Chapter 13
Oscillation

1. Mathematics

   (a) Definition

   (b) Free Oscillation

   (c) Damped Oscillation

      (i) Overdamping

      (ii) Critical damping

      (iii) Underdamping

   (d) Forced Oscillation.

      (i) Driven Oscillation of Pure Harmonic Oscillator

      (ii) Driven Oscillation of Real Harmonic Oscillator

      (iii) Average Power Dissipation

2. Simple Harmonic Oscillation

   (a) Spring

      (i) Equation of Motion.

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$ a, b, c are constant

$b = 0$ free oscillation

$F = -kx$ $x = x - x_0$

$\downarrow$ equilibrium

$$m\frac{d^2x}{dt^2} = -kx$$

$(a = m;\ c = k,\ b = 0)$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \omega^2 = \frac{k}{m} > 0$$

$$= -\omega^2 x \qquad \omega^2 = \frac{k}{m} = \frac{c}{a}$$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t$$

On the other hand

$$a\frac{d^2x}{dt} + cx = 0 \Rightarrow \frac{d^2x}{dt} = -\frac{c}{a}x$$

If $\frac{c}{a} < 0$, then $\omega^2 = -\frac{c}{a} > 0$

$$\frac{d^2x}{dt^2} = +\omega^2 x$$

$$\Rightarrow \quad x(t) = Ae^{\omega t} + Be^{-\omega t}$$

A, B are determined by the initial conditions.

Damped Oscillation

$$a \frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Ansatz $\qquad x(t) = e^{-bt/2a} f(t)$

$$\frac{dx}{dt} = e^{-bt/2a}\frac{df}{dt} + e^{-bt/2a}\left(-\frac{b}{2a}\right)f$$

$$\frac{d^2x}{dt} = e^{-bt/2a}\frac{d^2f}{dt^2} + e^{-bt/2a}\left(-\frac{b}{2a}\right)\frac{df}{dt}$$

$$+ e^{-bt/2a}\left(-\frac{b}{2a}\right)\frac{df}{dt} + e^{-bt/2a}\left(-\frac{b}{2a}\right)^2 f$$

$$LHS = e^{-bt/2a} \times \Big[a\frac{d^2f}{dt^2} - \cancel{\frac{b}{2}\frac{df}{dt}} - \cancel{\frac{b}{2}\frac{df}{dt}} + \frac{b^2}{4a} f$$

$$+ \cancel{b\frac{df}{dt}} - \frac{b^2}{2a} f + cf\Big] = 0$$

$$\Rightarrow \qquad a\frac{d^2f}{dt^2} = \frac{b^2 - 4ac}{4a} f$$

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$$F = -kx - "b"\dot{x}$$

$$m\ddot{x} = -kx - "b"x'$$

$$\underbrace{m}_{a}\ddot{x} + \underbrace{"b"}_{b}x' + \underbrace{k}_{c}x = 0$$

$a, c > 0$

Damped Oscillation (Overdamping)

$$\underbrace{\frac{b^2 - 4ac}{4a^2}}_{= \omega^2} > 0 \quad \Rightarrow \quad \omega < \frac{b}{2a}$$

$$\frac{d^2 f}{dt^2} = \omega^2 t \quad \Rightarrow \quad f = Ae^{\omega t} + Be^{-\omega t}$$

$$\Rightarrow \quad x(t) = e^{-\frac{b}{2a}t}\left[Ae^{\omega t} + Be^{-\omega t}\right]$$

A, B are constants to be determined by initial condition

$\Rightarrow$ The motion of the displacement is always decreasing

Critical Damping

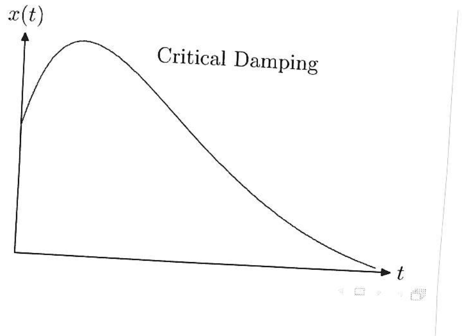
$$b^2 - 4ac = 0 \quad \Rightarrow \quad b^2 = 4ac$$

$$\Rightarrow \quad \frac{d^2 f}{dt^2} = 0$$

$$\downarrow$$

$$f(t) = A + Bt$$

$$\Rightarrow \quad x(t) = e^{-bt/2a} (A + Bt)$$



Underdamping

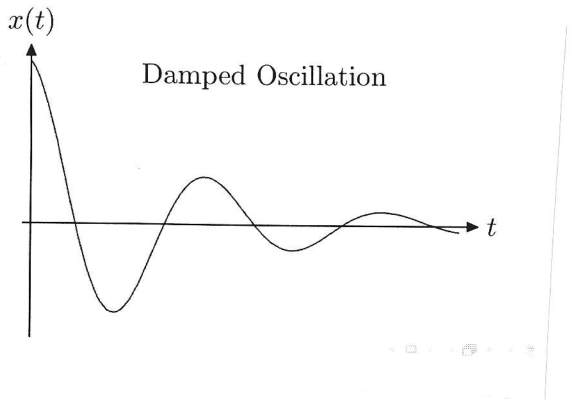
$$\frac{b^2-4ac}{4a^2} < 0$$

Let $\quad \omega^2 = \dfrac{4ac-b^2}{4a^2} > 0$, then $\dfrac{d^2f}{dt^2} = -\omega^2 f$

$$f(t) = A\sin\omega t + B\cos\omega t$$

$$\Rightarrow \quad x(t) = e^{-bt/2a}\,(A\sin\omega t + B\cos\omega t)$$

A, B are constants to be determined by initial conditions



Inhomogeneous D.E.

The inhomogeneous term $F(t)$ represents an external driven force

$$a\ddot{x} + b\dot{x} + cx = F(t)\,.$$

We are mainly interested in a periodically driven force $F_0 \cos \omega t$ where ω is the external angular frequency which can varied over a range. The general solution is the homogeneous solution plus a particular solution

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} + p(t)\,,$$

so that

$$a\ddot{p} + b\dot{p} + cp = F(t)\,.$$

Driven Oscillation of Pure Harmonic Oscillator

The inhomogeneous D.E. is

$$\ddot{x} + \omega_0^2 x = (F_0/m)\cos\omega t\,.$$

(i) For $\omega \neq \omega_0$ we can take $p(t) = D\cos\omega t$. Then

$$-\omega^2 D\cos\omega t + \omega_0^2 D\cos\omega t = (F_0/m)\cos\omega t\,,$$

giving

$$D = \frac{F_0}{m(\omega_0^2 - \omega^2)}\,,$$

So the particular solution represents an "in phase motion" with the external driven periodic force.

The general solutions for

$$\ddot{x} + \omega_0^2 x = 0$$

$$\Rightarrow \quad x_g = A\sin\omega t + B\cos\omega t$$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t + \frac{F_0}{m(\omega_0^2-\omega^2)}$$

(ii) For $\omega = \omega_0$ we would take $p(t) = Dt \sin \omega t$,

$$(-\omega^2 Dt \sin \omega t + 2\omega D \cos \omega t) + \omega^2 Dt \sin \omega t = (F_0/m) \cos \omega t \,,$$

giving

$$D = \frac{F_0}{2m\omega} \,.$$

So the particular solution at resonance ($\omega = \omega_0$) gives a displacement growing with time, absorbing energy from the driven force. In order to attain this, part of the velocity must be in phase with the external force. So there is an abrupt phase change at resonance.

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$$p(t) = Dt \sin\omega t$$

$$\frac{dp}{dt} = Dt\,\omega\cos\omega t + D\sin\omega t$$

$$\frac{d^2p}{dt^2} = Dt(-\omega^2)\sin\omega t + D\omega\cos\omega t + D\omega\cos\omega t$$

$$\Rightarrow \quad (-\cancel{\omega^2 Dt\sin\omega t} + 2\omega D\cos\omega t) + \cancel{\omega^2 Dt\sin\omega t}$$

$$= \frac{F_0}{m}\cos\omega t$$

$$\Rightarrow \quad D = \frac{F_0}{2m\omega}$$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t + \frac{F_0}{2m\omega}t\sin\omega t$$

Driven Oscillation of a Real Oscillator

$$F = -kx - \underset{\substack{\uparrow \\ \lambda}}{b}v + F_0 \cos\omega t$$

$$m\frac{d^2x}{dt^2} + \underset{\substack{\updownarrow \\ \lambda}}{b}\frac{dx}{dt} + kx = F_0 \cos\omega t$$

$$x(t) = \underset{\substack{\downarrow \\ \text{solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{equation}}}{x_g(t)} + \underset{\substack{\downarrow \\ \text{particular} \\ \text{solution}}}{p(t)}$$

$$m\frac{d^2p}{dt^2} + \lambda\frac{dp}{dt} + kp = F_0 \cos\omega t$$

$$m\ddot{p} + \overset{b}{\lambda}\dot{p} + kp - F_0 \cos\omega t = 0 \quad \Rightarrow \quad m\ddot{p} + \lambda\dot{p} + \underset{\omega_0^2}{\frac{k}{m}} mp - F_0 \cos\omega t$$

$$p(t) = C \sin(\omega t + \delta)$$

$$\dot{p}(t) = C\omega \cos(\omega t + \delta)$$

$$\ddot{p}(t) = C(-\omega^2) \sin(\omega t + \delta)$$

$$(\omega_0^2 - \omega^2)\, mC \sin(\omega t + \delta) + \lambda\omega c \cos(\omega t + \beta) - F_0 \cos\omega t = 0 \qquad (A)$$

$$A \equiv \omega t + \delta, \quad B = \delta$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos\omega t = \cos\delta \cos(\omega t + \delta) + \sin\delta \sin(\omega t + \delta)$$

Substitute back into A

$$[(\omega_0^2 - \omega^2) C.. - F_0 \sin\delta] \sin(\omega t + \delta) + (\lambda\omega c - F_0 \cos\delta) \cos(\omega t + \delta) = 0$$

$$\Rightarrow \begin{aligned} (\omega_0^2 - \omega^2)\, Cm &= F_0 \sin\delta \qquad (B) \\ \lambda\omega c &= F_0 \cos\delta \end{aligned}$$

$$(\omega_0^2 - \omega^2)\, C^2 m^2 = F_0^2 \sin^2\delta$$

$$\lambda^2\omega^2 C^2 = F_0^2 \cos^2\delta$$

$$C^2 [(\omega_0^2 - \omega^2) m^2 + \lambda^2\omega^2] = F_0^2$$

$$C = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2) m^2 + \lambda^2\omega^2}}$$

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\omega_0^2 - \omega^2}{\lambda\omega}$$

The mathematical problem has been solved

$$x(t) = e^{-bt/2a} f(t) + p(t)$$

Time Average Steady-State Power Transferred into the Oscillator from the Deriving Force

$$\langle P \rangle = \frac{1}{T}\int_0^T p(t)\,dt = \frac{1}{T}\int_0^T F_{DR}(t)\,v(t)\,dt$$

$$p(t) \downarrow F(t)\,v(t), \qquad v(t) \downarrow \frac{dx}{dt}$$

$$\downarrow$$

$$p = \frac{dW}{dt}$$

$$\begin{cases} T = \text{period} \\ \omega T = 2\pi \end{cases}$$

$$= \frac{1}{T}\int_0^T F_0 \cos\omega t \; v(t)\,dt$$

$$v(t) \downarrow C\omega\cos(\omega t + \delta)$$

$$\cos(A+B) = \cos A\cos B - \sin A \sin B \qquad A = \omega t,\ B = \delta$$

$$\cos(\omega t + \delta) = \cos\omega t \cos\delta - \sin\omega t \sin\delta$$

$$F_0 \cos\delta = C\lambda\omega$$

$$F_0 \sin\delta = (\omega_0^2 - \omega^2)Cm \qquad \Rightarrow F_0^2 = C^2[\lambda^2\omega^2 + m^2(\omega_0^2-\omega^2)^2]$$

$$= \frac{1}{T}\int_0^T F_0\cos\omega t\,[C\omega\cos\omega t\cos\delta - C\omega\sin\omega t\sin\delta]\,dt$$

$$= \frac{1}{T}F_0 C\omega\cos\delta\int_0^T \cos^2\omega t\,dt - \frac{1}{T}F_0\sin\delta\int_0^T \cos\omega t\sin\omega t\,dt$$

$$= \frac{F_0 C\omega\cos\delta}{2}$$

$$\int_0^T \frac{1}{2}(\sin 2\omega t)\, dt \quad = 0$$

$$\sin 2\omega t \downarrow \sin\omega t \cos\omega t$$

$$\underbrace{\cos^2\omega t - \sin^2\omega t}_{\cos 2\omega t} = 2\cos^2\omega t - 1$$

$$\frac{1}{T}\int_0^T \cos^2\omega t \, dt = \frac{1}{T}\int_0^T \left[\frac{1}{2}(1+\cos 2\omega t\right] dt$$

$$\int_0^T \sin 2\omega t \, dt$$

$$\omega T = 2\pi$$

$$\omega t = x \qquad t = 0 \quad x = 0$$

$$t = T \quad x = \omega T = 2\pi$$

$$\omega dt = dx$$

$$dt = \frac{1}{\omega} dx$$

$$= \frac{1}{\omega}\int_0^{2\pi} \sin x \, dx$$

$$= \frac{1}{\omega}\left[-\cos x\right]_0^{2\pi} = 0$$

$$\int_0^T \cos 2\omega t \, dt = 0$$ using the same argument

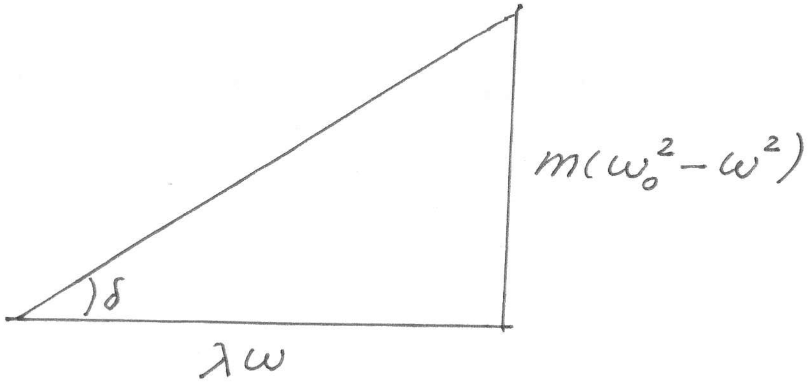
$$\Rightarrow \frac{1}{T}\int_0^T \cos^2\omega t \, dt = \frac{1}{2}$$

(See Figure in MISN-0-31 P. 5)

$$\frac{1}{T}\int_0^T \cos\omega t \sin\omega t \, dt = 0$$

Elimating the Phase Angle

$$\tan\delta = \frac{\omega_0^2 - \omega^2}{\lambda\omega}$$



$$\cos\delta = \frac{\lambda\omega}{\sqrt{\lambda^2\omega^2 + m^2(\omega^2 - \omega_0^2)^2}}$$

$$\langle p \rangle = \frac{F_0\ \omega\ c\ \cos\delta}{2} \qquad c = \frac{F_0}{\sqrt{\lambda^2\omega^2 + m^2(\omega_0^2 - \omega^2)}}$$

$$= \frac{F_0\ \omega c\ \lambda\omega}{2\sqrt{\lambda^2\omega^2 + m^2(\omega_0^2 - \omega^2)}}$$

$$= \frac{F_0^2\omega^2\lambda}{2[m^2(\omega_0^2 - \omega^2)^2 + \lambda^2\omega^2]}$$

Forced Harmonic Oscillation of a Real Oscillator (one dimensional)

$$\underset{\overset{\downarrow}{ma}}{F} = -kx - \lambda v + F_0 \cos\omega t$$

$$m\ddot{x} + \lambda\dot{x} + kx = F_0 \cos\omega t$$

Solution : $x(t) = \underset{\overset{\downarrow}{e^{-bt/2a} f(t)}}{x_g(t)} + \underset{\overset{\downarrow}{C\sin(\omega t + \delta)}}{p(t)}$

Overdamping $f(t) = Ae^{\omega t} + Be^{-\omega t}$

Critical $f(t) = A + Bt$

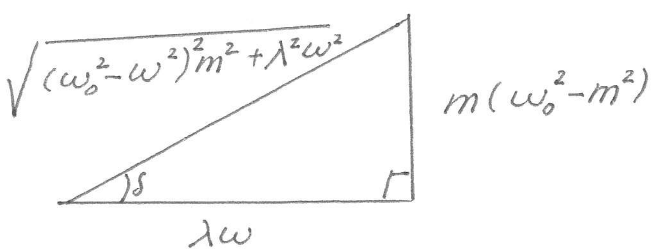
Underdamping $f(t) = A\sin\omega t + B\cos\omega t$

As t increase, the $p(t)$ dominates

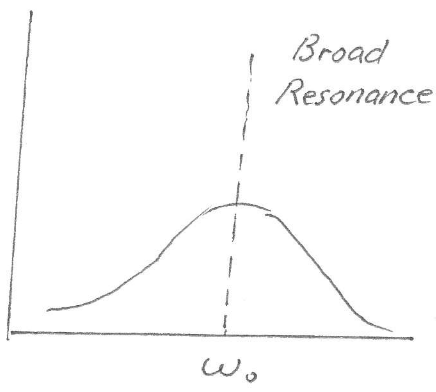
$$x(t) \longrightarrow \underset{\overset{\downarrow}{\text{steady state solution}}}{p(t)} = C\sin(\omega t + \delta)$$

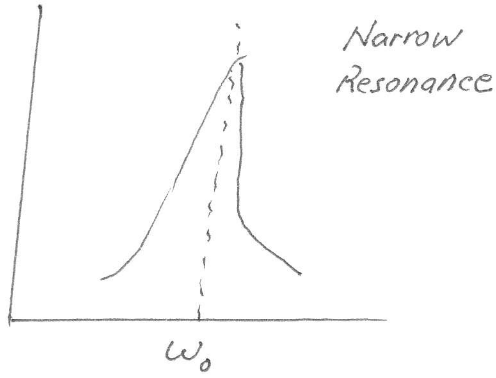
$$C = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)m^2 + \lambda^2\omega^2}} \qquad \omega_0^2 = \frac{k}{m}$$

$$\tan\delta = \frac{\omega_0^2 - \omega^2}{\lambda\omega}$$



$\langle P \rangle$ ↓ function of ω



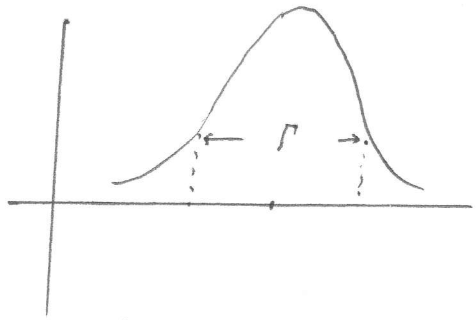


Maximum occur at $\omega = \omega_0$

$\omega_{1/2}$

Definition

$$\langle P \rangle_{\omega_{1/2}} = \frac{1}{2}\langle P \rangle_{\omega_0}$$



$$\frac{F_0^2\,\omega_{1/2}^2\,\lambda}{m^2(\omega_0^2-\omega_{1/2}^2)^2+\lambda^2\omega_{1/2}^2} = \frac{1}{2}\,\frac{F_0^2\,\omega_0^2\,\lambda}{\lambda^2\,\omega_0^2}$$

$$\Rightarrow \quad \omega_0^2 - \omega_{1/2}^2 = \pm \lambda\, \omega_{1/2}/m$$

$$\parallel$$

$$(\omega_0 + \omega_{1/2})(\omega_0 - \omega_{1/2}) = \pm \lambda\, \omega_{1/2}/m$$

$\omega_0 \sim \omega_{1/2}$ (narrow width approximation)

$$\downarrow$$

$$\omega_0 + \omega_{1/2} \approx 2\omega_0$$

$$\Rightarrow \quad (\omega_0 - \omega_{1/2})\, 2\omega_0 \cong \pm \lambda\, \omega_0$$

$$\omega_{1/2} = \omega_0 \pm \underbrace{\lambda/m}_{\Gamma}$$

λ 大 $\rightarrow$ distribution wide

$$\langle P \rangle_\omega = \frac{F_0^2\,\Gamma/8m}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

$$\langle P \rangle_{\omega_0} = \frac{F_0^2}{2\Gamma m}$$

Examples of Simple Harmonic Motion

(1) Spring

(2) Particle Moving on a circle with radius A and constant angular velocity ω.

(3) A Single Pendulum

$$m\ell \frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -(g/\ell)\sin\theta$$

For $\theta \ll 1$ $\quad \sin\theta \sim \theta$

$$\frac{d^2\theta}{dt^2} \cong -\underbrace{\left(\frac{g}{\ell}\right)}_{\omega^2}\theta$$

$$\omega = \sqrt{\frac{g}{\ell}}.$$

$$\sin\theta \simeq \theta - \frac{1}{3!}\theta^3 \Rightarrow \text{aharmonic}$$

$\downarrow$

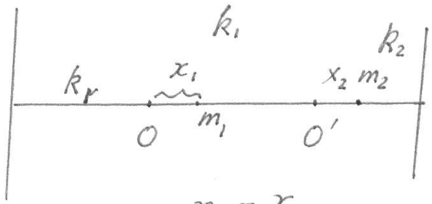
can be solved by perturbation method

Extension

Coupled oscillation

Parall $k = k_1 + k_2$

Series $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$



$x_2 - x_1$

m_1 $-k_1 x_1$
$+k(x_2 - x_1)$

m_2 $-k(x_2 - x_1)$
$-k_2 x_2$

$$m_1 \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1$$

$$m_2 \frac{d^2 x_2}{dt} = -k_2 x_2 - k(x_2 - x_1)$$

$m_1 = m_2 = m$, $k_1 = k_2$

$$\begin{cases} m \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1 \\ m \frac{d^2 x_2}{dt^2} = -k_1 x_2 - k(x_2 - x_1) \end{cases}$$

coupled

(+) $$m \frac{d^2}{dt^2}\underbrace{(x_1 + x_2)} = -k_1 \underbrace{(x_1 + x_2)}$$

(−) $$m \frac{d^2}{dt^2}\underbrace{(x_1 - x_2)} = -(2k + k_1)\underbrace{(x_1 - x_2)}$$

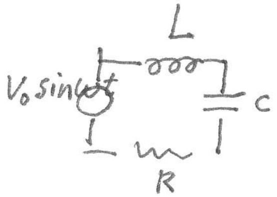
decoupled

normal mode

n wide

Two, Three dimensional problem

RLC Circuits (P. 926)



$$V_0 \sin\omega t - L\frac{d^2Q}{dt^2} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

$$F_0 \sin\omega t - m\frac{d^2x}{dt^2} - b\frac{dx}{dt} - kx = 0$$

对应

Q	x
$\frac{1}{C}$	k
R	b
L	m
$V_0 \sin\omega t$	$F_0 \sin\omega t$
$\frac{1}{\sqrt{LC}}$	$\sqrt{\frac{k}{m}}$