpha}{1+\epsilon}$$

$(-)$ $$x = -\frac{\epsilon\alpha}{1-\epsilon^2} - \frac{\alpha}{1-\epsilon^2} = -\frac{\alpha(1+\epsilon)}{1-\epsilon^2} = -\frac{\alpha}{1-\epsilon}$$

$$x = -\frac{\epsilon\alpha}{1-\epsilon^2}$$

$$y = \pm \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

$$a = \frac{\epsilon\alpha}{1-\epsilon^2} + \frac{\alpha}{1+\epsilon} = \frac{\epsilon\alpha + \alpha(1-\epsilon)}{1-\epsilon^2}$$

$$= \frac{\alpha}{1-\epsilon^2}$$

$$b = \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

Conservation of angular momentum
(with respect to the Sun)

At perihelion

$$v_p = \frac{L}{m}\,\frac{1+\epsilon}{\alpha}$$

$$\frac{1}{2} m v_p^2 - \frac{k}{r_a} = E$$

$$\frac{1}{2m}\,\frac{L^2(1+\epsilon)^2}{\alpha^2} - \frac{k(1+\epsilon)}{\alpha} = E$$

At aphelion

$$\frac{1}{2m}\,\frac{L^2(1-\epsilon)^2}{\alpha^2} - \frac{k(1-\epsilon)}{\alpha} = E$$

$$\frac{1}{2m}\,\frac{L^2}{\alpha^2}(4\epsilon) - \frac{k\,2\epsilon}{\alpha} = 0$$

$$\frac{L^2}{\alpha^2 m} - k = 0 \qquad \alpha$$

$$L^2 - \alpha m k = 0$$

$$\alpha = \frac{L^2}{mk}$$

$$\frac{1}{2m}\,\frac{L^2(1-\epsilon)^2}{\underset{\underset{\frac{L^2}{mk}}{\parallel}}{\alpha}\cdot\alpha} - \frac{k(1-\epsilon)}{\alpha} = E$$

$$\frac{(1-\epsilon)^2}{2} - (1-\epsilon) = \frac{\alpha E}{k} = \frac{L^2}{mk^2} E$$

$$1 - \cancel{2\epsilon} + \epsilon^2 - 2 + \cancel{2\epsilon} = \frac{L^2}{mk^2} E$$

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2}$$

Furthermore $\langle r \rangle = a$.

$$\pi ab = \frac{L}{2m} T$$

$$\Rightarrow \quad \frac{2m\pi ab}{L} = T$$

$$\Rightarrow \quad T^2 = \frac{4m^2}{L^2} \pi^2 a^2 b^2$$

$$b^2 = \frac{\alpha^2}{1-\epsilon^2}$$

$$a^2 = \frac{\alpha^2}{(1-\epsilon^2)^2}$$

$$\frac{b^2}{a^2} = (1-\epsilon^2)$$

$$b^2 = (1-\epsilon^2) a^2$$

$$T^2 = \frac{4m}{L^2} \pi^2 a^3 (1-\epsilon^2) a$$

$$(1-\epsilon^2) a = \alpha = \frac{L^2}{mk}$$

$$\Rightarrow \quad T^2 = \frac{4m^2}{L^2} \pi^2 a^3 \frac{L^2}{mk} \qquad k = GmM$$

$$\frac{T^2}{a^3} = \frac{4\pi^2 m}{GMm} = \frac{4\pi^2}{GM}$$

Angular momentum

The Angular momentum $\boldsymbol{L}$ is defined as

$$\boxed{\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}}.$$

The cross product of two vectors $\boldsymbol{a} \times \boldsymbol{b}$ is a "vector", with magnitude $ab \sin\theta$ (area of the the parallelogram bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$) with orientation defined by a right-hand rule.

The torque $\boldsymbol{N}$ defined as $\boldsymbol{r} \times \boldsymbol{F}$ causes change of angular momentum just as $\boldsymbol{F}$ causes change of linear momentum.

$$\boxed{\frac{d\boldsymbol{L}}{dt} = \cancel{\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p}} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{N}}.$$

1'

Convention of the cross product

We associate the cross product $\boldsymbol{a} \times \boldsymbol{b}$ as a vector of magnitude $ab \sin\theta$ and direction perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ as a right-handed screw from $\boldsymbol{a}$ to $\boldsymbol{b}$.

Mathematically we can write

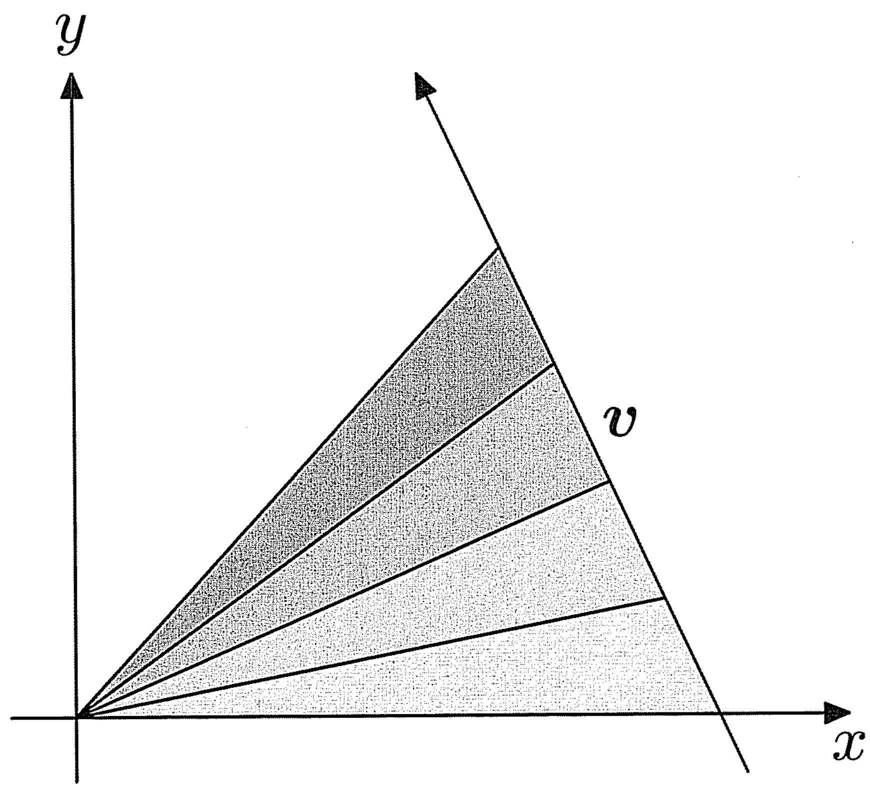
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} ,$$

or tensorially

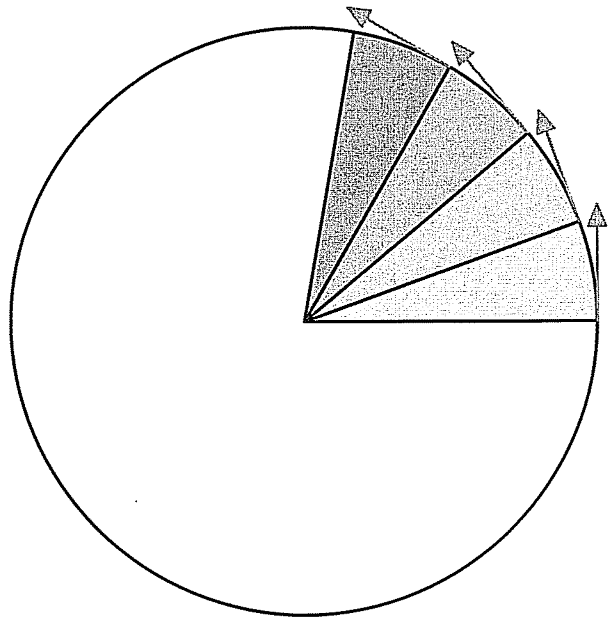
$$(\boldsymbol{a} \times \boldsymbol{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k \, .$$

Area Law for a free particle

Obviously, for a free particle with $\boldsymbol{F} = 0$ angular momentum is conserved. Pictorially this implies the area law.



Uniform circular motion and the area Law



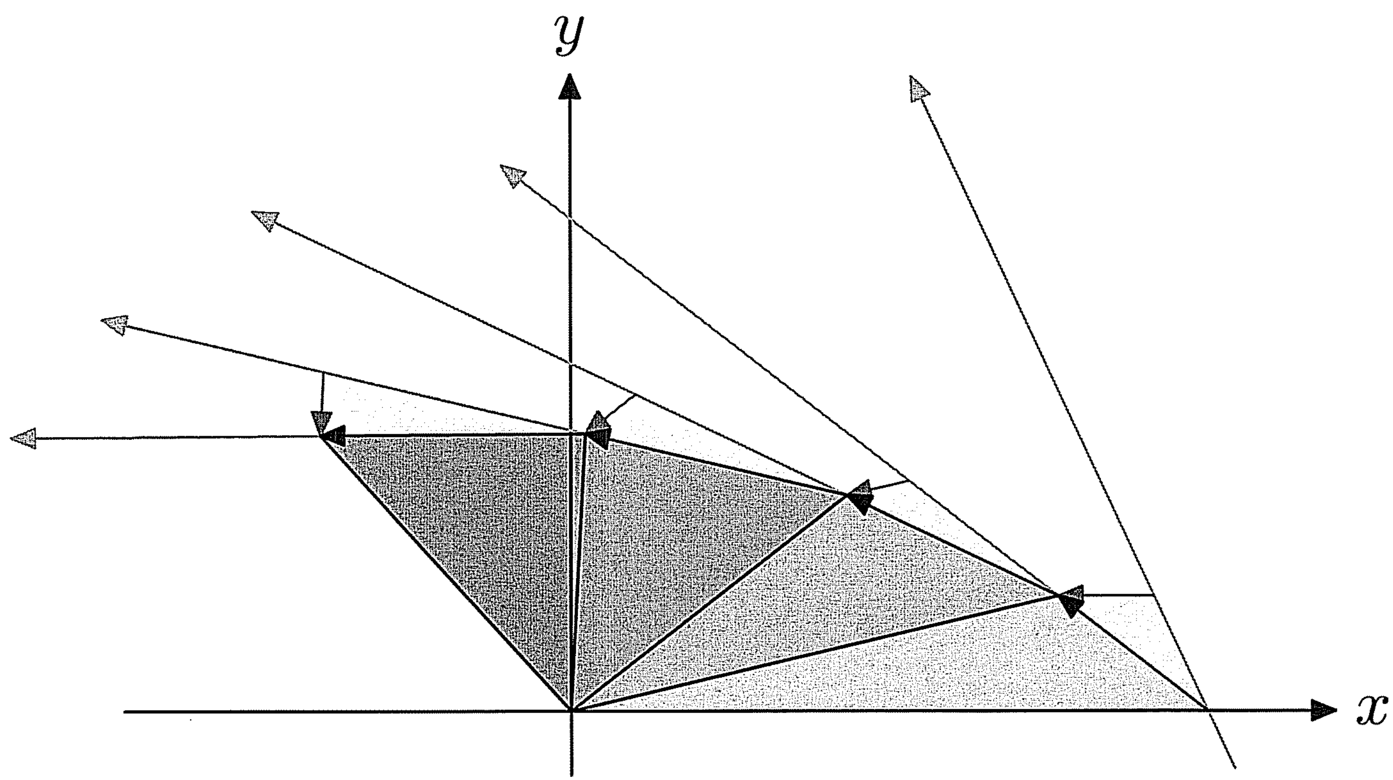
Since

$$L = mr(r\omega)\,, \qquad A = \frac{1}{2}r^2(\omega t)\,,$$

therefore, $$\boxed{\frac{dA}{dt} = \frac{L}{2m}}\,.$$

Area Law for central force

By definition the central force is defined as $\boldsymbol{F} = f(r)\hat{\boldsymbol{r}}$, which always tugs radially. $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F} = 0$, and so the area law still holds.



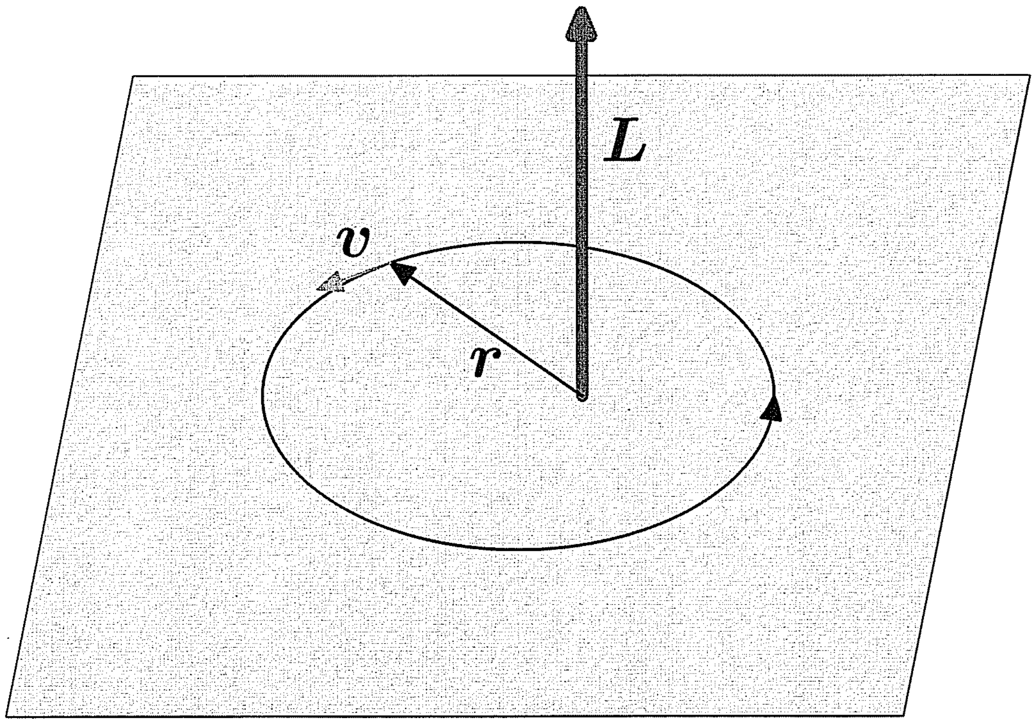
Angular Momentum Conservation and the Central force

Since

$$
\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{p})\,,\\
&= \cancel{\frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p}} + \boldsymbol{r}\times\frac{d\boldsymbol{p}}{dt}\,,\\
&= \boldsymbol{r}\times(f(r)\hat{\boldsymbol{r}})\,,\\
&= 0\,.
\end{aligned}
$$

Planar Motion

Since $\boldsymbol{L}$ is a vector, the conservation of its magnitude leads to the area law while the conservation of its direction supposes that the motion is essentially planar, in a plane perpendicular to $\boldsymbol{L}$.



6'

Newton's equation in the polar co-ordinates

We want to solve

$$m\frac{d^2\boldsymbol{r}}{dt^2} = F(r)\hat{\boldsymbol{r}} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}\,.$$

where $\boldsymbol{r}$ is the relative radius vector from the Sun of mass M to the planet of mass m. To begin with, we consider the attractive force as a force field, i.e. the Sun is assumed to be stationary at the centre. Correction to the motion of the Sun will be discussed later. The equations of motion are then

$$m(\ddot{r} - r\dot{\theta}^2) = F_r\,,$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0\,.$$

The second equation can be integrated easily

$$\boxed{mr^2\dot{\theta} = \text{constant} = L},$$

just a restatement of the conservation of angular momentum.

Effective Potential

In polar co-ordinates the kinetic energy is given as

$$\text{K.E.} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) .$$

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Angular Momentum

$\vec{L} = \vec{r} \times \vec{p}$ definition

$\vec{p} = m\vec{v}$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

$$= \underbrace{(yP_z - zP_y)}_{L_x}\hat{i} + \underbrace{(zP_x - xP_z)}_{L_y}\hat{j} + \underbrace{(xP_y - P_x y)}_{L_z}\hat{k}$$

- Definition depend on the reference point
- $\vec{L}$ is a vector, having three components
- $\vec{L}$ $\perp$ to both $\vec{r}$ and $\vec{p}$ (property of cross product)

$\downarrow$

$\vec{L}$ $\perp$ to the plane formed by $\vec{r}$ and $\vec{p}$

$\downarrow$

$\vec{r}, \vec{p}$ lies on the plane $\perp$ to $\vec{L}$

(Problem (i) of the Middle Term Examination.

Time-dependence of $\vec{L}$

$\Rightarrow \frac{d\vec{L}}{dt}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{= 0} + \underbrace{\vec{r} \times \frac{d\vec{p}}{dt}}_{= \vec{r} \times \vec{F}}$$

$$= \vec{r} \times \vec{F} = \vec{\tau} = \text{torque}$$

$$\frac{d}{dt} L_x = \frac{d}{dt}(yP_z - zP_y)$$

$$= y\frac{dP_z}{dt} + \underbrace{P_z\frac{dy}{dt}}_{= \frac{1}{m}P_z P_y} - z\frac{dP_y}{dt} - \underbrace{\frac{dz}{dt}P_y}_{\frac{1}{m}P_z P_y}$$

$= yF_z - zF_y \quad = (\vec{r}\times\vec{F})_x$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}$$

key equation

If $\quad \vec{r}\times\vec{F} = 0 \quad$ then $\quad \vec{L} = \vec{L}_0$

constant, independent of time

$\Rightarrow$ conservation of angular momentum

$\vec{r}, \vec{p}$ always lie in the plane $\perp$ to $\vec{L}_0$

the particle moves in a plane

two dimensional problem.

Central force $\quad \vec{F} \parallel \vec{r}$

$\Rightarrow \quad \vec{r}\times\vec{F} = 0$

$\vec{L}_0$ is a fixed vector, determined by initial condition.

Kepler's Problem

The Sun-Earth system

Assume the Sun is at rest and located at the origin

$$\vec{F} = -G\frac{M_S M_E}{r^2}\hat{r}$$

universal gravitational force

$\vec{F}$ is central, conservative force

angular momentum conservation

mechanical energy conservation

分類:
編號:
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It is natural to use polar coordinate

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{L} = \vec{r} \times m\vec{v} = mr\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$\hat{r} \times \hat{r} = 0, \quad \hat{r} \times \hat{\theta} = \hat{k}$$

$$\vec{L} = mr^2\dot{\theta}\hat{k} = L_0\hat{k}$$

$$\Rightarrow \quad mr^2\dot{\theta} = L_0$$

$$\dot{\theta} = \frac{L}{mr^2}$$

Consequence of angular momentum conservation

(mechanical)

Conservative force $\Rightarrow$ conservation of energy

$$K.E. + P.E. = E$$

$\hookrightarrow$ total mechanical energy. determined by initial conditions

$$P.E. = -G\frac{M_s M_E}{r}$$

P.E. is chosen to be zero as $r \to \infty$

$$K.E. = \frac{1}{2}m\vec{v}\cdot\vec{v}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{v}\cdot\vec{v} = \dot{r}^2 + (r\dot{\theta})^2$$

$$\frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - \frac{GM_sM_E}{r} = E$$

energy conservation

$$\Rightarrow \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{L_0}{mr^2}\right)^2\right] - \frac{GM_sM}{r} = E$$

$$\Rightarrow \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{1}{2}m\frac{L_0^2}{m^2r^2} - \frac{GM_sM_E}{r}}_{\frac{1}{2}\frac{L_0^2}{mr^2} - \frac{GM_sM_E}{r}} = E$$

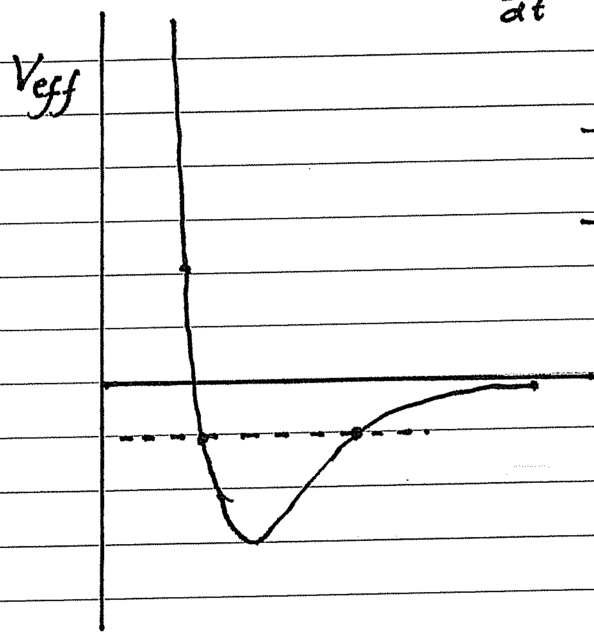
$$\underbrace{\frac{1}{2}\frac{L_0^2}{mr^2} - \frac{GM_sM_E}{r}}_{V_{eff}(r)}$$

$$\frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{F(r)}{\frac{L_0}{mr^2}} = \frac{dr}{d\theta}$$

give directly the orbit

$$\frac{dr}{d\theta} = H(r)$$

function of r



$E < 0 \quad \Rightarrow$ bound motion.

$$\Rightarrow \quad \frac{dr}{dt} = F(r)$$

$$\frac{dr}{F(r)} = dt$$

The third problem of the middle term examination

$\Rightarrow$ $r(t)$ can be obtained by integration
the integration constant is determined by initial conditions

$$\dot{\theta} = \frac{L_0}{mr^2} = G(t)$$

$\Rightarrow$ $\theta(t)$ can be obtained by integration

$r(t)$, $\theta(t)$ can be combined by eliminating

$t \Rightarrow r(\theta)$
$\hookrightarrow$ orbit

Effective Potential

In polar co-ordinates the kinetic energy is given as

$$\text{K.E.} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) .$$

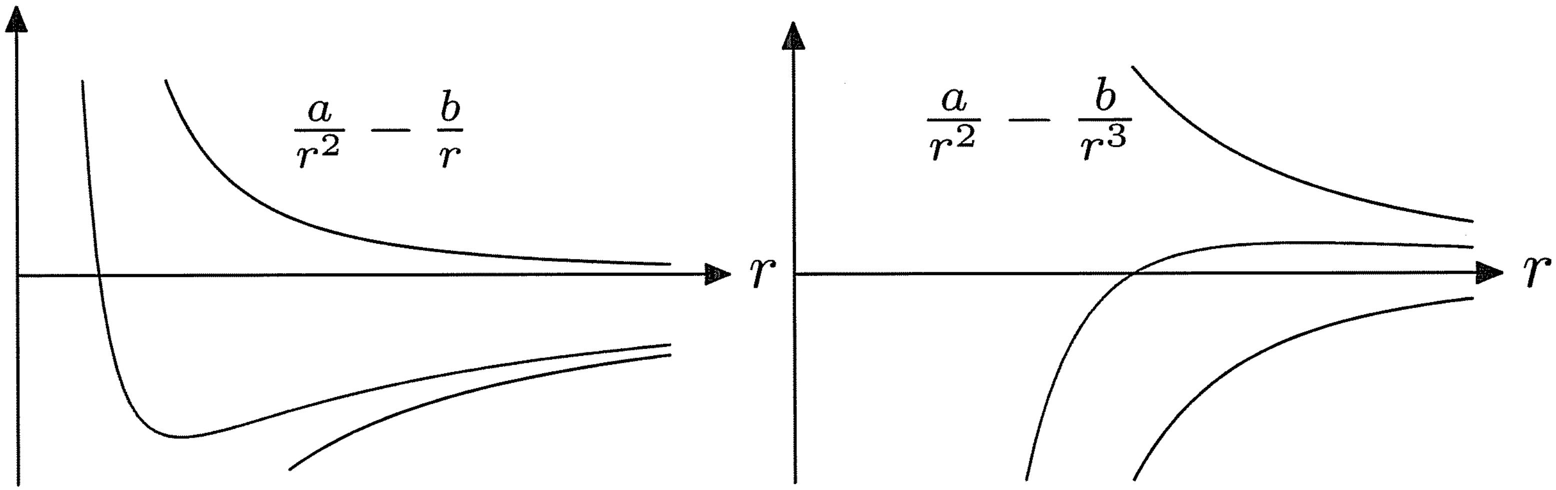
Immediately, we can evoke the energy conservation equation

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E .$$

where the constant E represents the total energy. The energy equation involves only the variable r, we can scoop up the two r dependent terms as the effective potential

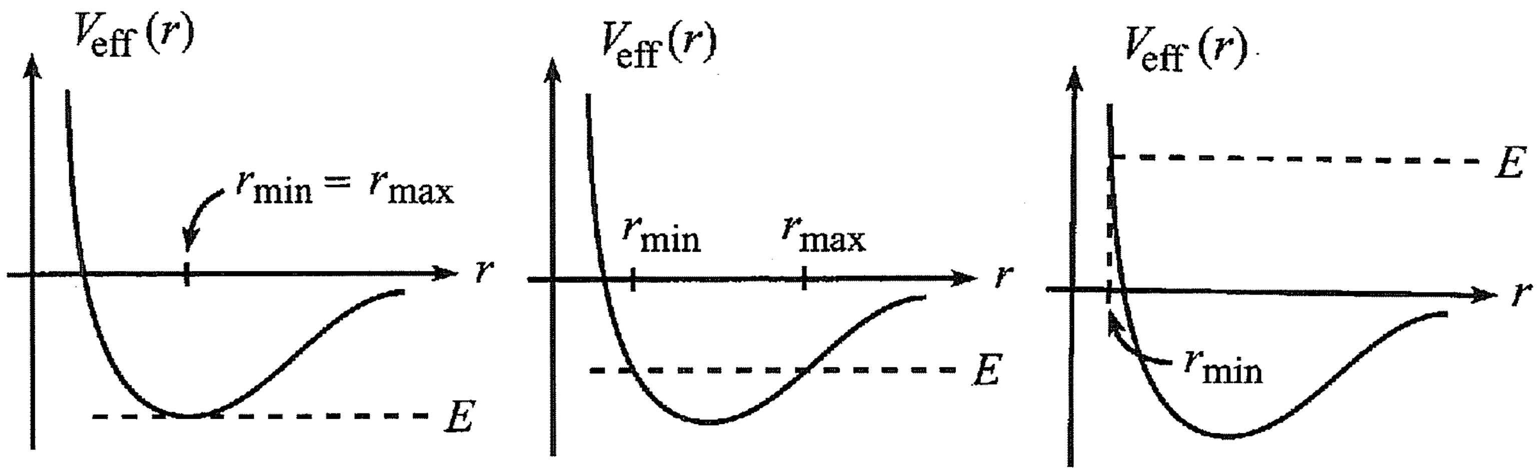
$$V_{\text{eff}} = \frac{L^2}{2mr^2} + V(r) .$$

Two typical cases:



- The left diagram shows a repulsive core and an attractive tail.
- The right diagram shows an attractive core and repulsive tail.

A typical effective potential for the Kepler problem is of the form



- For $E = V_{\text{min}}$ the motion is a circle.
- For $0 > E > V_{\text{min}}$ the motion is bounded within $r_{\text{max}} > r > r_{\text{min}}$.
- For $E \geq 0$ the motion is unbounded from $r = \infty$ to $r = r_{\text{min}}$.

//

Kepler Problem

Now take

$$V(r) = -\frac{GMm}{r} = -\frac{k}{r}\,.$$

The radial velocity is given by

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E + \frac{k}{r}\right) - \frac{L^2}{m^2r^2}}\,.$$

Solving this will give the $r(t)$ trajectory. We shall discuss this later. Here we shall going to get $r(\theta)$, its locus in space. We can write

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{L}{mr^2}\frac{dr}{d\theta}\,.$$

So we have to integrate

$$d\theta = \frac{(L/r^2)dr}{\sqrt{2m\left(E + \frac{k}{r}\right) - \frac{L^2}{r^2}}}.$$

It is convenient to employ the variable $u = 1/r$ so that $du = -1/r^2 dr$, and

$$\begin{aligned} d\theta &= \frac{-L du}{\sqrt{2m\left(E + ku\right) - L^2u^2}}, \\ &= \frac{-du}{\sqrt{\left(\frac{2mE}{L^2} + \frac{m^2k^2}{L^4}\right) - \left(u - \frac{mk}{L^2}\right)^2}}. \end{aligned}$$

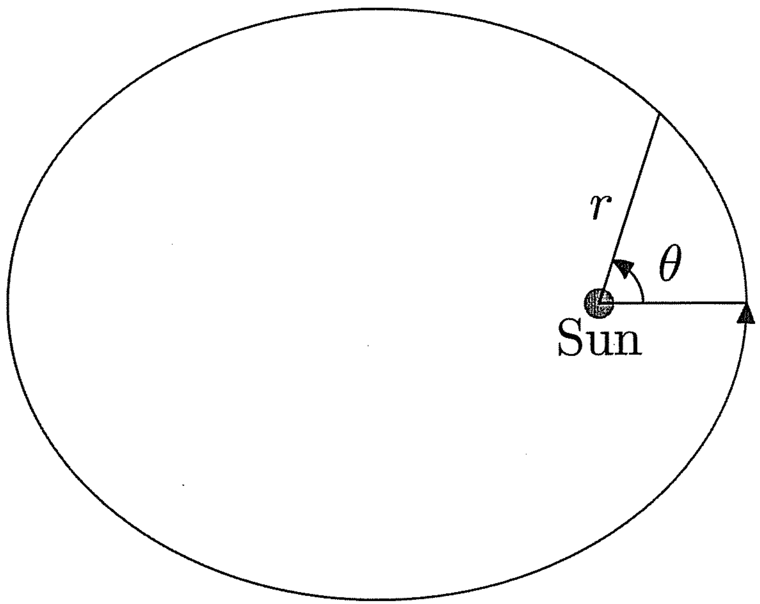
This is a standard integral. The result is

$$\theta - \theta_0 = \cos^{-1} \frac{\frac{L^2}{mk}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EL^2}{mk^2}}}.$$

Setting $\theta_0 = 0$ and writing

$$\alpha = \frac{L^2}{mk}, \qquad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}},$$

we have the orbit equation



$$\boxed{\frac{\alpha}{r} = 1 + \epsilon\cos\theta}$$

Bertrand Theorem: $-k/r$ and $\frac{1}{2}kr^2$ are the only two potentials that have closed orbits for all bounded states.

Notice that α has the dimension of a length.
By definition $\epsilon \geq 0$ and

- If $E = -mk^2/(2L^2) = -k/(2\alpha)$, then $\epsilon = 0$ and the motion is a circle.
- If $0 > E > -k/(2\alpha)$, then $1 > \epsilon > 0$ and the motion is bounded, called the bounded state (ellipse).
- If $E \geq 0$, then $\epsilon \geq 1$ and the motion is unbounded, called the scattering state (parabola or hyperbola).

The parameter ϵ will be proved to be the eccentricity of a conic section.

Orbits as Conic Section with centre at one focus

Take Cartesian co-ordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,.$$

The orbit equation will be

$$\begin{aligned} \alpha &= r + \epsilon x\,, \\ (\alpha - \epsilon x)^2 &= r^2\,, \\ \alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 &= x^2 + y^2\,. \end{aligned}$$

Obviously,

$\epsilon = 0$	Circle ,
$1 > \epsilon > 0$	Ellipse ,
$\epsilon = 1$	Parabola ,
$\epsilon > 1$	Hyperbola .

The equation of the ellipse is thus

$$\boxed{\frac{\left(x+\dfrac{\epsilon\alpha}{1-\epsilon^2}\right)^2}{\left(\dfrac{\alpha}{1-\epsilon^2}\right)^2}+\frac{y^2}{\dfrac{\alpha^2}{(1-\epsilon^2)}}=1}\,.$$

Hence

$$a=\frac{\alpha}{1-\epsilon^2}\,,$$
$$b=\frac{\alpha}{\sqrt{1-\epsilon^2}}\,.$$

and

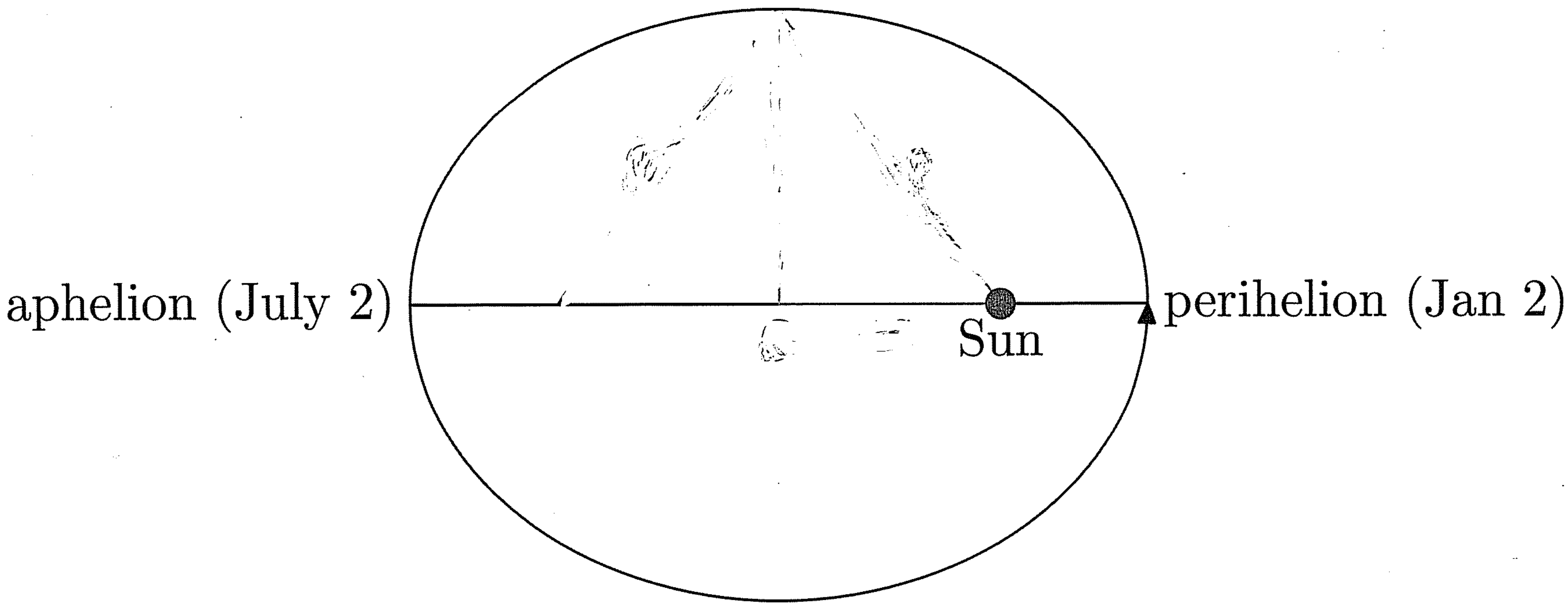
$$a^2-b^2=\epsilon^2a^2\,.$$

Sun at the focus

For bounded motion

$$r_{\mathsf{min}}(\text{perihelion}) = \frac{\alpha}{1+\epsilon}, \qquad r_{\mathsf{max}}(\text{aphelion}) = \frac{\alpha}{1-\epsilon}.$$

$\epsilon = 0.01674$, for the Earth's orbit



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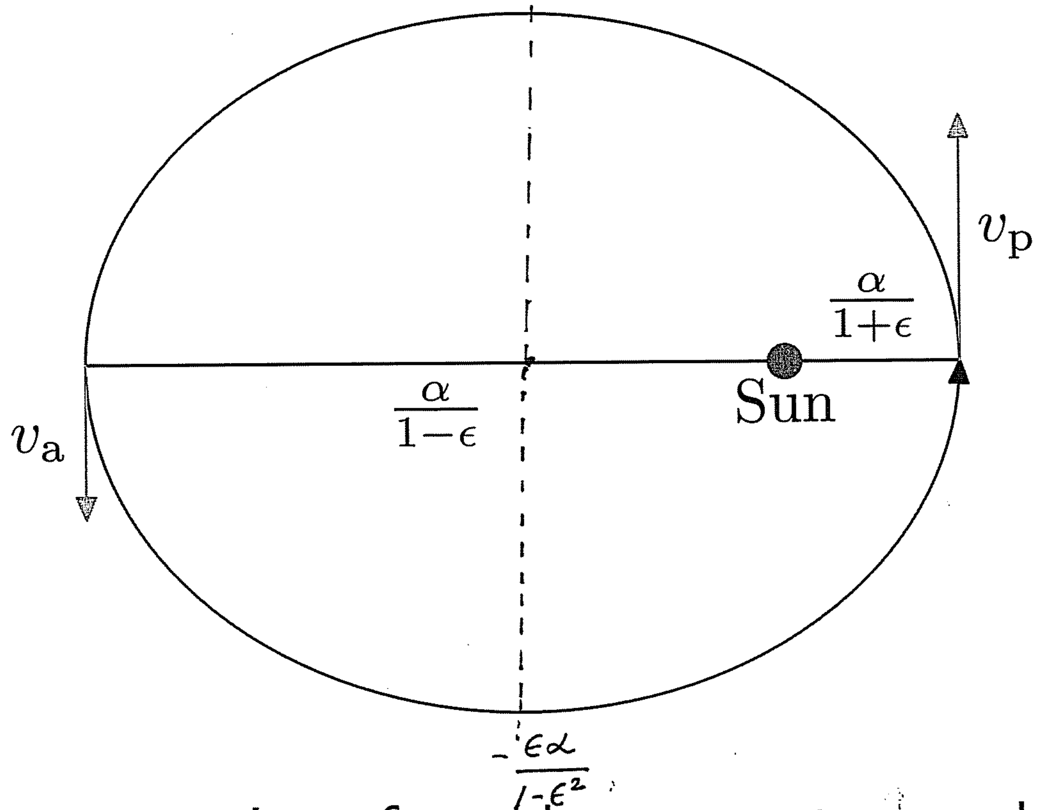
Simple Derivation of ϵ and α



Consider the conservation of angular momentum and energy at perihelion and aphelion.

$$v_a \frac{\alpha}{1-\epsilon} = v_p \frac{\alpha}{1+\epsilon} = \frac{L}{m},$$

$$\frac{1}{2m}\frac{L^2(1-\epsilon)^2}{\alpha^2} - \frac{k(1-\epsilon)}{\alpha} = E\,, \qquad (B)$$

$$\frac{1}{2m}\frac{L^2(1+\epsilon)^2}{\alpha^2} - \frac{k(1+\epsilon)}{\alpha} = E\,. \checkmark \qquad (A)$$

Subtracting, we get

$$\boxed{\alpha = \frac{L^2}{mk}}\,.$$

Substituting back, we get the expression

$$\frac{(1-\epsilon)^2}{2} - (1-\epsilon) = \frac{\alpha E}{k}\,,$$

$$\boxed{\epsilon^2 = 1 + \frac{2EL^2}{mk^2}}\,.$$

24

Kepler third Law

If T is the period, and from $dA/dt = L/2m$, we have

$$\pi ab = \frac{L}{2m} T \,,$$

Therefore

$$T^2 = \frac{4m^2}{L^2} \pi^2 a^3 (1 - \epsilon^2) a \,,$$

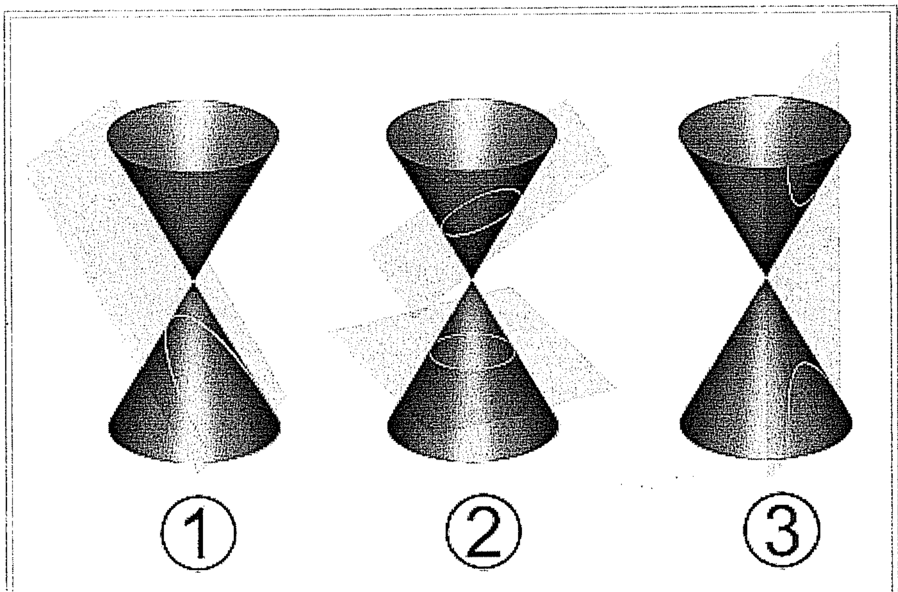
With $(1-\epsilon^2)a = \alpha = L^2/(mk)$ we got

$$\boxed{\frac{T^2}{a^3} = \frac{4\pi^2 \cancel{m}}{GM\cancel{m}}} \,.$$

Conic section

From Wikipedia, the free encyclopedia

In mathematics, a **conic section** (or just **conic**) is a curve obtained by intersecting a cone (more precisely, a right circular conical surface) with a plane. In analytic geometry, a conic may be defined as a plane algebraic curve of degree 2. It can be defined as the locus of points whose distances are in a fixed ratio to some point, called a ***focus***, and some line, called a ***directrix***.



Types of conic sections:
1. Parabola
2. Circle and ellipse
3. Hyperbola

The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, and is of sufficient interest in its own right that it is sometimes called the fourth type of conic section.

The conic sections were named and studied as long ago as 200 BC, when Apollonius of Perga undertook a systematic study of their properties.

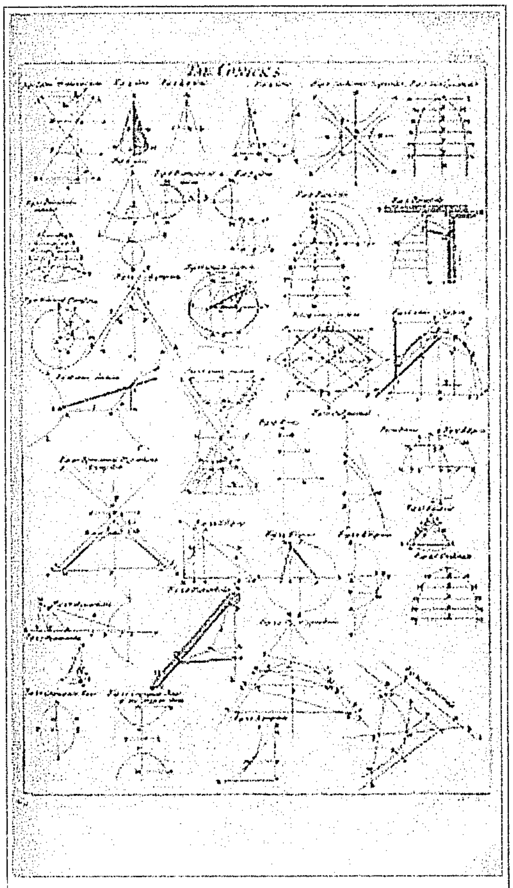


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Table of conics, *Cyclopaedia*, 1728

History

Menaechmus

It is believed that the first definition of a conic section is due to Menaechmus. This work does not survive, however, and is only known through secondary accounts. The definition used at that time differs from the one commonly used today in that it requires the plane cutting the cone to be perpendicular to one of the lines that generate the cone as a surface of revolution (a generatrix). Thus the shape of the conic is determined by the angle formed at the vertex of the cone (between two opposite generatrices): If the angle is acute then the conic is an ellipse; if the angle is right then the conic is a parabola; and if the angle is obtuse then the conic is a hyperbola. Note that the circle cannot be defined this way and was not considered a conic at this time.

Euclid is said to have written four books on conics but these were lost as well.[1] Archimedes is known to have studied conics, having determined the area bounded by a parabola and an ellipse. The only part of this work to survive is a book on the solids of revolution of conics.

Apollonius of Perga

The greatest progress in the study of conics by the ancient Greeks is due to Apollonius of Perga, whose eight volume *Conic Sections* summarized the existing knowledge at the time and greatly extended it. Apollonius's major innovation was to characterize a conic using properties within the plane and intrinsic to the curve; this greatly simplified analysis. With this tool, it was now possible to show that any plane cutting the cone, regardless of its angle, will produce a conic according to the earlier definition, leading to the definition commonly used today.

Pappus is credited with discovering importance of the concept of a focus of a conic, and the discovery of the related concept of a directrix.

Al-Kuhi

An instrument for drawing conic sections was first described in 1000 CE by the Islamic mathematician Al-Kuhi.[2][3]

Omar Khayyám

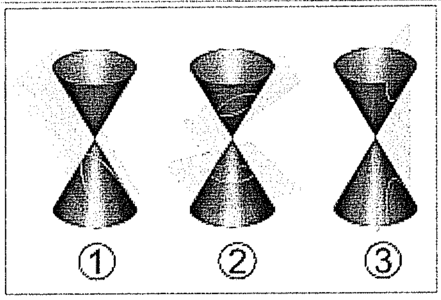
Apollonius's work was translated into Arabic (the technical language of the time) and much of his work only survives through the Arabic version. Persians found applications to the theory; the most notable of these was the Persian[4] mathematician and poet Omar Khayyám who used conic sections to solve algebraic equations.

Europe

Johannes Kepler extended the theory of conics through the "principle of continuity", a precursor to the concept of limits. Girard Desargues and Blaise Pascal developed a theory of conics using an early form of projective geometry and this helped to provide impetus for the study of this new field. In particular, Pascal discovered a theorem known as the hexagrammum mysticum from which many other properties of conics can be deduced. Meanwhile, René Descartes applied his newly discovered Analytic geometry to the study of conics. This had the effect of reducing the geometrical problems of conics to problems in algebra.

Features

The three types of conics are the ellipse, parabola, and hyperbola. The circle can be considered as a fourth type (as it was by Apollonius) or as a kind of ellipse. The circle and the ellipse arise when the intersection of cone and plane is a closed curve. The circle is obtained when the cutting plane is parallel to the plane of the generating circle of the cone – for a right cone as in the picture at the top of the page this means that the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a parabola. In the remaining case, the figure is a hyperbola. In this case, the plane will intersect *both* halves (*nappes*) of the cone, producing two separate unbounded curves.

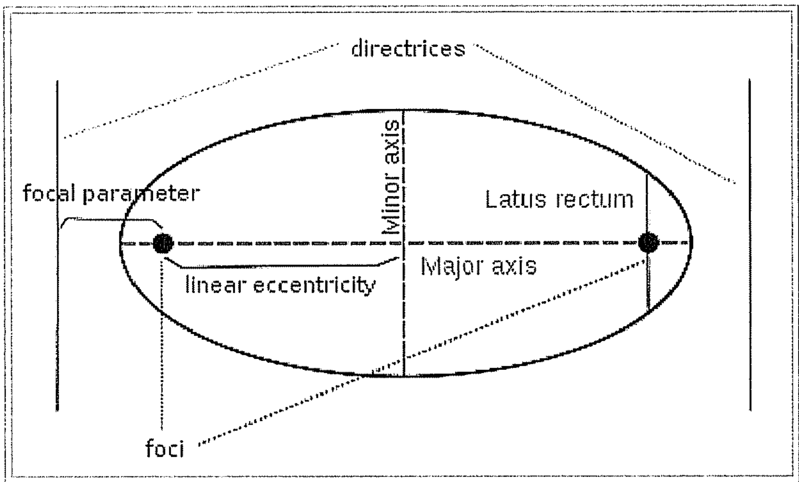


Conics are of three types: parabolas (1), ellipses, including circles (2), or hyperbolas (3).

Various parameters are associated with a conic section, as shown in the following table. (For the ellipse, the table gives the case of $a>b$, for which the major axis is horizontal; for the reverse case, interchange the symbols a and b. For the hyperbola the east-west opening case is given. In all cases, a and b are positive.)

conic section	**equation**	**eccentricity (e)**	**linear eccentricity (c)**	**semi-latus rectum (ℓ)**	**focal parameter (p)**
circle	$x^2+y^2=a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{a^2-b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2-b^2}}$
parabola	$y^2=4ax$	1	a	$2a$	$2a$
hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\sqrt{1+\frac{b^2}{a^2}}$	$\sqrt{a^2+b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2+b^2}}$

Conic sections are exactly those curves that, for a point F, a line L not containing F and a non-negative number e, are the locus of points whose distance to F equals e times their distance to L. F is called the focus, L the directrix, and e the **eccentricity**.



Conic parameters in the case of an ellipse

The **linear eccentricity** (c) is the distance between the center and the focus (or one of the two foci).

The **latus rectum** (2ℓ) is the chord parallel to the directrix and passing through the focus (or one of the two foci).

The **semi-latus rectum** (ℓ) is half the latus rectum.

The **focal parameter** (p) is the distance from the focus (or one of the two foci) to the directrix.

The following relations hold:

- $pe = \ell$
- $ae = c.$

Properties

Just as two (distinct) points determine a line, five points determine a conic. Formally, given any five points in the plane in general linear position, meaning no three collinear, there is a unique conic passing through them, which will be non-degenerate; this is true over both the affine plane and projective plane. Indeed, given any five points there is a conic passing through them, but if three of the points are collinear the conic will be degenerate (reducible, because it contains a line), and may not be unique; see further discussion.

Irreducible conic sections are always "smooth". More precisely, they never contain any inflection points. This is important for many applications, such as aerodynamics, where a smooth surface is required to ensure laminar flow and to prevent turbulence.

Intersection at infinity

An algebro-geometrically intrinsic form of this classification is by the intersection of the conic with the line at infinity, which gives further insight into their geometry:

- ellipses intersect the line at infinity in 0 points – rather, in 0 real points, but in 2 complex points, which are conjugate;
- parabolas intersect the line at infinity in 1 double point, corresponding to the axis – they are tangent to the line at infinity, and close at infinity, as distended ellipses;
- hyperbolas intersect the line at infinity in 2 points, corresponding to the asymptotes – hyperbolas pass through infinity, with a twist. Going to infinity along one branch passes through the point at infinity corresponding to the asymptote, then re-emerges on the other branch at the other side but with the inside of the hyperbola (the direction of curvature) on the other side – left vs. right (corresponding to the non-orientability of the real projective plane) – and then passing through the other point at infinity returns to the first branch. Hyperbolas can thus be seen as ellipses that have been pulled through infinity and re-emerged on the other side, flipped.

Degenerate cases

For more details on this topic, see Degenerate conic.

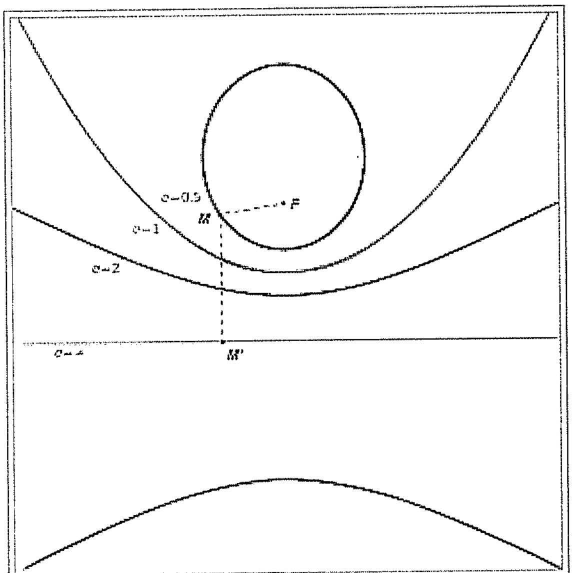
There are five degenerate cases: three in which the plane passes through apex of the cone, and three that arise when the cone itself degenerates to a cylinder (a doubled line can occur in both cases).

When the plane passes through the apex, the resulting conic is always degenerate, and is either: a point (when the angle between the plane and the axis of the cone is larger than tangential); a straight line (when the plane is tangential to the surface of the cone); or a pair of intersecting lines (when the angle is smaller than the tangential). These correspond respectively to degeneration of an ellipse, parabola, and a hyperbola, which are characterized in the same way by angle. The straight line is more precisely a *double* line (a line with multiplicity 2) because the plane is tangent to the cone, and thus the intersection should be counted twice.

Where the cone is a cylinder, i.e. with the vertex at infinity, cylindric sections are obtained;[5] this corresponds to the apex being at infinity. Cylindrical sections are ellipses (or circles), unless the plane is vertical (which corresponds to passing through the apex at infinity), in which case three degenerate cases occur: two parallel lines, known as a ribbon (corresponding to an ellipse with one axis infinite and the other axis real and non-zero, the distance between the lines), a double line (an ellipse with one infinite axis and one axis zero), and no intersection (an ellipse with one infinite axis and the other axis imaginary).

Eccentricity, focus and directrix

The four defining conditions above can be combined into one condition that depends on a fixed point F (the ***focus***), a line L (the ***directrix***) not containing F and a nonnegative real number e (the *eccentricity*). The corresponding conic section consists of the locus of all points whose distance to F equals e times their distance to L. For $0 < e < 1$ we obtain an ellipse, for $e = 1$ a parabola, and for $e > 1$ a hyperbola.



Ellipse (e=1/2), parabola (e=1) and hyperbola (e=2) with fixed focus F and directrix (e=∞).

For an ellipse and a hyperbola, two focus-directrix combinations can be taken, each giving the same full ellipse or hyperbola. The distance from the center to the directrix is a / e, where a is the semi-major axis of the ellipse, or the distance from the center to the tops of the hyperbola. The distance from the center to a focus is $a e$.

In the case of a circle, the eccentricity $e = 0$, and one can imagine the directrix to be infinitely far removed from the center. However, the statement that the circle consists of all points whose distance to F is e times the distance to L is not useful, because we get zero times infinity.

The eccentricity of a conic section is thus a measure of how far it deviates from being circular.

For a given a, the closer e is to 1, the smaller is the semi-minor axis.

Generalizations

Conics may be defined over other fields, and may also be classified in the projective plane rather than in the affine plane.

Over the complex numbers ellipses and hyperbolas are not distinct, since there is no meaningful difference between 1 and -1; precisely, the ellipse $x^2 + y^2 = 1$ becomes a hyperbola under the substitution $y = iw$, geometrically a complex rotation, yielding $x^2 - w^2 = 1$ – a hyperbola is simply an ellipse with an imaginary axis length. Thus there is a 2-way classification: ellipse/hyperbola and parabola. Geometrically, this corresponds to intersecting the line at infinity in either 2 distinct points (corresponding to two asymptotes) or in 1 double point (corresponding to the axis of a parabola), and thus the real hyperbola is a more suggestive image for the complex ellipse/hyperbola, as it also has 2 (real) intersections with the line at infinity.

In projective space, over any division ring, but in particular over either the real or complex numbers, all non-degenerate conics are equivalent, and thus in projective geometry one simply speaks of "a conic" without specifying a type, as type is not meaningful. Geometrically, the line at infinity is no longer special (distinguished), so while some conics intersect the line at infinity differently, this can be changed by a projective transformation – pulling an ellipse out to infinity or pushing a parabola off infinity to an ellipse or a hyperbola.

In other areas of mathematics

The classification into elliptic, parabolic, and hyperbolic is pervasive in mathematics, and often divides a field into sharply distinct subfields. The classification mostly arises due to the presence of a quadratic form (in two variables this corresponds to the associated discriminant), but can also correspond to eccentricity.

Quadratic form classifications:

quadratic forms
: Quadratic forms over the reals are classified by Sylvester's law of inertia, namely by their positive index, zero index, and negative index: a quadratic form in n variables can be converted to a diagonal form, as $x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+l}^2$, where the number of $+1$ coefficients, k, is the positive index, the number of -1 coefficients, l, is the negative index, and the remaining variables are the zero index m, so $k + l + m = n$. In two variables the non-zero quadratic forms are classified as:

- $x^2 + y^2$, – positive-definite (the negative is also included), corresponding to ellipses,
- x^2 – degenerate, corresponding to parabolas, and
- $x^2 - y^2$ – indefinite, corresponding to hyperbolas.

In two variables quadratic forms are classified by discriminant, analogously to conics, but in higher dimensions the more useful classification is as *definite,* (all positive or all negative), *degenerate,* (some zeros), or *indefinite* (mix of positive and negative but no zeros). This classification underlies many that follow.

curvature

The Gaussian curvature of a surface describes the infinitesimal geometry, and may at each point be either positive – elliptic geometry, zero – Euclidean geometry (flat, parabola), or negative – hyperbolic geometry; infinitesimally, to second order the surface looks like the graph of $x^2 + y^2$,, x^2 (or 0), or $x^2 - y^2$.. Indeed, by the uniformization theorem every surface can be taken to be globally (at every point) positively curved, flat, or negatively curved. In higher dimensions the Riemann curvature tensor is a more complicated object, but manifolds with constant sectional curvature are interesting objects of study, and have strikingly different properties, as discussed at sectional curvature.

Second order PDEs

Partial differential equations (PDEs) of second order are classified at each point as elliptic, parabolic, or hyperbolic, accordingly as their second order terms correspond to an elliptic, parabolic, or hyperbolic quadratic form. The behavior and theory of these different types of PDEs are strikingly different – representative examples is that the Laplacian is elliptic, the heat equation is parabolic, and the wave equation is hyperbolic.

Eccentricity classifications include:

Möbius transformations

Real Möbius transformations (elements of $PSL_2(\mathbf{R})$ or its 2-fold cover, $SL_2(\mathbf{R})$) are classified as elliptic, parabolic, or hyperbolic accordingly as their half-trace is $0 \leq |\operatorname{tr}|/2 < 1, |\operatorname{tr}|/2 = 1,$ or $|\operatorname{tr}|/2 > 1,$ mirroring the classification by eccentricity.

Variance-to-mean ratio

The variance-to-mean ratio classifies several important families of discrete probability distributions: the constant distribution as circular (eccentricity 0), binomial distributions as elliptical, Poisson distributions as parabolic, and negative binomial distributions as hyperbolic. This is elaborated at cumulants of some discrete probability distributions.

Cartesian coordinates

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section – though it may be degenerate, and all conic sections arise in this way. The equation will be of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ with } A, B, C \text{ not all zero.}$$

As scaling all six constants yields the same locus of zeros, one can consider conics as points in the five-dimensional projective space $\mathbf{P}^5$.

Discriminant classification

The conic sections described by this equation can be classified with the discriminant[6]

$$B^2 - 4AC.$$

If the conic is non-degenerate, then:

- if $B^2 - 4AC < 0$, the equation represents an ellipse;
  - if $A = C$ and $B = 0$, the equation represents a circle, which is a special case of an ellipse;
- if $B^2 - 4AC = 0$, the equation represents a parabola;
- if $B^2 - 4AC > 0$, the equation represents a hyperbola;
  - if we also have $A + C = 0$, the equation represents a rectangular hyperbola.

To distinguish the degenerate cases from the non-degenerate cases, let Δ be the determinant of the 3×3 matrix $[A, B/2, D/2\ ; B/2, C, E/2\ ; D/2, E/2, F\ ]$: that is, $\Delta = (AC - B^2/4)F + BED/4 - CD^2/4 - AE^2/4$. Then the conic section is non-degenerate if and only if $\Delta \neq 0$. If $\Delta=0$ we have a point ellipse, two parallel lines (possibly coinciding with each other) in the case of a parabola, or two intersecting lines in the case of a hyperbola.[7]:p.63

Moreover, in the case of a non-degenerate ellipse (with $B^2 - 4AC < 0$ and $\Delta\neq0$), we have a real ellipse if $C\Delta < 0$ but an imaginary ellipse if $C\Delta > 0$. An example is $x^2 + y^2 + 10 = 0$, which has no real-valued solutions.

Note that A and B are polynomial coefficients, not the lengths of semi-major/minor axis as defined in some sources.

Matrix notation

Main article: Matrix representation of conic sections

The above equation can be written in matrix notation as

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + Dx + Ey + F = 0.$$

The type of conic section is solely determined by the determinant of middle matrix: if it is positive, zero, or negative then the conic is an ellipse, parabola, or hyperbola respectively (see geometric meaning of a quadratic form). If both the eigenvalues of the middle matrix are non-zero (i.e. it is an ellipse or a hyperbola), we can do a transformation of variables to obtain

$$\begin{pmatrix} x-a \\ y-c \end{pmatrix}^T \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} x-a \\ y-c \end{pmatrix} = G$$

where a,c, and G satisfy $D + 2aA + Bc = 0$, $E + 2Cc + Ba = 0$, and $G = Aa^2 + Cc^2 + Bac - F$.

The quadratic can also be written as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} . \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0.$$

If the determinant of this 3×3 matrix is non-zero, the conic section is not degenerate. If the determinant equals zero, the conic is a degenerate parabola (two parallel or coinciding lines), a degenerate ellipse (a point ellipse), or a degenerate hyperbola (two intersecting lines).

Note that in the centered equation with constant term G, G equals minus one times the ratio of the 3×3 determinant to the 2×2 determinant.

As slice of quadratic form

The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rearranged by taking the affine linear part to the other side, yielding

$$Ax^2 + Bxy + Cy^2 = -(Dx + Ey + F).$$

In this form, a conic section is realized exactly as the intersection of the graph of the quadratic form $z = Ax^2 + Bxy + Cy^2$ and the plane $z = -(Dx + Ey + F)$. Parabolas and hyperbolas can be realized by a horizontal plane ($D = E = 0$), while ellipses require that the plane be slanted. Degenerate conics correspond to degenerate intersections, such as taking slices such as $z = -1$ of a positive-definite form.

Eccentricity in terms of parameters of the quadratic form

When the conic section is written algebraically as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the eccentricity can be written as a function of the parameters of the quadratic equation.[8] If $4AC = B^2$ the conic is a parabola and its eccentricity equals 1 (if it is non-degenerate). Otherwise, assuming the equation represents either a non-degenerate hyperbola or a non-degenerate, non-imaginary ellipse, the eccentricity is given by

$$e = \sqrt{\frac{2\sqrt{(A-C)^2+B^2}}{\eta(A+C)+\sqrt{(A-C)^2+B^2}}}$$

where η= 1 if the determinant of the 3×3 matrix is negative or η= -1 if that determinant is positive.

Standard form

Through change of coordinates these equations can be put in standard forms:

- Circle: $x^2+y^2=a^2$
- Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
- Parabola: $y^2=4ax,\ x^2=4ay$
- Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ \frac{x^2}{b^2}-\frac{y^2}{a^2}=-1$
- Rectangular Hyperbola: $xy=c^2$

Such forms will be symmetrical about the x-axis and for the circle, ellipse and hyperbola symmetrical about the y-axis.
The rectangular hyperbola however is only symmetrical about the lines $y=x$ and $y=-x$. Therefore its inverse function is exactly the same as its original function.

These standard forms can be written as parametric equations,

- Circle: $(a\cos\theta, a\sin\theta)$,
- Ellipse: $(a\cos\theta, b\sin\theta)$,
- Parabola: $(at^2, 2at)$,
- Hyperbola: $(a\sec\theta, b\tan\theta)$ or $(\pm a\cosh u, b\sinh u)$.
- Rectangular hyperbola: $\left(ct, \frac{c}{t}\right)$

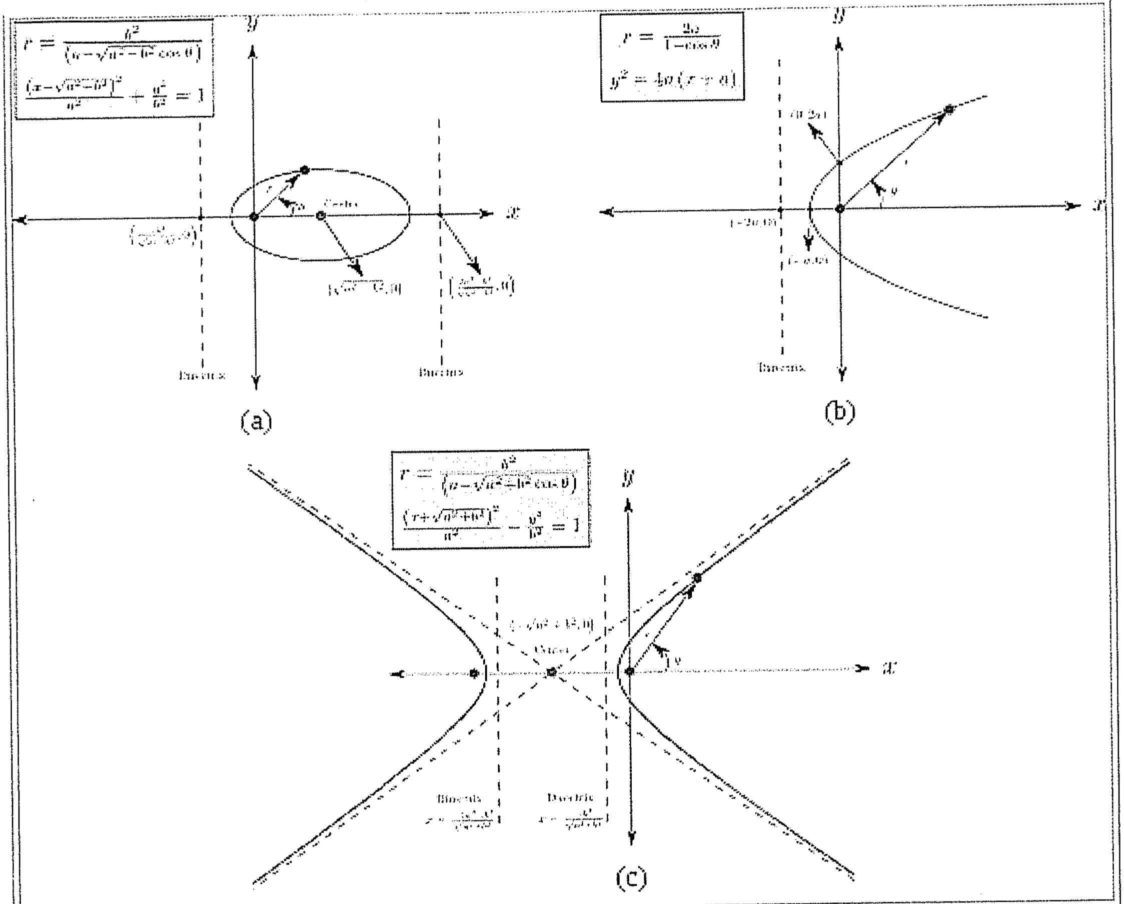
Invariants of conics

The trace and determinant of $\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$ are both invariant with respect to both rotation of axes and translation of the plane (movement of the origin).[9]

The constant term F is invariant under rotation only.

Modified form

For some practical



Three different types of conic sections. Focal-points corresponding to all conic sections are placed at the origin.

applications, it is important to re-arrange the standard form so that the focal-point can be placed at the origin. The mathematical formulation for a general conic section is then given in the polar form by

$$r = \frac{l}{1 - e\cos\theta}$$

and in the Cartesian form by

$$\sqrt{x^2 + y^2} = (l + ex)$$

$$\Rightarrow \left(\frac{x - \frac{le}{1-e^2}}{\frac{l}{1-e^2}}\right)^2 + \frac{(1-e^2)\,y^2}{l^2} = 1$$

From the above equation, the **linear eccentricity** (c) is given by $c = \left(\frac{le}{1-e^2}\right)$.

From the general equations given above, different conic sections can be represented as shown below:

- Circle: $x^2 + y^2 = r^2$

- Ellipse: $\frac{\left(x-\sqrt{a^2-b^2}\right)^2}{a^2}+\frac{y^2}{b^2}=1$
- Parabola: $y^2=4a\,(x+a)$
- Hyperbola: $\frac{\left(x+\sqrt{a^2+b^2}\right)^2}{a^2}-\frac{y^2}{b^2}=1$

Homogeneous coordinates

In homogeneous coordinates a conic section can be represented as:

$$A_1x^2+A_2y^2+A_3z^2+2B_1xy+2B_2xz+2B_3yz=0.$$

Or in matrix notation

$$\begin{bmatrix}x & y & z\end{bmatrix}\cdot\begin{bmatrix}A_1 & B_1 & B_2\\ B_1 & A_2 & B_3\\ B_2 & B_3 & A_3\end{bmatrix}\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0.$$

The matrix $M=\begin{bmatrix}A_1 & B_1 & B_2\\ B_1 & A_2 & B_3\\ B_2 & B_3 & A_3\end{bmatrix}$ is called *the matrix of the conic section.*

$\Delta=\det(M)=\det\left(\begin{bmatrix}A_1 & B_1 & B_2\\ B_1 & A_2 & B_3\\ B_2 & B_3 & A_3\end{bmatrix}\right)$ is called the determinant of the conic section.

If $\Delta = 0$ then the *conic section* is said to be *degenerate*; this means that the conic section is either a union of two straight lines, a repeated line, a point or the empty set.

For example, the conic section $\begin{bmatrix}x & y & z\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$ reduces to the union of two lines:

$$\{x^2-y^2=0\}=\{(x+y)(x-y)=0\}=\{x+y=0\}\cup\{x-y=0\}.$$

Similarly, a conic section sometimes reduces to a (single) repeated line:

$$\{x^2+2xy+y^2=0\}=\{(x+y)^2=0\}=\{x+y=0\}\cup\{x+y=0\}=\{x+y=0\}.$$

$\delta=\det\left(\begin{bmatrix}A_1 & B_1\\ B_1 & A_2\end{bmatrix}\right)$ is called the discriminant of the conic section. If $\delta = 0$ then the *conic section* is a parabola, if $\delta < 0$, it is an hyperbola and if $\delta > 0$, it is an ellipse. A conic section is a circle if $\delta > 0$ and $A_1 = A_2$ and $B_1 = 0$, it is an rectangular hyperbola if $\delta < 0$

and $A_1 = -A_2$. It can be proven that in the complex projective plane $\mathbf{CP}^2$ two conic sections have four points in common (if one accounts for multiplicity), so there are never more than 4 intersection points and there is always one *intersection point* (possibilities: four distinct intersection points, two singular intersection points and one double intersection points, two double intersection points, one singular intersection point and 1 with multiplicity 3, 1 intersection point with multiplicity 4). If there exists at least one intersection point with multiplicity > 1, then the two conic sections are said to be tangent. If there is only one intersection point, which has multiplicity 4, the two conic sections are said to be osculating.[10]

Furthermore each straight line intersects each conic section twice. If the intersection point is double, the line is said to be tangent and it is called the tangent line. Because every straight line intersects a conic section twice, each conic section has two points at infinity (the intersection points with the line at infinity). If these points are real, the conic section must be a hyperbola, if they are imaginary conjugated, the conic section must be an ellipse, if the conic section has one double point at infinity it is a parabola. If the points at infinity are (1,i,0) and (1,-i,0), the conic section is a circle. If a conic section has one real and one imaginary point at infinity or it has two imaginary points that are not conjugated it is neither a parabola nor an ellipse nor a hyperbola.

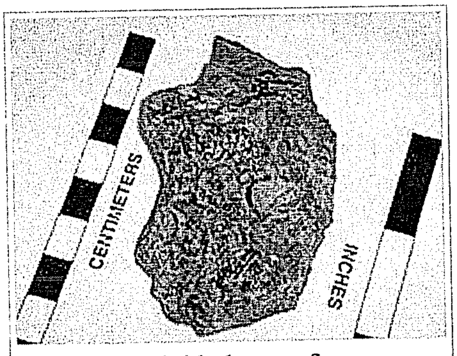
Polar coordinates

In polar coordinates, a conic section with one focus at the origin and, if any, the other on the x-axis, is given by the equation

$$r = \frac{l}{1 \pm e\cos\theta},$$

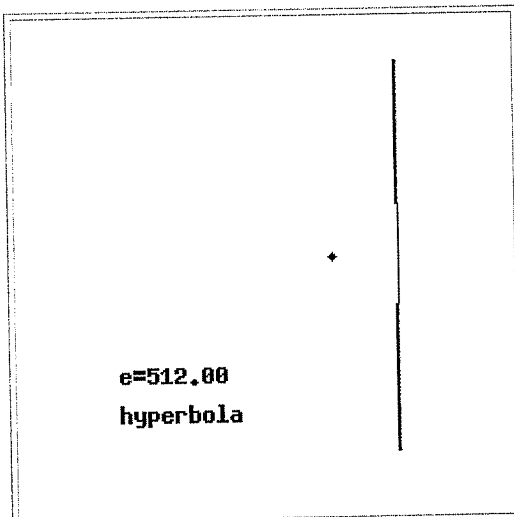
where e is the eccentricity and l is the semi-latus rectum (see below). As above, for $e = 0$, we have a circle, for $0 < e < 1$ we obtain an ellipse, for $e = 1$ a parabola, and for $e > 1$ a hyperbola.

Applications



The paraboloid shape of Archeocyathids produces conic

Conic sections are important in astronomy: the orbits of two massive objects that interact according to Newton's law of universal gravitation are conic sections if



Development of the conic section as the

sections on rock faces	their common center of mass is	eccentricity e increases

considered to be at rest. If they are bound together, they will both trace out ellipses; if they are moving apart, they will both follow parabolas or hyperbolas. See two-body problem.

In projective geometry, the conic sections in the projective plane are equivalent to each other up to projective transformations.

For specific applications of each type of conic section, see the articles circle, ellipse, parabola, and hyperbola.

For certain fossils in paleontology, understanding conic sections can help understand the three-dimensional shape of certain organisms.

Intersecting two conics

The solutions to a two second degree equations system in two variables may be seen as the coordinates of the intersections of two generic conic sections. In particular two conics may possess none, two or four possibly coincident intersection points. The best method of locating these solutions exploits the homogeneous matrix representation of conic sections, i.e. a 3x3 symmetric matrix which depends on six parameters.

The procedure to locate the intersection points follows these steps:

- given the two conics C_1 and C_2 consider the pencil of conics given by their linear combination $\lambda C_1 + \mu C_2$
- identify the homogeneous parameters (λ,μ) which corresponds to the degenerate conic of the pencil. This can be done by imposing that $\det(\lambda C_1 + \mu C_2) = 0$, which turns out to be the solution to a third degree equation.
- given the degenerate conic C_0, identify the two, possibly coincident, lines constituting it
- intersects each identified line with one of the two original conic; this step can be done efficiently using the dual conic representation of C_0
- the points of intersection will represent the solution to the initial equation system

See also

- Focus (geometry), an overview of properties of conic sections related to the foci
- Lambert conformal conic projection
- Matrix representation of conic sections
- Quadrics, the higher-dimensional analogs of conics
- Quadratic function
- Rotation of axes
- Dandelin spheres

- Projective conics
- Elliptic coordinates
- Parabolic coordinates
- Director circle

Notes

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External links

- Derivations of Conic Sections (http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=196&bodyId=60) at Convergence (http://mathdl.maa.org/convergence/1/)
- Conic sections (http://xahlee.org/SpecialPlaneCurves_dir/ConicSections_dir/conicSections.html) at Special plane curves (http://xahlee.org/SpecialPlaneCurves_dir/specialPlaneCurves.html) .
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- Determinants and Conic Section Curves (http://math.fullerton.edu/mathews/n2003/ConicFitMod.html)
- Occurrence of the conics. Conics in nature and elsewhere (http://britton.disted.camosun.bc.ca/jbconics.htm) .
- Conics (http://www.mathacademy.com/pr/prime/articles/conics/index.asp) . An essay on conics and how they are generated.
- See Conic Sections (http://www.cut-the-knot.org/proofs/conics.shtml) at cut-the-knot (http://www.cut-the-knot.org) for a sharp proof that any finite conic section is an ellipse and Xah Lee (http://xahlee.org/PageTwo_dir/more.html) for a similar treatment of other conics.
- Cone-plane intersection (http://www.mathworks.com/matlabcentral/fileexchange/19631) MATLAB code
- Eight Point Conic (http://math.kennesaw.edu/~mdevilli/eightpointconic.html) at Dynamic Geometry Sketches (http://math.kennesaw.edu/~mdevilli/JavaGSPLinks.htm)
- An interactive Java conics grapher; uses a general second-order implicit equation. (http://www.geogebra.org/en/upload/files/nikenuke/conics04b.html)

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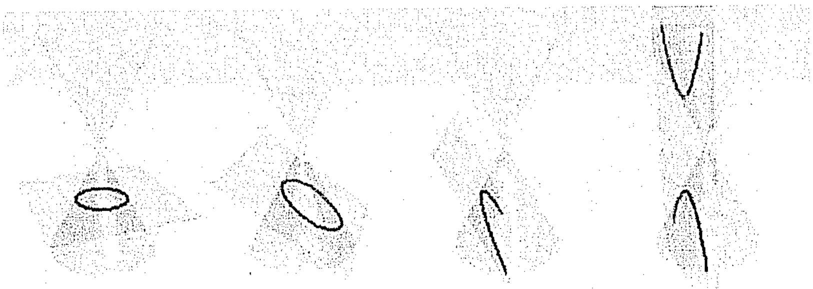
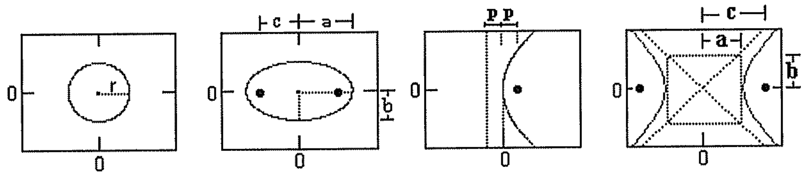
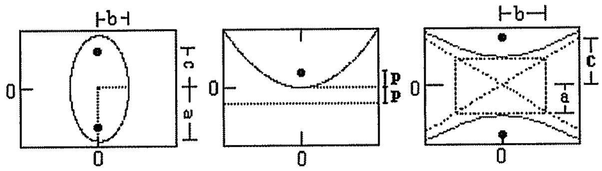


Circle Ellipse (h) Parabola (h) Hyperbola (h)

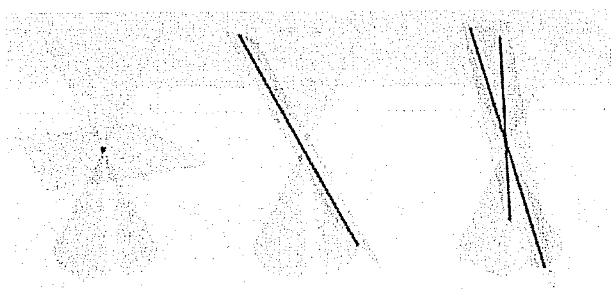


Definition: A conic section is the intersection of a plane and a cone.

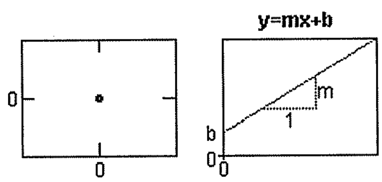
Ellipse (v) Parabola (v) Hyperbola (v)



By changing the angle and location of intersection, we can produce a circle, ellipse, parabola or hyperbola; or in the special case when the plane touches the vertex: a point, line or 2 intersecting lines.



Point Line Double Line



The General Equation for a Conic Section:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The type of section can be found from the sign of: $B^2 - 4AC$

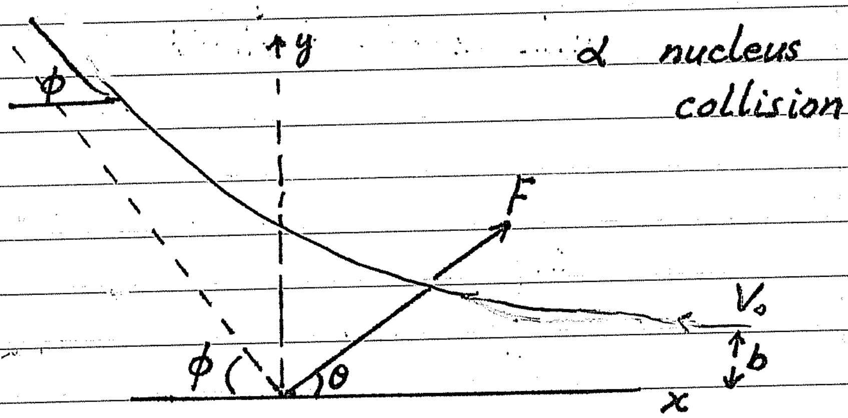
If $B^2 - 4AC$ is...	then the curve is a...
< 0	ellipse, circle, point or no curve.
$= 0$	parabola, 2 parallel lines, 1 line or no curve.
> 0	hyperbola or 2 intersecting lines.

The Conic Sections. For any of the below with a center (j, k) instead of (0, 0), replace each x term with (x-

j) and each y term with (y-k).

	Circle	**Ellipse**	**Parabola**	**Hyperbola**
Equation (horiz. vertex):	$x^2 + y^2 = r^2$	$x^2 / a^2 + y^2 / b^2 = 1$	$4px = y^2$	$x^2 / a^2 - y^2 / b^2 = 1$
Equations of Asymptotes:				$y = \pm (b/a)x$
Equation (vert. vertex):	$x^2 + y^2 = r^2$	$y^2 / a^2 + x^2 / b^2 = 1$	$4py = x^2$	$y^2 / a^2 - x^2 / b^2 = 1$
Equations of Asymptotes:				$x = \pm (b/a)y$
Variables:	r = circle radius	a = major radius (= 1/2 length major axis) b = minor radius (= 1/2 length minor axis) c = distance center to focus	p = distance from vertex to focus (or directrix)	a = 1/2 length major axis b = 1/2 length minor axis c = distance center to focus
Eccentricity:	0		c/a	c/a
Relation to Focus:	p = 0	$a^2 - b^2 = c^2$	p = p	$a^2 + b^2 = c^2$
Definition: is the locus of all points which meet the condition...	distance to the origin is constant	sum of distances to each focus is constant	distance to focus = distance to directrix	difference between distances to each foci is constant
Related Topics:	Geometry section on Circles			

Rutherford Scattering



Initial condition

$r \to \infty \qquad y = b, \quad x = \infty \qquad b = $ impact parameter

$v_y = 0 \qquad V_x = V_0$

$$\vec{F} = \frac{k}{r^2}\hat{r}$$

This is a central force $\Rightarrow \; mr^2 \frac{d\theta}{dt} = mv_0 b$

angular momentum is conserved

$$m\frac{dV_y}{dt} = F_y = F\sin\theta = \frac{k\sin\theta}{r^2}$$

$$\frac{dV_y}{dt} = \frac{k}{m}\frac{\sin\theta}{v_0 b}\frac{d\theta}{dt}$$

$$\int_0^{v_0\sin\phi} dV_y = \frac{k}{mv_0 b}\int_0^{\pi-\phi}\sin\theta\, d\theta \qquad V = V_0 \text{ outgoing}$$

$$v_0\sin\phi = \frac{k}{mv_0 b}\{-[\cos(\pi-\phi) - \cos 0^\circ]\}$$

$$= \frac{k}{mv_0 b}(1+\cos\phi)$$

$$v_0\, 2\sin\frac{\phi}{2}\cos\frac{\phi}{2} = \frac{k}{mv_0 b}\cos^2\frac{\phi}{2}$$

$$\cot(\phi/2) = \frac{mv_0^2 b}{k} \qquad \phi = 180^\circ \quad b = 0$$

$2\pi b\, db$

Paul A. Tipler

physics for scientists and engineers

Fourth Edition

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Oscillations and Waves

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W.H. FREEMAN AND COMPANY/WORTH PUBLISHERS

8-8 Rocket Propulsion

Rocket propulsion is a striking example of the conservation of momentum in action. The mathematical description of rocket propulsion can become quite complex because the mass of the rocket changes continuously as it burns fuel and expels exhaust gas. The easiest approach is to compute the change in the momentum of the total system (including the exhaust gas) for some time interval and use Newton's law in the form $F_{ext} = dP/dt$, where F_{ext} is the net force acting on the rocket.

Consider a rocket moving with speed v relative to the earth (Figure 8-44). If the fuel is burned at a constant rate, $R = |dm/dt|$, the rocket's mass at time t is

$$m = m_0 - Rt \qquad 8\text{-}35$$

where m_0 is the initial mass of the rocket. The momentum of the system at time t is

$$P_i = mv$$

At a later time $t + \Delta t$, the rocket has expelled gas of mass $R\,\Delta t$. If the gas is exhausted at a speed u_{ex} *relative to the rocket*, the velocity of the gas relative to the earth is $v - u_{ex}$. The rocket then has a mass $m - R\,\Delta t$ and is moving at a speed $v + \Delta v$ (Figure 8-45).

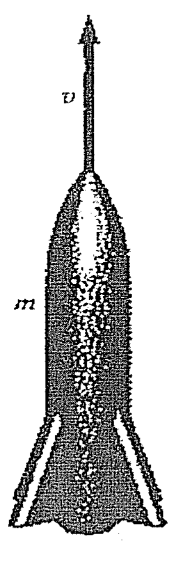


Figure 8-44

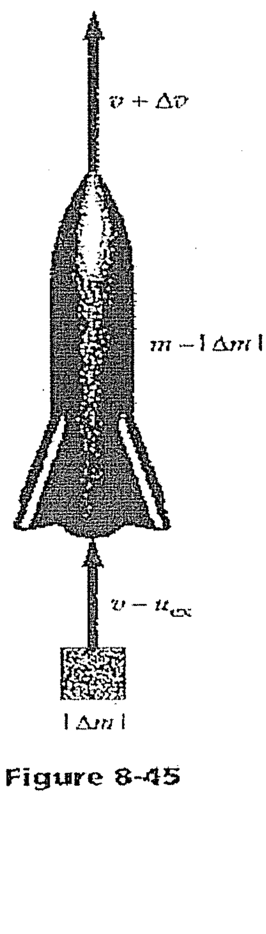


Figure 8-45

The momentum of the system at $t + \Delta t$ is

$$P_f = (m - R\,\Delta t)(v + \Delta v) + R\,\Delta t(v - u_{ex})$$
$$= mv + m\,\Delta v - v\,R\,\Delta t - R\,\Delta t\,\Delta v + v\,R\,\Delta t - u_{ex}\,R\,\Delta t$$
$$\approx mv + m\,\Delta v - u_{ex}\,R\,\Delta t$$

where we have dropped the term $R\,\Delta t\,\Delta v$, which is the product of two very small quantities, and therefore negligible compared with the others. The change in momentum is

$$\Delta P = P_f - P_i = m\,\Delta v - u_{ex}\,R\,\Delta t$$

and

$$\frac{\Delta P}{\Delta t} = m\frac{\Delta v}{\Delta t} - u_{ex}R \qquad \text{8-36}$$

As Δt approaches zero, $\Delta v/\Delta t$ approaches the derivative dv/dt, which is the acceleration. For a rocket moving upward near the surface of the earth, $F_{ext} = -mg$. Setting $dP/dt = F_{ext} = -mg$ gives us the **rocket equation:**

$$m\frac{dv}{dt} = Ru_{ex} + F_{ext} = Ru_{ex} - mg \qquad \text{8-37}$$

Rocket equation

or

$$\frac{dv}{dt} = \frac{Ru_{ex}}{m} - g = \frac{Ru_{ex}}{m_0 - Rt} - g \qquad \text{8-38}$$

The quantity Ru_{ex} is the force exerted on the rocket by the exhausting fuel. This is called the **thrust**:

$$F_{th} = Ru_{ex} = \left|\frac{dm}{dt}\right| u_{ex} \tag{8-39}$$

Definition—Rocket thrust

Equation 8-38 is solved by integrating both sides with respect to time. For a rocket starting at rest at $t = 0$, the result is

$$v = -u_{ex} \ln\left(\frac{m_0 - Rt}{m_0}\right) - gt \tag{8-40}$$

as can be verified by taking the time derivative of v. The **payload** of a rocket is the final mass, m_f, after all the fuel has been burned. The **burn time** t_b is given by $m_f = m_0 - Rt_b$, or

$$t_b = \frac{m_0 - m_f}{R} \tag{8-41}$$

Thus, a rocket starting at rest with mass m_0, and payload of m_f, attains a final speed

$$v_f = -u_{ex} \ln \frac{m_f}{m_0} - gt_b \tag{8-42}$$

Final speed of rocket

assuming the acceleration of gravity to be constant.

ction & Fields ; We shall come back to this point after we have studied the gravitional field.

vitational interaction

Gravitational Interaction

$$\vec{F} = G\frac{mm'}{r^2}\hat{r}$$

$m \; \vec{F} \qquad \vec{F} \; m'$

$\vec{F} = m\vec{a}$

$\Rightarrow$ describe the motion of the system if initial conditions are given.

iscovery of the law of gravitation

Understanding of planetary motion.

Greek, describe planetry motion relative to a frame of reference attached to the earth.

Copernius " " " " " " " to the sun.
(16 th century) $\Rightarrow$ motion of the planets had a simple description

Now we know the reason why the sun is a better choice.

i.e. the sun, the largest body in our planetary system $\Rightarrow$ it is practically

$M_{\odot} \gtrsim 10^3 M_{\text{any planet}}$

coincident with the center of mass system. This justifies its choice as center of reference, since it is, practically, an inertial frame.

Tycho Brahe. these debats about the nature of the motion of the planets would best be resolved if the actual position of the planets were measured sufficiently accurately.

To find something out, it is better to perform some careful experiments than to carry on deep philosophical arguments.

$\Rightarrow$ made many, many observations

Kepler's three laws;

(1) The planets describe elliptical orbits; with the sun at one focus.

(2) The position vector of any planet relative to the sun sweeps out equal areas of its ellipse in equal times (Law of area)

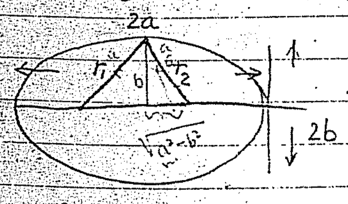
(3) $P^2 = k\, r_{ave}^3$

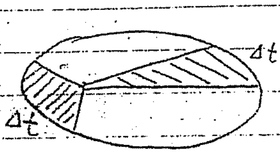
period of revolution.

$r_1 + r_2 = \text{const}$

$2\pi\sqrt{\frac{6.38\times10^3\times10^3}{9.8}}$

$= 2\pi\sqrt{\frac{6.38}{9.8}}\,10^3$





$\vec{F}=0$

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ r_1+r_2=2a \end{cases}$$

Galileo ; principle of inerta ; if something is moving with nothing touching it and completely undistribed, it will go on forever, coasting at a uniform speed in a straight line.

分類：
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總號：

eivton: force ↳ change the motion of a body

Force is needed to change the direction of motion of a body

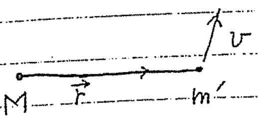
Change the direction of motion of a body sideways

⇒ a force has been applied

Stone attached to the string and is whirling around in a circle

⇒ we have to pull on the string

ntral force ⇒ the force needed to control the motion of a planet around the sun is not a force around the sun but toward the sun.



$\vec{r}_{init}$ and $\vec{v}_{init}$ defines a plane

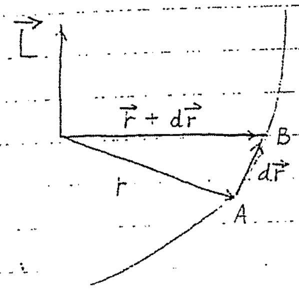
$\vec{r}\times\vec{v}$ is $\perp$ to both

No components of $\vec{r}$ and $\vec{v}$ in the direction of $\vec{r}\times\vec{v}$ (let called z)

$M \gg m'$

$\vec{F} \parallel \vec{r}$ ⇒ no acceleration along $\vec{r}\times\vec{v}$ direction

⇒ $\vec{r}, \vec{v}$ always only have components in the plane $\perp$ to z axis



$$d\vec{A} = \frac{1}{2}\,\vec{r}\times d\vec{r}$$

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\,\vec{r}\times\frac{d\vec{r}}{dt} = \text{const}$$

$$\vec{r}\times\vec{v} = \text{const}$$

$$\vec{L} = \text{const}$$

⇒ requires $\vec{\tau} = \vec{r}\times\vec{F} = 0$

This is satisfied if the force is central

Low of force

Particular case of an ellipse is a circle

$$F = \frac{mv^2}{r} \qquad P = \frac{2\pi r}{v}$$

↑ period

$$v = \frac{2\pi r}{P}$$

$$F = \frac{m\cdot\left(\frac{2\pi r}{P}\right)^2}{r}$$

$$= \frac{m\,\frac{4\pi r^2}{P^2}}{r} = \frac{m\,4\pi r}{P^2}$$

Kepler's third law — $P^2 = kr^3$

$$\Rightarrow F = \frac{m\,4\pi r}{kr^3}$$

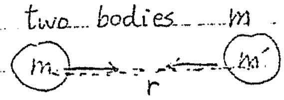
$$F \propto \frac{1}{r^2}$$

⇒ force is central inversely proportional to r^2

Universal gravitational interaction apply to any two bodies m and m'

$$F = \gamma\,\frac{mm'}{r^2}$$

↑ gravitional constant



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gravitional mass and inertial mass

So far we have two definition of mass.

Let us discuss in some detail

$$F = \gamma \frac{m_g m_g'}{r^2}$$ m_g = ability to produce gravitional force
similar to charge

gravitional mass

How to measure

$\cdot m_{g_1}$ $\cdot m_{g_2}$ $$\frac{F_{g_1}}{F_{g_2}} = \frac{m_{g_1}}{m_{g_2}} \quad (1)$$

$\cdot M$

Inertial mass $F = m_i a$ can be measured if F is known.

Take the M = earth

$a_1 = a_2 = g \rightarrow$ experimental result

$$F_{g_1} = m_{i_1} a_1$$

$$F_{g_2} = m_{i_2} a_2$$

$$\frac{F_{g_1}}{F_{g_2}} = \frac{m_{i_1}}{m_{i_2}} \quad (2)$$

Compare (1) and (2)

$$\frac{m_{g_1}}{m_{g_2}} = \frac{m_{i_1}}{m_{i_2}}$$

Choose $m_{g_1} = K m_{i_1}$

then $m_{g_2} = K m_{i_2}$

We can choose the unit such that $m_i = m_g = m$

$$\Rightarrow F = \gamma \frac{mm'}{r^2}$$

$$\gamma = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$$

Cavendish experiment : support the idea of universality

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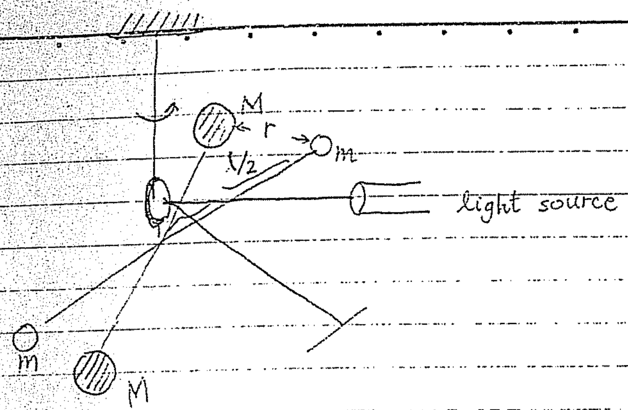


$\tau = k\theta$ analogy to $F = kx$

torsional constant

$\tau = 2\ \dfrac{\gamma Mm}{r^2}\cdot\dfrac{l}{2} = Fl = k\theta$

$P = 2\pi\sqrt{\dfrac{I}{k}}$ $\gamma = \dfrac{k\theta r^2}{Mml}$

period of torisonal oscillation

$I = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2$

Thus, one can first check the law of gravitiona

(1) M doubled θ doubled

(2) m " θ "

(3) r doubled θ is reduced by a factor of $\frac{1}{4}$

$\Rightarrow$ can also be used to determine γ

Gravitational potential energy

⇒ It is a conservative force

$\int_1^2 \vec{F} \cdot d\vec{s}$ is independent of path

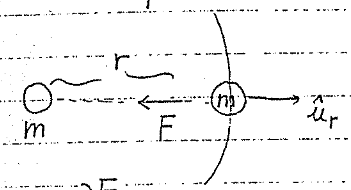
⇒ usefulness of the concept of potential energy

Advantages:

1) It is a scalar quantity ⇒ easier to handle.

2) It is most useful in discussing the boundness of the motion

$$\vec{F} = -\gamma \frac{mm'}{r^2} \hat{u}_r$$



move along the circle

⇒ no work has to be done

⇒ they have the same potential energy

⇒ potential energy is a function of r only

$$F_r = -\frac{\partial E_p}{\partial r}$$

$$\gamma \frac{mm'}{r^2} = \frac{dE_p}{dr}$$

$$E_p = -\gamma mm' \frac{1}{r} + C$$

Set $E_p = 0$ at $r = \infty$

$$C = 0$$

$$\therefore \quad E_p = -\gamma \frac{mm'}{r}$$

note $\frac{1}{r}$ potential energy

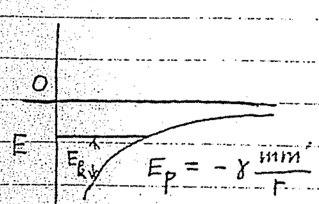
$$E = \frac{1}{2} m v^2 + \frac{1}{2} m' v'^2 - \frac{\gamma mm'}{r}$$

$$E = \sum_{\text{All particles}} \frac{1}{2} m_i v_i^2 - \sum_{\text{All pairs}} \frac{\gamma m_i m_j}{r_{ij}}$$

Boundness of the motion

Assume: $m' \gg m$

$$E = \frac{1}{2} m v^2 - \frac{\gamma mm'}{r}$$



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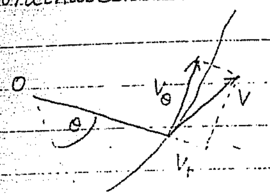
ation of orbits in inverse-square-law force fields

$$L = mr^2 \frac{d\theta}{dt}$$

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} + E_p(r)$$

ar coordinates



$$\vec{V} = \vec{V}_r + \vec{V}_\theta$$

$$\vec{L} = m\vec{r} \times \vec{V} = m\vec{r} \times \vec{v}_\theta$$

$$|\vec{v}_\theta| = r\frac{d\theta}{dt} \qquad |\vec{v}_r| = \frac{dr}{dt}$$

$$L = mr^2 \frac{d\theta}{dt}$$

$$E = \frac{1}{2} m V^2 + E_p(r) = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} mr^2 \left(\frac{d\theta}{dt}\right)^2 + E_p(r)$$

$$= \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{\alpha}{r} \qquad \boxed{\alpha \equiv \gamma m m'}$$

$$\Downarrow$$

$$\boxed{d\theta = \frac{L}{mr^2} dt} \qquad \frac{dr}{dt} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2\alpha}{m}}$$

$$dt = \frac{dr}{\sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2\alpha}{mr}}}$$

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}} \sin^{-1} \frac{bx+c}{x(b^2-4ac)^{1/2}} \qquad \begin{bmatrix} c < 0 \\ b^2 > 4ac \end{bmatrix}$$

$$d\theta = \frac{L\, dr}{mr^2 \sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2\alpha}{mr}}}$$

$c = 0$

$$\Rightarrow \int \frac{dx}{x^{3/2}\sqrt{ax+b}}$$

$a = 0$

$$\Rightarrow \int \frac{dx}{x\sqrt{ax+b}}$$

$$\therefore \theta - \theta_0 = \frac{L}{m} \int \frac{dr}{r^2 \sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2\alpha}{mr}}}$$

$$= \frac{L}{m} \int \frac{dr}{r\sqrt{\frac{2E}{m} r^2 + \frac{2\alpha}{m} r - \frac{L^2}{m^2}}}$$

Look up the integration table $\Rightarrow$

Let $\beta = \dfrac{L^2}{m\alpha}$ $\qquad \varepsilon \equiv \sqrt{1 + 2E\beta/\alpha} \leftarrow$ eccentricity

$$\theta - \theta_0 = \int \frac{\beta\, dr}{\varepsilon r^2 \sqrt{1 - [1/\varepsilon - (\beta/\varepsilon r)]^2}}$$

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pf: $\theta - \theta_0 = \int \frac{\beta\, dr}{\varepsilon r^2 \sqrt{1-(\frac{1}{\varepsilon} - \beta/\varepsilon r)^2}}$

$$= \int \frac{L}{m\alpha} \frac{dr}{r^2\left[\varepsilon^2 - (1-\frac{\beta}{r})^2\right]^{1/2}}$$

$$= \frac{L}{m} \int \frac{1}{r^2\left[\frac{\alpha^2}{L^2}(\varepsilon^2 - (1-\frac{\beta}{r})^2)\right]^{1/2}}$$

Compare the coefficient inside the square root.

(1) $\frac{\alpha^2}{L^2}(\varepsilon^2 - 1) = \frac{\alpha^2}{L^2}\left(\frac{2E\beta}{\alpha}\right) = \frac{\alpha^2}{L^2} \cdot 2E \cdot \frac{L^2}{m\alpha} = \frac{2E}{m}$

$(\frac{1}{r})$ $2\frac{\alpha^2}{L^2}\frac{\beta}{r} = 2\frac{\alpha^2}{L^2}\frac{(\frac{L^2}{m\alpha})}{r} = \frac{2\alpha}{mr}$

$(\frac{1}{r^2})$ $-\frac{\alpha^2}{L^2}\frac{\beta^2}{r^2} = -\frac{\alpha^2}{L^2}\frac{\frac{L^4}{m^2\alpha^2}}{r^2} = -\frac{L^2}{m^2 r^2}$

$$\theta - \theta_0 = \int \frac{L\, dr}{mr^2\sqrt{\frac{2E}{m} + \frac{2\alpha}{mr} - \frac{L^2}{m^2r^2}}}$$

Define $x = \frac{1}{\varepsilon} - \frac{\beta}{\varepsilon r}$

$dx = \frac{\beta}{\varepsilon r^2} dr$

$dr = \frac{\varepsilon r^2}{\beta} dx$

$\theta - \theta_0 = \int \frac{\beta}{\varepsilon r^2} \cdot \frac{\varepsilon r^2}{\beta} \frac{dx}{\sqrt{1-x^2}}$

$= \int \frac{dx}{\sqrt{1-x^2}}$

Define $x = \sin y$

$dx = \cos y\, dy = \sqrt{1-x^2}\, dy$

$dy = \frac{dx}{\sqrt{1-x^2}}$

$\theta - \theta_0 = \int dy = y = \sin^{-1} x = \sin^{-1}\left(\frac{1}{\varepsilon} - \frac{\beta}{\varepsilon r}\right)$

$\Rightarrow \sin(\theta - \theta_0) = \frac{1}{\varepsilon} - \frac{\beta}{\varepsilon r}$

$\frac{\beta}{r} = 1 - \varepsilon \sin(\theta - \theta_0)$

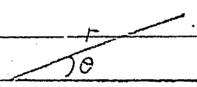
$r(\theta) = \frac{\beta}{1 - \varepsilon \sin(\theta - \theta_0)}$

$\varepsilon < 1$ ellipse

$\varepsilon > 1$ hyperbola, denominator $\to 0$ $\Rightarrow r \to \infty$

Require that $\theta = 0°$ to get the largest $r(0) = \frac{\beta}{1 + \varepsilon \sin\theta_0}$

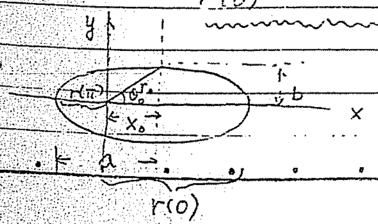
It is obvious that it is maximum when $\theta_0 = -\frac{\pi}{2}$



$r(\theta) = \frac{\beta}{1 - \varepsilon \sin(\theta + \frac{\pi}{2})}$

$= \frac{\beta}{1 - \varepsilon \cos\theta}$

$\Rightarrow$ $r(\theta) = \frac{\beta}{1 - \varepsilon\cos\theta}$



$x = r\cos\theta$ $\quad \begin{cases} x_0 = r_0 \cos\theta_0 \\ y_0 = r_0 \sin\theta_0 \end{cases}$

$y = r\sin\theta$

$x' = x - x_0$

$y' = y - y_0$

Want to show $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$

Major axis: $a = \frac{1}{2}\left[r(0) + r(\pi)\right]$ $\quad r(0) = \frac{\beta}{1-\varepsilon}, \quad r(\pi) = \frac{\beta}{1+\varepsilon}$

$$= \frac{1}{2}\beta\left[\frac{1}{1-\varepsilon} + \frac{1}{1+\varepsilon}\right]$$

$$= \frac{\beta}{1-\varepsilon^2}$$

$b \rightarrow$ minor axis

$$x_0 = a - r(\pi) = \frac{\beta}{1-\varepsilon^2} - \frac{\beta}{1+\varepsilon}$$

$$= \frac{\beta\varepsilon}{1-\varepsilon^2}$$

$$= r_0 \cos\theta_0 = \underbrace{\frac{\beta}{1-\varepsilon\cos\theta_0}}_{r_0} \cos\theta_0$$

from the equation of the orbit

$$\frac{\beta\varepsilon}{1-\varepsilon^2} = \frac{\beta}{1-\varepsilon\cos\theta_0}\cos\theta_0$$

It is thus obvious that $\underline{\varepsilon = \cos\theta_0}$

$$y_0 = b = r_0 \sin\theta_0$$

$$= \frac{\beta}{1-\varepsilon\cos\theta_0}\sqrt{1-\cos^2\theta_0}$$

$$= \frac{\beta}{1-\varepsilon^2}\sqrt{1-\varepsilon^2} = \frac{\beta}{\sqrt{1-\varepsilon^2}}$$

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Proof: $\left[\frac{1}{a}(x - x_0)\right]^2$

$$= \left\{\left[\underbrace{\frac{\beta\cos\theta}{1-\varepsilon\cos\theta}}_{x} - \underbrace{\frac{\beta\varepsilon}{1-\varepsilon^2}}_{x_0}\right] \Big/ \frac{\beta}{1-\varepsilon^2}\right\}^2$$

$$= \left\{(1-\varepsilon^2)\frac{\cos\theta}{1-\varepsilon\cos\theta} - \varepsilon\right\}^2 = \underset{(i)}{\frac{(1-\varepsilon^2)^2\cos^2\theta}{(1-\varepsilon\cos\theta)^2}} - \underset{(ii)}{\frac{2\varepsilon(1-\varepsilon^2)\cos\theta}{1-\varepsilon\cos\theta}} + \underset{(iii)}{\varepsilon^2}$$

$$\left[\frac{y'}{b}\right]^2 = \left[\frac{\beta\sin\theta}{1-\varepsilon\cos\theta} \Big/ \underset{b}{\frac{\beta}{\sqrt{1-\varepsilon^2}}}\right]^2$$

$y' = y = r\sin\theta$

$$= \underset{(iv)}{(1-\varepsilon^2)\frac{\sin^2\theta}{(1-\varepsilon\cos\theta)^2}}$$

$$(i) + (iv) = \frac{1}{(1-\varepsilon\cos\theta)^2}\left[\cos^2\theta - 2\varepsilon^2\cos^2\theta + \varepsilon^4\cos^2\theta + \sin^2\theta - \varepsilon^2\sin^2\theta\right]$$

$$= \frac{1}{(1-\varepsilon\cos\theta)^2}\left[1 - \varepsilon^2 - \varepsilon^2\cos^2\theta + \varepsilon^4\cos^2\theta\right]$$

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$$= \frac{1}{(1-\varepsilon\cos\theta)^2}\left[1-\varepsilon^2-\varepsilon^2(1-\varepsilon^2)\cos^2\theta\right]$$

$$= \frac{1}{(1-\varepsilon\cos\theta)^2}(1-\varepsilon^2)\left[1-\varepsilon^2\cos^2\theta\right]$$

$$= \frac{(1-\varepsilon^2)(1+\varepsilon\cos\theta)}{(1-\varepsilon\cos\theta)}$$

(v)

$$(ii)+(v) = \frac{1}{1-\varepsilon\cos\theta}\left\{-2\varepsilon\cos\theta+2\varepsilon^3\cos\theta+1-\varepsilon^2+\varepsilon\cos\theta-\varepsilon^3\cos\theta\right]$$

(i)'' + (iv)

$$= \frac{1}{1-\varepsilon\cos\theta}\left[1-\varepsilon\cos\theta-\varepsilon^2(1-\varepsilon\cos\theta)\right]$$

$$= (1-\varepsilon^2)$$

$$\therefore (i)+(ii)+(iii)+(iv) = 1-\varepsilon^2+\varepsilon^2 = 1$$

$$\Rightarrow \left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2 = 1$$

$$a = \frac{\beta}{1-\varepsilon^2} = \frac{\beta}{\frac{2E\beta}{\alpha}} \qquad \varepsilon = \sqrt{1+2E\beta/\alpha}$$

$$= -\frac{\alpha}{2E} = \frac{\alpha}{2|E|}$$

$$b = \frac{\beta}{\sqrt{1-\varepsilon^2}} = \frac{L^2/m\alpha}{\sqrt{\frac{2EL^2/m\alpha}{\alpha}}} = \frac{L}{\sqrt{2m|E|}}$$

$E<0$

$\varepsilon = \cos\theta_0 \qquad \therefore 0<\varepsilon<1$

$\varepsilon = 0 \Rightarrow a=b$ circle

$E>0$ Go back to

$$\theta-\theta_0 = \int\frac{L\,dr}{mr^2\sqrt{\frac{2E}{m}+\frac{2\alpha}{mr}-\frac{L^2}{m^2r^2}}}$$

Here $E>0$, carry out the integration with $\varepsilon>1$

$\Rightarrow$ hyperbola

Two branches

$a/b = \sqrt{1-\varepsilon^2}$

$E=0 \Rightarrow \varepsilon=1$

$\Rightarrow a, b \Rightarrow \infty$

$\Rightarrow$ parabola

$a^2-b^2 = \frac{\beta^2}{1-\varepsilon^2}\left[\frac{\varepsilon^2}{1-\varepsilon^2}\right]$

lation between L, E and a, ε

$$a = \frac{\beta}{1-\varepsilon^2}$$

$$\beta = \frac{L^2}{m\alpha}$$

$$a = \frac{\frac{L^2}{m\alpha}}{1-\varepsilon^2}$$

$$\Rightarrow L^2 = m\alpha(1-\varepsilon^2)\,a$$

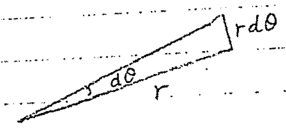
$$a = \frac{\alpha}{2|E|} \quad \Rightarrow |E| = \frac{\alpha}{2a}$$

epler's third law is valid for elliptical orbits

$$mr^2\frac{d\theta}{dt} = L$$

$$r^2\,d\theta = \frac{L}{m}\,dt$$

$$\text{area} = \frac{1}{2}r\cdot r\,d\theta = \frac{1}{2}r^2\,d\theta$$



$$\therefore \text{Total area} = \int_0^{2\pi}\frac{1}{2}r^2\,d\theta = \frac{L}{2m}\int_0^{P}dt = \frac{P\,L}{2m}$$

see bottom of the page

$$= \frac{1}{2}\int_0^{2\pi}\frac{\beta^2}{(1+\varepsilon\cos\theta)^2}\,d\theta = \frac{\pi\beta^2}{(1-\varepsilon^2)^2}\sqrt{1-\varepsilon^2} = \pi a^2\sqrt{1-\varepsilon^2}$$

Area of the ellipse

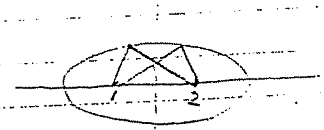
$$S = \pi ab = \pi a^2\sqrt{1-\varepsilon^2}$$

$$\therefore \pi^2 a^4(1-\varepsilon^2) = \frac{P^2L^2}{4m^2} = \frac{P^2}{4m^2}\,\underbrace{m\alpha(1-\varepsilon^2)a}_{L^2}$$

$$\pi^2 a^3 = \frac{1}{4}\gamma m' P^2$$

m': mass of the sun

$$\langle r\rangle \propto a$$



This can be seen from symmetry consideration

$r_1 + r_2 = 2a$ defines an ellipse

$$r_1 + r_2 = 2a$$

From symmetry consideration

$$\langle r\rangle = a$$

$$\therefore P^2 \propto \langle r\rangle^3 \quad \text{(This is Kepler's third law)}$$

$$\frac{1}{2}\int_0^{2\pi}\frac{\beta^2}{(1+\varepsilon\cos\theta)^2}\,d\theta = \frac{1}{2}\beta^2\frac{\varepsilon\sin x}{(\varepsilon^2-1)(1+\varepsilon\cos\theta)}\Big|_0^{2\pi} - \frac{1}{2}\beta^2\frac{1}{\varepsilon^2-1}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

$$= \frac{1}{2}\frac{\beta^2}{1-\varepsilon^2}\cdot\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} = \frac{1}{2}\beta^2\frac{1}{1-\varepsilon^2}\cdot\frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\frac{(1-\varepsilon)\tan\frac{\theta}{2}}{\sqrt{1-\varepsilon^2}}\Big|_0^{2\pi}$$

$$= \frac{1}{2}\beta^2\frac{2\pi}{(1-\varepsilon^2)^2}\sqrt{1-\varepsilon^2} = \frac{\pi\beta^2}{(1-\varepsilon^2)^2}\sqrt{1-\varepsilon^2}$$

After we have established the fact that the orbit is an ellipse we only have to show that $x_0^2+y_0^2=a^2$ to prove that the origin of the polar coordinate (i.e., the sun) is at one focus

$$x_0^2+y_0^2=\frac{\beta^2\varepsilon^2}{(1-\varepsilon^2)^2}+\frac{\beta^2}{(1-\varepsilon^2)}$$

$$=\frac{\beta^2\varepsilon^2+\beta^2-\beta^2\varepsilon^2}{(1-\varepsilon^2)^2}=\frac{\beta^2}{(1-\varepsilon^2)^2}=a^2$$

Thus the proof

Perturbation of planetary motion

Assumption in previous discussion: motion of a planet around the sun was not affected by the other planets and heavenly bodies

Presence of other planets introduce perturbations in a planet's orbit.

These perturbations ⇒ celestial mechanics

(i) advance of the perihelion: elliptical orbit of a planet is not closed but the major axis of the ellipse rotates very slowly around the focus when the sun is located.

(ii) periodic variation of the eccentricity of the ellipse about its average value tational Field

ribe the interactions, we introduce the concept of field.

d ⇒ physical property extended over a region of space and described by a function of position and time

iteraction between particles,

) Particle produce around it a corresponding field
this field in turn acts on a second particle to produce the required interaction.

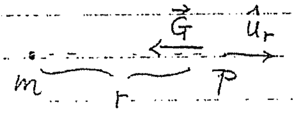
i) The second particle produces its own field, which acts on the second particle resulting in a mutual interaction

Other type of field: pressure field, temperature, density field etc.

Vector field, scalar field

$$\vec{F} = -\frac{\gamma m m'}{r^2}\hat{u}_r$$

$$\vec{G}_P = \frac{\vec{F}}{m'} = -\frac{\gamma m}{r^2}\hat{u}_r$$



Several field

$$\vec{G} = \vec{G}_1 + \vec{G}_2 + \vec{G}_3 + \cdots$$

Line of force

direction of the field is tangent to the line that passes through the point

density $\propto$ strength of the field

Examples: See PP. 376 - 377 for illustration

Experimental method of defining a field

⇒ use of test body

avitional potential:

$$V = \frac{E_p}{m'}$$ at a certain point in a gravitional field a mass m' has a potential energy E_p ⇒ gravitional potential V

$$V = -\frac{\gamma m}{r}$$

Several sources

$$V = -\gamma \sum_i \frac{m_i}{r_i}$$

$$\vec{F} = -\nabla E_p \Rightarrow F_s = -\frac{\partial E_p}{\partial s}$$

$$\vec{F} = m'\vec{G} \qquad E_p = mV$$

$$\vec{G} = -\nabla V \Rightarrow G_s = -\frac{\partial V}{\partial s}$$

G_s = component of $\vec{G}$ in the direction of the displacement ds.

Advantage: scalar field, easy to handle

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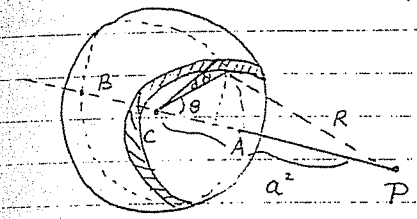
Equipotential surface: joining the points at which the gravitational potential has the same value.

Equipotential surfaces are $\perp$ to the line of force $\Rightarrow$ the proof is obvious.

Gravitational Field due to a spherical body

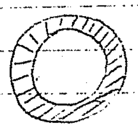
Spherical shell with mass m



m in the strip

$R \equiv$ distance from strip from point P

$m = \underbrace{\text{mass per unit}}_{\frac{m}{4\pi a^2}}$ area $\cdot$ area



area = length of the circle $\times$ width

$\quad 2\pi a \sin\theta \qquad\qquad \widetilde{a\,d\theta}$

mass of the strip $= 2\pi a \sin\theta \; a\,d\theta \cdot \dfrac{m}{4\pi a^2}$

$= \frac{1}{2} m \sin\theta \, d\theta$

$$dV = -\frac{\gamma \cdot \frac{1}{2} m \sin\theta \, d\theta}{R}$$

$$R^2 = a^2 + r^2 - 2ar\cos\theta$$

$$2R\,dr = 2ar\sin\theta\,d\theta$$

$$\sin\theta\,d\theta = \frac{R\,dR}{ar}$$

$$dV = -\frac{\gamma m}{2ar}\,dR$$

$$V = -\int_{r-a \to A}^{r+a \to B} \frac{\gamma m}{2ar}\,dR$$

$$= -\frac{\gamma m}{r} \qquad \text{for } r > a \quad \Rightarrow \vec{G} = -\frac{\gamma m}{r^2}\hat{u}_r$$

$$V = -\int_{a-r}^{a+r} \frac{\gamma m}{2ar}\,dR$$

$$= -\frac{\gamma m}{a} \qquad \text{for } r < a \quad \Rightarrow \vec{G} = 0$$

For a solid sphere

Uniform solid sphere

$r > a$

$$\vec{G} = \sum \text{shell} = -\hat{u}_r \frac{\gamma}{r^2} \underbrace{\sum_{\text{shell}} m}_{\text{~~total mass~~ of the sphere}} = -\frac{\gamma M}{r^2}\hat{u}_r$$

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$\Rightarrow$ for particle outside the sphere, the field is the same as the field produced by a mass M located at the center of the sphere

$r < a$

$\vec{G} = \sum$ shell contribution

$$= -\hat{u}_r \frac{\gamma}{r^2} \sum_{\substack{\text{shells with} \\ \text{radius less} \\ \text{than } r}} m$$

Total mass $= M = \rho \cdot \frac{4}{3}\pi a^3$

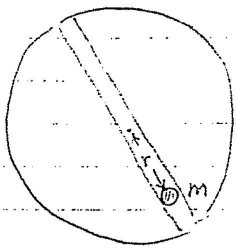
Mass with radius $< r = \rho \cdot \frac{4}{3}\pi r^3$

$$\therefore \sum_{\substack{\text{shells with} \\ \text{radius} < r}} m = \frac{M}{\frac{4}{3}\pi a^3} \frac{4}{3}\pi r^3 = M \frac{r^3}{a^3}$$

$$\vec{G} = -\hat{u}_r \frac{\gamma}{r^2} M \frac{r^3}{a^3}$$

$$= -\frac{\gamma M r}{a^3} \hat{u}_r$$

$\vec{G}$ is the same if $r = a$, which can be readily checked



$$F = -m' \frac{\gamma M r}{a^3} \hat{u}_r \quad \Rightarrow \text{ simple harmonic motion}$$

$$= -k r m \hat{u}_r \qquad M = \text{mass of the earth}$$

$$\Rightarrow \quad k = \frac{\gamma M m'}{a^3}$$

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m'}{m\gamma M/a^3}} = 2\pi \sqrt{\frac{a^3}{M\gamma}}$$

$$= 2\pi \sqrt{\frac{(6.37 \times 10^6 \text{ m})^3}{5.98 \times 10^{24} \cdot 6.67 \times 10^{-11}}} \text{ sec}$$

$$= 2\pi \sqrt{\frac{(6.37)^3}{5.98 \times 6.67} 10^5} \text{ sec}$$

$$= 2\pi \sqrt{\frac{(6.37)^3 \times 10^{-1}}{5.98 \times 6.67}} \times 10^3 \text{ sec}$$

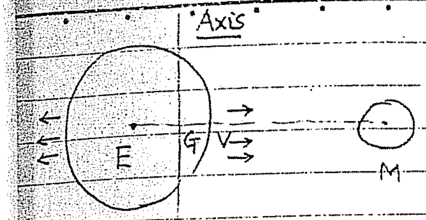
$$= 5050 \text{ sec} = 84.2 \text{ min}$$

Time for delivery of the mail $= 42.1$ min

Note the result is independent of mass.

Tide 潮汐

Earth & Moon be the only pair of bodies in existence



$F(r) \propto \frac{1}{r^2}$

$\frac{\partial F}{\partial r} \propto \frac{1}{r^3}$

$\frac{1}{60^2}$

* $\frac{1}{59^2} - \frac{1}{60^2} = 0.000,009.49 \sim \frac{2}{60^3}$ $\boxed{\frac{2}{60^3}}$ $\boxed{\Delta F = \frac{\partial F}{\partial r}\Delta R}$

Tide

Why the effect of the moon is more important than the effect of the sun

Pre-Newtonian reasonings

(i) The moon pulls the water up under it and mades the tide and since the earth spins underneath that makes the tide at one station go up and down every 24 hours

Actually the tide goes up and down in 12 hours.

ii) High tide should be on the other side of the earth because, so they argued, the moon pulls the earth away from the water!

Actually it works like this

The pull of the moon for the earth and for the water is "balanced" at the center. But the water which is closer to the moon is pulled more than the average and which is far away from it is pulled less than the average

Furthermore the water can flow while the more rigid earth cannot

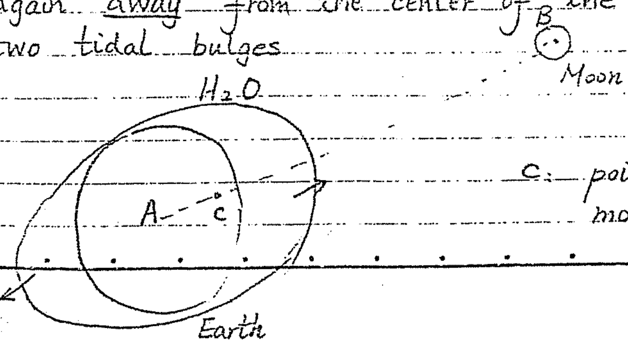
⇒ "tide" is due to the combination of the two things

Centrifugal force $= \frac{mv^2}{r} = \frac{mr^2\omega^2}{r} = \underline{mr\omega}$

The moon's attraction on the far side is weaker and the "centrifugal force" is stronger. The net result is an imbalance of the water in the direction away from the center of the earth.

On the near side, the attraction is stronger, the "centrifugal force" is weaker and the imbalance is in the opposite direction in space, but again away from the center of the earth

⇒ two tidal bulges



c: point around which earth and moon rotate

Chapter 8

GRAVITATION

According to Newton's law of universal gravitation each pair of particles in the universe is mutually attracted with a force proportional to the product of their masses, inversely proportional to the square of the distance between them, and directed along the line joining them. The proportionality constant G in the gravitational force law is known as Newton's constant. Although G is the least precisely measured fundamental constant, known only to one part in 10^4, its constancy is very well checked by careful analyses of solar system motions to better than one part in 10^{12} per year, which corresponds to a variation of no more than one percent over the age of the universe. Newtonian physics provides a nearly complete understanding of the motions of the planets, satellites, stars, galaxies and the universe as a whole. Indeed, it has only been in this century that a few tiny discrepancies have been uncovered whose explanation requires the more complete theory of gravity provided by Einstein's general relativity.

8.1 Attraction of a Spherical Body: Newton's Theorem

The statement of Newton's law of gravity applies to the attraction between two point masses, whereas celestial bodies are roughly spherical collections of particles. The theorem, first shown by Newton, that a spherically symmetric body acts as if its mass is concentrated at its center, is an essential step in the application of the law of gravitation to celestial mechanics. A corollary is that a particle located in a spherical mass distribution at a radius r from the center of the distribution experiences a net gravitational force only from the mass $M(r)$ within the radius r and the net force is as if $M(r)$ were located at $r = 0$. We give a proof of Newton's theorem using the concept of potential energy.

The gravitational potential energy between two point masses m and M separated by a distance r is

$$V(r) = -\frac{GMm}{r} \tag{8.1}$$

The corresponding force on m due to M is given by

$$\mathbf{F} = -\nabla V(r) = GMm\frac{d}{dr}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r} \tag{8.2}$$

where $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$. It is convenient to define the gravitational force on m as $\mathbf{F} = m\mathbf{g}$, where $\mathbf{g}$ is the *acceleration of gravity* at the position of m, independent of the value of m. (The fact that any mass at a given position in a gravitational field has the same acceleration $\mathbf{g}$ is known as the *equivalence principle.*) Correspondingly the gravitational potential energy is defined as $V = m\Phi$, where Φ is the gravitational *potential*, so

$$\mathbf{g} = -\nabla\Phi \tag{8.3}$$

From (8.2) the gravitational *potential* due to a mass M at a distance r is

$$\Phi(\mathbf{r}) = -\frac{GM}{r} \tag{8.4}$$

We first calculate the gravitational potential due to a uniform spherical shell of mass M at a distance R from the center of the shell, as illustrated in Fig. 8-1. To begin, we evaluate the potential energy of a circular ring element of mass dM shown in Fig. 8-1. If the radius of the shell is a, the surface mass density is

$$\sigma = \frac{M}{4\pi a^2} \tag{8.5}$$

The circular ring element has differential area $dA = 2\pi(a\sin\theta)(ad\theta)$ and mass

$$\begin{aligned} dM &= (2\pi a^2 \sin\theta d\theta)\sigma \\ &= \frac{M}{2}\sin\theta d\theta \end{aligned} \tag{8.6}$$

The distance r from dM to the point where the potential is being evaluated is given by the law of cosines

$$r^2 = a^2 + R^2 - 2aR\cos\theta \tag{8.7}$$

By differentiation we obtain

$$rdr = aR\sin\theta d\theta = 2aR\frac{dM}{M}$$

so that dM can be expressed as

$$dM = \frac{Mrdr}{2aR} \tag{8.8}$$

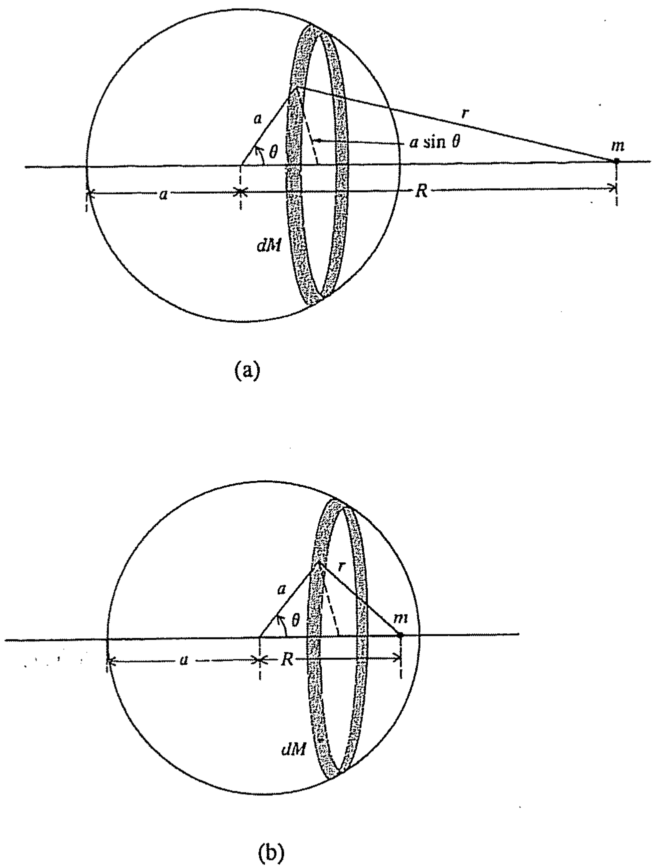


FIGURE 8-1. Gravitational attraction of a point mass m and a differential ring element dM on a spherical shell of mass M, with (a) m outside the shell and (b) m inside the shell.

The potential due to the ring mass dM is

$$d\Phi(r) = -\frac{G\,dM}{r} = -\frac{GM}{2aR}dr \tag{8.9}$$

The contributions of all ring elements on the shell are obtained by integration over r

$$\Phi(r) = \int_{r_{\min}}^{r_{\max}} d\Phi(r) = -\frac{GM}{2aR}(r_{\max} - r_{\min}) \tag{8.10}$$

We see from Fig. 8-1 that $r_{\max} = R+a$ and $r_{\min} = |R-a|$ and thus when r is outside the shell

$$r_{\max} - r_{\min} = (R+a) - (R-a) = 2a \tag{8.11}$$

whereas when m is inside the shell,

$$r_{\max} - r_{\min} = (R+a) - (a-R) = 2R \tag{8.12}$$

Thus the potential is

$$\Phi(R) = \begin{cases} -\dfrac{GM}{R} & R > a \qquad (8.13) \\[2ex] -\dfrac{GM}{a} & R < a \qquad (8.14) \end{cases}$$

Since $\Phi(r)$ is constant inside the shell, $\mathbf{g}$ vanishes there. When r is outside the shell, the potential in (8.13) is as if the mass M of the shell were concentrated at the center of the shell. Since a spherically symmetric solid body can be represented as a collection of concentric spherical shells, the gravitational force on m due to a spherical body is as if the total mass M were concentrated at the center of the sphere. Newton's theorem follows: the gravitational force of any spherically symmetric distribution of matter at a distance R from the center is the same as if all the mass within the sphere of radius R were concentrated at the center.

8.2 The Tides

When a body moves in a non-uniform gravitational field, it is subjected to tide-generating forces. These shearing forces may even tear the body apart—this is a possible origin of the rings of Saturn.

The acceleration of the body $\mathbf{a}_B$ is the total gravitational force on its component masses divided by its total mass. (If the body is spherically symmetric, then the result of Newton's theorem and the "action equals reaction" principle is that $\mathbf{a}_B$ is simply the value of $\mathbf{g}(\mathbf{r})$ at the center of the body.) If we use coordinates centered on the body (*i.e.*, "falling with the body") the gravitational field becomes $\mathbf{g}(\mathbf{r}) - \mathbf{a}_B$. If we separate $\mathbf{g}$ into the part due to the body itself $\mathbf{g}_{\text{self}}$ (which vanishes at the center of the body) and to the part due to external masses $\mathbf{g}_{\text{ext}}$, then the gravitational field in the frame fixed on the body is $\mathbf{g}_{\text{self}} + (\mathbf{g}_{\text{ext}} - \mathbf{a}_B)$. The second term, $(\mathbf{g}_{\text{ext}} - \mathbf{a}_B)$, is the tidal field.

Tidal forces on a planet are maximum along a line to the external force center and give two high tides on opposite sides of the planet. For a planet in a circular orbit about the sun the origin of the double tide is easily explained by the following argument. The forces acting on a mass m are the attractive gravitational force GmM/r^2 and the repulsive centrifugal force $m\omega^2 r$ due to the revolution of the planet about the sun. At the CM of the planet the gravity force exactly balances the centrifugal force since there is no radial acceleration in a circular orbit. At the point closest to the sun, the sun's gravitational attraction is larger than at the CM and the centrifugal force is smaller, giving a net tidal force in the direction of the sun. At the farthest point on the planet from the sun the centrifugal force exceeds that of gravity and there is a tidal force directed away from the sun.

The ocean tides on earth are caused by the variation from place to place of the gravitational attraction due to the moon and the sun. The atmosphere, the ocean, and the solid earth all experience tidal forces, but only the effects on the ocean are commonly observed. To estimate the gross features of the midocean tides, we begin with a static theory in which the rotation of the earth about its axis is neglected. The daily rotation of the earth will be invoked later to explain the propagation of the tides.

To calculate the tide-generating force, we consider the acceleration of a small mass m on the ocean's surface under the combined influence of the gravitational attraction of the earth and a distant mass M, as shown in Fig. 8-2. The coordinates of the masses m, M_E, M in an inertial frame are represented by the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively. For convenience, we denote the relative coordinates of the masses by

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{R} &= \mathbf{r}_2 - \mathbf{r}_3 \\ \mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_3 = \mathbf{R} + \mathbf{r} \end{aligned} \tag{8.15}$$

With this notation, the motion of m and M_E due to gravitational forces is determined by

$$m\ddot{\mathbf{r}}_1 = -\frac{GmM_E\hat{\mathbf{r}}}{r^2} - \frac{GmM}{d^2}\hat{\mathbf{d}} \tag{8.16}$$

$$M_E\ddot{\mathbf{r}}_2 = -\frac{GM_EM}{R^2}\hat{\mathbf{R}} \tag{8.17}$$

By dividing the first equation by m, the second equation by M_E, and then

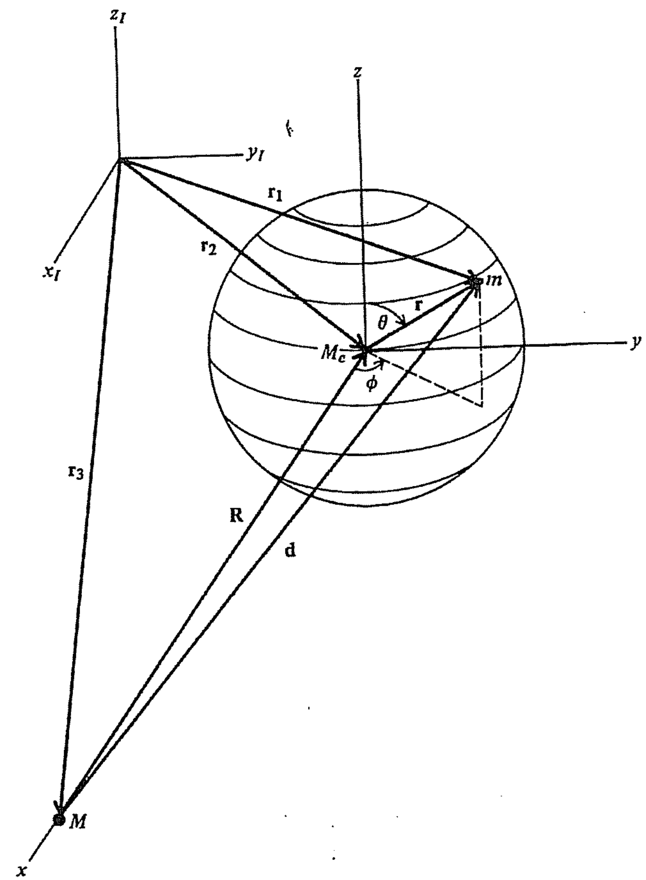


FIGURE 8-2. Location of a point on the earth's surface and a distant mass M in an inertial frame and an earth-centered frame.

subtracting, we find the equation of motion for the relative coordinate $\mathbf{r}$.

$$\ddot{\mathbf{r}} = -\frac{GM_E\hat{\mathbf{r}}}{r^2} - GM\left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{R}}}{R^2}\right) \tag{8.18}$$

This result could have been directly obtained from (6.22). The first term on the right-hand side of (8.18) is the central gravity force of the earth on a particle of unit mass. The second term is the tide-generating force per unit mass due to the presence of the distant mass M. The tide-generating force is the difference between the forces on the surface of the earth and at the center of the earth. The direction and relative magnitude of the tide-generating force due to M are plotted in Fig. 8-3 for points around the earth's equator. The effect of this force is to produce the two tidal bulges which, as the earth rotates, are observed twice daily as high tides.

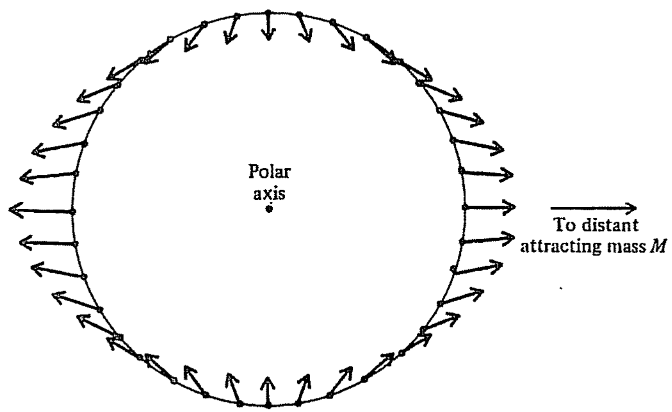


FIGURE 8-3. Tide-generating force on the surface of the earth at the equator due to a distant mass.

If the tidal forces are small compared to the gravitational force on the CM and the distance to the external force center is large compared to the planetary radius we can approximate (8.18) as follows. By (8.15) we can express the second factor of (8.18) as

$$\begin{aligned}\frac{\hat{\mathbf{d}}}{d^2}-\frac{\hat{\mathbf{R}}}{R^2}&=\frac{\mathbf{d}}{d^3}-\frac{\mathbf{R}}{R^3}=\frac{\mathbf{R}+\mathbf{r}}{d^3}-\frac{\mathbf{R}}{R^3}\\&=\mathbf{R}\left(\frac{1}{d^3}-\frac{1}{R^3}\right)+\frac{\mathbf{r}}{d^3}\end{aligned}\tag{8.19}$$

We form the square of d

$$\begin{aligned}d^2&=R^2+r^2+2\mathbf{R}\cdot\mathbf{r}\\d&=R\left(1+\frac{2\mathbf{R}\cdot\mathbf{r}}{R^2}+\frac{r^2}{R^2}\right)^{1/2}\end{aligned}\tag{8.20}$$

Then for $R \gg r$ we apply the binomial expansion $(1+\beta)^n \simeq 1+n\beta+\cdots$, with $\beta=\mathbf{R}\cdot\mathbf{r}/R^2$ and $n=1/2$, and retain only leading terms

$$d\simeq R\left(1+\frac{\mathbf{R}\cdot\mathbf{r}}{R^2}+\cdots\right)\tag{8.21}$$

The quantity d^{-3} in (8.19) can be approximated by

$$\begin{aligned}\frac{1}{d^3}&\simeq\frac{1}{R^3}\left(1-\frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\right)\\&=\frac{1}{R^3}-\frac{3\hat{\mathbf{R}}\cdot\mathbf{r}}{R^4}\end{aligned}\tag{8.22}$$

where the binomial expansion with $n=-3$ has been applied. To first order in $\mathbf{r}$ (8.19) becomes

$$\begin{aligned}\frac{\hat{\mathbf{d}}}{d^2}-\frac{\hat{\mathbf{R}}}{R^2}&=\mathbf{R}\left(\frac{1}{d^3}-\frac{1}{R^3}\right)+\frac{\mathbf{r}}{R^3}\\&\simeq-\frac{3(\mathbf{R}\cdot\mathbf{r})}{R^4}+\frac{\mathbf{r}}{R^3}\\&\simeq\frac{1}{R^3}\left[-3\hat{\mathbf{R}}\left(\hat{\mathbf{R}}\cdot\mathbf{r}\right)+\mathbf{r}\right]\end{aligned}\tag{8.23}$$

In our choice of coordinate system in Fig. 8-2, $\hat{\mathbf{R}}=-\hat{\mathbf{x}}$ and thus

$$\frac{\hat{\mathbf{d}}}{d^2}-\frac{\hat{\mathbf{R}}}{R^2}\simeq\frac{1}{R^3}\left(-3x\hat{\mathbf{x}}+\mathbf{r}\right)\tag{8.24}$$

In this approximation the tidal acceleration of (8.18) is

$$\ddot{\mathbf{r}}=-\frac{GM_E\hat{\mathbf{r}}}{r^2}+\frac{GM}{R^3}(3x\hat{\mathbf{x}}-\mathbf{r})\tag{8.25}$$

Since gravitational forces are conservative this force per unit mass can be derived from a potential and we may write

$$\ddot{\mathbf{r}}\equiv-\boldsymbol{\nabla}_{\mathbf{r}}\Phi\tag{8.26}$$

where $\boldsymbol{\nabla}_{\mathbf{r}}$ means the gradient with respect to the vector $\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ whose origin is at the center of the earth. It is easy to guess that the potential whose negative gradient is the right side of (8.25) is

$$\Phi=-\frac{GM_E}{r}-\frac{GM}{R^3}\left(\frac{3}{2}x^2-\frac{1}{2}r^2\right)\tag{8.27}$$

Since $x=r\sin\theta\cos\phi$, we have

$$\Phi=-\frac{GM_E}{r}-\frac{GM}{r}\left(\frac{r}{R}\right)^3\left(\frac{3}{2}\sin^2\theta\cos^2\phi-\frac{1}{2}\right)\tag{8.28}$$

For equilibrium of the ocean surface, the net tangential force on m must vanish. Equivalently, the potential at any point on the ocean's surface must be constant. We choose the constant to be $\Phi(\mathbf{r})=-GM_E/R_E$, where R_E is the undistorted spherical radius of the earth (i.e., when the distant M is absent). Using this condition in (8.28) gives

$$r - R_E = \frac{M}{M_E}\frac{r^3 R_E}{R^3}\left(\frac{3}{2}\sin^2\theta\cos^2\phi - \frac{1}{2}\right) \tag{8.29}$$

Since the height of the tidal displacement

$$h(\theta,\phi) \equiv r - R_E \tag{8.30}$$

is quite small compared with R_E, (8.29) gives

$$h(\theta,\phi) \simeq \frac{M}{M_E}\frac{R_e^4}{R^3}\left(\frac{3}{2}\sin^2\theta\cos^2\phi - \frac{1}{2}\right) \tag{8.31}$$

For a given colatitude angle θ in (8.31), the high tides occur at $\phi = 0$ and $\phi = \pi$, and low tides occur at $\phi = \pi/2$ and $\phi = 3\pi/2$. The difference in height between high and low tide, known as the *tidal range*, is

$$\Delta h = \frac{3}{2}\frac{M}{M_E}\frac{R_e^4}{R^3}\sin^2\theta \tag{8.32}$$

The tidal displacement h is largest at $\theta = 90°$ (on the equator). The tidal distortion is illustrated in Fig. 8-4. The tide for an ocean devoid of continents has a prolate spheroid shape (football-like), with the major axis in the direction of the distant mass. The calculation of such an ideal tide was first made by Newton in 1687.

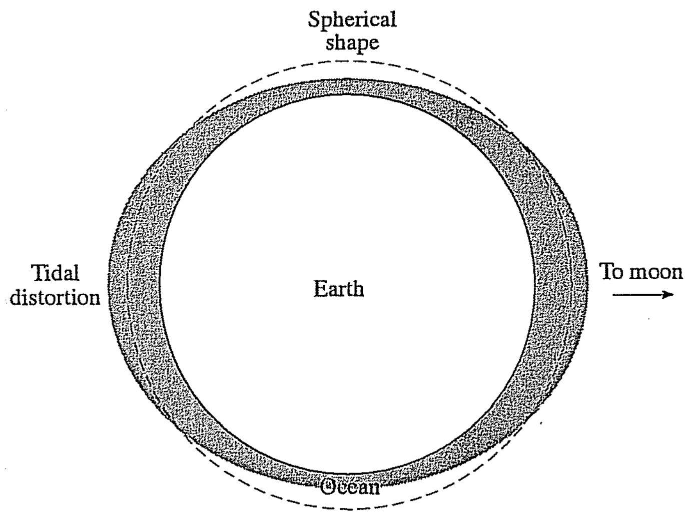


FIGURE 8-4. Tidal distortion at the earth's equator on an exaggerated scale.

The preceding discussion applies to the tidal forces induced by a single astronomic body. If there are two tide-producing bodies the net tide is the superposition of the separate tides. (If the bodies are not collinear with the planet, the total tidal shape is not axially symmetric but a triaxial ellipsoid instead.) From (8.31) the ratio of the maximum heights of the lunar (L) and solar ($\odot$) tides on earth is

$$\frac{h_L}{h_\odot} = \left(\frac{M_L}{M_\odot}\right)\left(\frac{a_E}{a_L}\right)^3 \tag{8.33}$$

where a_L is the earth-moon distance and a_E is the earth-sun distance. The numerical value of this ratio is

$$\frac{h_L}{h_\odot} = \frac{(1/81.5)M_E}{\left(\frac{1}{3}\times 10^6\right)M_E}\left(\frac{1.5\times 10^8\ \text{km}}{3.8\times 10^5\ \text{km}}\right)^3 = 2.2 \tag{8.34}$$

Thus the sun's tidal effect is smaller than the moon's, but it is not negligible. When the sun and moon are lined up (new or full moon), an especially large tide results (spring tide), and when they are at right angles (first or last quarter moon), their tidal effects partially cancel (neap tide). The diagram in Fig. 8-5 illustrates these orientations of the moon relative to the earth and sun.

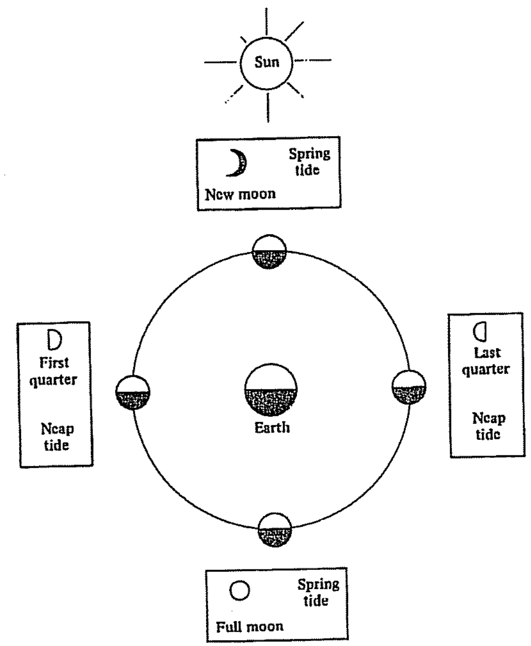


FIGURE 8-5. Relation of the phases of the moon to the tides on earth.

The tidal range due to the moon at a point on the earth-moon axis can be calculated from (8.32). We get

$$\Delta h\left(\theta=\frac{\pi}{2}\right)=\frac{3}{2}\left(\frac{1}{81.5}\right)\left(\frac{6,371}{384,000}\right)^3(6{,}371\times 10^3)=0.56\,\mathrm{m}$$

This figure agrees roughly with the measured tidal difference in midocean. As the earth rotates about its own axis, the tidal maxima, which lie on the earth-moon axis, will pass a given point on the earth's surface approximately two times a day. More precisely, since the orbital rotation of the moon about the earth (with period of $27\frac{1}{3}$ days) is in the same sense as the earth's own rotation (with period 24 h), two tidal maxima pass a given spot on earth every $(24 + 24/27\frac{1}{3})$ h. Thus high tide occurs every 12 h and 26.5 min, and high tide is observed about 53 min later each day.

The two high tides are not of the same height because of the inclination of the earth's axis to the normal of the moon's orbital plane about the earth. In the Northern Hemisphere the high tide which occurs closest to the moon is higher, as illustrated in Fig. 8-6.

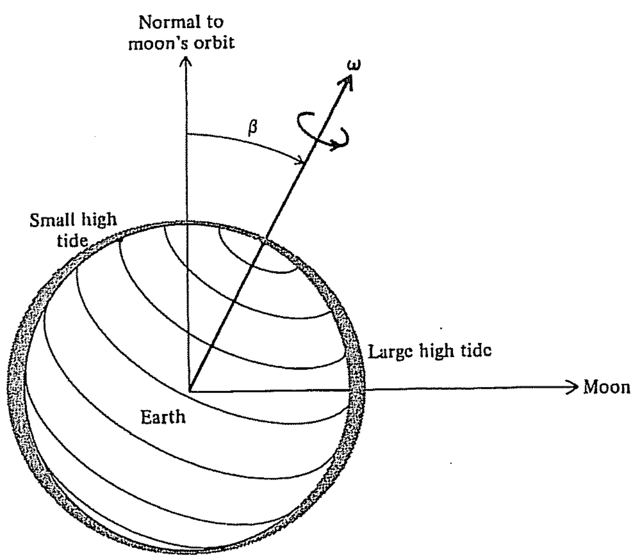


FIGURE 8-6. Effect of the inclination angle β of the earth's axis to the moon's orbital plane on the heights of tides. β varies from 17° to 29° as the moon's elliptical orbit precesses slowly about the normal to the plane of the earth's heliocentric orbit.

The tides are in reality more complicated than described above. Along coastal regions the configuration of the land masses and the ocean bottom cause considerable amplification or suppression of the tidal range. Over the world, tidal ranges vary as much as twenty meters.

The friction of the moving tidal waves against ocean bottoms and the continental shorelines dissipates energy at a rate estimated at 7 billion horsepower. To supply this energy, the earth's rotation about its axis slows down at the rate of 4.4×10^{-8} s per day. The cumulative time over a century is about 28 s. This gradual lengthening of the day is confirmed by the observation that various astronomical events such as eclipses seem to run systematically ahead of calculations based on observations over preceding centuries.

8.3 Tidal Evolution of a Planet-Moon System

The earth-moon system has very little external torque acting upon it on the average. The total angular momentum of the system is thus nearly constant. The consequence of angular momentum conservation is that the moon spirals outward about a half a centimeter each month as the earth's rotation is slowed by tidal friction. Ultimately the moon's distance will increase by over forty percent of its present value and our day will lengthen by a factor of about 50. The moon will then remain stationary above one spot on the earth.

To see this, we make the following simplifications, which are sufficiently accurate to represent the physical situation.

1. The spin angular momentum $\mathbf{S}=I\omega$ of the earth is parallel to the orbital angular momentum $\mathbf{L}$ of the moon about the earth. (The earth's spin precesses about the normal to the ecliptic plane with a period of 26,000 years and the plane of the moon's orbit about the earth precesses similarly with a period of about 19 years, so the average values of both $\mathbf{S}$ and $\mathbf{L}$ are perpendicular to the ecliptic plane—the plane of earth's orbit around the sun).
2. The total angular momentum

$$\mathbf{J}=\mathbf{L}+\mathbf{S}=(L+S)\hat{\mathbf{L}} \tag{8.35}$$

is constant (we are neglecting the solar tidal drag).

3. The moon's orbit about the earth is circular and lies in the ecliptic plane (point 1 above).

4. The moon is much less massive than the earth and the moon's spin angular momentum is negligible.

In a reference frame with the earth at rest at the origin, the energy of the earth-moon system is

$$E = \frac{1}{2}mv^2 - \frac{\alpha}{r} + \frac{1}{2}I\omega^2 \tag{8.36}$$

where m is the mass of the moon, v is its velocity, r is its distance from the earth, $\alpha = GmM_E$, and I is the moment of inertia of the earth about its spin axis. It is useful to express E in terms of the angular momenta. The last term in (8.36) is the spin energy of the earth $S^2/(2I)$, where $S = I\omega$. The first two terms in (8.36) are the orbital energy of the moon, which can be expressed in terms of $L = mvr$ by using the circular-orbit balance of gravitational and centrifugal forces

$$m\frac{v^2}{r} = \frac{\alpha}{r^2} \tag{8.37}$$

We obtain

$$E = -\frac{m\alpha^2}{2L^2} + \frac{S^2}{2I} \tag{8.38}$$

Because the total angular momentum $J = L + S$ is conserved, we can express S as $J - L$ and thus get E expressed in terms of one independent variable quantity, L

$$E = -\frac{m\alpha^2}{2L^2} + \frac{(J-L)^2}{2I} \tag{8.39}$$

If tidal friction is present the energy E (kinetic plus potential) of the system as well as L and S are not constant. The ultimate state of this system will be the state of lowest energy. The extreme values of E with J held fixed are determined by

$$0 = \frac{dE}{dL} = \frac{M_L\alpha^2}{L^3} - \frac{(J-L)}{I} \tag{8.40}$$

Using (8.38) and $S = J - L$ this condition can be expressed as

$$\frac{L}{M_L r^2} = \frac{S}{I} \tag{8.41}$$

The left-hand side is the orbital angular velocity Ω and the right-hand side is the spin angular velocity ω so the condition (8.41) of extreme energy at fixed total angular momentum is simply *corotation*

$$\Omega = \omega \tag{8.42}$$

In general, for fixed total angular momentum J about the CM, the state of minimum energy of an isolated system is rigid rotation. (Another example is the state of water in an isolated spinning bucket. Eventually the water rotates as a rigid body with the same angular velocity as the bucket.)

At present $\Omega < \omega$ for the earth-moon system. In Fig. 8-7 we plot Ω and ω as a function of r. There are two solutions for corotation.

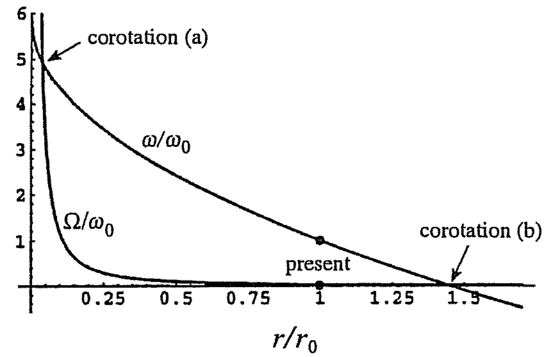


FIGURE 8-7. The spin angular velocity ω and the orbital angular velocity Ω for the earth-moon system as a function of orbital angular momentum. The subscript 0 denotes present value.

In Fig. 8-8 the energy of the earth-moon-system is plotted versus r. The two extrema correspond to the corotation points of Fig. 8-7. Case (a) is an unstable equilibrium; the bulk of the angular momentum is in the spin of the earth. Case (b) is a stable equilibrium; the bulk of the angular momentum is in the orbit of the moon. In Fig. 8-9 a more detailed plot of the energy is shown for the more immediate past and future. In the past the spin angular momentum S was larger and the orbital angular momentum L was smaller, corresponding to a higher energy for the system.

The earth's day is lengthening by 4.4×10^{-8} s/day, which corresponds to an angular acceleration of

$$\dot{\omega} = -0.85 \times 10^{-21}\ \text{rad/s}^2 \tag{8.43}$$

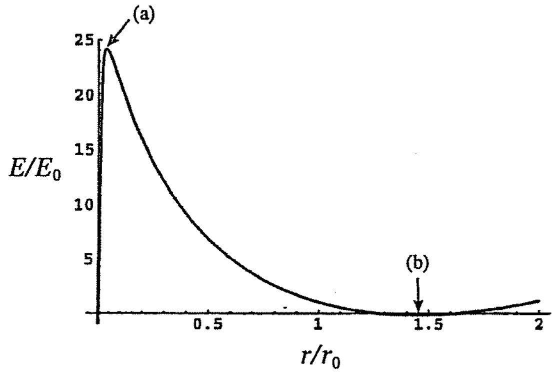


FIGURE 8-8. The energy of the earth-moon system versus the moon's orbital angular momentum for constant total angular momentum J. Here E_0 and r_0 are the present values. The labels (a) and (b) refer to the corotation points of Fig. 8-7.

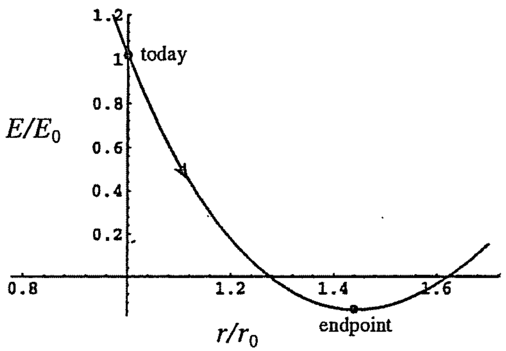


FIGURE 8-9. A blow-up of Fig. 8-8 near the present time. As the energy of the earth-moon system decreases due to tidal drag the moon's distance r increases.

Using (8.38), $L+S$ constant, and $S = I\omega$, we obtain

$$\frac{\dot{r}}{r} = \frac{2\dot{L}}{L} = -\frac{2\dot{S}}{L} = -\frac{2\dot{\omega}}{\omega}\frac{S}{L} \tag{8.44}$$

Then using the present values we find

$$\dot{r} \simeq 0.4 \text{ cm/month} \tag{8.45}$$

Thus the moon is spiraling outward roughly one-half centimeter per revolution. This process will continue until the energy reaches minimum at $r = 1.44 r_0$, where r_0 is the present earth-moon separation. At this point corotation is achieved and lunar tidal drag vanishes. (From then on, solar tidal drag evolves the system.)

The torque between the earth and the moon that transfers S to L is caused by the tidal friction. The earth's rotation acts to drag the tidal bulge ahead of the line between the earth and moon, as shown in Fig. 8-10. The lead angle Δ can be calculated by equating the torque applied by the moon to the tidal bulge (which depends on Δ; it obviously vanishes for $\Delta = 0°$ or $90°$) to the torque implied by $\dot{\omega}$,

$$N = I\dot{\omega} \tag{8.46}$$

The tidal torque on a volume element of water is

$$dN_{\text{tide}} = \rho_{H_2O}\left(R_E^2 \sin\theta \, d\theta \, d\phi\right) h(\theta,\phi) \left(-\frac{\partial \Phi_{\text{tide}}}{\partial \phi}\right) \tag{8.47}$$

where ρ_{H_2O} is the density of water and $-\frac{\partial \Phi_{\text{tide}}}{\partial \phi}$ is the torque per unit mass. From (8.31) for a tide displaced by an angle Δ as in Fig. 8-10,

$$h(\theta,\phi) = \frac{M_L}{2M_c}\left(\frac{R_E}{R}\right)^2 R_c \left[3\sin^2\theta \cos^2(\phi-\Delta) - 1\right] \tag{8.48}$$

We then integrate (8.47) over the surface of the earth to obtain

$$N_{\text{tide}} = -\frac{6}{5}\frac{M_L G}{M_c}\rho_{H_2O} R_c^2 \left(\frac{R_c}{R}\right)^2 \sin 2\Delta \tag{8.49}$$

Equating the two torques (8.46) and (8.49) using (8.43) gives the angle Δ that the tide leads the direction to the moon

$$\Delta \simeq 10 \text{ degrees} \tag{8.50}$$

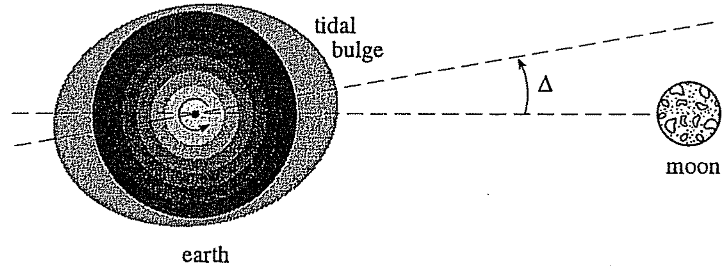


FIGURE 8-10. Earth-moon system as seen from above the north pole. Friction drags the tidal bulge ahead of the moon. This drag is opposed by a torque due to the moon's attraction.

In the past the moon was closer to the earth. At the present recession rate of 0.4 cm per month, two billion years ago the moon would have been at about three quarters of its present distance. The tidal height would have been double that at present and the increased tidal bulge would have caused larger tidal friction. On the other hand, differences in continental configurations and ocean levels might have decreased tidal drag in the distant past.

The moon could never have been closer than the *Roche limit*. According to this limit a moon having the same density as the planet will be pulled apart by tidal forces at distances closer than $R = 2.44R_E$. Most astronomical bodies are held together by their self-gravity, which is stronger for large bodies than the chemical forces that hold rocks together. As a satellite comes within the Roche limit tidal forces overcome the self-gravity and the satellite falls apart.

8.4 General Relativity: The Theory of Gravity

Einstein's theory of general relativity is a theory of gravity. At this level we do not have the mathematical tools to completely discuss the theory because it is expressed most naturally in the language of metric differential geometry. We can however illustrate some of the physical ideas which underlie general relativity and explore a few instances in which it differs from Newtonian gravity. These differences can be dramatic in very intense gravity fields.

A. The Principle of Equivalence

There are two aspects of mass: inertia as it appears in the second law and a proportionality constant in the gravity force. The equivalence of the two has the important consequence that all objects fall equally in a gravity field. Newton tested this hypothesis by verifying that pendulum bobs made of different materials have the same period to roughly 1 part in a thousand. Modern tests of the equivalence principle have improved this limit to one part in 10^{12}.

This remarkable equivalence led Albert Einstein to propose that locally (*i.e.*, at any given point) one cannot distinguish between the acceleration of a reference frame (*e.g.*, in an elevator) and gravitational force. Free fall is indistinguishable from being located in a gravity-free region. The value of $\mathbf{g} = -\nabla\Phi$ is a frame-dependent quantity. The general principle of relativity requires that in free fall all physical laws reduce to those in an inertial frame.

B. Gravitational Frequency Shift

One of the most direct implications of the principle of equivalence is that "higher clocks run faster." According to Einstein's theory, if waves are emitted on the earth with frequency ν as in Fig. 8-11(a) they arrive a distance h below with frequency ν' where

$$\nu' = \nu\left(1 + \frac{gh}{c^2}\right) \tag{8.51}$$

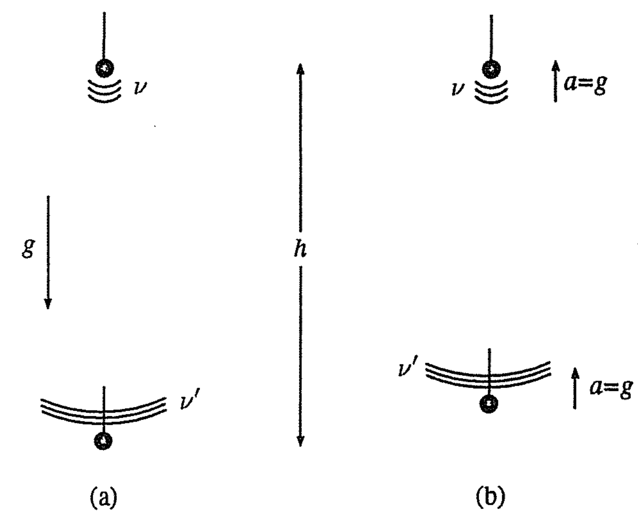


FIGURE 8-11. Gravitational frequency shift. In (a) the lower observer receives waves at higher frequency. In (b) the equivalence principle relates this shift to a Doppler shift. The clocks are supported against gravity in (a), and accelerated in (b), by strings.

To see how this comes about we invoke the principle of equivalence. The same frequency shift will occur in the situation of Fig. 8-11(b), where both emitter and receiver are being accelerated upward with $a = g$ in a region far from the earth. Consider waves emitted at $t = 0$ with frequency ν. When these waves reach the receiver a time h/c later, the receiver is moving faster by velocity $\Delta v = g(h/c)$ than the emitter at the time of emission. The waves will therefore be Doppler shifted to a higher rod. The wheel D rotates with constant angular velocity ω_0 (see the figure). The wheel D is a homogeneous thin circular disk of mass m and radius r. The rod S goes through the center of D.

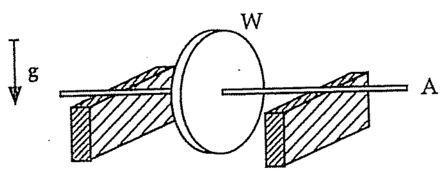
The other end of S is fastened to the mid-point O of a thin, rigid horizontal axis A, which can turn without friction in two bearings, B and C, that are fixed in the laboratory. The axle A has length a, and its mass can be ignored in this problem.

The rod S may thus turn in a vertical plane perpendicular to the horizontal axis A. Initially, S is held horizontal, and at rest. At a certain time the rod is released and S begins to turn in the vertical plane through O. The initial angular velocity of the rod is thus zero.

Neglect all frictional effects (in particular, the disk D maintains its angular velocity ω_0).

(1) Find the angular velocity Ω of the rod S at the moment the disk D passes through its lowest position (where S is vertical).
(2) Find the forces $\mathbf{F}_B$ and $\mathbf{F}_C$ by which the axle bearing B and C respectively act on the axle A, when S passes through its lowest position (the angular velocity vector Ω of the rod is in the direction OC).

Problem 13.4.



A wheel W is shaped as a homogeneous, flat, circular disk. The wheel is fastened to a thin axle A that passes through the center of the wheel. We ignore the mass of A.

The axle A rests in two bearings in such a way that the system can rotate without friction around a horizontal axis. Note that A is supported only from below! The radius R of the disk is $R = 50$ cm, and the distance between the two bearings is 1 m. The CM of the disk is exactly at the mid-point between the two bearings.

The wheel is fastened to the axle A in such a way that the normal to the disk of the wheel forms an angle of $\theta = 1°$ with the axle A. The acceleration of gravity is $g = 9.8\ \mathrm{m\,s^{-2}}$. The wheel is now set in motion, performing ν revolutions per second.

14. The Motion of the Planets

We shall now apply Newtonian mechanics to the study of the motion of planets.

Look at the sky on a dark, clear night. Seen from the Earth, the stars and the planets appear to be fastened inside a huge sphere, called the celestial sphere. The celestial sphere appears to rotate, once per day, around an axis passing through the north pole and the south pole of the Earth.

The stars are far away from the Earth. Therefore the stars seem to have fixed positions on the celestial sphere. The stars do not change positions relative to one another over periods of time comparable to a human life time.

The planets, on the other hand, are rather close to us. Even though from one night to the next, a planet appears to be fastened on the celestial sphere, over periods of weeks the line of sight, even to an outer planet, does change noticeably relative to the stars.

The roots of modern science are found in the study of the motion of the planets. Therefore we begin with some remarks of a historical nature.

14.1 Tycho Brahe

On the island of Hven, between Sweden and Denmark, Brahe built, in the years 1576–1597, an astronomical research institution of historical significance. Brahe understood the significance of precise astronomical observations, and he had the means to construct the necessary instruments. Before the time of Brahe, the positions of the heavenly bodies were known with the precision of 10 minutes of arc (10′). Brahe improved the precision significantly and reached an accuracy of 1 to 2 minutes of arc, which is close to the limit of precision obtainable with the unaided eye.

The years 1576–1597, in which Brahe performed his measurements, will be remembered as one of the most decisive periods in the history of science, and indeed in the history of man. This is impressive, especially when we recall that the naked-eye observational methods of Brahe were deemed to become obsolete after half a score of years. In 1609, Galileo pointed a telescope one might add that if the observations of the planets made by Brahe had been even more precise, if these observations had disclosed the "irregularities" in for instance the motion of Mars (irregularities caused by gravitational fields from the other planets), Kepler might not have uncovered his three simple laws for the orbits of the planets. The gravitational law of Newton might have been more difficult to find, and the history of man had changed. Such speculations may be considered entertaining, but are not particularly useful. Seen from the viewpoint of physics the remarks merely illustrate that it is important to find the essential aspects of experimental or observational material.

14.2 Kepler and the Orbit of Mars

Contrary to Brahe, Kepler initially accepted the heliocentric model of the solar system, as described by N. Copernicus. The Sun is at rest in the center of the system, with the planets moving in circular orbits around the Sun.

Originally Kepler looked for a connection between forces and the structure of the solar system. He realized very soon that the periods of the planets increase with the distance of the planet from the Sun. It was Kepler's belief that the increase in the periods of the planets was connected to a force from the Sun, a force that decreased with distance.

Kepler did not succeed in connecting the motions in the solar system to the concept of force. According to historians of science, he had a quite clear understanding of the importance of this task. The title of Kepler's principal book (published in the year 1609) was:

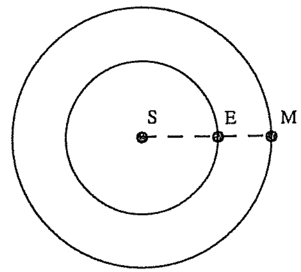
A New Astronomy Based on Causation
or
A Physics of the Sky
Derived from Investigations of the Motions of the Star Mars.
Founded on Observations made by the Nobleman Tycho Brahe.

If Kepler did not succeed in explaining the dynamics of planetary motions, he did succeed in using the empirical material of Brahe in a masterly way. Kepler condensed this material into an elegant form, a form that became decisive for the work of Newton. Through a nearly superhuman effort of calculation, Kepler succeeded in showing that planets move around the Sun, not in circular orbits with the Sun in the center, but in elliptical orbits with the Sun in one focus.

The difficult problem confronting Kepler was to determine the orbit of a planet relative to the Sun, based solely on observed *directions* to the planet,

14.2.1 The Length of a Martian Year

The time needed for a planet to complete one revolution around the Sun is called the *sidereal* period. The sidereal period for a planet cannot be directly observed, but it can be determined as shown below. Assume that the orbits of the Earth (E) and Mars (M) are circles with the Sun (S) in the center. The planet Mars is said to be in opposition, when the Sun, the Earth, and



Mars lie on a straight line, and Mars is closer to the Earth than to the Sun (see the figure).

The time between two successive oppositions is called the *synodic* period of the planet. The synodic period can be determined by observation. When the synodic period is known the sidereal period may be calculated as follows.

Let the sidereal period of Mars be T and the synodic period S. The sidereal period of the Earth – i.e., one year – is called A. The quantities T, S, and A are the respective periods measured in days (1 day = 24 h).

Mars is an outer planet relative to Earth. In the time between two successive oppositions, i.e., during one synodic period, Mars moves 360^0 less relative to the Sun than the Earth moves in the same time.

In one sidereal period Mars moves 360° relative to the Sun. In one day Mars moves $360°/T$ relative to the Sun.

In one synodic period Mars moves $(S/T)360°$ relative to the Sun.

During one synodic period for Mars, the Earth moves $(S/A)360°$ relative to the Sun.

The equation to determine T is thus:

$$\frac{S}{T}360° = \frac{S}{A}360° - 360° ,$$

or

$$\frac{1}{T} = \frac{1}{A} - \frac{1}{S} ,$$

From Brahes measurements Kepler knew that the synodic period for Mars – i.e.. the time between two successive oppositions – was $S = 779.8$ days.

$$\frac{1}{T} = \frac{1}{365.24} - \frac{1}{779.8} = 0.0014555 \text{ days}^{-1} ,$$
$$T = 687 \text{ days.}$$

To trace its orbit around the Sun, Mars thus needs 687 days = 1.88 years.

Kepler actually started by determining the orbit of the Earth. To do this he used his knowledge about the sidereal period of Mars (687 days) to identify the dates on which Mars was back in a given point in its orbit.

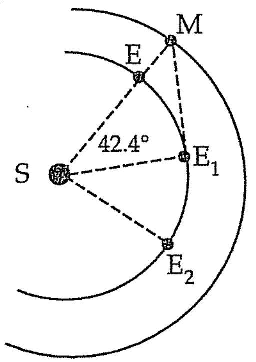


Fig. 14.1.

We shall show only the principles used by Kepler. Consider Figure 14.1. Let the point M mark an opposition of the planet Mars. It will take Mars 687 days to return to the point M in the orbit. During 687 days the Earth has completed $687/365 \approx 1.88$ revolutions around the Sun. The Earth has thus moved $1.882 \times 360° = 677.6°$ in its orbit, or 42.4° less than two complete revolutions. The Earth will then be located in the point E_1, as shown on Figure 14.1, i.e., 42.4° "behind" Mars. After an additional 687 days Mars will again be in the point M, while the Earth will be in the point marked E_2 on Figure 14.1 (assuming circular orbits).

From each successive complete revolutions of the planet Mars, Kepler was able to find one point of the orbit of the Earth.

Brahe had observed Mars for more than 20 years. The observations included ten oppositions of Mars.

Using the method outlined above, Kepler was able to construct the orbit of the Earth relative to the Sun (relative to the heliocentric reference frame!). Kepler found that, within the precision of observation, the orbit of the Earth was a circle, but with the essential feature that the Sun was *not* located at the center of the circle.

By plotting the position of the Earth at various dates, Kepler discovered that the Earth does not move with the same speed all year round. The Earth

The radius vector from the Sun to the planet sweeps out equal areas in equal amounts of time.

As we have already seen (Chapter 10), this law is a direct consequence of the conservation of angular momentum in a central field of force.

Now Kepler knew the orbit of the Earth. The next problem was – based on the observations of Brahe and on the knowledge of the orbit of the Earth – to find the orbit of Mars relative to the Sun.

14.2.2 The Orbit of the Planet Mars

Kepler utilized the fact that he knew the length of a Martian year (sidereal period, 687 days).

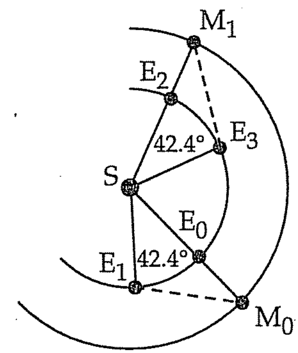


Fig. 14.2. Kepler's determination of the orbit of the planet Mars. (SE_0M_0) is one opposition of Mars and (SE_2M_1) is another

Kepler again used the oppositions of Mars. Consider the opposition marked by the line SE_0M_0 in Figure 14.2. The line of sight to Mars relative to the stars was known from the measurements of Brahe. When Mars, 687 days later, again is in the point marked M_0, the planet is *not* in opposition, because the Earth will now be in the point marked E_1, i.e., 42.4° "behind" Mars. Kepler knew the orbit of the Earth, and the position of the Earth at any given date was now also known. That means: Kepler knew the date on which the Earth was in the point of its orbit marked E_1. The position of Mars, i.e. the line of sight to Mars on the day the Earth was in E_1 could be found from the tables of Brahe. Kepler was able to calculate the angle SE_1M_0. Using the distance from the Sun to the Earth as unit of length, Kepler found one point in the orbit of Mars by triangulation.

Using the ten oppositions studied by Brahe, Kepler was able to determine ten points in the orbit of Mars. Kepler tried to find a circle passing through

It is interesting to note that before Brahe the positions of the planets were known with an accuracy of 10′. The improvement of the precision of observation to about 2′ was therefore decisive. If – before Brahe – one had tried to fit a circle to the observation points it would have been considered successful.

Kepler had confidence in the data of Brahe, and he took the decisive step. The hypothesis of circular orbits must be rejected. Instead of conserving "the ideal circular orbits" Kepler chose to believe in observations.

The author Stefan Zweig has written a book with the title *Sternstunden der Menschheit*. Here is a *Sternstunde* in our history, far more important than any described by Stefan Zweig. A description of Kepler's achievements cannot be found in the book by Stefan Zweig.

Kepler found that the ten points of the orbit of Mars could be fitted to a curve known to the mathematicians for a long time: an ellipse. Johannes Kepler had thus reached the law that has become known as Kepler's first law:

> The orbit of a planet relative to the Sun lies in a fixed plane containing the Sun, and each planet moves around the Sun in an elliptical orbit with the Sun in one focus.

Through the work of Brahe and Kepler the solar system had disclosed one of its deepest secrets.

The study of the solar system has resulted in several other decisive advances in physics: the law of gravity, the finite velocity of light (the "lingering of light", Ole Rømer), and the rotation of the perihelion of the elliptical orbit of the planet Mercury. Somewhere in the solar system – perhaps on Mars – we may find the key to the greatest riddle of the natural sciences: the origin of life itself.

The reason why it was not possible to fit the points of observation of Mars into a circular orbit around the Sun is the substantial eccentricity ("flatness") of the Martian ellipse. For the precise definition of eccentricity, see below. The eccentricity of the elliptical orbit of Mars is $e = 0.09$, which is five times larger than the eccentricity of the elliptical orbit of the Earth, and more than twelve times the eccentricity of the orbit of Venus. It is, however, important to note that even for Mars the deviation from the circular form is small.

Kepler's third law was published – among several more obscure results – in the year 1619:

> The square of the period of revolution of a planet is proportional to the third power of the greatest semi axis of the ellipse.

14.2.3 Determination of Absolute Distance in the Solar System

The sidereal period of revolution T_p for a planet may be determined via the observation of the synodic period. From Kepler's third law, the semi-major axis a_p for the planetary orbit (ellipse, see below) may then be found using the astronomical unit as the basic measure of distance. One astronomical unit (1 AU) is defined as the mean value of the distance of the Earth from the Sun. Measuring T_p in years we have

$$\frac{T_\mathrm{p}^2}{a_\mathrm{p}^3} = \frac{T_\mathrm{E}^2}{a_\mathrm{E}^3} = \frac{1^2}{1^3} = 1\,.$$

The absolute distances in the solar system, i.e., distances measured in meters, can be found only when one distance – for instance the distance from Earth to Venus, or from Earth to Mars – has been determined.

The problem has not been simple to solve. Today one can measure the distance say, from the Earth to Venus, with high precision by means of radar signals reflected from the surface of Venus.

Historically the problem was first solved by triangulation. From two points on the Earth, a large distance apart, the direction of the line of sight to a planet is measured. The two directions of the line of sight will then form a certain angle, which is larger the closer the planet is to the Earth. The difficulties with this measurement is obviously the small value of the angle between the lines of sight. The angle between two lines of sight from the Earth to the Moon may be about 1^0. The angle between two lines of sight from the Earth to even the nearest planets will never be more than 1′ (1 arc minute).

Mars is closest to the Earth when in opposition. In the most favorable oppositions Mars is 0.37 AU from the Earth. Venus may come even closer (0.26 AU). When Mars is in opposition the illuminated half sphere of Mars is facing the Earth. Therefore Mars is easy to observe during an opposition. The orbit of Venus lies within the orbit of the Earth. Therefore when Venus is closest to the Earth, Venus will have its dark side facing the Earth. Venus is therefore impossible to observe when it is closest to Earth unless the planet passes in front of the solar disk. Venus will then be observable as a small dark spot against the large luminous disk of the Sun. This phenomenon is called a transit of Venus. The orbits of the Earth and the orbit of Venus lie nearly in the same plane, but not exactly so. As a rule Venus will bypass the Sun. A Venus transit is a rare phenomenon. They come in pairs. There were two in the 19th century (1874 and 1882), and the next pair is 2004 and 2012.

From the first two of the mentioned Venus transits a triangulation measurement was made. Based on this the AU was estimated to be between 147×10^6 km and 149×10^6 km

Eros may be as small as 0.15 AU. A favorable opposition of Eros took place in 1930. At this opposition the astronomical unit was determined as 149.7×10^6 km. The present value is 1 AU $= 149.598 \times 10^6$ km.

Kepler published his laws as unexplained facts. The full dynamical consequences of these laws were recognized by Newton, after he had formulated his general laws of motion. We shall show that gravitational attraction, i.e., Newton's law of gravity is implied by Kepler's laws. After this we shall demonstrate the converse: Kepler's three laws are consequences of Newtonian mechanics and the law of gravity.

Before proceeding, we shall briefly review some results related to the geometry of conic sections.

14.3 Conic Sections

Detailed descriptions of conic sections may be found in books on geometry or calculus. Here we give a rudimentary introduction.

The curves obtained by intersecting a cone with a plane which does not pass through the vertex of the cone, are called conic sections. If the plane

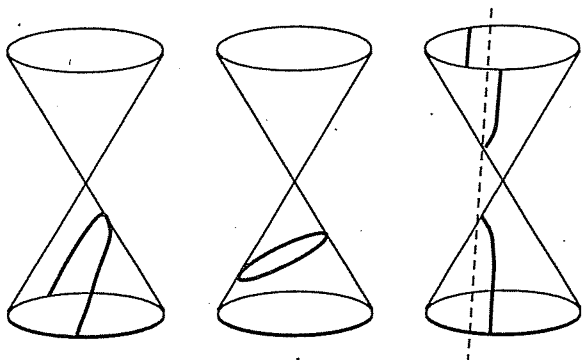


Fig. 14.3. Conic sections obtained by intersecting a cone with a plane: (a) parabola, (b) ellipse, (c) hyperbola

intersecting the cone is parallel to a generator of the cone (Figure 14.3a), the conic section becomes a parabola. Otherwise the curve produced is called an ellipse or a hyperbola, depending on whether the plane intersects one portion of the cone (Figure 14.3b) or both portions (Figure 14.3c). A circle is a special case of an ellipse.

The three types of nondegenerate conic sections may be characterized

$$\frac{\mathrm{PF}}{\mathrm{Pl}} = e\,, \tag{14.1}$$

where PF is the distance from P to F and Pl is the distance from P to l. The line l, which is called the directrix, does not pass through F.

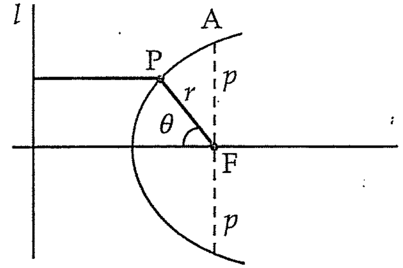


Fig. 14.4. The equation for conic sections using polar coordinates

Let $2p$ be the length of the chord perpendicular to the axis of the conic section and passing through the focus F. By choosing the point P at the endpoint of the chord, e.g., at the point A (see Figure 14.4) the defining equation becomes

$$p = e(\mathrm{Fl})\,. \tag{14.2}$$

The quantity p is called the parameter of the conic section.

Measuring the angle from the symmetry axis as in the figure and using (14.1) and (14.2) we find

$$r = \mathrm{FP} = e(\mathrm{Pl}) = e(\mathrm{Fl} - r\cos\theta)\,,$$

$$r = e\left(\frac{p}{e} - r\cos\theta\right) = p - er\cos\theta\,.$$

Finally we get

$$r = \frac{p}{1 + e\cos\theta}\,. \tag{14.3}$$

The expression (14.3) is the equation for a conic section for all three cases.

We get an ellipse for $0 \le e < 1$, a hyperbola for $1 < e$, and a parabola for $e = 1$. In Figure 14.4 only a part of the curve close to F has been shown. In this way all three cases may be said to be included in the figure.

For the ellipse ($e < 1$) the angle θ may take all values from 0 to 2π. Since $e < 1$ the denominator can never become zero. For the parabola ($e = 1$) we have: $r \to \infty$ for $\theta \to \pi$ (or $-\pi$). For the hyperbola one obtains all points on the branch considered, when θ is limited to $|\,\theta\,| < \theta_0$, where $\cos\theta_0 \equiv -1/e$; $\pi/2 < \theta_0 < \pi$. One finds $r \to \infty$ for $|\,\theta\,| \to \theta_0$, which give the directions of the asymptotes. (If θ runs through the intervals $\theta_0 < |\,\theta\,| < \pi$ one gets

Concluding: for any value of $e \geq 0$ and $p > 0$, (14.3) describes a conic section. The result (14.3) also includes the circle, which is the special case $e = 0$. In general, e determines the *shape* of the conic section, and p determines the *size*.

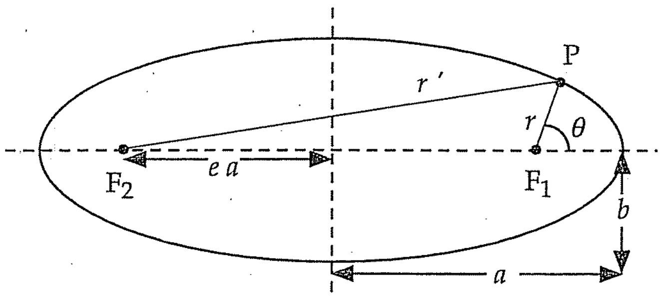


Fig. 14.5. The ellipse is the set of points where the distances to F_1 and F_2 have a constant sum

The ellipse can also be defined as the set of points P where the distances from two fixed points (the *foci*) have a constant sum (see Figure 14.5).

The major axis has length $2a$, the minor axis $2b$. The distance between the foci is $e \cdot 2a$, where – as we shall see below – e is the eccentricity. From the definition we obtain

$$r + r' = 2a\,.$$

Furthermore (see Figure 14.5),

$$(r')^2 = (2ea + r\cos\theta)^2 + (r\sin\theta)^2\,.$$

Using these two equations we obtain

$$r = \frac{a(1-e^2)}{1+e\cos\theta}\,. \tag{14.4}$$

By comparing (14.4) and (14.3) we find $p = a(1-e^2)$.

The perihelion (smallest value of r) and the aphelion (largest value of r) are determined by

$$\theta = 0 \text{ (perihelion) } \Rightarrow r_{\min} = \frac{p}{1+e} = a(1-e)\,,$$

$$\theta = \pi \text{ (aphelion) } \Rightarrow r_{\max} = \frac{p}{1-e} = a(1+e)\,.$$

From this

$$\frac{r_{\max}}{r_{\min}} = \frac{1+e}{1-e}\,, \tag{14.5}$$

The connection between semi-major axis a, semi-minor axis b, and the eccentricity is

$$b = a\sqrt{1-e^2}\,. \tag{14.6}$$

If we – instead of the angle θ – use the angle $\varphi \equiv \pi - \theta$ as the polar angle we will have

$$r = \frac{a(1-e^2)}{1-e\cos\varphi} = \frac{p}{1-e\cos\varphi}\,. \tag{14.7}$$

The perihelion is at $\varphi = \pi$.

14.4 Newton's Law of Gravity Derived from Kepler's Laws

Newton's gravitational law is contained within Kepler's three laws. In Chapter 1 we demonstrated this for the special case of uniform circular motion. Below we present the general calculations for elliptic orbits.

Kepler's first law was formulated in two steps. Kepler first showed that the orbit of a given planet lies in a fixed plane containing the center of the Sun; then that the planet moves in an ellipse with the Sun at one focus.

We consider motion in a fixed plane. The problem is to deduce the $1/r^2$ dependence of gravitational attraction only from the observed motion of the planets.

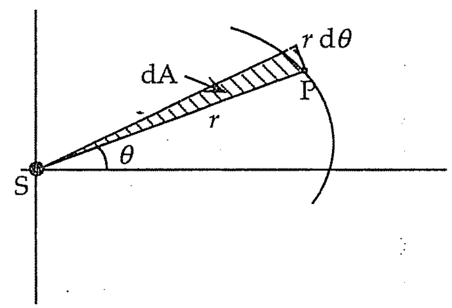


Fig. 14.6. The orbit of a planet, P, around the Sun, S

We start by calculating the acceleration of a planet moving in an elliptical orbit with the Sun in one focus. The element of area dA (see Figure 14.6) in polar coordinates (with origin in the Sun) is

Kepler's second law states that the area velocity is constant and we denote the constant by $h/2$. We have

$$\mathrm{d}A = \frac{1}{2}h\mathrm{d}t\,, \quad \text{or} \quad \dot{A} = \frac{1}{2}h\,.$$

From Kepler's second law it thus follows that

$$2\dot{A} = r^2\dot{\theta} = h\,. \tag{14.9}$$

We seek the acceleration vector for a Kepler orbit. In polar coordinates the two components of the acceleration are as follows (see the Appendix):

Radial component:

$$a_r = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \ddot{r} - r\dot{\theta}^2\,. \tag{14.10}$$

Angular component:

$$a_\theta = \frac{1}{r}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\right] = 2\dot{r}\dot{\theta} + r\ddot{\theta}\,. \tag{14.11}$$

It will be useful to introduce a substitution. Instead of r we shall use $u \equiv 1/r$ as a new variable.

From Kepler's first law we know the shape of the curve $r = r(\theta)$ or $u = u(\theta)$. The goal is to find the two components of the acceleration, a_r and a_θ, through $u = u(\theta)$, i.e., through the equation for the orbit. With this purpose in mind we eliminate the time from (14.10) and (14.11), i.e., we express $\dot{r}, \ddot{r}, \dot{\theta}$, and $\ddot{\theta}$ by $u, (\mathrm{d}u/\mathrm{d}\theta)$ and $(\mathrm{d}^2u/\mathrm{d}\theta^2)$.

From (14.9)

$$\dot{\theta} = \frac{h}{r^2} = hu^2\,. \tag{14.12}$$

Differentiating (14.12) with respect to time:

$$\ddot{\theta} = 2hu\dot{u} = 2hu\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2h^2u^3\frac{\mathrm{d}u}{\mathrm{d}\theta}\,. \tag{14.13}$$

Moreover, because $(\mathrm{d}u/\mathrm{d}\theta) = -(1/r^2)(\mathrm{d}r/\mathrm{d}\theta)$, we have

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}\,. \tag{14.14}$$

The quantities $\dot{\theta}$, $\ddot{\theta}$, and $\dot{r}$ are now expressed by $u = u(\theta)$.

We can then determine a_θ:

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 2\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)hu^2 + \frac{1}{u}2h^2u^3\frac{\mathrm{d}u}{\mathrm{d}\theta}\,,$$

$$a_\theta = 0\,.$$

Based on Kepler's second law we have shown that the acceleration of the planet is directed in a radial direction (i.e., towards the Sun). This result, well known from circular motion, is thus also true for a general elliptical orbit.

We now seek a_r. From (14.10) we see that we have to find $\ddot{r}$ expressed by $u = u(\theta)$.

$$\begin{aligned} \ddot{r} &= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\,. \end{aligned} \tag{14.15}$$

By inserting (14.15) and (14.12) into (14.10) we obtain

$$a_r = -h^2u^2\left[\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right]. \tag{14.16}$$

From Kepler's second law we have found the radial acceleration, expressed by the area velocity constant h, and the equation of the orbit $u = u(\theta)$.

From Kepler's first law

$$r = \frac{p}{1 + e\cos\theta} \quad \text{(an ellipse)},$$

or

$$u = u(\theta) = \frac{1}{p}\left[1 + e\cos\theta\right].$$

Thus

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{e}{p}\cos\theta\,.$$

Inserting $(\mathrm{d}^2u/\mathrm{d}\theta^2)$ and u into (14.16) we finally obtain

$$a_r = -\frac{h^2}{p}\frac{1}{r^2}\,. \tag{14.17}$$

Conclusion. The acceleration of the planet is directed towards the Sun (the minus sign), and the acceleration is inversely proportional to the square of the distance from the Sun.

We proceed to apply Kepler's third law, in order to demonstrate that the constant h^2/p can depend only on the physical nature of the Sun, i.e., the value for h^2/p is the same for all planets.

Kepler's third law may be written

$$\frac{a^3}{T^2} = C\,,$$

By integration over a complete revolution (14.9) becomes

$$2A = hT\,. \tag{14.18}$$

The area A of an ellipse is $A = \pi ab$, where a and b are the semi-major and semi-minor axes respectively. Furthermore,

$$p = a(1 - e^2) \quad \text{and} \quad b = a\sqrt{1 - e^2} = \frac{p}{\sqrt{1 - e^2}}\,.$$

From (14.18) we therefore get, using $b^2 = ap$,

$$T^2 = \left(\frac{2}{h}A\right)^2 = \left(\frac{2\pi}{h}\right)^2 a^2 b^2 = 4\pi^2 a^3 \frac{p}{h^2}\,. \tag{14.19}$$

From (14.19) – and applying Kepler's third law, $a^3/T^2 = C$ – we find that

$$\frac{h^2}{p} = 4\pi^2 \frac{a^3}{T^2} = 4\pi^2 C\,.$$

We may then finally write (see 14.17)

$$a_r = -\frac{4\pi^2 C}{r^2}\,, \tag{14.20}$$

where C is the same for all planets, i.e., C depends (at most) on properties of the Sun only.

From Kepler's three laws we have computed the acceleration of the planet, and seen that it depends only on the distance of the planet from the Sun: The acceleration is directed towards the Sun, and the acceleration is inversely proportional to the square of the distance from the Sun.

Newton added a decisive new feature to these results in the form of a theoretical interpretation of the derived formula. Newton introduced the Sun as the *cause* of the acceleration of the planets, and this guided him to the fundamentally new idea about *universal gravitation* (see Chapter 1).

This most surprising step, rightfully admired by both Newton's contemporaries and by later generations, was Newton's linking of the fall of bodies towards the Earth with the motion of celestial bodies.

The interaction that makes an apple fall to the ground also holds the Moon in its orbit around the Earth.

In Chapter 8 we proved that the Earth acts gravitationally as if all of its mass was concentrated in the center. From (14.20) we know that the acceleration near the surface of the Earth is (ρ = radius of the Earth)

$$g = \frac{4\pi^2 C'}{\rho^2}\,.$$

orbit of the Moon: $C' = r^3/T^2$, where r = radius of the lunar orbit and T is the sidereal period of revolution of the Moon. Introducing numerical values, Newton found the gravitational acceleration g near the surface of the Earth:

$$g = \frac{4\pi^2 r^3}{\rho^2 T^2} = 9.8\ \mathrm{m\,s^{-2}}\,,$$

which is in accordance with the observed value. The greatest achievement in the history of man was completed.

14.5 The Kepler Problem

The derivation of Kepler's three laws, setting out from Newtonian mechanics and the law of gravitational attraction, is called the Kepler problem. The solution of this problem is one of the jewels of theoretical physics.

We start by deriving Kepler's first law. We first solve the so-called one-

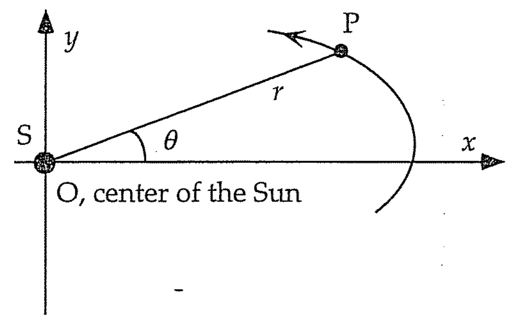


Fig. 14.7. A planet moving around the Sun

body problem. It is assumed that the Sun is fixed in the origin of the coordinate system. We furthermore neglect the gravitational interactions between the planets. We thus consider one planet moving in the gravitational field of the Sun, which is at rest in the origin of an inertial system (the heliocentric reference frame).

The angular momentum $\mathbf{L}_0$ of the planet around O is

$$\mathbf{L}_0 = \mathbf{r} \times m\dot{\mathbf{r}} = \mathbf{r} \times m\mathbf{v}\,.$$

In a central force field the angular momentum is a constant of the motion. The plane spanned by $\mathbf{r}$ and $\mathbf{v}$ is the plane of motion for the planet. The plane of motion is fixed perpendicular to $\mathbf{L}_0$, and passes through the center of the Sun.

We use polar coordinates. The mass of the planet is m. We write the equation of motion of the planet:

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r}\,. \tag{14.21}$$

The term on the right is the force on the planet. The equation of motion may be written as:

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{C}{r^2}\frac{\mathbf{r}}{r}\,, \tag{14.22}$$

where $C \equiv -GMm$.

In the form (14.22) the equation is more general: for attractive forces C is negative (gravitational forces, electron in the Coulomb field of a proton). For repulsive forces C is positive (two electrically charged particles with the same sign of the charge).

In polar coordinates the acceleration is

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \equiv \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\mathbf{e}_\theta\,,$$

with $\mathbf{e_r}$ being the unit vector along the radius vector, and $\mathbf{e}_\theta$ the unit vector perpendicular to radius vector. The equation of motion (14.22) written in two components, one along $\mathbf{e_r}$ and one along $\mathbf{e}_\theta$, becomes

$$m(\ddot{r} - r\dot{\theta}^2) = \frac{C}{r^2}\,, \tag{14.23}$$

$$m\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0\,. \tag{14.24}$$

From (14.24)

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0\,,$$

or, after integration,

$$mr^2\dot{\theta} = L\,, \tag{14.25}$$

where L is the magnitude of the angular momentum of the planet relative to O. The magnitude of the angular momentum, $L = |\,\mathbf{L}\,|$, is constant and determined by the initial conditions. We have thus introduced a constant of motion into the process of integration.

From (14.25) we find

$$\dot{\theta} = \frac{L}{mr^2}\,. \tag{14.26}$$

Introducing $\dot{\theta}$ into (14.23) gives

$$\ddot{r} - \frac{L^2}{m^2r^3} = \frac{C}{mr^2}\,. \tag{14.27}$$

of the planet, but in the shape of the orbit. In other words: We are interested in determining r as a function of θ, not r as a function of t.

We eliminate t from (14.27) by using (14.26). First we determine

$$\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t}\,, \quad \text{then } \ddot{r} \equiv \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\,,$$

both expressed by θ instead of t.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{L}{mr^2}\,, \tag{14.28}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{L}{mr^2}\right),$$

or

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{L^2}{m^2r^4}\left[\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - \frac{2}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]. \tag{14.29}$$

We introduce a new variable $u(\theta) = 1/r(\theta)$. The reason for introducing u as variable instead of r is that the parenthesis in (14.29) is close to being equal to $(\mathrm{d}^2u/\mathrm{d}\theta^2)$. We find

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\,,$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{1}{r^2}\left[\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - \frac{2}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right].$$

Using these results we obtain

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} \equiv \ddot{r} = -\frac{L^2}{m^2r^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\,. \tag{14.30}$$

Introducing (14.30) into (14.27) and using $1/r = u$ we obtain the following differential equation for $u = u(\theta)$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{Cm}{L^2}\,. \tag{14.31}$$

This differential equation has the same form as the equation describing the oscillation of a mass at the end of a spring, hanging in the gravitational field of the Earth (see Example 2.3).

The solution of (14.31) is

$$u = A\cos(\theta + \varphi_0) - \frac{Cm}{L^2}\,. \tag{14.32}$$

The quantities A and φ_0 are constants of integration. By the substitution of

The integration constant φ_0 describes the orientation of the orbit in the plane. By choosing the polar axis in a suitable way we can obtain $\varphi_0 = 0$. The result (14.32) may thus be written as follows:

$$\frac{1}{r} = A\cos\theta - \frac{Cm}{L^2}\,. \tag{14.33}$$

We proceed by introducing another constant of integration: The mechanical energy E of the planet. The planet moves in a conservative field of force. The energy is therefore conserved.

We express the integration constant A through E.

$$E \equiv \frac{1}{2}mv^2 + \frac{C}{r} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{C}{r}\,. \tag{14.34}$$

Using the expressions for $\dot{r}$ (14.28) and $\dot{\theta}$ (14.26) we obtain

$$E = \frac{1}{2}m\left(\frac{L^2}{m^2r^4}\right)\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right] + \frac{C}{r}\,. \tag{14.35}$$

The total mechanical energy E is – in (14.35) – expressed in terms of the parameters of the orbit. Using (14.33) we find an equation connecting E and A. From (14.33)

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r^2 A\sin\theta\,.$$

Introducing $\mathrm{d}r/\mathrm{d}\theta$ in (14.35) gives

$$E = \frac{1}{2}m\left(\frac{L^2}{m^2r^4}\right)(r^4A^2\sin^2\theta + r^2) + \frac{C}{r}\,.$$

By using (14.33) again we find

$$E = \frac{1}{2}\frac{L^2}{m}A^2 - \frac{C^2m}{2L^2}\,,$$

or

$$A = \frac{Cm}{L^2}\left(1 + \frac{2EL^2}{C^2m}\right)^{1/2}\,. \tag{14.36}$$

Consider again (14.33). Introducing (14.36) and using $C \equiv -GMm$ we obtain the expression for the orbit of the planet in polar coordinates and expressed by two constants of the motion, L and E:

$$\frac{1}{r} = \frac{Gm^2M}{L^2}\left[1 - \left(1 + \frac{2EL^2}{G^2m^3M^2}\right)^{1/2}\cos\theta\right]\,. \tag{14.37}$$

Equation (14.37) describes an ellipse with perihelion for $\theta = \pi$ (see Section

$$\frac{1}{r} = \frac{Gm^2M}{L^2}\left[1 - \left(1 + \frac{2EL^2}{G^2m^3M^2}\right)^{1/2}\cos(\theta + \pi)\right]\,.$$

We prefer this choice of φ_0 and write our final result as

$$\frac{1}{r} = \frac{Gm^2M}{L^2}\left[1 + \left(1 + \frac{2EL^2}{G^2m^3M^2}\right)^{1/2}\cos\theta\right] \tag{14.38}$$

$$\frac{1}{r} = \frac{1}{p}\left[1 + e\cos\theta\right]\,. \tag{14.39}$$

Equation (14.39) is the equation, in polar coordinates, for a conic section.

Our result, (14.38), describes a conic section with parameter

$$p = \frac{L^2}{Gm^2M}\,,$$

and eccentricity

$$e = \sqrt{1 + \frac{2EL^2}{G^2m^3M^2}}\,.$$

The total energy,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}\,,$$

may be either negative, positive, or zero.

From (14.38) we conclude:

1. For $E < 0$, $e < 1$, we have an ellipse, $e = 0$ corresponds to a circular motion
2. For $E > 0$, $e > 1$, we have a hyperbola
3. For $E = 0$, $e = 1$, we have a parabola

> We have shown that Kepler's first law follows from Newton's second law, in combination with the law of gravitational attraction.

Many other important consequences concerning the motion of celestial bodies may be read from (14.38). Let us conclude.

A planet, a comet, an asteroid, or any heavenly body whatsoever, governed by the gravitational field of the Sun, will traverse an orbit that is a conic section. The form of the conic section is determined solely by the total mechanical energy E, given by the initial conditions. If $E \geq 0$ the body is not bound to the solar system. The orbit is a hyperbola, or for $E = 0$, a parabola. If a body with $E \geq 0$ passes "close to the Sun", i.e., if such a body appears in the solar system at all, it will happen only once. No comet with $E > 0$ has been observed until now. For $E < 0$ we have a "bound state" of the planet.

The angular momentum $\mathbf{L}$ is likewise a constant of the motion, given

14.5.1 Derivation of Kepler's 3rd Law from Newton's Law of Gravity

In Chapter 10 we proved that Kepler's second law is a consequence of conservation of angular momentum in a central force field. We shall now show that Kepler's third law also follows from Newtonian mechanics.

Kepler's third law is

$$T^2 = \frac{1}{C}a^3 \equiv ka^3 \,.$$

The constant of proportionality k is the same for all planets.

Kepler's second law can be written as

$$\frac{dA}{dt} = \frac{L}{2m} \,. \tag{14.40}$$

To prove the third law we have to introduce T, the sidereal period, into (14.40). By integration over a complete revolution,

$$A = \frac{LT}{2m} \,, \tag{14.41}$$

where $A = \pi ab$ (the area of the ellipse).

From (14.41)

$$T^2 = \left(\frac{2m}{L}\right)^2 A^2 = \left(\frac{2m}{L}\right)^2 \pi^2 a^2 b^2 \,. \tag{14.42}$$

From Section 14.2

$$b^2 = a^2(1 - e^2) \,,$$

and

$$p = a(1 - e^2) = \frac{L^2}{Gm^2 M} \,.$$

From (14.42) we obtain

$$T^2 = \frac{4\pi^2}{GM}a^3 = ka^3 \,. \tag{14.43}$$

The constant $k = 4\pi^2/GM$ depends only on the mass of the Sun, and is thus the same for all planets. Kepler's third law has been derived from Newtonian mechanics.

From (14.43)

$$T = (ka^3)^{1/2} \,,$$

or

$$\log a = \frac{2}{3}\log T + B, \quad \text{where } B \text{ is a constant.}$$

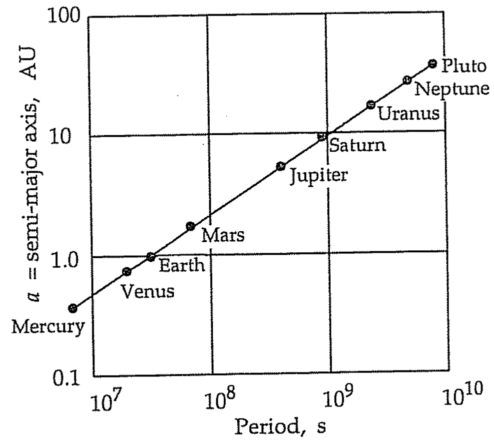


Fig. 14.8. Kepler's third law for the solar system, $\log a = \frac{2}{3}\log T + B$, $B \equiv$ constant. Based on *Berkeley Physics Course*

Kepler's third law is a consequence of the universal law of gravity and Newton's laws of motion. The law is valid also for elliptical orbits of moons moving around planets. The mass M of the Sun is then replaced by the mass of the given planet.

Newton tested the validity of Kepler's third law on the four Jupiter moons known to him. Newton knew the periods of revolution of the moons of Jupiter with fairly good accuracy.

The table below shows the radius ρ in the orbits of the moons; $\rho = r/R_j$ is measured in units of the radius R_j of Jupiter. The table furthermore gives the period of revolution T for the moons, and finally (ρ^3/T^2). Kepler's third law is seen to be valid to a high order of accuracy.

The radius in the orbits of the moons, as given in the table, is measured in units of the radius of Jupiter. The knowledge of the absolute distances in the solar system was limited at the time of Newton, and the size of Jupiter was consequently not known.

	r/R_j	T (s)	ρ^3/T^2 (s^{-2})
Io	5.58	1.53×10^5	7.4×10^{-9}
Europa	8.88	3.07×10^5	7.5×10^{-9}
Ganymede	14.16	6.19×10^5	7.5×10^{-9}
Callisto	24.90	1.45×10^6	7.4×10^{-9}

14.6 The Effective Potential

In this section we shall briefly describe another procedure for the integration of the equation of motion for a planet moving in the gravitational field of the Sun.

The angular momentum $\mathbf{L}$ of the planet is assumed to be different from zero ($L \equiv |\,\mathbf{L}\,| = 0$ corresponds to the planet moving along a radius vector, away from or into the Sun).

From Example 10.2 it is known that the total energy of the planet may be written as follows:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \,. \tag{14.44}$$

The term $L^2/2mr^2$ is called the *centrifugal potential energy*.

We look for the differential equation of the orbit. The magnitude of the angular momentum is

$$L = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \,.$$

The energy E and the magnitude of the angular momentum L are known constants of motion.

We transform differentiation with respect to time into differentiation with respect to θ:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta} \,,$$

Furthermore, we apply the variable transformation $u = 1/r$. This means that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta} \,.$$

Equation (14.44) may thus be rewritten as

$$\frac{1}{2}\frac{L^2}{m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{1}{2}\frac{L^2}{m}u^2 - GMmu = E \,. \tag{14.45}$$

This is a differential equation for $u = u(\theta)$. The equation may be simplified by differentiation with respect to θ and division by $(L^2/m)(\mathrm{d}u/\mathrm{d}\theta)$. We find

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{GMm^2}{L^2} \,.$$

This is the equation integrated in Section 14.5. The solution may be written – with a suitable choice of polar axes – as $u = A\cos\theta + 1/p$, where we have introduced $p \equiv L^2/GMm^2$.

Instead of the integration constant A we shall use $A \equiv e/p$, where e is a new integration constant. The possible orbits then have the form

$$r = \frac{p}{\quad} \,.$$

It is convenient to express the integration constant e – which obviously is the eccentricity – by means of E. By differentiating $u = (1 + e\cos\theta)/p$ with respect to θ and inserting the result into equation (14.45) we find

$$E = \frac{G^2M^2m^3}{2L^2}(e^2 - 1) \,,$$

or

$$e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \,,$$

which is identical to the result found previously.

14.7 The Two-Body Problem

We have solved the so-called one-body problem: a material particle moves in a central field of force of the type $1/r^2$. The model corresponds to a planet moving in the gravitational field of the Sun, where the Sun is assumed to be fixed at the origin.

Below we investigate the motion of two spherically symmetrical bodies moving in astronomical space. The two bodies are assumed to move exclusively under their mutual gravitational interaction. The problem is called the two-body problem. As a model one may think of the Sun of mass M and one of its planets, say Jupiter, of mass m. Consider Figure 14.9.

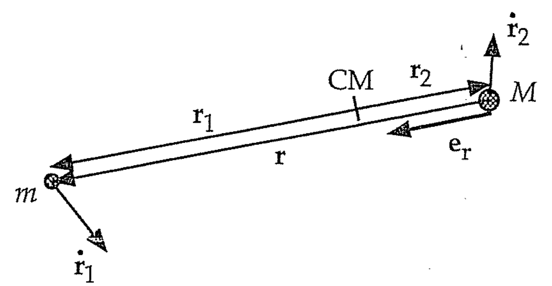


Fig. 14.9. The two-body problem

The center of mass, CM, of the system will be either "at rest" or move with constant velocity, because no external forces are acting on the system. Seen from the point of view of the original Newtonian mechanics this means that CM is either at rest in absolute space (astronomical space) or moves with a constant velocity relative to that space. The modified form of Newtonian mechanics are now in logical difficulties: we look for the motion of the Sun the description. The astronomical two-body problem illustrates the profound difficulties connected with the choice of inertial systems.

We choose CM as the origin for an inertial system (see Figure 14.9):

$$M\mathbf{r}_2 + m\mathbf{r}_1 = 0\,.$$

The radius vector to m measured from M is denoted $\mathbf{r}$:

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2\,.$$

A unit vector in the direction of $\mathbf{r}$ is denoted $\mathbf{e}_r$:

$$\mathbf{e}_r \equiv \frac{\mathbf{r}}{r} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{r}\,.$$

The equations of motion for each of the two bodies are

$$m\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\mathbf{e}_r\,,$$

$$M\frac{d^2\mathbf{r}_2}{dt^2} = \frac{GMm}{r^2}\mathbf{e}_r\,.$$

If we add these two equations we reach the not surprising conclusion that the total momentum of the system is a constant.

Our real aim is to obtain a differential equation for $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Transferring the masses to the right side of the equations and subtracting the second equation from the first we get

$$\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\mathbf{r}_2}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\left[\frac{1}{m} + \frac{1}{M}\right]\mathbf{e}_r\,,$$

or

$$\frac{mM}{M+m}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\mathbf{e}_r\,.$$

The reduced mass μ is defined as

$$\mu \equiv \frac{mM}{m+M}\,,$$

or

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}\,. \tag{14.46}$$

We obtain the following differential equation for $\mathbf{r}$, the radius vector from M to m:

$$\mu\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{G(M+m)\mu}{r^2}\mathbf{e}_r\,.$$

This differential equation has the same form as the equation we solved for the one-body problem. We have reached a fundamental result: the motion of

mass μ as defined in (14.46). The two-body problem is reduced to a one-body problem for the motion of a mass μ in the gravitational field of a mass of magnitude $M+m$.

Note that m and M enter the problem in a completely symmetrical way. We could equally well have used $(-\mathbf{r})$ for a description of the motion.

We shall briefly show how the two constants of motion, the angular momentum $\mathbf{L}$ and the mechanical energy E may be expressed by the reduced mass μ.

See again Figure 14.9. Consider first the total angular momentum $\mathbf{L}$ with respect to CM:

$$\mathbf{L}_{\mathrm{CM}} = \mathbf{r}_2 \times M\dot{\mathbf{r}}_2 + \mathbf{r}_1 \times m\dot{\mathbf{r}}_1\,.$$

Eliminate M by means of the definition of CM:

$$-M\mathbf{r}_2 = m\mathbf{r}_1\,.$$

We get

$$\begin{aligned} \mathbf{L}_{\mathrm{CM}} &= -\mathbf{r}_2 \times m\dot{\mathbf{r}}_1 + \mathbf{r}_1 \times m\dot{\mathbf{r}}_1 \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times m\dot{\mathbf{r}}_1\,. \end{aligned}$$

The term $m\dot{\mathbf{r}}_1$ may be rewritten:

$$\begin{aligned} m\dot{\mathbf{r}}_1 &= \frac{m(m+M)}{m+M}\dot{\mathbf{r}}_1 \\ &= \frac{m}{m+M}(m\dot{\mathbf{r}}_1 + M\dot{\mathbf{r}}_1) \\ &= \frac{m}{m+M}(-M\dot{\mathbf{r}}_2 + M\dot{\mathbf{r}}_1) \\ &= \mu\dot{\mathbf{r}}\,. \end{aligned}$$

The total angular momentum for the two masses in their motion around CM is

$$\mathbf{L}_{\mathrm{CM}} = \mathbf{r} \times \mu\dot{\mathbf{r}}\,.$$

We conclude: the angular momentum $\mathbf{L}_{\mathrm{CM}}$ may be calculated *as if* the mass μ moved around M. For this calculation the mass M may be taken to be at rest in an inertial frame.

Next we calculate the total mechanical energy:

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{r}}_1^2 + \frac{1}{2}M\dot{\mathbf{r}}_2^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 + \frac{1}{2}M\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 - \frac{GMm}{r}\,, \end{aligned}$$

or, using $\dot{\mathbf{r}}_2 = -(m/M)\dot{\mathbf{r}}_1$, Making use of

$$\dot{\mathbf{r}}_1 = \frac{M}{m+M}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2) = \frac{M}{m+M}\dot{\mathbf{r}},$$

we find

$$E = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{GMm}{r} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{G(M+m)\mu}{r}.$$

We conclude that the total energy E may be calculated *as if* the mass μ moved around M. For this calculation the mass M may be taken to be at rest in an inertial frame.

14.7.1 The Two-Body Problem and Kepler's 3rd Law

The differential equations describing the one-body problem and the two-body problem are of similar form. With an easily understandable notation the equations may be written as

$$\begin{aligned} \text{one-body:} \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= -\frac{GM}{r^2}\frac{\mathbf{r}}{r}, \\ \text{two-body:} \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= -\frac{G(M+m)}{r^2}\frac{\mathbf{r}}{r}. \end{aligned}$$

For the one-body problem we found Kepler's third law:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}.$$

For the two-body problem the corresponding expression becomes

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}.$$

The ratio T^2/a^3 is thus not exactly the same for all planets, due to the fact that m varies from planet to planet. Due to the large mass of the Sun compared to planetary masses the deviations from planet to planet are small.

14.8 Double Stars: The Motion of the Heliocentric Reference Frame

Many stars are double stars, i.e., two neighboring stars moving under their mutual gravitational interaction. For simplicity we assume that the two stars move in circles around their common CM.

The distance from CM to mass M is called R, and the distance from CM to m is denoted r ($mr = MR$).

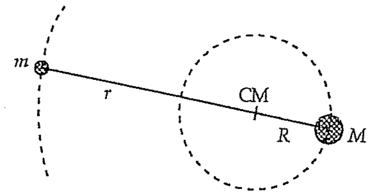


Fig. 14.10. Double stars

moving around M. The radius in that circular motion is called ρ. We have: $\rho = r + R$.

$$\mu\rho\omega^2 = G\frac{Mm}{\rho^2}, \quad \mu \equiv \frac{Mm}{M+m}.$$

For the angular frequency we find

$$\omega^2 = \frac{G(M+m)}{\rho^3}.$$

The period of revolution is determined by $\omega T = 2\pi$.

Relative to what do the two stars move? It would be absurd to use the heliocentric reference frame!

Our own solar system is "nearly a double star system". The mass of Jupiter dominates the planetary system. In units of the mass of the Earth the masses in the solar system are:

Sun: 332 946
Jupiter: 317.9
Saturn: 95.2
The rest of the planets together: 33.7

Neglecting the mass of all the planets except the mass of Jupiter, we can estimate the position of the CM of the solar system. The distance of Jupiter from the Sun is

$$5.2 \text{ AU} = 5.2 \times 1.5 \times 10^8 \text{ km},$$

$$R_{\mathrm{CM}} \approx \frac{m_{\mathrm{J}}}{M_{\mathrm{S}}} r_{\mathrm{J}} \approx 744\,750 \text{ km}.$$

The radius of the Sun is 700 000 km. The CM of the solar system is thus located about 50 000 km above the surface of the Sun.

A coordinate frame with its origin in the center of the Sun, i.e., the heliocentric reference frame, has an acceleration relative to the CM of the solar system. The CM of the solar system moves around the center of the galaxy. The Sun is about 30 000 light years or 3×10^{22} cm from the galactic center.

$$T \approx 8 \times 10^{15}\ \mathrm{s}.$$

The acceleration of the CM relative to the galactic center is thus

$$a = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2} \approx 1.9 \times 10^{-6}\ \mathrm{cm\,s^{-2}}.$$

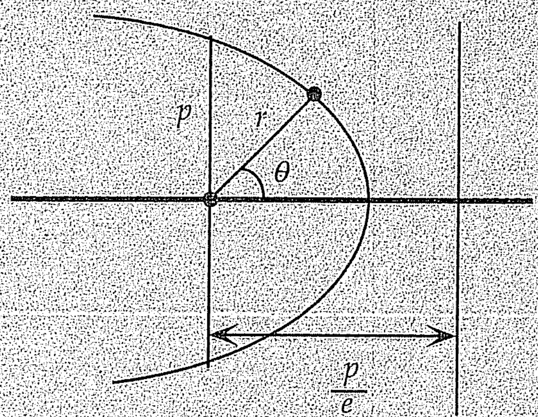
The tides in the freely falling heliocentric reference frame are so small that we have not been able to measure them (yet). Therefore we use the heliocentric reference frame as a local inertial reference frame.

From measurements on radioactive isotopes in minerals in meteorites (and in rocks from the Moon and the Earth) we know that the solar system formed 4.6×10^9 years ago. The solar system has completed

$$\frac{4.6 \times 10^9}{2.5 \times 10^8} \approx 18$$

revolutions around the center of the galaxy, since the system was born. What did we meet on this long journey? Supernova explosions? Interstellar clouds?

14.9 Review: Kepler Motion



Conic sections:

$$\frac{1}{r} = \frac{1}{p}[1 + e\cos\theta]$$

$$\frac{1}{r} = \frac{Gm^2M}{L^2}\left[1 + \left(1 + \frac{2EL^2}{G^2m^3M^2}\right)^{\frac{1}{2}}\cos\theta\right]$$

Conservation laws:

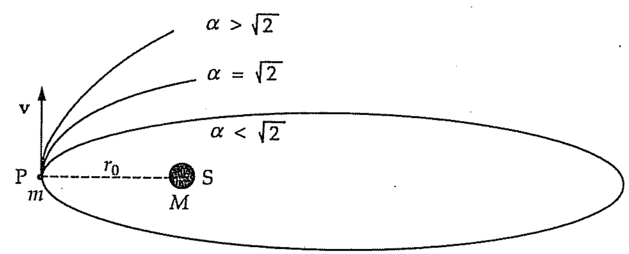
$$L = mr^2\dot{\theta} = \text{constant}$$

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r} = \text{constant}$$

14.10 Examples

In this section we discuss a few examples of motion in the solar system.

Example 14.1. Planetary Orbits and Initial Conditions. We consider a family of possible orbits for a planet around the Sun, S. The Sun has the mass M. The planet, P, with mass m, is imagined to be started with a velocity always perpendicular to the line SP, but with various values of the magnitude of the initial velocity. The initial distance between S and P is r_0.



We begin by determining the magnitude of the velocity necessary for a uniform circular motion:

$$v_0 = \sqrt{\frac{GM}{r_0}}.$$

The planet is then imagined to be started with an arbitrary magnitude of the velocity v_P (still perpendicular to SP). We introduce the ratio

$$\alpha \equiv \frac{v_P}{v_0}.$$

The value $\alpha = 1$ thus corresponds to a circular orbit. Below we shall show that

$$\begin{aligned} &\text{for } \alpha = \sqrt{2} \text{ the orbit is a parabola,}\\ &\text{for } \alpha < \sqrt{2} \text{ the orbit is an ellipse,}\\ &\text{for } \alpha > \sqrt{2} \text{ the orbit is a hyperbola} \end{aligned}$$

$$
\begin{aligned}
E &= \frac{1}{2}mv_p^2 - \frac{GMm}{r_0} \\
&= \frac{1}{2}mv_0^2\alpha^2 - \frac{GMm}{r_0} \\
&= \frac{1}{2}(\alpha^2-1)mv_0^2 + \frac{1}{2}mv_0^2 - \frac{GMm}{r_0}\,.
\end{aligned}
$$

The last two terms in this expression form the energy E_0 in a circular orbit.

$$E = E_0 + \frac{1}{2}(\alpha^2-1)mv_0^2\,.$$

Question: Show that the total energy in a circular orbit may be written $E_0 = -\frac{1}{2}mv_0^2$.

We finally obtain

$$E = E_0(2-\alpha^2)\,,$$

or, as E_0 is negative,

$$E = (\alpha^2-2)\mid E_0\mid\,.$$

From this result we see that:

$$
\begin{array}{lllll}
\text{for } \alpha & > & \sqrt{2} & E>0 & \text{(hyperbola)}, \\
\text{for } \alpha & = & \sqrt{2} & E=0 & \text{(parabola)}, \\
\text{for } \alpha & < & \sqrt{2} & E<0 & \text{(ellipse)}.
\end{array}
$$

The shape of the orbit is determined not only by Newton's laws but also by the initial conditions. This fact makes it possible to find out something about the origin of the solar system. The fact that the planetary orbits lie nearly in the same plane has something to do with the initial conditions of the system, and is not dictated by the laws of force and motion. See Example 10.2. △

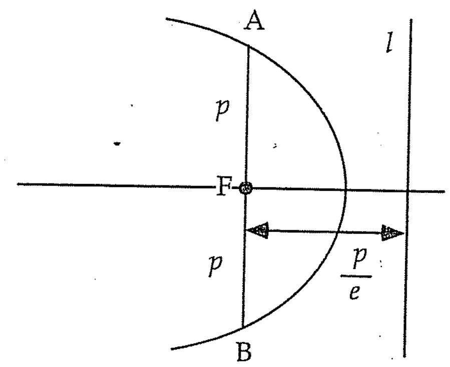
Example 14.2. Shape and Size of Planetary Orbits. Consider the figure (see also Figure 14.4 for the definition of conic sections).

We draw the chord through the focal point F and perpendicular to the axis of the conic section. The points where the chord intersects the conic section are denoted A and B. All conic sections with the same value of the parameter p pass through A and B. The shape of the conic section is determined by the eccentricity, which again is determined by the distance $d = p/e$ to the directrix l.

We have (for a celestial body):

$$p = \frac{L^2}{GMm^2}\,, \quad e = \sqrt{1 + \frac{2EL^2}{G^2m^3M^2}}\,.$$

From this, the orbit for all celestial bodies with the same magnitude of



The eccentricity, and consequently the shape of the conic section, is also determined by the total energy E.

Question. Does the value of m influence the size and shape of the orbit?

Answer. No. We proceed to show that the semi-major axis of an elliptical orbit depends only on E.

For an ellipse $p = a(1-e^2)$. From p and e as given above we get (note $E<0$)

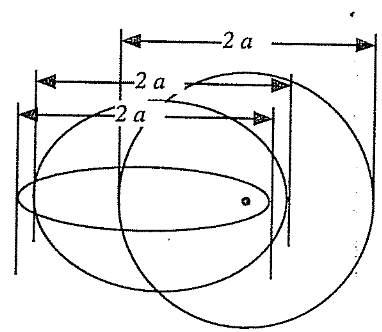
$$a = \frac{GMm}{2(-E)}\,,$$

or

$$E = -\frac{GMm}{2a}\,.$$

For a circle, $a = \frac{1}{2}(r+r) = r$.

Consider the following figure showing elliptical orbits with the same semi-major axis a.



Of all planetary orbits, with the same angular momentum, the circle has the lowest energy E. This is seen from

$$\frac{2EL^2}{G^2M^2m^3}+1=e^2\,,$$

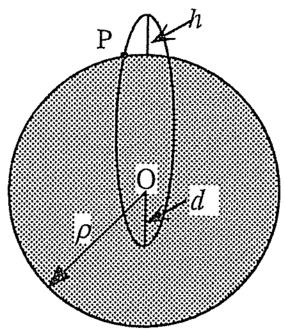
or

$$E=\frac{(e^2-1)G^2M^2m^3}{2L^2}\,.$$

For a circular orbit $e^2=0$.

$\triangle$

Example 14.3. Motion Near the Surface of the Earth



The trajectory of a cannonball near the surface of the Earth is – neglecting air resistance – a parabola. The approximation made is that the acceleration due to gravity is a constant vector.

The approximation is very good indeed, but strictly speaking we should consider the top point in the orbit as the aphelion (apogee) in an elongated ellipse, where the center 0 of the Earth is in one of the foci of the ellipse.

Assume that a cannonball is fired with the initial velocity v_0 and from the point P. Assume further more that the Earth is at rest in an inertial frame. The total energy in an elliptical orbit is determined exclusively by the semi-major axis a. The energy E is

$$E=\frac{1}{2}mv_0^2-\frac{GMm}{\rho}=-\frac{GMm}{2a}\,,$$

where

$$\rho \equiv \text{radius of the Earth,}$$
$$M \equiv \text{mass of the Earth,}$$

The major axis $2a$ is only slightly different from the radius of the Earth, ρ. We write

$$2a=\rho+\triangle=\rho\left(1+\frac{\triangle}{\rho}\right)\,.$$

Let us estimate the magnitude of $\triangle$:

$$\frac{1}{2}mv_0^2-\frac{GMm}{\rho}=-\frac{GMm}{\rho}\left(1+\frac{\triangle}{\rho}\right)^{-1}\,,$$

$$\frac{1}{2}mv_0^2-\frac{GMm}{\rho}\approx-\frac{GMm}{\rho}\left(1-\frac{\triangle}{\rho}\right)\,.$$

We thus obtain

$$\triangle\approx\frac{v_0^2}{2GM/\rho^2}=\frac{v_0^2}{2g}\,,$$

where g is the acceleration of gravity at the surface of the Earth. We have that $\triangle=h+d$ (see the figure).

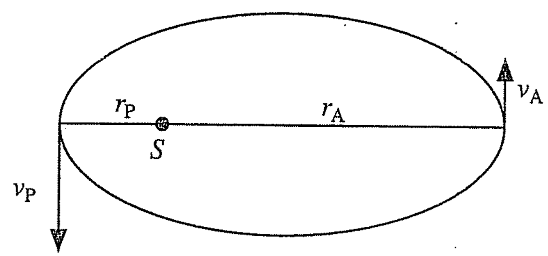
If the start velocity of the cannonball is $v=1\ \mathrm{km\,s^{-1}}$ we find that

$$\triangle\cong\frac{v^2}{2g}\approx 51\ \ \text{km.}$$

The shape of the elliptic orbit depends not only on the speed v_0 but also on the firing angle.

Note. Close to the perihelion it is difficult to distinguish a "long ellipse" from a parabola. For instance, many of the comets observed until now are in orbits with excentricities very close to 1, i.e., many comets are in orbits that are nearly parabolic. $\triangle$

Example 14.4. Velocities in an Elliptical Orbit



The velocity of a planet in the perihelion (perigæum) is v_P and the corresponding velocity in the aphelion (apogæum) is v_A. Both velocities are perpendicular to the axis of the ellipse, and measured relative to the heliocentric

From conservation of angular momentum:

$$m r_{\mathrm{A}} v_{\mathrm{A}} = m r_{\mathrm{P}} v_{\mathrm{P}} \,,$$

$$v_{\mathrm{A}} = \frac{r_{\mathrm{P}}}{r_{\mathrm{A}}} v_{\mathrm{P}} = \frac{1-e}{1+e} v_{\mathrm{P}} \,.$$

From conservation of energy:

$$\frac{1}{2} m v_{\mathrm{A}}^2 - \frac{GMm}{a(1+e)} = \frac{1}{2} m v_{\mathrm{P}}^2 - \frac{GMm}{a(1-e)} \,.$$

By using $v_{\mathrm{A}} = (1-e) v_{\mathrm{P}}/(1+e)$ we obtain

$$v_{\mathrm{P}} = \sqrt{\frac{GM}{a} \frac{1+e}{1-e}} \,,$$

$$v_{\mathrm{A}} = \sqrt{\frac{GM}{a} \frac{1-e}{1+e}} \,.$$

For the planet Earth:

$$e = 0.0167, \qquad a = 1 \text{ AU} = 149.6 \times 10^6 \text{ km},$$

$$\sqrt{\frac{GM}{a}} = 29.78 \text{ km s}^{-1}, \qquad \sqrt{\frac{1+e}{1-e}} = 1.0168,$$

$$v_{\mathrm{P}} = 30.3 \text{ km s}^{-1}, \qquad v_{\mathrm{A}} = 29.3 \text{ km s}^{-1}$$

For the planet Mars:

$$e = 0.0933, \qquad a = 1.524 \text{ AU},$$

$$\sqrt{\frac{GM}{a}} = 24.24 \text{ km s}^{-1}, \qquad \sqrt{\frac{1+e}{1-e}} = 1.098,$$

$$v_{\mathrm{P}} = 26.6 \text{ km s}^{-1}, \qquad v_{\mathrm{A}} = 22.1 \text{ km s}^{-1}.$$

△

Example 14.5. Hohman Orbit to Mars. When a spaceship is sent to another planet, the ship is first placed in a so-called parking orbit around the Earth. To enter the transfer orbit, the spaceship must leave the parking orbit and escape the gravitational field of the Earth.

The rocket engine delivers the thrust necessary for placing the spaceship in the interplanetary orbit. Exactly when the rocket engines should be started depends on the relative position of the Earth and the planet of destination.

In calculating the transfer orbit from the Earth to another planet we shall make a series of simplifying assumptions. We ignore the binding energy of

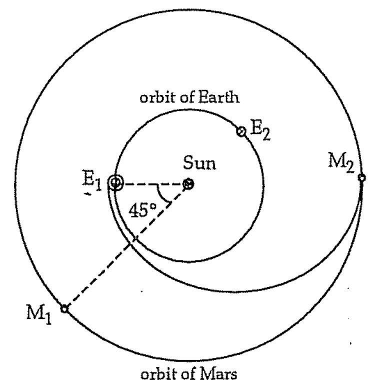


Fig. 14.11. Hohman orbit to Mars. Launch window: E_1 Earth at launch; E_2 Earth at arrival; M_1 Mars at launch; M_2 Mars at arrival

We shall briefly discuss a journey from the Earth to Mars along the so-called *Hohman orbit*, named after the astronomer who first calculated this transfer orbit.

The launch should take place as the spacecraft is on the dark side of the Earth. The velocity of the spacecraft in the parking orbit is then in the same direction as the velocity of the Earth in its orbit around the Sun.

Let us now assume that the spacecraft is nearly free of the gravitational field of the Earth, i.e., we neglect the gravitational field of the Earth. The velocity of the spacecraft in the heliocentric reference frame is assumed to be the same as the orbital velocity of the Earth in this frame ($\approx$ 30 km/s). As we shall demonstrate below only a rather modest increase in the velocity of the spacecraft relative to the heliocentric reference frame is necessary to bring the craft into an elliptical orbit towards Mars.

The Hohman orbit is tangential to the orbit of the Earth at launch, and tangential to the orbit of Mars at arrival. The Hohman orbit is thus a semi-elliptical orbit, whose perihelion coincides with the orbit of the Earth and whose aphelion coincides with the orbit of Mars.

The exact calculation of a Hohman orbit is involved, particularly due to the fact that the plane of the orbit of Mars is slightly tilted relative to ecliptica ($i = 1°51'$).

The essential aspects of the determination of the transfer orbit are nevertheless present in the calculations below, in terms of our simplified model.

Assume that the orbits of the Earth and Mars are in the same plane, the ecliptica. Furthermore, assume that the Earth and Mars perform uniform circular motions around the Sun, with the Sun located in the common center of the orbits.

The radius in the orbit of the Earth is 1 AU, and the radius in the orbit of Mars is 1.52 AU. With the assumption of a circular orbit, the velocity of

The semi-major axis for the Hohman orbit becomes

$$a_H = \frac{1+1.52}{2} = 1.26 \text{ AU}.$$

The trip to Mars along the Hohman orbit may be determined from Kepler's third law. The time for one complete revolution in a Hohman orbit is denoted T_H. Let T_H be measured in years. The time for a complete revolution of the Earth (one year) is called $T_E = 1$ year, Then, from Kepler's third law:

$$\frac{T_H^2}{a_H^3} = \frac{T_E^2}{a_E^3} = \frac{1^2}{1^3} = 1,$$

$$T_H = a_H^{3/2} = 1.414 \text{ years}.$$

The travel time τ to Mars corresponds to one half revolution:

$$\tau = 258 \text{ days}.$$

The sidereal time of revolution for Mars is 687 days. As the spaceship has moved along the Hohman orbit, Mars has moved

$$360^\circ \frac{258}{687} \cong 135^\circ .$$

If Mars is 45° ahead of the Earth at launch, the spaceship will meet Mars at the point where the Hohman orbit touches the orbit of Mars.

The velocity at launch. The spaceship is in the perihelion of the Hohman orbit at launch. The initial velocity should then be

$$v_P = \sqrt{\frac{GM}{a_H}\frac{1+e}{1-e}} = \sqrt{\frac{GM}{a_H}\frac{r_A}{r_P}},$$

where $a_H = 1.26$ AU.

For the Earth we know that

$$\sqrt{\frac{GM}{a_E}} = 29.8 \text{ km s}^{-1}.$$

For v_P we obtain

$$v_P = 29.8\sqrt{\frac{1}{1.26}}\sqrt{\frac{1.52}{1}} = 32.7 \text{ km s}^{-1}.$$

When the rocket engine has released the spaceship from the gravitational field of the Earth, the ship has the same velocity as the Earth relative to the heliocentric frame, i.e., 29.8 km s^{-1}. By means of the rocket engine the spaceship should be given an increase in velocity of Δv, where

When this has taken place, the ship will "fall" along the Hohman orbit to Mars, guided by the gravitational field of the Sun.

At the arrival to Mars, the spaceship is in the apehelion of the Hohman orbit. The velocity of the ship is then

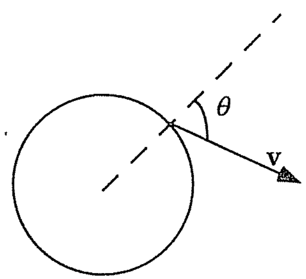
$$v_A = v_P\frac{r_P}{r_A} = 32.7\frac{1}{1.52} \cong 21.5 \text{ km s}^{-1}.$$

If the spaceship is bound to enter an orbit around Mars the rockets must adjust the velocity of the ship to the velocity of the planet.

Even if the Hohman orbit is inexpensive from the point of view of fuel, it will not be used in the manned expedition to Mars, due to the long time of flight. △

14.11 Problems

Problem 14.1.



Assume that the Earth is at rest in an inertial frame. A rocket is started, not along the vertical, but in a direction forming an angle θ with the vertical.

(1) Calculate the magnitude of the start velocity v_0, when it is assumed that the rocket just escapes the gravitational field of the Earth.

Remark: the launch facilities of the European Space Agency are located in South America. Why not in, say, northern Norway?

(2) This question deals with the escape velocity from the solar system from a point in the orbit of the Earth: Assume that a rocket interacts only with the gravitational field of the Sun. Determine the smallest velocity v relative to the sun that a spacecraft should be given at the distance of 1 AU from the Sun, so that the spacecraft leaves the solar system.

mechanical energy E and the magnitude L of the angular momentum for the planet, and show explicitly that the excentricity e is zero. Use

$$e^2 = 1 + \frac{2EL^2}{G^2 m^3 M^2},$$

M = mass of Sun,
m = mass of planet,
G = gravitational constant.

Problem 14.3. In a double star system (also called a binary star) one of the stars has the mass $m = 3 \times 10^{30}$ kg and the other has the mass $M = 4 \times 10^{30}$ kg.

Each of the stars performs a uniform circular motion around the center of mass (CM) of the system and relative to an inertial frame. The stars may be considered as mass points. The distance between the stars is 10^{13} m.

(1) Determine the angular velocity ω of the motion of the stars.
(2) Determine the magnitude of the total inner angular momentum of the system, i.e., determine $L_{CM} = |\ \mathbf{L}_{CM}\ |$, the angular momentum relative to the CM of the system.

Problem 14.4. Consider the Earth–Moon as an isolated two body system. The Earth and the Moon are assumed to move in circular orbits around the center of mass (CM) of the system.

(1) Determine the position of the center of mass (CM) for the Earth–Moon system.
(2) Determine the orbital speed and the orbital period of revolution of the Moon. (The CM is assumed to be at rest in an inertial system.)

Problem 14.5. Comet Halley orbits the Sun in an elliptical orbit. At perihelion, the distance of the comet from the Sun is 87.8 $\times 10^6$ km. At aphelion the distance from the Sun is 5280 $\times 10^6$ km.

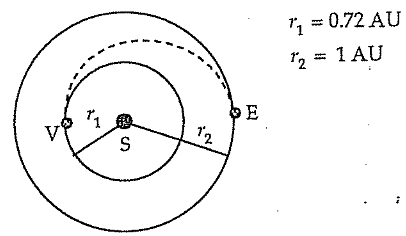
(1) Calculate the period of the comet.
(2) Calculate the speed of the comet relative to the heliocentric reference system when the comet is in the perihelion (V_P) and when the comet is in the aphelion (V_A).

Problem 14.6. The first artificial satellite, the *Sputnik 1*, was launched on October 4, 1957. Sputnik 1 had a perihelion of 227 km above the surface of the Earth. The speed at perihelion was 8 km s^{-1}, measured relative to the

(1) Determine the height above the surface of the Earth that Sputnik 1 had at aphelion.
(2) Determine the orbital period of revolution for Sputnik 1.

Problem 14.7. This problem deals with a Hohman transfer orbit to Venus.

Assume that the Earth and Venus move in the same plane (ecliptica) and in circular orbits around the Sun. The radius of the orbit of Venus is 0.72 AU. (1 AU = 1.5×10^8 km). Compare the present problem with Example 14.5. The



spacecraft is in a parking orbit around the Earth. The launch into a Hohman orbit to Venus (an inner planet) occurs when the spacecraft emerges onto the sunlit side of the Earth. The initial velocity of the spacecraft includes two contributions: the orbital velocity of the Earth about the Sun plus the orbital velocity of the spacecraft around the Earth. When the spacecraft is on the sunlit side of the Earth these contributions are in opposite directions. A rocket thrust in the direction of the orbital motion of the craft around the Earth will allow the spacecraft to escape from the gravitational field of the Earth.

Once the spacecraft is essentially free of the influence of the Earth, the spacecraft will move in an elliptical orbit around the Sun, with an initial speed v_0 relative to the heliocentric reference frame. Note: $2a = 1.72$ AU.

(1) Determine v_0 such that the spacecraft enters a Hohman transfer orbit to Venus (compare with Example 14.5). Show that $v_0 < v_E$, where v_E is the orbital speed of the Earth around the Sun.
(2) Determine the travel time τ to Venus.
(3) Determine the speed v_1 of the spacecraft when it reaches Venus. Show that $v_1 > v_V$, where v_V is the orbital speed of Venus in the heliocentric reference frame.
(4) Discuss the relative positions at launch of Earth and Venus necessary for a Hohman transfer orbit to be realized.

9.2 Solar system data

Solar data

equatorial radius	$R_\odot$	=	$6.960\times10^{8}\,\mathrm{m}$	=	$109.1R_\oplus$
mass	$M_\odot$	=	$1.9891\times10^{30}\,\mathrm{kg}$	=	$3.32946\times10^{5}M_\oplus$
polar moment of inertia	$I_\odot$	=	$5.7\times10^{46}\,\mathrm{kg\,m^2}$	=	$7.09\times10^{8}I_\oplus$
bolometric luminosity	$L_\odot$	=	$3.826\times10^{26}\,\mathrm{W}$		
effective surface temperature	$T_\odot$	=	$5770\,\mathrm{K}$		
solar constant[a]			$1.368\times10^{3}\,\mathrm{W\,m^{-2}}$		
absolute magnitude	M_V	=	$+4.83$; M_{bol}	=	$+4.75$
apparent magnitude	m_V	=	-26.74; m_{bol}	=	-26.82

[a]Bolometric flux at a distance of 1 astronomical unit (AU).

Earth data

equatorial radius	$R_\oplus$	=	$6.378\,14\times10^{6}\,\mathrm{m}$	=	$9.166\times10^{-3}R_\odot$
flattening[a]	f	=	$0.003\,353\,64$	=	$1/298.183$
mass	$M_\oplus$	=	$5.9742\times10^{24}\,\mathrm{kg}$	=	$3.0035\times10^{-6}M_\odot$
polar moment of inertia	$I_\oplus$	=	$8.037\times10^{37}\,\mathrm{kg\,m^2}$	=	$1.41\times10^{-9}I_\odot$
orbital semi-major axis[b]	1 AU	=	$1.495979\times10^{11}\,\mathrm{m}$	=	$214.9R_\odot$
mean orbital velocity			$2.979\times10^{4}\,\mathrm{m\,s^{-1}}$		
equatorial surface gravity	g_e	=	$9.780327\,\mathrm{m\,s^{-2}}$		(includes rotation)
polar surface gravity	g_p	=	$9.832186\,\mathrm{m\,s^{-2}}$		
rotational angular velocity	ω_e	=	$7.292115\times10^{-5}\,\mathrm{rad\,s^{-1}}$		

[a]f equals $(R_\oplus - R_{\mathrm{polar}})/R_\oplus$. The mean radius of the Earth is $6.3710\times10^{6}\,\mathrm{m}$.
[b]About the Sun.

Moon data

equatorial radius	R_m	=	$1.7374\times10^{6}\,\mathrm{m}$	=	$0.27240R_\oplus$
mass	M_m	=	$7.3483\times10^{22}\,\mathrm{kg}$	=	$1.230\times10^{-2}M_\oplus$
mean orbital radius[a]	a_m	=	$3.84400\times10^{8}\,\mathrm{m}$	=	$60.27R_\oplus$
mean orbital velocity			$1.03\times10^{3}\,\mathrm{m\,s^{-1}}$		
orbital period (sidereal)			$27.32166\,\mathrm{d}$		
equatorial surface gravity			$1.62\,\mathrm{m\,s^{-2}}$	=	$0.166g_e$

[a]About the Earth.

Planetary data[a]

	$M/M_\oplus$	$R/R_\oplus$	T(d)	P(yr)	a(AU)
Mercury	0.055 274	0.382 51	58.646	0.240 85	0.387 10
Venus[b]	0.815 00	0.948 83	243.018	0.615 228	0.723 35
Earth	1	1	0.997 27	1.000 04	1.000 00
Mars	0.107 45	0.532 60	1.025 96	1.880 93	1.523 71
Jupiter	317.85	11.209	0.413 54	11.861 3	5.202 53
Saturn	95.159	9.449 1	0.444 01	29.628 2	9.575 60
Uranus[b]	14.500	4.007 3	0.718 33	84.746 6	19.293 4
Neptune	17.204	3.882 6	0.671 25	166.344	30.245 9
Pluto[b]	0.002 51	0.187 36	6.387 2	248.348	39.509 0

M	mass
R	equatorial radius
T	rotational period
P	orbital period
a	mean distance
$M_\oplus$	$5.9742\times10^{24}\,\mathrm{kg}$
$R_\oplus$	$6.378\,14\times10^{6}\,\mathrm{m}$
1 d	86400 s
1 yr	$3.15569\times10^{7}\,\mathrm{s}$
1 AU	$1.495979\times10^{11}\,\mathrm{m}$

[a]Using the osculating orbital elements for 1998. Note that P is the instantaneous orbital period, calculated from the planet's daily motion. The radii of gas giants are taken at 1 atmosphere pressure.
[b]Retrograde rotation.

about the other principal axis is often called the *tennis racket theorem.* The conclusions of this theorem can be readily demonstrated by throwing this book (with a rubber band around it) or other oblong object into the air with a spin about one of the principal axes. The detailed nature of the spin about the stable axes is similar to the free symmetric top discussed in the next section.

7.8 The Earth as a Free Symmetric Top

Since the earth is nearly spherical in shape, the gravitational torques exerted on the earth by the sun and the moon are quite small. To a good approximation the rotational motion can therefore be described by Euler's equations with no external torques. Since the earth is nearly axially symmetric, the principal moments of inertia for the two axes in the equatorial plane are equal.

$$I_1 = I_2 = I \tag{7.116}$$

The third principal axis with moment of inertia I_3 is along the polar symmetry axis. From (7.88) the differential equations for the earth's motion in an earth-based coordinate frame are

$$\begin{aligned} \dot{\omega}_1 + \frac{I_3 - I}{I}\omega_3\omega_2 &= 0 \\ \dot{\omega}_2 - \frac{I_3 - I}{I}\omega_1\omega_3 &= 0 \\ \dot{\omega}_3 &= 0 \end{aligned} \tag{7.117}$$

Any rigid body which obeys this set of torque-free equations is called a *free axially symmetric top.* The exact solution to this coupled set of equations is easily obtained. The last equation above implies that ω_3 is constant.

$$\omega_3(t) = \omega_3(0) = \omega_3 \tag{7.118}$$

The equations (7.117) can be solved using the method of (7.97)–(7.101). The solution is

$$\begin{aligned} \omega_1(t) &= a\cos(\Omega t + \alpha) \\ \omega_2(t) &= a\sin(\Omega t + \alpha) \end{aligned} \tag{7.119}$$

where

$$\Omega = \omega_3\left(\frac{I_3 - I}{I}\right) \tag{7.120}$$

The magnitude of the angular-velocity vector $\boldsymbol{\omega}$ is

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^3} = \sqrt{a^2 + \omega_3^2} \tag{7.121}$$

Since the components ω_1 and ω_2 in (7.119) trace out a circle of radius a while ω_3 and ω remain constant, an observer on the earth sees the angular-velocity vector precesses uniformly about the symmetry axis with angular velocity Ω, as shown in Fig. 7-16.

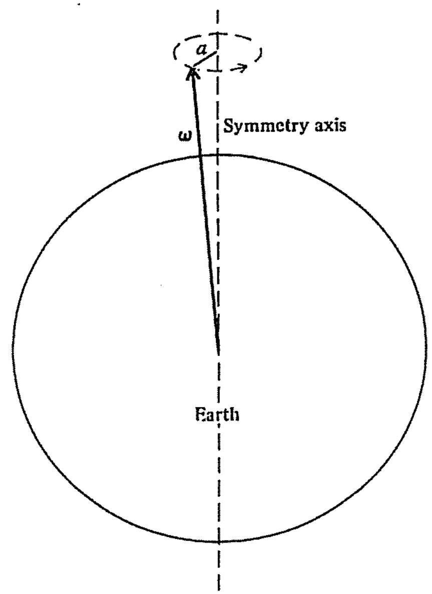


FIGURE 7-16. Precession of the earth's spin about the symmetry axis.

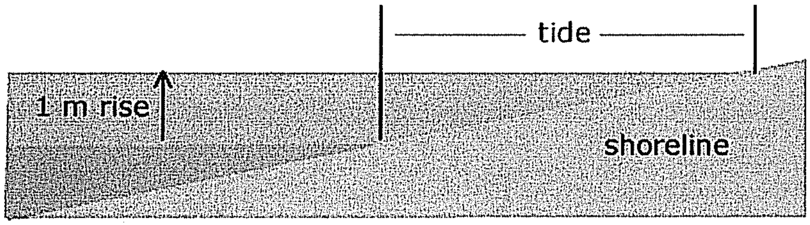
The period of precession of $\boldsymbol{\omega}$ about the earth's symmetry axis is

$$\tau = \frac{2\pi}{\Omega} = \left(\frac{I}{I_3 - I}\right)\frac{2\pi}{\omega_3} \tag{7.122}$$

For the earth, since $2\pi/\omega_3 = 1$ day, the period of precession in days is determined by the moment-of-inertia ratio. For an earth of uniform density and oblate spheroidal shape, the value of this ratio, calculated from the measured radii of the earth, is

$$\frac{I}{I_3 - I} \approx 300 \tag{7.123}$$

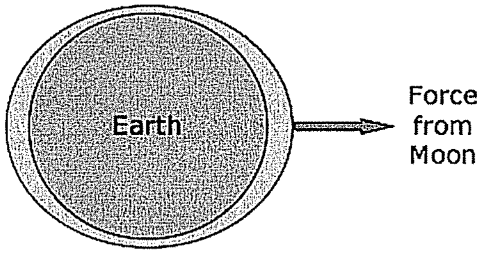
Although the earth becomes more dense toward its center, the moment-of-inertia ratio is not appreciably changed from the uniform-density result.



1 meter rise results in several meter rise in tide

High tide every 12 hours 25 minutes

Since the Earth rotates on its axis, the Moon appears to orbit the Earth and is over head every 24 hours and 50 minutes. The extra 50 minutes is a result of the Moon's 27 day actual orbit around the Earth.



Force from Moon pulls ocean toward it

Although the Moon is overhead every 24 hours and 50 minutes, the high tide comes every 12 hours and 25 minutes. One high tide corresponds to when the Moon is overhead and the other high tide is when the Moon is on the opposite side of the Earth.

Cause of tides on both sides

Since the tides are primarily caused by the gravitation of the Moon acting on the oceans and pulling the surface of the water toward the Moon, you would think the shape of the oceans would be pulled toward the Moon, as opposed to having a high tide on both sides of the Earth. In fact, the configuration seems counter-intuitive.

Simple explanation

A simple explanation for the double tides is that normally a fluid or liquid in space will take on a spherical shape. When you pull or apply a force on one side, the sphere elongates into an oval shape.

Thus, when the Moon pulls the water toward it, the action causes a high tide or bulge on the side of the Earth facing the Moon. But also, the Moon is pulling on the Earth and causes it to move slightly toward it and away from the ocean on the opposite side. This results in the high tide on the side away from the Moon.

Although this explanation is somewhat correct, it really isn't very satisfying.

Theory of the tidal configuration

A more sophisticated explanation is the theory of the tidal configuration which states that the various parts of the Earth's ocean are attracted toward the Moon, according to their separation from the Moon, as well as the angle to the Moon's center. This is also called a gravitational differential.

The force of attraction of the water on the side of the Earth that is closer to the Moon is greater than that on the far side of the Earth. This is represented in the illustration below by the force-line arrows or vectors.

Orbital motion

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Escape velocity

Overview of Gravitational Escape Velocity
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Gravity

Gravity topics
Overview of Force of Gravity

Explanation of how Gravitation Causes Tides on Earth by Ron Kurtus - Succeed in Understanding Physics. Also refer to periodic, sea level, ocean, forces, Moon, Sun, gravitational differential, spring tides, neap tides, physical science, School for Champions.

Gravitation Causes Tides on Earth

by Ron Kurtus (7 September 2010)

Tides are periodic rise and fall of sea levels, as seen in a specific location on the shore. They are caused by the gravitational forces from the Moon and Sun that attract the ocean water toward them and away from other areas in the ocean.

The rotation of the Earth and the position of the Moon cause the level of the tide to change in a given location. There are two high and low tides each day.

Although you would think the rise in water would only occur on the side toward the Moon and Sun, high tides actually occur on opposite sides of the Earth, caused by a gravitational differential.

The orientation of the Moon and Sun with respect to the Earth determine when the highest and lowest tides occur, as well as when the moderate tides occur. At the times of the month when the Moon and Sun are aligned, their combined gravitational pull cause the highest tides. The lowest tides are seen at locations on Earth at right angles to the alignment of the Moon and Sun.

Questions you may have include:

- What causes tides?
- Why are there high tides on both sides of the Earth?
- What role does the alignment of the Moon and Sun have on the tides?

This lesson will answer those questions.

Useful tools: Metric-English Conversion | Scientific Calculator.

Gravitation and tides

If you live near the ocean, you have probably seen the rise and fall of the sea level that happens twice a day. When the sea level is above normal, it is called the high tide. Similarly, low tide is when the sea level on the shore is below normal.

Gravitation from Moon

The gravitational pull on the water from the Moon is the primary cause of the rising tide. Gravitation from the Sun also can contribute to the height of the tide. Centrifugal force on the water from the Earth's rotation also provides a small contribution to the tides.

The gravitational attraction between the Earth and the Moon is **F** = $1.99*10^{20}$ N (*See Gravitational Force Between Two Objects for the calculations*). That force is sufficient to slightly distort the solid surface of both objects toward each other.

Water level rises

Since shape of a body of water can easily be changed, the force from the Moon

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Overview of Gravity and Gravitation

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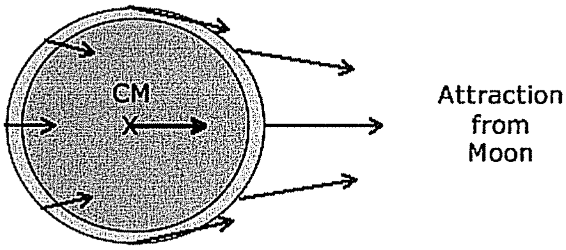
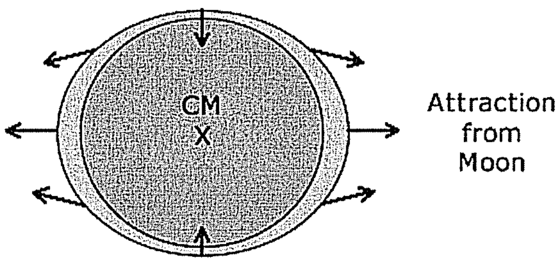


Moon attracts ocean and Earth toward it

But also, the Moon is attracting the mass of the Earth toward it. This can be approximated by considering the mass of the Earth concentrated at its center of mass (CM). This approximation is explained in the *Universal Gravitation Equation* lesson. The heavy vector represents the attraction of the Earth's mass toward the Moon.

If you subtract the force of attraction on the Earth's center of mass from each of the vectors or force lines to the Moon, the resulting forces on the ocean water are toward and away from the Moon on the ends and moving inward on the sides.



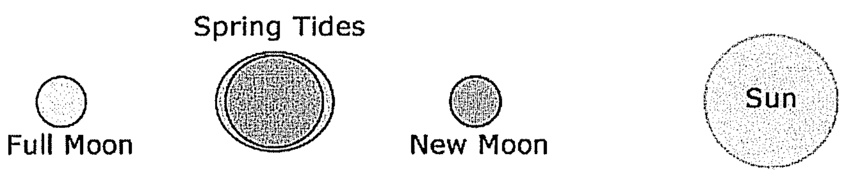
Subtraction of vectors results in double bulge

Tides and orientation of Moon and Sun

Although the gravitational pull from the Moon is the major factor in the creation of the tides, gravitation from the Sun also affects the height of the tide.

When the Sun and the Moon are aligned on the same side of the Earth, it is called a New Moon. With this configuration, the gravitational forces combine and cause a very high tide known as a *spring* tide. The name has nothing to do with the season and actually occurs slightly after the Moon is overhead, due to the inertia of the ocean and the rotation of the Earth.

When the Sun and Moon are on opposite sides of the Earth, each contributes a pull on the water, resulting in another spring tide. The two spring tides occur two weeks apart.



Alignment of Sun and Moon for spring tides

When the Moon is located at a right angle to the Sun with respect to the Earth, it is called the first quarter or third quarter Moon. In such a case, the difference between the high tide and low tide is much smaller, since the gravitational forces cancel each other. These low tides are called *neap* tides.

Since the orbit of the Moon around the Earth is elliptical, once every 1.5 years the Moon is closest to the Earth. This situation results in an unusually high tide called the *proxigean spring tide*.

Summary

High tides occur on opposite sides of the Earth, as do low tides, according to the theory of the tidal configuration. The orientation of the Moon and Sun with respect to the Earth determine when the highest and lowest tides occur, as well as when the moderate tides occur.

You have potential

Resources and references

Author's Credentials

The following resources provide information on this subject:

Websites

Saltwater Tides - Predictions of tides in various U.S. states

Moon Tides - How the Moon affects ocean tides

Tides - Wikipedia

Ocean Tides - NASA - Ocean Motion

Gravitational Tides - Astronomy 221 - Case Western Reserve University

Forces Involved in Making Tides - The Lobsterman's Page

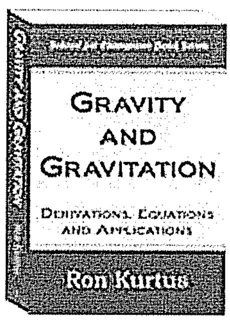
Gravitation Resources

Books

Top-rated books on STides

Top-rated books on Advanced Gravity Physics

Our new ***Gravity and Gravitation*** book is now available!



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Do you have any questions, comments, or opinions on this subject? If so, send an email with your feedback We will try to get back to you as soon as

Tides

M_E R_E

O'

$$\underset{\Delta m \omega^2 r_s}{\overset{F_i}{\parallel}} = \frac{G M_s \Delta m}{r_s^2}$$

1

$$F_1 = \frac{G M_s \Delta m}{(r_s - R_E)^2} = \frac{G M_s \Delta m}{r_s^2 \left(1 - \frac{R_E}{r_s}\right)^2}$$

$R_E \ll r_s$

$$= \frac{G M_s \Delta m}{r_s^2}\left(1 + \frac{2R_E}{r_s}\right)$$

$$F_1' = F_1 - F_i = \frac{2G \boxed{M_s} \Delta m}{\boxed{r_s^3}} R_E \quad > 0$$

$\Leftarrow$

2 $F_i - F_2 > 0$

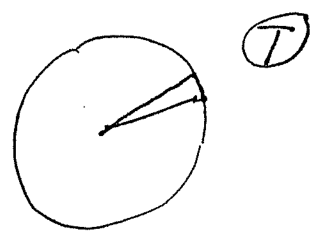
$\Rightarrow$ $$\frac{2 G M_s \Delta m}{r_s^3} R_E > 0$$

$$\frac{M_s}{r_s^3} < \frac{M_M}{r_M^3} \qquad \frac{h_L}{L_\odot} = 2.2$$

Lunar influence

3, 4

一天两次 $T \geq 24$ hr.



$$\left(24 + \frac{24}{27\frac{1}{3}}\right) h$$

53 min

$12 +$ 26 min

slow down

4.4×10^{-8} sec

100 year

28 sec.

More Angular Momentum

Physics 1425 Lecture 22

Michael Fowler, UVa

Torque as a Vector

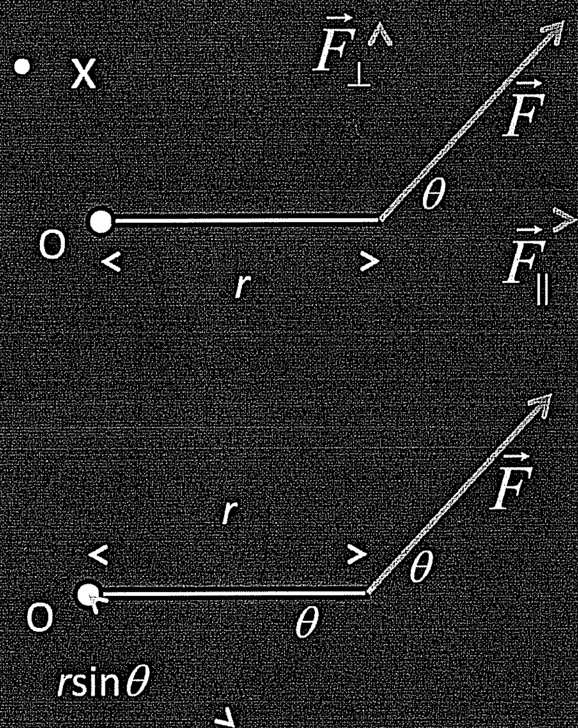
- Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.
- If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

  it's clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!
- BUT this vector $\vec{\tau}$, is, remember made of two other vectors: the force $\vec{F}$ and the place $\vec{r}$ where it acts!

More Torque...

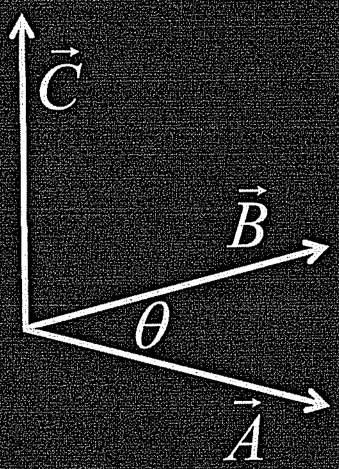
- Expressing the force vector $\vec{F}$ as a sum of components $\vec{F}_{\perp}$ ("fperp") perpendicular to the lever arm and $\vec{F}_{\parallel}$ parallel to the arm, it's clear that only $\vec{F}_{\perp}$ has leverage, that is, torque, about O.
  $\vec{F}_{\perp}$ has magnitude $F\sin\theta$, so $\tau = rF\sin\theta$.
- Alternatively, keep $\vec{F}$ and measure *its* lever arm about O: that's $r\sin\theta$.



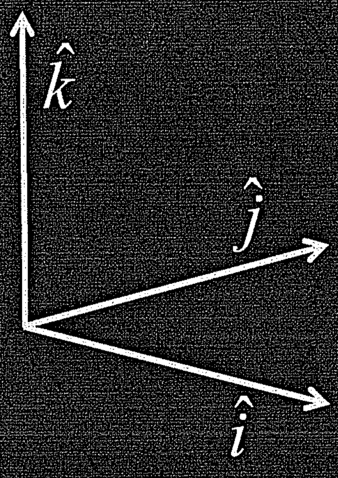
Definition: The Vector Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

- The magnitude C is $AB\sin\theta$, where θ is the angle between the vectors $\vec{A}, \vec{B}$.
- The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, and is your right thumb direction if your curling fingers go from $\vec{A}$ to $\vec{B}$.



The Vector Cross Product in Components



- Recall we defined the unit vectors $\hat{i}, \hat{j}, \hat{k}$ pointing along the *x*, *y*, *z* axes respectively, and a vector can be expressed as $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
- Now $\hat{i}\times\hat{i} = 0,\ \hat{i}\times\hat{j} = \hat{k},\ \hat{i}\times\hat{k} = -\hat{j}, \ldots$
- So

$$\vec{A}\times\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

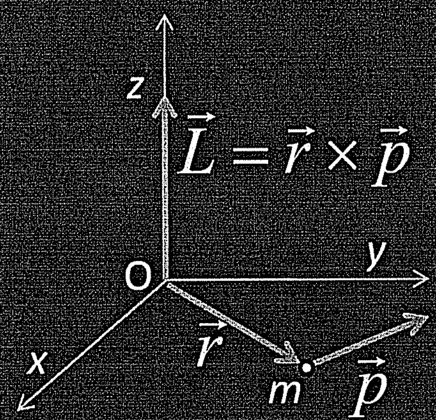
$$= \hat{i}\left(A_yB_z - A_zB_y\right) + \ldots$$

Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space*.



Viewing the *x*-axis as coming out of the slide, this is a "right-handed" set of axes: $\hat{i} \times \hat{j} = +\hat{k}$

Angular Momentum and Torque for a Particle

- Angular momentum <u>about the origin</u> of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Torque <u>about the origin</u>

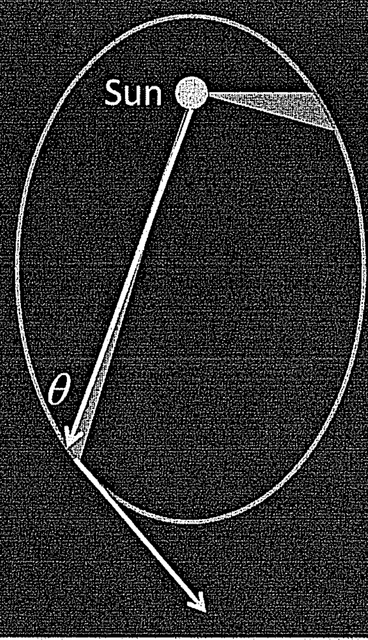
- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Kepler's Second Law

As the planet moves, a line from the planet to the center of the Sun sweeps out equal areas in equal times.

- In unit time, it moves through a distance $\vec{v}$.
- The area of the triangle swept out is ½*rv*sinθ (from ½ base x height)
- This is ½*L*/*m*, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler's Law is telling us <u>the angular momentum about the Sun is constant</u>: this is because the Sun's pull has *zero torque* about the Sun itself.



The base of the thin blue triangle is a distance *v* along the tangent. The height is the perp distance of this tangent from the Sun.

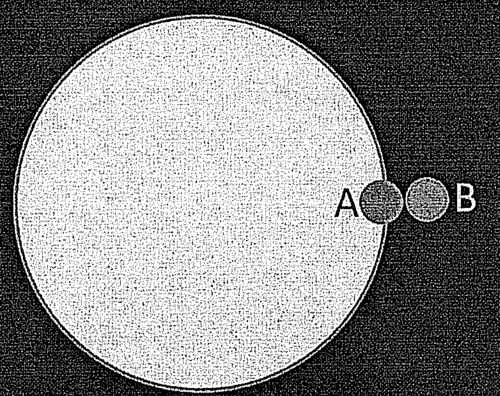
Guy on Turntable

- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, next to B, who's on the ground.
- A now walks around the edge until he's back with B.
- How far does he walk?

A. $2\pi R$

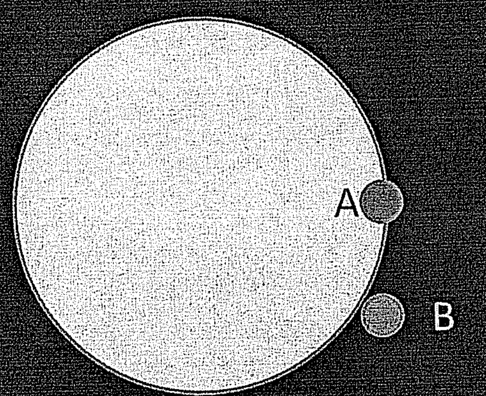
B. $2.5\pi R$

C. $3\pi R$



Guy on Turntable Catches a Ball

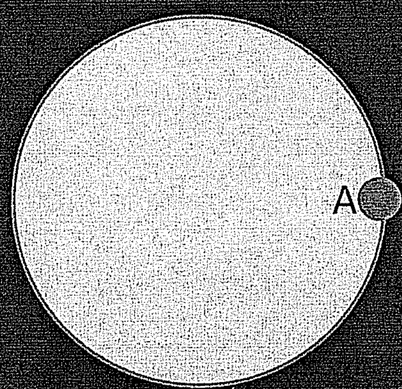
- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, at rest.
- B, who's on the ground, throws a ball weighing $0.1m$ at speed v to A, who catches it without slipping.
- What is the angular momentum of turntable + man + ball now?



A. 0.1mvR
B. (0.1/3.1)mvR
C. (0.1/5.1)mvR

Guy on Turntable Walks In

- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?



A. 12 rpm
B. 9 rpm
C. 6 rpm
D. 4 rpm

Reminder: Angular Momentum and Torque for a Particle...

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Lots of Particles

- Suppose we have particles acted on by external forces, and also acting on each other.
- The rate of change of angular momentum of one of the particles about a fixed origin O is:

$$d\vec{L}_i / dt = \vec{\tau}_{i\ \mathrm{int}} + \vec{\tau}_{i\ \mathrm{ext}}$$

- The internal torques come in equal and opposite pairs, so

$$d\vec{L} / dt = \sum_i d\vec{L}_i / dt = \sum_i \vec{\tau}_{i\ \mathrm{ext}}$$

Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

$$d\vec{L}/dt = \sum_i \vec{\tau}_i$$

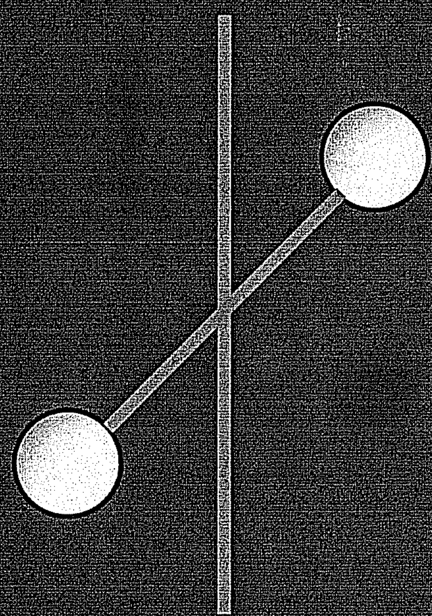
the vector sum of the applied torques, $\vec{L}$ and the $\vec{\tau}_i$ being measured about a fixed origin O.

- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$\boxed{d\vec{L}_{\text{CM}}/dt = \sum \vec{\tau}_{\text{CM}}}$$

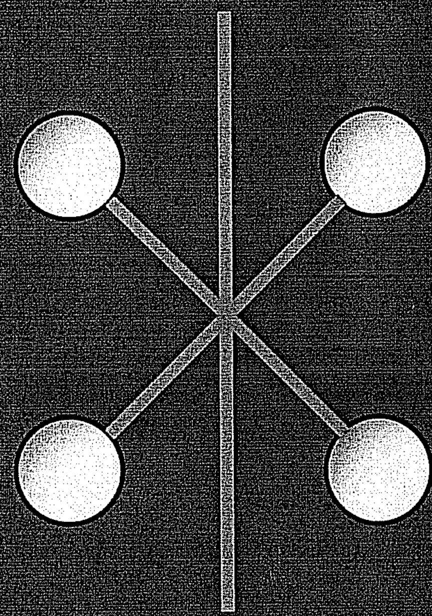
Angular Velocity and Angular Momentum Need not be Parallel

- Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Think about this at the instant the balls are in the plane of the slide—so is $\vec{L}$, but it's not vertical!



When *are* Angular Velocity and Angular Momentum Parallel?

- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there's an equal mass at the corresponding point on the other side.
- For this pair of masses, $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ *is* along the axis.
- (Check it out!)



Cavendish experiment

From Wikipedia, the free encyclopedia

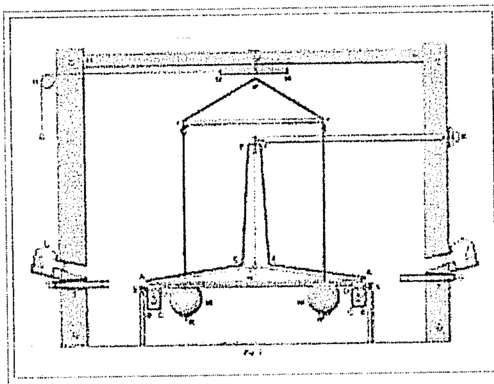
The **Cavendish experiment**, performed in 1797–98 by British scientist Henry Cavendish was the first experiment to measure the force of gravity between masses in the laboratory,[1] and the first to yield accurate values for the gravitational constant.[2][3] Because of the unit conventions then in use, the gravitational constant does not appear explicitly in Cavendish's work. Instead, the result was originally expressed as the specific gravity of the Earth,[4] or equivalently the mass of the Earth; and were the first accurate values for these geophysical constants. The experiment was devised sometime before 1783[5] by geologist John Michell,[6] who constructed a torsion balance apparatus for it. However, Michell died in 1793 without completing the work, and after his death the apparatus passed to Francis John Hyde Wollaston and then to Henry Cavendish, who rebuilt the apparatus but kept close to Michell's original plan. Cavendish then carried out a series of measurements with the equipment, and reported his results in the *Philosophical Transactions of the Royal Society* in 1798.[7]

Contents

- 1 The experiment
- 2 Did Cavendish determine G?
- 3 Derivation of G and the Earth's mass
- 4 See also
- 5 References
- 6 Notes
- 7 External links

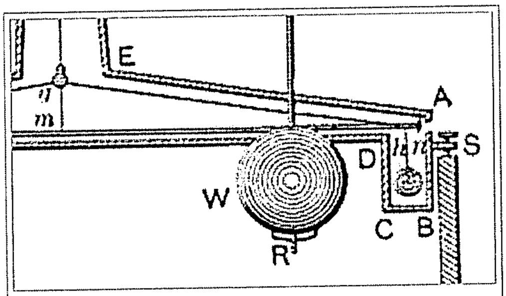
The experiment

The apparatus constructed by Cavendish was a torsion balance made of a six-foot (1.8 m) wooden rod suspended from a wire, with a 2-inch (51 mm) diameter 1.61-pound (0.73 kg) lead sphere attached to each end. Two 12-inch (300 mm) 348-pound (158 kg) lead balls were located near the smaller balls, about 9 inches (230 mm) away, and held in place with a separate suspension system.[8] The experiment measured the faint gravitational attraction between the small balls and the larger ones.



Vertical section drawing of Cavendish's t i b l i t ti l di th

The two large balls were positioned on alternate sides of the horizontal wooden arm of the balance. Their mutual attraction to the small balls caused the arm to rotate, twisting the wire supporting the arm. The arm stopped rotating when it reached an angle where the twisting force of the wire balanced the combined gravitational force of attraction between the large and small lead spheres. By measuring the angle of the rod, and knowing the twisting force (torque) of the wire for a given angle, Cavendish was able to determine the force between the pairs of masses. Since the gravitational force of the Earth on the small ball could be measured directly by weighing it, the they could be rotated into position next to the small balls by a pulley from outside. Figure 1 of Cavendish's paper.



Detail showing torsion balance arm (*m*), large ball (*W*), small ball (*x*), and isolating box (*ABCDE*).

Cavendish found that the Earth's density was 5.448 ± 0.033 times that of water (due to a simple arithmetic error, found in 1821 by F. Baily, the erroneous value 5.48 ± 0.038 appears in his paper).[9]

To find the wire's torsion coefficient, the torque exerted by the wire for a given angle of twist, Cavendish timed the natural oscillation period of the balance rod as it rotated slowly clockwise and counterclockwise against the twisting of the wire. The period was about 20 minutes. The torsion coefficient could be calculated from this and the mass and dimensions of the balance. Actually, the rod was never at rest; Cavendish had to measure the deflection angle of the rod while it was oscillating.[10]

Cavendish's equipment was remarkably sensitive for its time.[9] The force involved in twisting the torsion balance was very small, 1.74 x 10^{-7} N,[11] about 1/50,000,000 of the weight of the small balls[12] or roughly the weight of a large grain of sand.[13] To prevent air currents and temperature changes from interfering with the measurements, Cavendish placed the entire apparatus in a wooden box about 2 feet (0.61 m) thick, 10 feet (3.0 m) tall, and 10 feet (3.0 m) wide, all in a closed shed on his estate. Through two holes in the walls of the shed, Cavendish used telescopes to observe the movement of the torsion balance's horizontal rod. The motion of the rod was only about 0.16 inches (4.1 mm).[14] Cavendish was able to measure this small deflection to an accuracy of better than one hundredth of an inch using vernier scales on the ends of the rod.[15]

Cavendish's experiment was repeated by Reich (1838), Baily (1843), Cornu & Baille (1878), and many others. Its accuracy was not exceeded for 97 years, until C. V. Boys' 1895 experiment. In time, Michell's torsion balance became the dominant technique for measuring the gravitational constant (*G*), and most contemporary measurements still use variations of it. This is why Cavendish's experiment became *the* Cavendish experiment.[16]

Did Cavendish determine *G*?

The formulation of Newtonian gravity in terms of a gravitational constant did not become standard until long after Cavendish's time. Indeed, one of the first references to *G* is in 1873, 75 years after Cavendish's work.[17] Cavendish expressed his result in terms of the density of the Earth, and he referred to his experiment in correspondence as 'weighing the world'. Later authors reformulated his results in modern terms.[18][19][20] thus:

$$G = g\frac{R_{\mathrm{earth}}^2}{M_{\mathrm{earth}}} = \frac{3g}{4\pi R_{\mathrm{earth}}\rho_{\mathrm{earth}}}$$

After converting to SI units, Cavendish's value for the Earth's density, 5.448 g cm^{-3}, gives

$$G = 6.74 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2},$$

which differs by only 1% from the currently accepted value: $6.67428 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$.

Physicists, however, often use units where the gravitational constant takes a different form. The Gaussian gravitational constant used in space dynamics is a defined constant, and the Cavendish experiment can be considered as a measurement of the astronomical unit. In Cavendish's time, physicists used the same units for mass and weight, in effect taking g as a standard acceleration. Then, since R_{earth} was known, ρ_{earth} played the role of an inverse gravitational constant. The density of the Earth was hence a much sought-after quantity at the time, and there had been earlier attempts to measure it, such as the Schiehallion experiment in 1774.

For these reasons, physicists generally do credit Cavendish with the first measurement of the gravitational constant.[25][26][27][28][29]

Derivation of G and the Earth's mass

For the definitions of terms, see the drawing below and the table at the end of this section.

The following is not the method Cavendish used, but shows how modern physicists would use his results.[30][31][32] From Hooke's law, the torque on the torsion wire is proportional to the deflection angle θ of the balance. The torque is $\kappa\theta$ where κ is the torsion coefficient of the wire. However, the torque can also be written as a product of the attractive forces between the balls and the distance to the suspension wire. Since there are two pairs of balls, each experiencing force F at a distance $L / 2$ from the axis of the balance, the torque is LF. Equating the two formulas for torque gives the following:

$$\kappa\theta \ = LF$$

For F, Newton's law of universal gravitation is used to express the attractive force between the large and small balls:

$$F = \frac{GmM}{r^2}$$

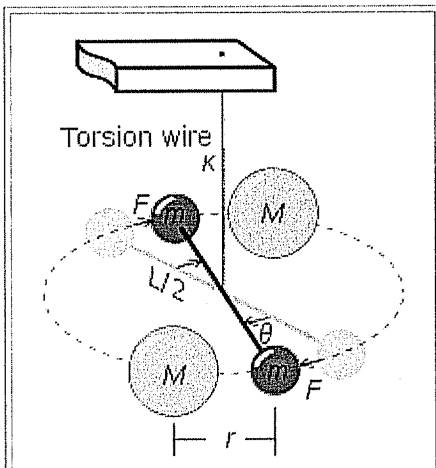


Diagram of torsion balance

Substituting F into the first equation above gives

$$\kappa\theta \ = L\frac{GmM}{r^2} \qquad (1)$$

To find the torsion coefficient (κ) of the wire, Cavendish measured the natural resonant oscillation period T of the torsion balance:

$$T = 2\pi\sqrt{I/\kappa}$$

Assuming the mass of the torsion beam itself is negligible, the moment of inertia of the balance is just due to the small balls:

$$I = m(L/2)^2 + m(L/2)^2 = 2m(L/2)^2 = mL^2/2,$$

and so:

$$T = 2\pi\sqrt{\frac{mL^2}{\ }}$$

Solving this for κ, substituting into (1), and rearranging for G, the result is:

$$G = \frac{2\pi^2 L r^2}{M T^2}\theta$$

Once G has been found, the attraction of an object at the Earth's surface to the Earth itself can be used to calculate the Earth's mass and density:

$$mg = \frac{GmM_{earth}}{R_{earth}^2}$$

$$M_{earth} = \frac{gR_{earth}^2}{G}$$

$$\rho_{earth} = \frac{M_{earth}}{4\pi R_{earth}^3/3} = \frac{3g}{4\pi R_{earth} G}$$

Definition of terms		
θ	radians	Deflection of torsion balance beam from its rest position
F	N	Gravitational force between masses M and m
G	$\mathrm{m^3 kg^{-1} s^{-2}}$	Gravitational constant
m	kg	Mass of small lead ball
M	kg	Mass of large lead ball
r	m	Distance between centers of large and small balls when balance is deflected
L	m	Length of torsion balance beam between centers of small balls
κ	$\mathrm{N\,m\,radian^{-1}}$	Torsion coefficient of suspending wire
I	$\mathrm{kg\,m^2}$	Moment of inertia of torsion balance beam
T	s	Period of oscillation of torsion balance
g	$\mathrm{m\,s^{-2}}$	Acceleration of gravity at the surface of the Earth
M_{earth}	kg	Mass of the Earth
R_{earth}	m	Radius of the Earth
ρ_{earth}	$\mathrm{kg\,m^{-3}}$	Density of the Earth

See also

- Schiehallion experiment

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- This article incorporates text from a publication now in the public domain: Chisholm, Hugh, ed (1911). *Encyclopædia Britannica* (11th ed.). Cambridge University Press.

Notes

1. ^ Boys 1894 (http://books.google.com/books?id=ZrloHemOmUEC&pg=PA355) p.355
2. ^ Encyclopaedia Britannica 1910 (http://books.google.com/books?id=DgTALFa3sa4C&pg=PA385) p.385 'The aim [of experiments like Cavendish's] may be regarded either as the determination of the mass of the Earth,...conveniently expressed...as its "mean density", or as the determination of the "gravitation constant", G'. Cavendish's experiment is generally described today as a measurement of G (Clotfelter 1987 p.210).
3. ^ Many sources state erroneously that this was the first measurement of G (or the Earth's density), such as Feynman, Richard P. (1963) (– Scholar search (http://scholar.google.co.uk/scholar?hl=en&lr=&q=author%3AFeynman+intitle%3ALectures+on+Physics%2C+Vol.1&as_publication=&as_ylo=&as_yhi=&btnG=Search)). *Lectures on Physics, Vol.1* (http://books.google.com/?id=k6MQrphL-NIC&pg=PA28) . Addison-Wesley. pp. 6–7. ISBN 0-201-02116-1. http://books.google.com/?id=k6MQrphL-NIC&pg=PA28. There were previous measurements, chiefly Bouguer (1740) and Maskelyne (1774), but they were very inaccurate (Poynting 1894 (http://books.google.com/books?id=dg0RAAAAIAAJ))(Encyclopedia Britannica 1910 (http://books.google.com/books?id=DgTALFa3sa4C&pg=PA385)).

of weighing the world'. Not clear whether 'earliest mention' refers to Cavendish or Michell.

6. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA59) , p.59 Cavendish gives full credit to Michell for devising the experiment
7. ^ Cavendish, H. 'Experiments to determine the Density of the Earth', *Philosophical Transactions of the Royal Society of London*, (part II) **88** p.469-526 (21 June 1798), reprinted in Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA59)
8. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA59) , p.59
9. ^ ***a b*** Poynting 1894 (http://books.google.com/books?id=dg0RAAAAIAAJ&pg=PA45) , p.45
10. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA64) , p.64
11. ^ Boys 1894 (http://books.google.com/books?id=ZrloHemOmUEC&pg=PA357) p.357
12. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA60) p. 60
13. ^ A 2 mm sand grain weighs ~13 mg. Theodoris, Marina (2003). "Mass of a Grain of Sand" (http://hypertextbook.com/facts/2003/MarinaTheodoris.shtml) . *The Physics Factbook.* http://hypertextbook.com/facts/2003/MarinaTheodoris.shtml.
14. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA99) , p. 99, Result table, (scale graduations = 1/20 in ≈ 1.3 mm) The total deflection shown in most trials was twice this since he compared the deflection with large balls on opposite sides of the balance beam.
15. ^ Cavendish 1798 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA63) , p.63
16. ^ McCormmach & Jungnickel 1996 (http://books.google.com/books?id=EUoLAAAAIAAJ&pg=PA341&sig=--1AlZ9rl_0AEL7h73LZvtK01S4) , p.341
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18. ^ Boys 1894 (http://books.google.com/books?id=ZrloHemOmUEC&pg=PA353) , p.330 In this lecture before the Royal Society, Boys introduces G and argues for its acceptance
19. ^ Poynting 1894 (http://books.google.com/books?id=dg0RAAAAIAAJ&pg=PA4) , p.4
20. ^ MacKenzie 1900 (http://books.google.com/books?id=O58mAAAAMAAJ&pg=PA1) , p.vi
21. ^ Clotfelter 1987
22. ^ McCormmach & Jungnickel 1996 (http://books.google.com/books?id=EUoLAAAAIAAJ&pg=PA336&sig=--1AlZ9rl_0AEL7h73LZvtK01S4) , p.337
23. ^ Hodges 1999 (http://www.public.iastate.edu/~lhodges/Michell.htm)
24. ^ Lally 1999
25. ^ Halliday, David; Resnick, Robert (1993). *Fundamentals of Physics* (http://books.google.com/?id=-AjnmJHPiKMC&pg=PA418) . John Wiley & Sons. pp. 418. ISBN 9780471147312. http://books.google.com/?id=-AjnmJHPiKMC&pg=PA418 'The apparatus used in 1798 by Henry Cavendish to measure the gravitational constant'
26. ^ Feynman, Richard P. (1963) (– Scholar search (http://scholar.google.co.uk/scholar?hl=en&lr=&q=author%3AFeynman+intitle%3ALectures+on+Physics%2C+Vol.1&as_publication=&as_ylo=&as_yhi=&btnG=Search)). *Lectures on Physics, Vol.1* (http://books.google.com/?id=k6MQrphL-NIC&pg=PA28) . Addison-Wesley. pp. 6–7. ISBN 0-201-02116-1. http://books.google.com/?id=k6MQrphL-NIC&pg=PA28 'Cavendish claimed he was weighing the Earth, but what he was measuring was the coefficient G...'
27. ^ Feynman, Richard P. (1967) (– Scholar search (http://scholar.google.co.uk/scholar?hl=en&lr=&q=author%3AFeynman+intitle%3AThe+Character+of+Physical+Law&as_publication=&as_ylo=&as_yhi=&btnG=Search)). *The Character of Physical Law* (http://books.google.com/?id=k6MQrphL-NIC&pg=PA28) . MIT Press. pp. 28. ISBN 0-262-56003-8. http://books.google.com/?id=k6MQrphL-NIC&pg=PA28 'Cavendish was able to measure the force, the two masses, and the distance, and thus determine the gravitational constant G'
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External links

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- Weighing the Earth (http://www.juliantrubin.com/bigten/cavendishg.html) - background and experiment

Categories: Physics experiments | 1790s in science | 1797 in science | 1798 in science

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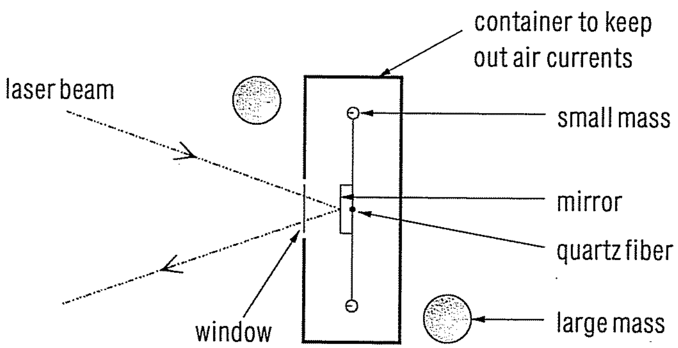
MISN-0-100

THE CAVENDISH EXPERIMENT



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THE CAVENDISH EXPERIMENT

by

P. Signell and V. Ross

1. Introduction 1

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a. Introduction 1
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Title: **The Cavendish Experiment**

Author: P. Signell and V. Ross, Dept. of Physics, Mich. State Univ

Version: 2/1/2000 Evaluation: Stage 1

Length: 1 hr; 12 pages

Input Skills:

1. Find the torque produced about a shaft by a given force and describe the twisting effect produced by that torque (MISN-0-34) or (MISN-0-416).
2. Define the equilibrium point in simple harmonic motion and explain how it is related to the spatial properties of the restoring force (MISN-0-25).

Output Skills (Knowledge):

K1. Describe and sketch the essentials of the Cavendish balance, communicating clearly how the apparatus works.

K2. Describe how the Cavendish experiment can be used to examine the validity of each of the three variables in Newton's law of gravitation.

External Resources (Optional):

1. I. Freeman, *Physics—Principles and Insights*, McGraw-Hill (1968). For availability, see this module's *Local Guide.*

Post-Options:

1. "Newton's Law of Gravitation" (MISN-0-101).
2. "Derivation of Newton's Law of Gravitation" (MISN-0-103).
3. "The Equivalence Principle: An Introduction to Relativistic Gravitation" (MISN-0-110).

THE CAVENDISH EXPERIMENT

by

P. Signell and V. Ross

1. Introduction

Newton's law of gravitation is certainly one of the greatest laws of the universe: the one that describes what holds together our earth, holds us to the earth, and holds our earth in its orbit about our sun. Observations indicate that it holds in the incredibly distant reaches of the universe exactly as it holds here on earth. How exciting it is, then, to be able to examine this great law in the laboratory with fairly simple apparatus! We will first describe the background for Cavendish's experiment and then show how it can be used to examine the gravitation law.

2. Historical Overview

2a. Introduction. One of the greatest adventures in the history of mankind has been the determination of the causes of night and day and of the seasons, and the regularities of motion of "the wanderers,"[1] the planets. Careful observations were recorded in many cultures, including that of the American Indian. However, the first statements of the simple mathematical characteristics of planetary motion were three laws proposed by Johannes Kepler[2] who built on Tycho Brahe's careful astronomical measurements and Copernicus's proposal that the sun is at the center of the solar system.

2b. Newton's Law of Gravitation. Then, in 1666, Isaac Newton published his law of universal gravitation.[3] Kepler's three laws for the motions of the planets in our solar system were now replaced by a single law which covered, as well, the force of gravity here on the earth and the motions of planetary moons, galaxies, binary stars, and our sun's constituents. Newton's law said that the gravitational force between two spherically symmetric objects is directly proportional to each of their

[1]So-called in ancient times because their positions moved against the background of stars.

[2]See "Derivation of Newton's Law of Gravitation" (MISN-0-103) for a discussion of Kepler's laws and a simplified version of Newton's derivation.

[3]See "Newton's Law of Gravitation" (MISN-0-101) for applications of the law to people, mountains, satellites, and planets.

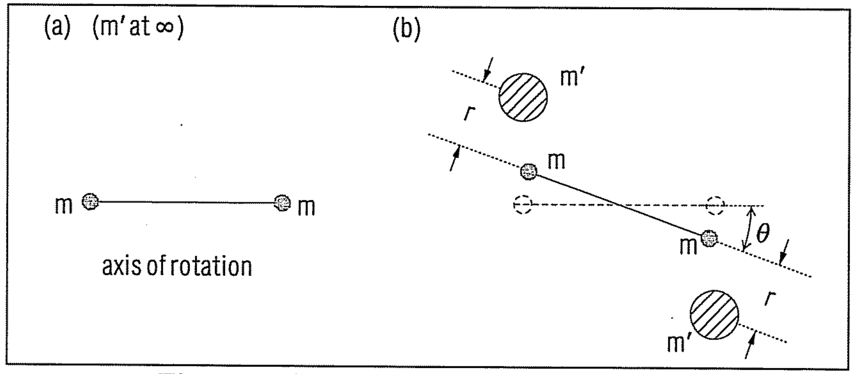

Figure 1. A schematic diagram of the Cavendish apparatus: (a) m' far away; (b) m' up close.

masses and inversely proportional to the square of the distance between their centers. That is,

$$F = G\frac{mm'}{r^2}, \qquad (1)$$

where G is some universal gravitational constant, m and m' are the two masses[4] and r is the distance between their centers. Here F is the force with which each of the masses is attracted to the other.[5]

2c. Newton Could Not Determine G. Newton checked his law mainly through ratios of forces[6] since he could not directly measure the gravitational constant G. For example, he set the force on an object at the earth's surface equal to its weight, mg, and found:

$$GM_E = gR_E^2, \qquad (2)$$

where R_E and M_E are the radius and mass of the earth. The value of R_E was known from the curvature of the horizon but M_E was not accessible to measurement. Thus Newton could only determine a numerical value for the product GM_E.

2d. Cavendish Makes the First Measurement of G. Over a century after Newton's formulation of the law of gravitation, Henry

[4]Mass can be defined in several ways. One method is given in (MISN-0-14), while the strange equivalence of gravitational and inertial mass is examined in (MISN-0-110).

[5]See (MISN-0-16) for a discussion of Newton's third law and the equality of the two forces.

[6]See (MISN-0-103) for a derivation of one of the ratios used by Newton to check the law of gravitation.

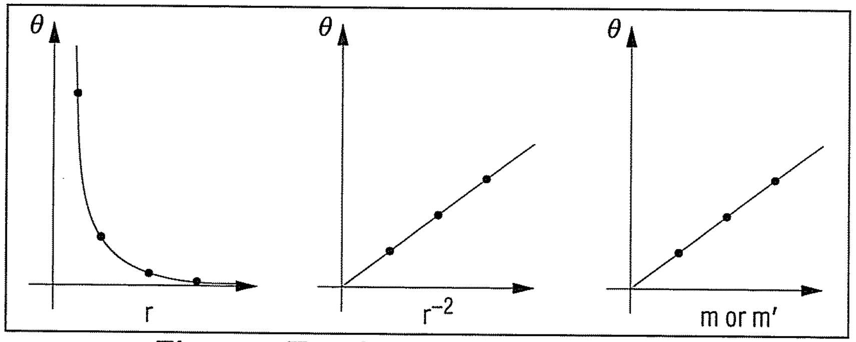

Figure 2. Hypothetical Cavendish data that would confirm Newton's law of gravitation.

Cavendish succeeded in directly measuring the gravitational force between two masses, thus enabling him to evaluate G and hence the masses of the earth,[7] moon[8] and sun.

3. The Cavendish Experiment

3a. Description of the Cavendish Apparatus. A good description of Cavendish's apparatus is given by Freeman[9] : "...A light rod with a small metal ball at each end [was] hung from a fixed point by means of a thin wire. When two massive lead spheres [were] brought close to the small balls, the gravitational forces of attraction [made] the suspended system turn slightly to a new position of equilibrium. The torque[10] with which the suspending wire [opposed] twisting [was] measured in a separate experiment..." Figure 1 shows a top view of the Cavendish apparatus at two stages during a measurement of the gravitational force between known masses. Figure 1a shows the position of the suspended light rod and small masses m when the large lead masses m' are far away ($r = \infty$). The top end of the suspending wire, the end toward the viewer, is rigidly clamped.

[7]The major motivation of finding M_E was that of determining the earth's average density so as to obtain information about the composition of its interior.

[8]See "Derivation of Newton's Law of Gravitation" (MISN-0-103) for the equation used.

[9]Ira M. Freeman, *Physics — Principles and Insights*, McGraw-Hill (1968). For availability, see this module's *Local Guide*.

[10]For a discussion of torque see "Torque and Angular Momentum in Circular Motion," MISN-0-34.

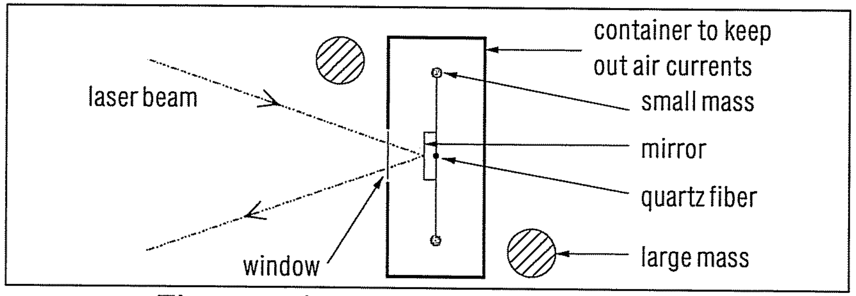

Figure 3. A top-view diagram of the modern student-lab Cavendish apparatus.[]

When the large masses m' are brought so close as to produce a significant force of gravitational attraction on the small masses, the small masses move toward the large masses, causing the rod to rotate. This rotation twists the lower end of the wire (see Fig. 1b).

3b. Gravitational Attraction Balanced by Restoring Torque. In the position shown in Fig. 1b with m' close to m, the gravitational attraction between the m-m' pairs is balanced by the restoring torque (or force) produced by the twisting wire:

$$F_{\text{gravitational}} = -F_{\text{restoring}}\,. \tag{3}$$

The values of m, m' and r are varied, producing various values for the gravitational force and hence producing various equilibrium angles of twist θ (see Fig. 1b). At this point we could plot the measurements as m vs. θ, m' vs. θ, and r^{-2} vs. θ. However, it is force that occurs in Newton's Law, not θ.

3c. Determining the Restoring Force. Here we describe how one obtains the factor for converting the measured equilibrium angles θ to force values F. It involves measuring, in a separate experiment, the frequency of free oscillations of the apparatus.

In practice experimenters make use of the fact that the suspension receives such a small amount of twisting that the arc-like displacement of each mass from its equilibrium position is linearly proportional to the restoring force in the suspension[11] :

$$F_{\text{restoring}} = -ks = -k\ell\theta\,, \tag{4}$$

[11]Here the "restoring force" is the force within the metal of the suspension that resists the twisting. It grows linearly with twist angle, opposing the force causing the twisting. When the twist angle is so large that the two opposing forces are equal,

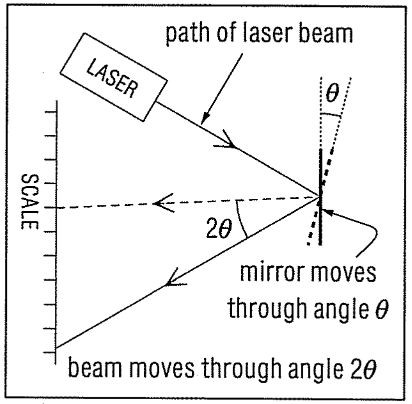

Figure 4. The angular deflection of the laser beam produced by the twisting mirror.

where ℓ is the length of either arm of the suspension and s is the displacement of the mass along an arc. Because of this linearity, when the suspended system is displaced from equilibrium and then released it will exhibit a simple harmonic twisting motion about an equilibrium angle[12] with an angular frequency of oscillation given by[13]

$$\omega^2 = k/m \,. \tag{5}$$

A simple measurement of the angular frequency of oscillation, along with measurements of m and ℓ, then gives the needed proportionality constant for converting θ values to F values:

$$F_{\text{restoring}} = -k\ell\theta = -(\omega^2 m\ell)\theta \,. \tag{6}$$

3d. Examination of Cavendish Data. Data of the type shown in Fig. 2 would support the form of Newton's law of gravitation. That is, since the force is seen to be linearly proportional to m, m', and r^{-2}, the form must be:

$$F = \text{ constant } \times mm'/r^2 \,. \tag{7}$$

the force on the suspension is zero. For more details see "Simple Harmonic Motion" (MISN-0-25) and for further discussion of restoring-force linearity with displacement see "Small Oscillation Technique" (MISN-0-28).

[12] "Simple harmonic motion" about an "equilibrium point" is an oscillating ("back and forth") motion where the position of the object is a sinusoidal function of time. The "frequency" of the oscillation is the number of complete motion cycles per unit time. See "Simple Harmonic Motion" (MISN-0-25).

[13] For a derivation of this relation see "Simple Harmonic Motion" (MISN-0-25) .

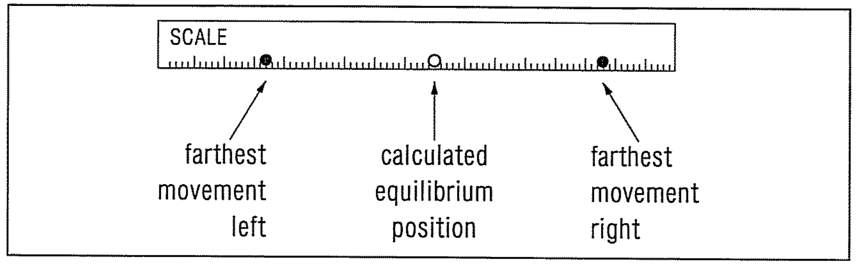

Figure 5. Finding the equilibrium position.

Then the measured force for any one combination of m, m', and r gives the proportionality constant G:

$$F = G\frac{mm'}{r^2} \,. \tag{8}$$

The current best measured value of G is:[14]

$$G = 6.6732(31) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \,. \tag{9}$$

3e. The Modern Student Cavendish Balance. In a modern student lab apparatus (see Fig. 3), the Cavendish balance is basically the same as the original one, except that the thin wire is replaced by a thin quartz fiber which produces a more consistent restoring torque. A mirror is attached to the quartz fiber and a laser beam is reflected off the mirror and onto a scale.[15] As the small masses oscillate back and forth around the axis, the fiber twists and the reflected laser beam moves back and forth along the scale.

Note that the law of reflection[16] requires the angular displacement of the beam to be twice that of the mirror (see Fig. 4). The scale is sometimes curved to make easier the conversion from the scale reading to the angle θ. By measuring the points traveled farthest to the left and right on the scale, the equilibrium point can be found (see Fig. 5). The large masses are moved near or away from the small masses by an arm which is pivoted in the center so that the distance, r, between the large and small mass will be the same on both ends of the balance beam.

[14] *Handbook of Chemistry and Physics*, 54th Edition, Chemical Rubber Co., CRC Press (1973).

[15] A laser beam is used because it is pencil-thin.

[16] See "The Rules of Geometrical Optics" (MISN-0-220).

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