效应。

*2.7 潮汐

潮汐是海水的周期性涨落现象。"昼涨称潮，夜涨称汐。"钱江大潮，高达数米，排山倒海，蔚为壮观（图 2.22）。这种现象由牛顿首先给出了正确的说明。它是月亮、太阳对海水的引力以及地球公转和自转的结果。它的解释是应用非惯性系分析物体受力的一个很好的例子。

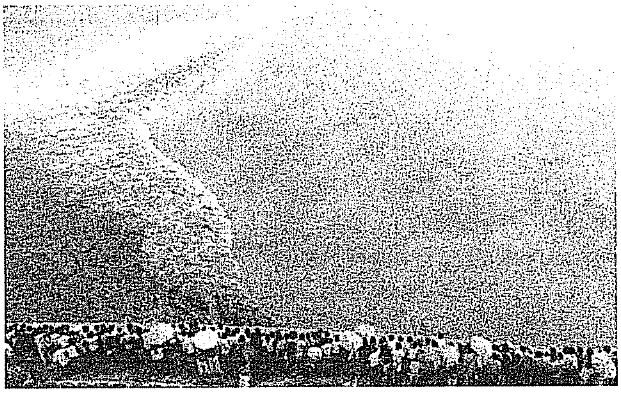

图 2.22 人潮争睹钱江潮（农历八月十八日）（潮水向图右方涌进）

1. 引潮力的计算

为分析方便起见，设想地球是一个均匀球体，表面为一层海水全面覆盖。以 M_E 表示地球的质量，R_E 表示地球的半径。先考虑太阳对海水的引力效果。

在太阳参考系内观察，地球的运动是公转和自转的合成运动，公转可看成是平动。以 r_S 表示太阳到地心的距离，以 ω 表示公转的角速度。这种平动的圆周运动使地球上各处都有指向太阳的向心加速度 $a_n=\omega^2 r_S$。不失其一般性，分析地心正好通过太阳坐标系 x 轴时的情况。图 2.23 画出了在太阳坐标系中地球受力和运动情况。

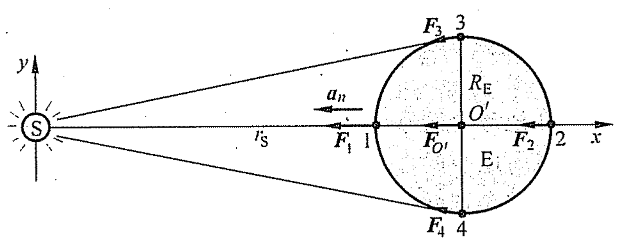

图 2.23 太阳坐标系中地球的平动

下面转入地心参考系。由于对太阳参考系有加速度 a_n，所以地心参考系是一个非惯性系，相对于此参考系，地球上任何物体除了受真实力外，都受到与 a_n 方向相反的惯性力 F_i。选地心参考系的 x' 和 y' 轴分别与太阳参考系的 x 轴和 y 轴平行，在地心参考系内的情况如图 2.24 所示，其中各处惯性力均与 x' 轴平行。

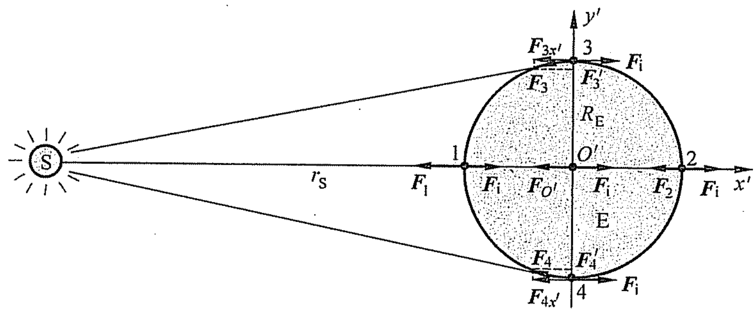

图 2.24　地心参考系内的太阳引力和惯性力

先考虑地心 O'处一小块质量为 Δm 的物质。它受的惯性力为

$$F_i = \Delta m\omega^2 r_S \tag{2.32}$$

它受太阳的引力为

$$F_{O'} = \frac{GM_S\Delta m}{r_S^2} \tag{2.33}$$

其中 M_S 为太阳的质量。它还受到地球本身其他部分的力(引力和化学结合力，主要是引力)，但由于球对称性，这些力的合力为零。这样，由于 Δm 在地心参考系中静止，所以 $\boldsymbol{F}_i$ 和 $\boldsymbol{F}_{O'}$ 相互平衡。因而有

$$F_i = \Delta m\omega^2 r_S = \frac{GM_S\Delta m}{r_S^2} \tag{2.34}$$

现在考虑地球表面离太阳最近的“1”处的质量为 Δm 的海水。它受的太阳的引力为

$$F_1 = \frac{GM_S\Delta m}{(r_S - R_E)^2}$$

由于 $R_E \ll r_S$，取一级近似，上式可写成

$$F_1 = \frac{GM_S\Delta m}{r_S^2}\left(1 + \frac{2R_E}{r_S}\right) \tag{2.35}$$

此处 Δm 海水所受惯性力仍由式(2.34)给出。将 F_1 和 F_i 相比可知 F_1 较大，视其差值为一力，其大小为

$$F_1' = F_1 - F_i = \frac{2GM_S\Delta m}{r_S^3}R_E \tag{2.36}$$

方向背离地心。这就是在地球上观察到的“引潮力”，它将使此处海水凸起形成涨潮，直到地球其他部分对 Δm 的指向地心的力和这一引潮力平衡为止。

再考虑地球表面离太阳最远的“2”处的质量为 Δm 的海水。它受太阳的引力为

$$F_2 = \frac{GM_S\Delta m}{(r_S + R_E)^2}$$

仍取一级近似，得

$$F_2 = \frac{GM_S\Delta m}{r_S^2}\left(1 - \frac{2R_E}{r_S}\right) \tag{2.37}$$

这力比式(2.34)给出的惯性力 F_i 为小，其差值代表一背离地心的力，大小为

$$F_2' = F_1 - F_2 = \frac{2GM_S\Delta m}{r_S^3}R_E \tag{2.38}$$

这就是此处观察到的引潮力。它也将使此处的海水凸起形成涨潮，直到地球其他部分对 Δm 的指向地心的力和这一引潮力平衡为止。

再考虑地球表面上"3"，"4"两处质量为 Δm 的海水。它们受太阳的引力为

$$F_3 = F_4 = \frac{GM_S\Delta m}{r_S^2 + R_E^2}$$

此力平行于 x' 轴的分量，取一级近似，为

$$F_{3x'} = F_{4x'} = \frac{GM_S\Delta m}{r_S^2}$$

此分力正好和式(2.34)给出的惯性力平衡。上述引力沿 y' 轴的分量，取一级近似为

$$F_3' = F_4' = \frac{GM_S\Delta m}{r_S^3}R_E \tag{2.39}$$

其方向指向地心。这一分力将压迫此处的海水向地心下降，我们可姑且称之为"压潮力"。

以上只分析了 4 处海水受力情况，更详细的分析给出的引潮力和压潮力的分布如图 2.25 所示。由于这种力的分布，地球表面的海水也就呈现了凸起和压下的形状。

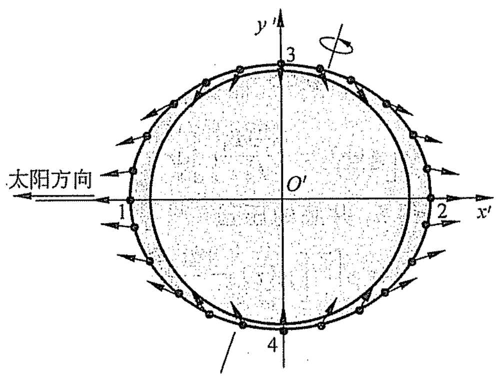

图 2.25 地球表面引潮力的分布

以上只考虑了地球公转的影响。实际上，地球还绕着地轴自转。在自转的任一时刻，地球表面的海水均有如图 2.24 所示的形状。地球自转一周的时间是一天。地球上各点每转一周，离太阳最远和最近各一次。图 2.25 所示的"静态"形状就转化为每天有两次涨潮，即朝夕各一次了。

这里顺便指出，如果引力源的质量(式(2.36)和式(2.38)中的 M_S)很大，当另一星体靠近它运行时，由于 r_S 很小，引潮力可能大到将该星体撕碎。1994 年的天文奇观——苏梅克-列维 9 号彗星撞击木星时，彗星是以 20 余块碎块撞到木星上的。这些碎块就是该彗星在靠近木星时被引潮力撕碎而形成的。

2. 潮高的计算

下面利用牛顿设计的一种方法来计算潮高。如图 2.26 所示，设想在地球内沿 x' 和 y' 方向分别挖一个竖井直达地心相通。二井深度分别为 h_1 和 h_3，截面积为 dS，井内充

满水。

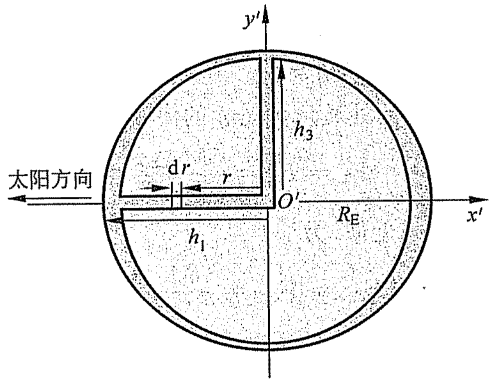

图 2.26 潮高计算用图

先计算 h_1 井中的水在地心处产生的压强 p_1。以 ρ 表示水的密度，视为常量。$\mathrm{d}r$ 一段内水的质量为 $\mathrm{d}m=\rho\mathrm{d}r\mathrm{d}S$，它受地球的引力为 $\mathrm{d}mg(r)=\rho g(r)\mathrm{d}r\mathrm{d}S$，其中 $g(r)$ 是在 r 处的重力加速度。此处的引潮力可用式(2.36)表示，只是用 r 取代其中的 R_E。由此可得 $\mathrm{d}r$ 一段水产生的压强

$$\begin{aligned}\mathrm{d}p_1&=\left[\rho g(r)\mathrm{d}r\mathrm{d}S-\frac{2GM_\mathrm{S}r}{r_\mathrm{S}^3}\rho\mathrm{d}r\mathrm{d}S\right]\Big/\mathrm{d}S\\&=\rho\left[g(r)-\frac{2GM_\mathrm{S}}{r_\mathrm{S}^3}r\right]\mathrm{d}r\end{aligned}$$

将此式对整个井深 h_1 积分，可得 h_1 井底的压强

$$p_1=\rho\int_0^{h_1}\left[g(r)-\frac{2GM_\mathrm{S}}{r_\mathrm{S}^3}r\right]\mathrm{d}r \tag{2.40}$$

同样的道理得出 h_3 井底的压强

$$p_3=\rho\int_0^{h_3}\left[g(r)+\frac{GM_\mathrm{S}}{r_\mathrm{S}^3}r\right]\mathrm{d}r \tag{2.41}$$

在稳定情况下，$p_1=p_3$，即

$$\int_0^{h_1}\left[g(r)-\frac{2GM_\mathrm{S}}{r_\mathrm{S}^3}r\right]\mathrm{d}r=\int_0^{h_3}\left[g(r)+\frac{GM_\mathrm{S}}{r_\mathrm{S}^3}r\right]\mathrm{d}r$$

移项可得

$$\int_0^{h_1}g(r)\mathrm{d}r-\int_0^{h_3}g(r)\mathrm{d}r=\int_0^{h_1}\frac{2GM_\mathrm{S}}{r_\mathrm{S}^3}r\mathrm{d}r+\int_0^{h_3}\frac{GM_\mathrm{S}}{r_\mathrm{S}^3}r\mathrm{d}r \tag{2.42}$$

此式左侧两积分可合并为 $\int_{h_3}^{h_1}g(r)\mathrm{d}r$。由于 h_1 和 h_3 都和地球半径 R_E 相差不多，$g(r)$ 就可取地球表面的重力加速度值 $g(R_\mathrm{E})=\dfrac{GM_\mathrm{E}}{R_\mathrm{E}^2}$。这样

$$\int_{h_3}^{h_1}g(r)\mathrm{d}r=(h_1-h_3)g(R_\mathrm{E})=\frac{GM_\mathrm{E}}{R_\mathrm{E}^2}\Delta h_\mathrm{S}$$

其中 $\Delta h_\mathrm{S}=h_1-h_3$，可视为潮高。

式(2.42)右侧可取 $h_1=h_3=R_E$ 而合并为

$$\int_0^{R_E}\frac{3GM_S}{r_S^3}r\mathrm{d}r=\frac{3GM_S}{2r_S^3}R_E^2$$

由此，式(2.42)给出

$$\frac{GM_E}{R_E^2}\Delta h_S=\frac{3GM_S}{2r_S^3}R_E^2$$

而潮高

$$\Delta h_S=\frac{3}{2}\frac{M_S}{M_E}\left(\frac{R_E}{r_S}\right)^3R_E \tag{2.43}$$

将 $M_S=1.99\times10^{30}$ kg，$M_E=5.98\times10^{24}$ kg，$R_E=6.4\times10^3$ km，$r_S=1.5\times10^8$ km 代入上式，可得太阳引起的潮——太阳潮——之高

$$\Delta h_S=0.25\ \mathrm{m}$$

上述分析同样可以用来分析月球在地球上引起的潮汐——太阴潮。与式(2.43)类似，太阴潮高为

$$\Delta h_M=\frac{3}{2}\frac{M_M}{M_E}\left(\frac{R_E}{r_M}\right)^3R_E \tag{2.44}$$

将月球质量 $M_M=7.35\times10^{22}$ kg，它到地心的距离 $r_M=3.8\times10^5$ km 代入上式，可得

$$\Delta h_M=0.56\ \mathrm{m}$$

实际上，潮高为 Δh_S 和 Δh_M 的矢量叠加。在朔日(新月)和望日(满月)，月球、太阳和地球几乎在同一直线上，太阳潮和太阴潮相加形成大潮，潮高可达 0.81 m(图 2.27(a))。在上弦月或下弦月时，月球和太阳对地球的方位垂直，二者相消一部分，形成小潮，潮高为 0.31 m(图 2.27(b))。一个月内大潮和小潮各出现两次。

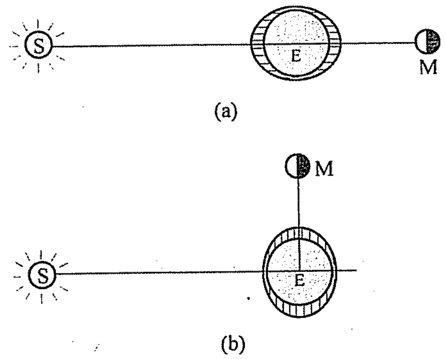

图 2.27 大潮和小潮

(a) 满月大潮；(b) 上弦小潮

和实际观测相比，以上潮高的计算值偏小，该计算值约适用于开阔的洋面。在海岸处的潮高和海岸的形状、海底的情况等有关。我国钱塘江口的排山倒海的大潮就和该处江口的喇叭形状有关。涨潮时由于水道越来越窄，致使海水越堆越高，遂形成特高潮的壮观。

由于潮水和地球固体表层的相对移动，其间摩擦力要阻碍地球的转动而导致地球自转速度的减小[①]。据计算，每过一个世纪，每一天要延长 28 秒。现代地学从珊瑚和牡蛎化石的生长线数判断，三亿多年前地球的一年约有 400 天，而现在只约有 365 天了。

① 参看张三慧. 潮汐是怎样使地球自转速度变慢的？物理与工程，2001，11(2)：6.

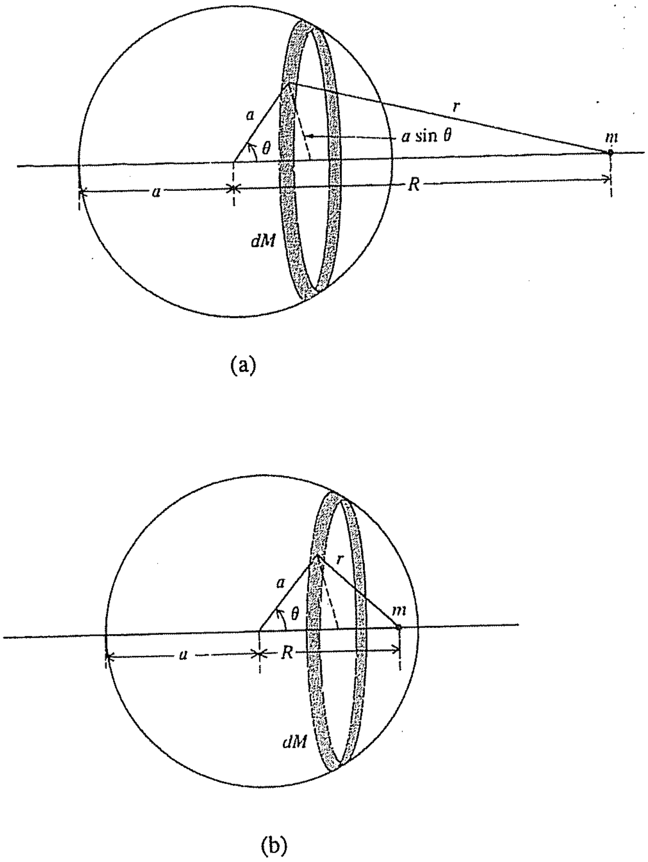

FIGURE 8-1. Gravitational attraction of a point mass m and a differential ring element dM on a spherical shell of mass M, with (a) m outside the shell and (b) m inside the shell.

The potential due to the ring mass dM is

$$d\Phi(r) = -\frac{G\,dM}{r} = -\frac{GM}{2aR}dr \tag{8.9}$$

The contributions of all ring elements on the shell are obtained by integration over r

$$\Phi(r) = \int_{r_{\min}}^{r_{\max}} d\Phi(r) = -\frac{GM}{2aR}(r_{\max} - r_{\min}) \tag{8.10}$$

We see from Fig. 8-1 that $r_{\max} = R+a$ and $r_{\min} = |R-a|$ and thus when

r is outside the shell

$$r_{\max} - r_{\min} = (R + a) - (R - a) = 2a \tag{8.11}$$

whereas when m is inside the shell,

$$r_{\max} - r_{\min} = (R + a) - (a - R) = 2R \tag{8.12}$$

Thus the potential is

$$\Phi(R) = \begin{cases} -\dfrac{GM}{R} & R > a \qquad (8.13) \\ -\dfrac{GM}{a} & R < a \qquad (8.14) \end{cases}$$

Since $\Phi(r)$ is constant inside the shell, $\mathbf{g}$ vanishes there. When r is outside the shell, the potential in (8.13) is as if the mass M of the shell were concentrated at the center of the shell. Since a spherically symmetric solid body can be represented as a collection of concentric spherical shells, the gravitational force on m due to a spherical body is as if the total mass M were concentrated at the center of the sphere. Newton's theorem follows: the gravitational force of any spherically symmetric distribution of matter at a distance R from the center is the same as if all the mass within the sphere of radius R were concentrated at the center.

8.2 The Tides

When a body moves in a non-uniform gravitational field, it is subjected to tide-generating forces. These shearing forces may even tear the body apart—this is a possible origin of the rings of Saturn.

The acceleration of the body $\mathbf{a}_B$ is the total gravitational force on its component masses divided by its total mass. (If the body is spherically symmetric, then the result of Newton's theorem and the "action equals reaction" principle is that $\mathbf{a}_B$ is simply the value of $\mathbf{g}(\mathbf{r})$ at the center of the body.) If we use coordinates centered on the body (*i.e.*, "falling with the body") the gravitational field becomes $\mathbf{g}(\mathbf{r}) - \mathbf{a}_B$. If we separate $\mathbf{g}$ into the part due to the body itself $\mathbf{g}_{\text{self}}$ (which vanishes at the center of the body) and to the part due to external masses $\mathbf{g}_{\text{ext}}$, then the gravitational field in the frame fixed on the body is $\mathbf{g}_{\text{self}} + (\mathbf{g}_{\text{ext}} - \mathbf{a}_B)$. The second term, $(\mathbf{g}_{\text{ext}} - \mathbf{a}_B)$, is the tidal field.

Tidal forces on a planet are maximum along a line to the external force center and give two high tides on opposite sides of the planet. For a planet in a circular orbit about the sun the origin of the double tide is easily explained by the following argument. The forces acting on a mass m are the attractive gravitational force GmM/r^2 and the repulsive centrifugal force $m\omega^2 r$ due to the revolution of the planet about the sun. At the CM of the planet the gravity force exactly balances the centrifugal force since there is no radial acceleration in a circular orbit. At the point closest to the sun, the sun's gravitational attraction is larger than at the CM and the centrifugal force is smaller, giving a net tidal force in the direction of the sun. At the farthest point on the planet from the sun the centrifugal force exceeds that of gravity and there is a tidal force directed away from the sun.

The ocean tides on earth are caused by the variation from place to place of the gravitational attraction due to the moon and the sun. The atmosphere, the ocean, and the solid earth all experience tidal forces, but only the effects on the ocean are commonly observed. To estimate the gross features of the midocean tides, we begin with a static theory in which the rotation of the earth about its axis is neglected. The daily rotation of the earth will be invoked later to explain the propagation of the tides.

To calculate the tide-generating force, we consider the acceleration of a small mass m on the ocean's surface under the combined influence of the gravitational attraction of the earth and a distant mass M, as shown in Fig. 8-2. The coordinates of the masses m, M_E, M in an inertial frame are represented by the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively. For convenience, we denote the relative coordinates of the masses by

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{R} &= \mathbf{r}_2 - \mathbf{r}_3 \\ \mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_3 = \mathbf{R} + \mathbf{r} \end{aligned} \tag{8.15}$$

With this notation, the motion of m and M_E due to gravitational forces is determined by

$$m\ddot{\mathbf{r}}_1 = -\frac{GmM_E\hat{\mathbf{r}}}{r^2} - \frac{GmM}{d^2}\hat{\mathbf{d}} \tag{8.16}$$

$$M_E\ddot{\mathbf{r}}_2 = -\frac{GM_EM}{R^2}\hat{\mathbf{R}} \tag{8.17}$$

By dividing the first equation by m, the second equation by M_E, and then

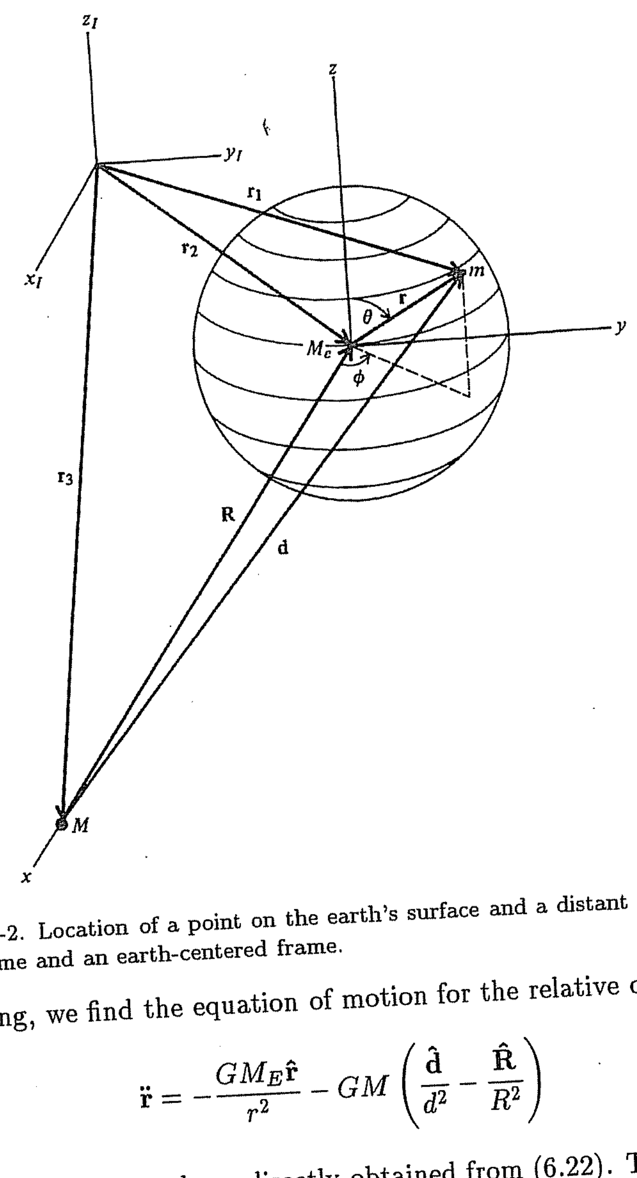

FIGURE 8-2. Location of a point on the earth's surface and a distant mass M in an inertial frame and an earth-centered frame.

subtracting, we find the equation of motion for the relative coordinate $\mathbf{r}$.

$$\ddot{\mathbf{r}} = -\frac{GM_E\hat{\mathbf{r}}}{r^2} - GM\left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{R}}}{R^2}\right) \tag{8.18}$$

This result could have been directly obtained from (6.22). The first term on the right-hand side of (8.18) is the central gravity force of the earth on a particle of unit mass. The second term is the tide-generating force per unit mass due to the presence of the distant mass M. The tide-generating force is the difference between the forces on the surface of the earth and at the center of the earth. The direction and relative magnitude of the tide-generating force due to M are plotted in Fig. 8-3 for points around the earth's equator. The effect of this force is to produce the two tidal bulges which, as the earth rotates, are observed twice daily as high tides.

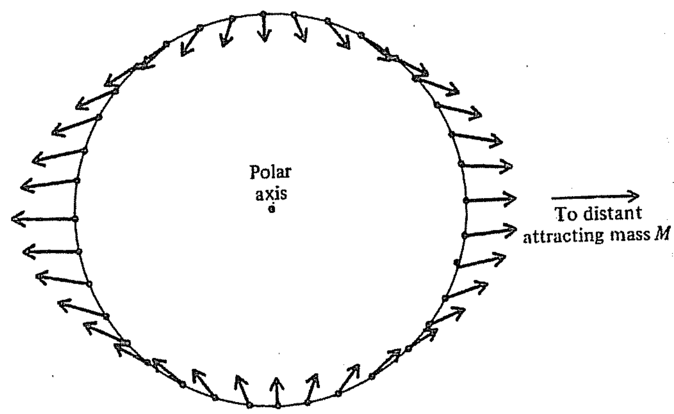

FIGURE 8-3. Tide-generating force on the surface of the earth at the equator due to a distant mass.

If the tidal forces are small compared to the gravitational force on the CM and the distance to the external force center is large compared to the planetary radius we can approximate (8.18) as follows. By (8.15) we can express the second factor of (8.18) as

$$\begin{aligned}\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{R}}}{R^2} = \frac{\mathbf{d}}{d^3} - \frac{\mathbf{R}}{R^3} &= \frac{\mathbf{R}+\mathbf{r}}{d^3} - \frac{\mathbf{R}}{R^3} \\ &= \mathbf{R}\left(\frac{1}{d^3} - \frac{1}{R^3}\right) + \frac{\mathbf{r}}{d^3}\end{aligned} \tag{8.19}$$

We form the square of d

$$\begin{aligned}d^2 &= R^2 + r^2 + 2\mathbf{R}\cdot\mathbf{r} \\ d &= R\left(1 + \frac{2\mathbf{R}\cdot\mathbf{r}}{R^2} + \frac{r^2}{R^2}\right)^{1/2}\end{aligned} \tag{8.20}$$

Then for $R \gg r$ we apply the binomial expansion $(1+\beta)^n \simeq 1 + n\beta + \cdots$, with $\beta = \mathbf{R}\cdot\mathbf{r}/R^2$ and $n = 1/2$, and retain only leading terms

$$d \simeq R\left(1 + \frac{\mathbf{R}\cdot\mathbf{r}}{R^2} + \cdots\right) \tag{8.21}$$

The quantity d^{-3} in (8.19) can be approximated by

$$\begin{aligned}\frac{1}{d^3} &\simeq \frac{1}{R^3}\left(1-\frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\right)\\ &= \frac{1}{R^3}-\frac{3\hat{\mathbf{R}}\cdot\mathbf{r}}{R^4}\end{aligned} \tag{8.22}$$

where the binomial expansion with $n=-3$ has been applied. To first order in $\mathbf{r}$ (8.19) becomes

$$\begin{aligned}\frac{\hat{\mathbf{d}}}{d^2}-\frac{\hat{\mathbf{R}}}{R^2} &= \mathbf{R}\left(\frac{1}{d^3}-\frac{1}{R^3}\right)+\frac{\mathbf{r}}{R^3}\\ &\simeq -\frac{3(\mathbf{R}\cdot\mathbf{r})}{R^4}+\frac{\mathbf{r}}{R^3}\\ &\simeq \frac{1}{R^3}\left[-3\hat{\mathbf{R}}\left(\hat{\mathbf{R}}\cdot\mathbf{r}\right)+\mathbf{r}\right]\end{aligned} \tag{8.23}$$

In our choice of coordinate system in Fig. 8-2, $\hat{\mathbf{R}}=-\hat{\mathbf{x}}$ and thus

$$\frac{\hat{\mathbf{d}}}{d^2}-\frac{\hat{\mathbf{R}}}{R^2}\simeq\frac{1}{R^3}\left(-3x\hat{\mathbf{x}}+\mathbf{r}\right) \tag{8.24}$$

In this approximation the tidal acceleration of (8.18) is

$$\ddot{\mathbf{r}} = -\frac{GM_E\hat{\mathbf{r}}}{r^2}+\frac{GM}{R^3}(3x\hat{\mathbf{x}}-\mathbf{r}) \tag{8.25}$$

Since gravitational forces are conservative this force per unit mass can be derived from a potential and we may write

$$\ddot{\mathbf{r}} \equiv -\nabla_{\mathbf{r}}\Phi \tag{8.26}$$

where $\nabla_{\mathbf{r}}$ means the gradient with respect to the vector $\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ whose origin is at the center of the earth. It is easy to guess that the potential whose negative gradient is the right side of (8.25) is

$$\Phi = -\frac{GM_E}{r}-\frac{GM}{R^3}\left(\frac{3}{2}x^2-\frac{1}{2}r^2\right) \tag{8.27}$$

Since $x=r\sin\theta\cos\phi$, we have

$$\Phi = -\frac{GM_E}{r}-\frac{GM}{r}\left(\frac{r}{R}\right)^3\left(\frac{3}{2}\sin^2\theta\cos^2\phi-\frac{1}{2}\right) \tag{8.28}$$

For equilibrium of the ocean surface, the net tangential force on m must vanish. Equivalently, the potential at any point on the ocean's surface must be constant. We choose the constant to be $\Phi(\mathbf{r})=-GM_E/R_E$, where R_E is the undistorted spherical radius of the earth (i.e., when the distant M is absent). Using this condition in (8.28) gives

$$r - R_E = \frac{M}{M_E}\frac{r^3 R_E}{R^3}\left(\frac{3}{2}\sin^2\theta\cos^2\phi - \frac{1}{2}\right) \tag{8.29}$$

Since the height of the tidal displacement

$$h(\theta,\phi) \equiv r - R_E \tag{8.30}$$

is quite small compared with R_E, (8.29) gives

$$h(\theta,\phi) \simeq \frac{M}{M_E}\frac{R_e^4}{R^3}\left(\frac{3}{2}\sin^2\theta\cos^2\phi - \frac{1}{2}\right) \tag{8.31}$$

For a given colatitude angle θ in (8.31), the high tides occur at $\phi = 0$ and $\phi = \pi$, and low tides occur at $\phi = \pi/2$ and $\phi = 3\pi/2$. The difference in height between high and low tide, known as the *tidal range*, is

$$\Delta h = \frac{3}{2}\frac{M}{M_E}\frac{R_e^4}{R^3}\sin^2\theta \tag{8.32}$$

The tidal displacement h is largest at $\theta = 90°$ (on the equator). The tidal distortion is illustrated in Fig. 8-4. The tide for an ocean devoid of continents has a prolate spheroid shape (football-like), with the major axis in the direction of the distant mass. The calculation of such an ideal tide was first made by Newton in 1687.

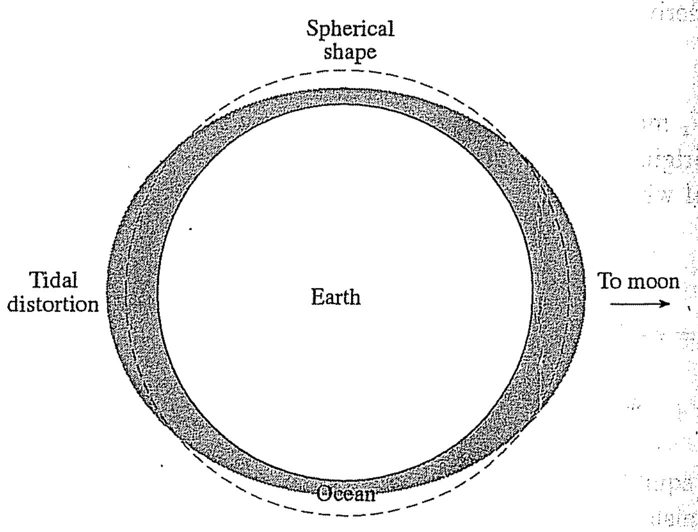

FIGURE 8-4. Tidal distortion at the earth's equator on an exaggerated scale.

The preceding discussion applies to the tidal forces induced by a single astronomic body. If there are two tide-producing bodies the net tide is the superposition of the separate tides. (If the bodies are not collinear with the planet, the total tidal shape is not axially symmetric but a triaxial ellipsoid instead.) From (8.31) the ratio of the maximum heights of the lunar (L) and solar ($\odot$) tides on earth is

$$\frac{h_L}{h_\odot} = \left(\frac{M_L}{M_\odot}\right)\left(\frac{a_E}{a_L}\right)^3 \tag{8.33}$$

where a_L is the earth-moon distance and a_E is the earth-sun distance. The numerical value of this ratio is

$$\frac{h_L}{h_\odot} = \frac{(1/81.5)M_E}{\left(\frac{1}{3}\times 10^6\right)M_E}\left(\frac{1.5\times 10^8\ \text{km}}{3.8\times 10^5\ \text{km}}\right)^3 = 2.2 \tag{8.34}$$

Thus the sun's tidal effect is smaller than the moon's, but it is not negligible. When the sun and moon are lined up (new or full moon), an especially large tide results (spring tide), and when they are at right angles (first or last quarter moon), their tidal effects partially cancel (neap tide). The diagram in Fig. 8-5 illustrates these orientations of the moon relative to the earth and sun.

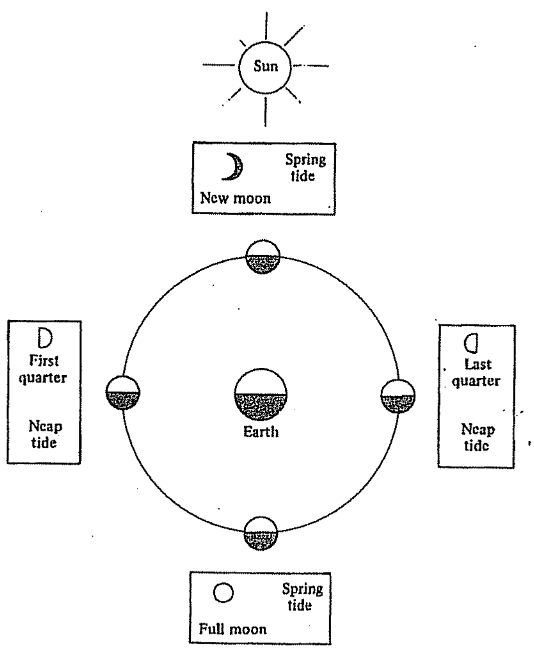

FIGURE 8-5. Relation of the phases of the moon to the tides on earth.

The tidal range due to the moon at a point on the earth-moon axis can be calculated from (8.32). We get

$$\Delta h\left(\theta = \frac{\pi}{2}\right) = \frac{3}{2}\left(\frac{1}{81.5}\right)\left(\frac{6,371}{384,000}\right)^3 (6,371 \times 10^3) = 0.56\ \mathrm{m}$$

This figure agrees roughly with the measured tidal difference in midocean. As the earth rotates about its own axis, the tidal maxima, which lie on the earth-moon axis, will pass a given point on the earth's surface approximately two times a day. More precisely, since the orbital rotation of the moon about the earth (with period of $27\frac{1}{3}$ days) is in the same sense as the earth's own rotation (with period 24 h), two tidal maxima pass a given spot on earth every $(24 + 24/27\frac{1}{3})$ h. Thus high tide occurs every 12 h and 26.5 min, and high tide is observed about 53 min later each day.

The two high tides are not of the same height because of the inclination of the earth's axis to the normal of the moon's orbital plane about the earth. In the Northern Hemisphere the high tide which occurs closest to the moon is higher, as illustrated in Fig. 8-6.

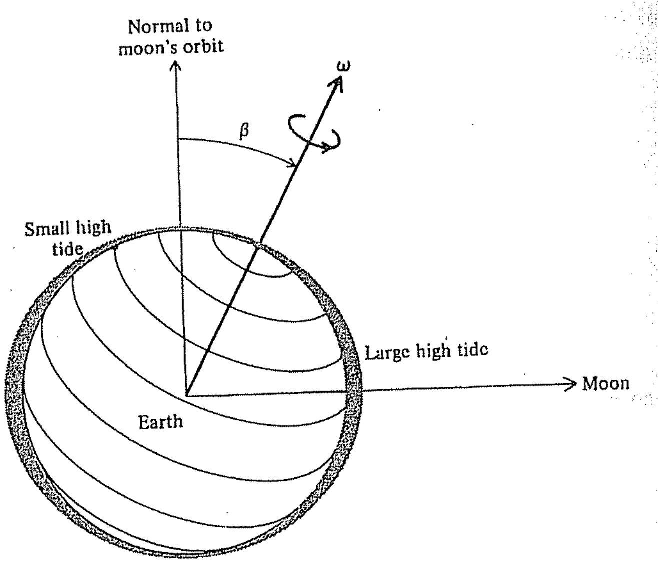

FIGURE 8-6. Effect of the inclination angle β of the earth's axis to the moon's orbital plane on the heights of tides. β varies from 17° to 29° as the moon's elliptical orbit precesses slowly about the normal to the plane of the earth's heliocentric orbit.

The tides are in reality more complicated than described above. Along coastal regions the configuration of the land masses and the ocean bottom cause considerable amplification or suppression of the tidal range. Over the world, tidal ranges vary as much as twenty meters.

The friction of the moving tidal waves against ocean bottoms and the continental shorelines dissipates energy at a rate estimated at 7 billion horsepower. To supply this energy, the earth's rotation about its axis slows down at the rate of 4.4×10^{-8} s per day. The cumulative time over a century is about 28 s. This gradual lengthening of the day is confirmed by the observation that various astronomical events such as eclipses seem to run systematically ahead of calculations based on observations over preceding centuries.

8.3 Tidal Evolution of a Planet-Moon System

The earth-moon system has very little external torque acting upon it on the average. The total angular momentum of the system is thus nearly constant. The consequence of angular momentum conservation is that the moon spirals outward about a half a centimeter each month as the earth's rotation is slowed by tidal friction. Ultimately the moon's distance will increase by over forty percent of its present value and our day will lengthen by a factor of about 50. The moon will then remain stationary above one spot on the earth.

To see this, we make the following simplifications, which are sufficiently accurate to represent the physical situation.

1. The spin angular momentum $\mathbf{S} = I\omega$ of the earth is parallel to the orbital angular momentum $\mathbf{L}$ of the moon about the earth. (The earth's spin precesses about the normal to the ecliptic plane with a period of 26,000 years and the plane of the moon's orbit about the earth precesses similarly with a period of about 19 years, so the average values of both $\mathbf{S}$ and $\mathbf{L}$ are perpendicular to the ecliptic plane—the plane of earth's orbit around the sun).
2. The total angular momentum
$$\mathbf{J} = \mathbf{L} + \mathbf{S} = (L + S)\hat{\mathbf{L}} \tag{8.35}$$
is constant (we are neglecting the solar tidal drag).
3. The moon's orbit about the earth is circular and lies in the ecliptic plane (point 1 above).

Feynman's Lecture

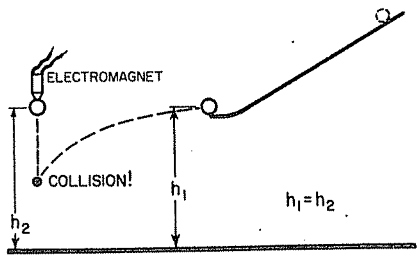

Fig. 7–3. Apparatus for showing the independence of vertical and horizontal motions.

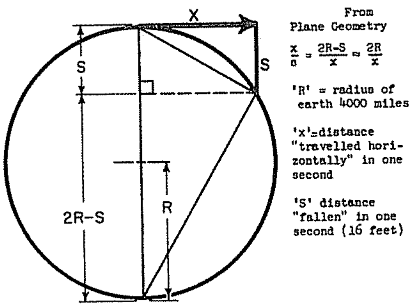

Fig. 7–4. Acceleration toward the center of a circular path. From plane geometry, $x/s = (2R - S)/x \approx 2R/x$, where R is the radius of the earth, 4000 miles; x is the distance "travelled horizontally" in one second; and S is the distance "fallen" in one second (16 feet).

explanation: the moon falls in the sense that *it falls away from the straight line that it would pursue if there were no forces.* Let us take an example on the surface of the earth. An object released near the earth's surface will fall 16 feet in the first second. An object shot out *horizontally* will also fall 16 feet; even though it is moving horizontally, it still falls the same 16 feet in the same time. Figure 7–3 shows an apparatus which demonstrates this. On the horizontal track is a ball which is going to be driven forward a little distance away. At the same height is a ball which is going to fall vertically, and there is an electrical switch arranged so that at the moment the first ball leaves the track, the second ball is released. That they come to the same depth at the same time is witnessed by the fact that they collide in midair. An object like a bullet, shot horizontally, might go a long way in one second—perhaps 2000 feet—but it will still fall 16 feet if it is aimed horizontally. What happens if we shoot a bullet faster and faster? Do not forget that the earth's surface is curved. If we shoot it fast enough, then when it falls 16 feet it may be at just the same height above the ground as it was before. How can that be? It still falls, but the earth curves away, so it falls "around" the earth. The question is, how far does it have to go in one second so that the earth is 16 feet below the horizon? In Fig. 7–4 we see the earth with its 4000-mile radius, and the tangential, straightline path that the bullet would take if there were no force. Now, if we use one of those wonderful theorems in geometry, which says that our tangent is the mean proportional between the two parts of the diameter cut by an equal chord, we see that the horizontal distance travelled is the mean proportional between the 16 feet fallen and the 8000-mile diameter of the earth. The square root of $(16/5280) \times 8000$ comes out very close to 5 miles. Thus we see that if the bullet moves at 5 miles a second, it then will continue to fall toward the earth at the same rate of 16 feet each second, but will never get any closer because the earth keeps curving away from it. Thus it was that Mr. Gagarin maintained himself in space while going 25,000 miles around the earth at approximately 5 miles per second. (He took a little longer because he was a little higher.)

Any great discovery of a new law is useful only if we can take more out than we put in. Now, Newton *used* the second and third of Kepler's laws to deduce his law of gravitation. What did he *predict?* First, his analysis of the moon's motion was a prediction because it connected the falling of objects on the earth's surface with that of the moon. Second, the question is, *is the orbit an ellipse?* We shall see in a later chapter how it is possible to calculate the motion exactly, and indeed one can prove that it should be an ellipse,* so no extra fact is needed to explain Kepler's *first* law. Thus Newton made his first powerful prediction.

The law of gravitation explains many phenomena not previously understood. For example, the pull of the moon on the earth causes the tides, hitherto mysterious. The moon pulls the water up under it and makes the tides—people had thought of that before, but they were not as clever as Newton, and so they thought there ought to be only one tide during the day. The reasoning was that the moon pulls the water up under it, making a high tide and a low tide, and since the earth spins underneath, that makes the tide at one station go up and down every 24 hours. Actually the tide goes up and down in 12 hours. Another school of thought claimed that the high tide should be on the other side of the earth because, so they argued, the moon pulls the earth away from the water! Both of these theories are wrong. It actually works like this: the pull of the moon for the earth and for the water is "balanced" at the center. But the water which is closer to the moon is pulled *more* than the average and the water which is farther away from it is pulled *less* than the average. Furthermore, the water can flow while the more rigid earth cannot. The true picture is a combination of these two things.

What do we mean by "balanced"? What balances? If the moon pulls the whole earth toward it, why doesn't the earth fall right "up" to the moon? Because the earth does the same trick as the moon, it goes in a circle around a point which is inside the earth but not at its center. The moon does not just go around the earth, the earth and the moon both go around a central position, each falling

* The proof is not given in this course.

toward this common position, as shown in Fig. 7–5. This motion around the common center is what balances the fall of each. So the earth is not going in a straight line either; it travels in a circle. The water on the far side is thrown out more by the "centrifugal force" than the average for the center of the earth, which is just balanced by the moon's attraction. The moon's attraction on the far side is weaker and the "centrifugal force" is stronger. The net result is an imbalance of the water in a direction away from the center of the earth. On the near side, the attraction from the moon is stronger, and because the radius vector is shorter, the "centrifugal force" is also weaker and the imbalance is in the opposite direction in space, but again *away* from the center of the earth. The net result is that we get *two* tidal bulges.

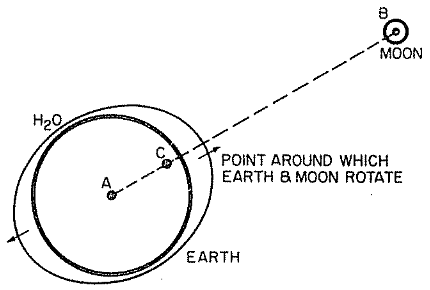

Fig. 7–5. The earth-moon system, with tides.

7–5 Universal gravitation

What else can we understand when we understand gravity? Everyone knows the earth is round. Why is the earth round? That is easy; it is due to gravitation. The earth can be understood to be round merely because everything attracts everything else and so it has attracted itself together as far as it can! If we go even further, the earth is not *exactly* a sphere because it is rotating, and this brings in centrifugal effects which tend to oppose gravity near the equator. It turns out that the earth should be elliptical, and we even get the right shape for the ellipse. We can thus deduce that the sun, the moon, and the earth should be (nearly) spheres, just from the law of gravitation.

What else can you do with the law of gravitation? If we look at the moons of Jupiter we can understand everything about the way they move around that planet. Incidentally, there was once a certain difficulty with the moons of Jupiter that is worth remarking on. These satellites were studied very carefully by Roemer, who noticed that the moons sometimes seemed to be ahead of schedule, and sometimes behind. (One can find their schedules by waiting a very long time and finding out how long it takes on the average for the moons to go around.) Now they were *ahead* when Jupiter was particularly *close* to the earth and they were *behind* when Jupiter was *farther* from the earth. This would have been a very difficult thing to explain according to the law of gravitation—it would have been, in fact, the death of this wonderful theory if there were no other explanation. If a law does not work even in *one place* where it ought to, it is just wrong. But the reason for this discrepancy was very simple and beautiful: it takes a little while to *see* the moons of Jupiter because of the time it takes light to travel from Jupiter to the earth. When Jupiter is closer to the earth the time is a little less, and when it is farther from the earth, the time is more. This is why moons appear to be, on the average, a little ahead or a little behind, depending on whether they are closer to or farther from the earth. This phenomenon showed that light does not travel instantaneously, and furnished the first estimate of the speed of light. This was done in 1656.

If all of the planets push and pull on each other, the force which controls, let us say, Jupiter in going around the sun is not just the force from the sun; there is also a pull from, say, Saturn. This force is not really strong, since the sun is much more massive than Saturn, but there is *some* pull, so the orbit of Jupiter should not be a perfect ellipse, and it is not; it is slightly off, and "wobbles" around the correct elliptical orbit. Such a motion is a little more complicated. Attempts were made to analyze the motions of Jupiter, Saturn, and Uranus on the basis of the law of gravitation. The effects of each of these planets on each other were calculated to see whether or not the tiny deviations and irregularities in these motions could be completely understood from this one law. Lo and behold, for Jupiter and Saturn, all was well, but Uranus was "weird." It behaved in a very peculiar manner. It was not travelling in an exact ellipse, but that was understandable, because of the attractions of Jupiter and Saturn. But even if allowance were made for these attractions, Uranus *still* was not going right, so the laws of gravitation were in danger of being overturned, a possibility that could not be ruled out. Two men, Adams and Leverrier, in England and France, independently,

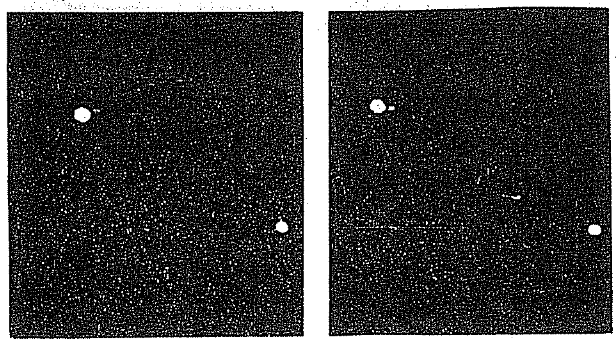

Fig. 7–6. A double-star system.

arrived at another possibility: perhaps there is *another* planet, dark and invisit which men had not seen. This planet, N, could pull on Uranus. They calculai where such a planet would have to be in order to cause the observed perturl tions. They sent messages to the respective observatories, saying, "Gentlem point your telescope to such and such a place, and you will see a new plane It often depends on with whom you are working as to whether they pay any att tion to you or not. They did pay attention to Leverrier; they looked, and th planet N was! The other observatory then also looked very quickly in the n few days and saw it too.

This discovery shows that Newton's laws are absolutely right in the sc system; but do they extend beyond the relatively small distances of the neai planets? The first test lies in the question, do *stars* attract *each other* as well planets? We have definite evidence that they do in the *double stars.* Figure ' shows a double star—two stars very close together (there is also a third stai the picture so that we will know that the photograph was not turned). The st are also shown as they appeared several years later. We see that, relative to "fixed" star, the axis of the pair has rotated, i.e., the two stars are going aroi each other. Do they rotate according to Newton's laws? Careful measureme of the relative positions of one such double star system are shown in Fig. 7 There we see a beautiful ellipse, the measures starting in 1862 and going all way around to 1904 (by now it must have gone around once more). Everytł coincides with Newton's laws, except that the star Sirius A is *not at the fo* Why should that be? Because the plane of the ellipse is not in the "plane of sky." We are not looking at right angles to the orbit plane, and when an ell is viewed at a tilt, it remains an ellipse but the focus is no longer at the same pl Thus we can analyze double stars, moving about each other, according to requirements of the gravitational law.

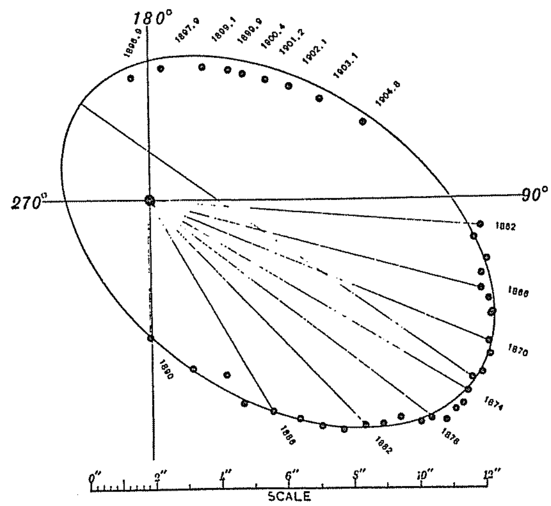

Fig. 7–7. Orbit of Sirius *B* with respect to Sirius *A*.

Myths about Gravity and Tides

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Introduction

The idea for this paper came as I was having a coffee in a beach-front cafe in North Miami Beach, watching the tide, and perusing a newspaper. Thanks to a small-time serendipity, I chanced upon the following piece of local news: *"During a full moon, the moon has a higher gravitational pull, creating a higher tide. Miami Beach psychiatrist Arnold Lieber says that pull affects oceans and people in a similar way, since the human body is mostly water."* [1]

Obviously, the above quoted text is another wonderful example of bad physics [2]. In reality, the Moon does not pull any harder during the full Moon, and a water content of a human body is entirely irrelevant to a question of tidal effect on humans. Yet a subject of tidal effects on small bodies of water comes up quite often, as evidenced by two additional examples.

"Are there ocean tides in fresh water or just the oceans?" is the question that I found in the "Ask Marilyn" column. *"There are tides everywhere on Earth, including not just oceans and lakes but also ground we stand on (which is factor in earthquakes) and the atmosphere we breathe. If you stood still long enough, there would even be tides in your tummy"*, answered Marilyn vos Savant. [3]

In a similar spirit, an astronomy teacher asserted in my local newspaper: *"Here in Southern Illinois as in much of the world, we experience two periods of high and low tides every day. (...) The tidal forces act on all bodies of water, not just oceans (...) At perigee (...) the moon's gravitational tug on Earth is at its maximum and so the waters of the world rise and fall to their maximum extent."* [4]

While statements in both examples seem plausible, in reality tidal effect on small bodies of water of the size of reader's tummy or lakes in Southern Illinois is negligible and therefore impossible to observe.

One should realize that the tidal effect is caused not only by the Moon, but also by the Sun. If the Moon had not existed, there still would have been ocean tides, but not as high, since the Sun contributes only about 30% of the tidal effect. Furthermore, and more importantly, it is not the magnitude of the gravitational tug *per se* that is responsible for a tidal mechanism, but rather a subtle difference in the gravitational tug on water at various parts of a basin.

Let us first discuss the magnitude of the gravitational tug. The center of Earth is at the average distance $d_s = 1.496\times10^{11}$ m from the center of the Sun. The gravitational pull of the Sun on 1 kg mass at the distance d_s is

$$a_s = \frac{\mathrm{GM_s}}{\mathrm{d_s^2}} = 6.04\times10^{-4}\, g \tag{1}$$

where $g = 9.81$ N/kg. Similar calculation for the Moon, an average distance $d_m = 3.84\times10^8$ m away, yields the following value for gravitational pull of the Moon on a unit mass at the center of Earth

$$a_m = \frac{\mathrm{GM_m}}{\mathrm{d_m^2}} = 3.39\times10^{-6}\, g \tag{2}$$

Comparing Eq.(1) and Eq.(2), we see that the Sun's gravitational pull per unit mass is about 178 times stronger than the Moon's gravitational pull, hardly a surprising result as Earth orbits the Sun, not the Moon. (Nevertheless, when I ask my students what pulls harder on Earth, the Moon or the Sun, they invariably choose the Moon...) Specifically, Earth's, Sun's and Moon's average gravitational pulls on a 80 kg person on the surface of Earth are about 785 N, 0.47 N, and 0.0027 N, respectively. Now, since Earth is in a free fall about Earth-Moon center of mass, which in turn is in a free fall about the Sun-Earth-Moon center of mass (located inside the Sun), no scale would register the last two pulls. To better see it, remember that if you were standing on a scale inside a freely-falling elevator or orbiting space shuttle, the reading on the scale would be zero, and you would experience apparent weightlessness.

The Source of Tides

Let us now discuss the difference in the gravitational tug on water at various parts of a basin. The gravitational pull of the Sun on 1 kg mass on the far side of Earth, a distance of $R = 6.37\times10^6$ m further away from Earth's center, is by the amount of Δa_s smaller than a_s, i.e.

$$\Delta a_s = \frac{\mathrm{GM_s}}{\mathrm{d_s^2}} - \frac{\mathrm{GM_s}}{(\mathrm{d_s}+R)^2} = \frac{2\mathrm{GM_s}R}{\mathrm{d_s^3}} = a_s\frac{2R}{d_s} = 0.515\times10^{-7}\, g \tag{3}$$

This is known as a differential pull. Note that in the second step in Eq.(3) we neglected corrections from higher order R/d_s terms. A similar calculation shows that the gravitational pull of the Sun per 1 kg mass at the point on Earth closest to the Sun is in turn greater than a_s, also by the amount Δa_s. This result could be also obtained by taking the differential of the gravitational force equation. Note that Δa_s is inversely proportional to the *cube* of the distance.

To see how tides come about, let us first ignore the existence of the Moon, and suppose the whole Earth is covered with ocean of equal depth everywhere; there is no dry land. The Earth is simply in free fall towards the Sun.

Consider the point C on Earth closest to the Sun and the point F on a far side of Earth. The Sun pulls harder on a unit mass at the point C, not as hard on a unit mass at Earth center O, and weaker yet on a unit mass at point F. The acceleration a_s of Earth as a whole in free fall towards the Sun is determined by the gravitational pull of the Sun on Earth's center. Hence the unit mass at C has a tendency to accelerate towards the Sun with acceleration $a_s + \Delta a_s$, i.e. more than the center of Earth, while a mass at the far side F has a tendency to accelerate towards the Sun with acceleration $a_s - \Delta a_s$, i.e. to lag behind the center of Earth.

This difference in Solar gravitational pulls would have lead to a disintegration of Earth, had Earth's own gravity been to weak to hold Earth together. To an observer on Earth it would have looked like rocks at point C and F were lifted away from the surface of Earth. Fortunately, Earth's own gravity pulls the unit masses at C and F towards Earth center O with a gravitational pull per unit mass equal to g. The combined effect is that if you drop a rock at the point C or F, rock's acceleration towards the center of Earth is $g - \Delta a_s$.

Another important part of the tidal effect is due to the fact that the Sun is at a finite distance from the Earth, and therefore Sun's pull at the point L (halfway from C towards F along Earth's surface) is not exactly parallel to the Sun's pull at the center O, but has a $\sin \alpha$ component toward the center of the Earth, where α is the angle made between the line from the Sun to the center of the Earth O and the line from the Sun to the point L, i.e. $\tan \alpha = R/d_s$. As a result, a rock dropped at the point L accelerates towards the center of Earth at the rate of $g + \Delta a_s/2$.

From the point of view of a hypothetical observer located at the center of the Earth, it appears that Earth's gravitational pull on a rock at C and F is reduced by the amount of Δa_s, while the pull at the point L is increased by $\Delta a_s/2$. In other words, while Earth's own gravity pulls the mass m down with the force mg everywhere on Earth's surface, there appears to be another force, the tidal force, pulling the mass m at points C and F <u>up</u> with magnitude $m\Delta a_s$, and <u>down</u> at point L with magnitude $m\Delta a_s/2$, see Figure 1. This is the source of the tidal mechanism, and the origin of the water bulge at C and F. (The tidal mechanism is nicely illustrated in conceptual physics textbooks [5,6].)

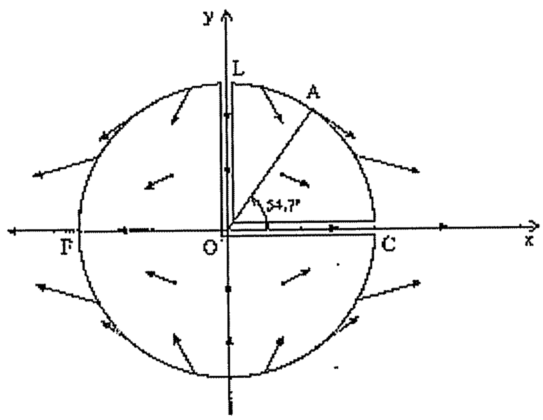

Figure 1. The tidal mechanism, see text. The x-axis points toward the Sun. The Earth is in a free fall toward the Sun. The resulting tidal force causes high tides at the point C closest to the Sun and point F furthest from the Sun, and a low tide at the point L. Note that at the point A the tidal force is tangent to the surface of the Earth.

The lunar tidal effect is calculated in an analogous way. Again, one has to realize that Earth is in a free fall towards the Earth-Moon center of mass. The difference between the Moon's pull on a unit mass at the center of Earth and at the closest/farthest point on Earth is

$$\Delta a_m = a_m \frac{2R}{d_m} = 1.13 \times 10^{-7} g \qquad (4)$$

While the solar gravitational pull is 178 times stronger than lunar gravitational pull, the tidal pull is proportional to the inverse cube of the distance, and the ratio of d_s/d_m is 390, thus making the solar tidal pull only 178/390 = 0.46 of the lunar tidal pull, $\Delta a_s = 0.46\ \Delta a_m$. This could be also seen by a direct comparison of values in Eq.(2) and Eq.(4). Since $\Delta a_m/(\Delta a_m + \Delta a_s) = 69\%$, we see that it is the Moon that dominates the tidal mechanism. But if the Moon had been only twice as far from Earth as it is now, the Moon's tidal pull would have decreased 8 times and become 4 times weaker than the Sun's tidal pull.

When the Sun, Earth and the Moon all lie along a straight line, as at new and full Moon, the Sun's and the Moon's tidal forces pull in the same direction and cause high tides to be higher than average, and low tides to be lower than average. These stronger tides are called spring tides. With the Moon in first or last quarter, the tidal force of the Moon acts in a direction perpendicular to the Sun's tidal force. This makes the tides smaller than average, and they are called neap tides.

First Approximation

Let us first consider the tidal action of the Moon on a hypothetical ocean uniformly covering the whole Earth. For simplicity, I'll neglect effects related to Earth's rotation, as they do not affect our conclusions. The water bulge takes a shape of a football (spheroid or ellipsoid of revolution), and for a not rotating Earth the major axis of the football would point towards the Moon. All water on the surface of the water bulge has equal potential energy, i.e. the water surface is an equipotential. As the Earth is in a free fall, and the water bulge is shaped by the tidal force and Earth's gravity, the equpotential in question in addition to Earth's gravity involves a tidal potential, traditionally called W_2 in literature [7].

At a low-tide point L on the water bulge, the tidal force is directed into Earth, while at a high-tide points C and F, the lunar tidal force is directed up, c.f. Figure 1. If we choose the reference frame with the origin at the center of Earth, the x-axis toward the Moon, and y-axis toward the low-tide point L, c.f. Figure 1, then the Moon's tidal force per unit mass anywhere inside the Earth could be parametrized as

$$\begin{aligned} \Delta a_{m,x} &= a_m \frac{2x}{d_m} \\ \Delta a_{m,y} &= -a_m \frac{y}{d_m} \end{aligned} \qquad (5)$$

where a_m is given by Eq.(2). It is perhaps interesting to note that at the point labeled A in Figure 1, the tidal force is purely tangential to Earth's surface. A straightforward calculation shows that the point A is located at the angle $\theta = 54.7°$ away from the x-axis.

Moving the unit mass of water from the low-tide point L to the point C requires a positive work by a tidal force. To calculate this work Newton considered two imaginary wells, one running from the low-tide point L to the center of Earth, and the other from the high-tide point C to the center O. When the unit mass is moved from a low tide point L to the center O to the high-tide point C, the average tidal force along the displacement in the tunnel LO is $\Delta a_{av} = 0.25\Delta a_m$, and in the tunnel OC it is $\Delta a_{av} = 0.50\Delta a_m$, where Δa_m is given by Eq.(4). Hence, the tidal force does a positive work equal to $0.25\Delta a_m R + 0.50\ \Delta a_m R = 0.75\Delta a_m R$, and the tidal potential energy per unit mass decreases by the amount

$$\Delta W_2 = -0.75\ \Delta a_m R \tag{6}$$

This decrease in the tidal gravitational potential must be compensated by the increase of the gravitational potential caused by a rise Δh_m of the water surface with respect to center of Earth, if the water surface is to remain an equipotential. Hence we have

$$g\Delta h_m - 0.75\,\Delta a_m R = 0 \tag{7}$$

Solving for Δh_m, we obtain the value of 0.54 m for the rise of the water level in a homogenous ocean caused by the lunar tidal force. A similar calculation for the solar tidal force yields $\Delta h_s = 0.46\times\Delta h_m = 0.25$ m. Thus during a spring tide one could expect the tidal rise of the water level by $\Delta h = \Delta h_m + \Delta h_s = 0.79$ m.

Mathematical Digression About the Tidal Potential

Results obtained above could be more formally arrived at beginning with a concept of the tidal potential. To this end we need to recognize that the tidal force per unit mass given by Eq.(5) could be written as a gradient of tidal potential energy per unit mass,

$$\begin{aligned} \Delta a_{m,x} &= -\frac{\partial}{\partial x} W_2 \\ \Delta a_{m,y} &= -\frac{\partial}{\partial y} W_2 \end{aligned} \tag{8}$$

where the tidal potential is obviously

$$W_2(x,y) = -\frac{GM_m}{d_m^3}(x^2 - \frac{1}{2}y^2) \tag{9}$$

It is convenient to use polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and rewrite the tidal potential as

$$W_2(r,\theta) = -\frac{GM_m}{d_m^3} r^2 P_2(\cos\theta) \tag{10}$$

where P_2 is the Legendre polynomial, $P_2(z) = \frac{1}{2}(3z^2 - 1)$. The main result given by Eq.(6) now simplifies to

$$\Delta W_2 = W_2(R,0°) - W_2(R,90°) = -0.75\,\Delta a_m R \qquad (11)$$

Real Tides

Since in reality Earth rotates and is not uniformly covered with an ocean of a constant depth, analysis of ocean tides is very difficult. Oceans have complicated shapes, varying depths and floor configurations. Shorelines are quite complex. All these factors contribute to unusual local variations of ocean tides [8]. In some places there is only one tide per day. Large basins open to ocean (sounds, gulfs, bays) may exhibit an enormous resonant behavior. There are places where there are no observable tides, and other, where the water rises as much as 51 ft (Minas Basin, Bay of Fundy). In other, there is little tide in a sense of a rise and fall, but strong currents flow periodically back and forth. The high water may not occur when the Moon is overhead, i.e. tides are not necessarily in sync with the differential forces, but may lag behind by several hours. Duration of a high and low water is also affected by local conditions. But for a given location the high water will always lag the passage of the Moon by a fixed amount of hours (this is known as "the establishment of the port" [9].)

What about the tides in smaller bodies of water? The symbol R in Eq.(4) is now replaced by Δd, the difference between a distance to the center of the Moon from the closest and furthest point of the water basin under consideration.

For Great Lakes, the difference in distance from the Moon to various points on a surface of the water is much smaller than radius of the Earth, but tides of amplitude of about two inches could be still observed. But local lakes are so small that all points on the surface of the water are practically at the same distance from the Moon, hence no observable tides exist. Likewise there are no tides in a swimming pool, a bathtub, cup of coffee, or a tummy. Indeed, for a shallow puddle of some 20 m in diameter, the value of Δd due to curvature of Earth is $\Delta d = 1.3\times10^{-3}$ m, and Eq.(4) yields $\Delta a_m = 10^{-17}$ g, an unimaginably small value. Hence, no tides in puddles.

As already mentioned above, not only water, but also atmosphere and land are subject to a tidal action and hence experience tidal effects. Because continents are much more rigid than oceans, the effect is much smaller. Nevertheless, parts of continent may rise and fall as much as 0.40m (16 inches) when the Moon passes overhead. So a swimming pool with the water in it may rise few inches, but because the entire land area moves up and down together, we don't readily observe this effect.

Negligible and Small Effects

It is perhaps interesting to estimate the tidal stretch of a human body. Assuming $\Delta d = H = 1.7$ m, one obtains $\Delta a_m = 3\times10^{-13}$ N/kg for the Moon's tidal contribution, and at a high tide the combined effect of the Moon and the Sun yields $\Delta a = \Delta a_m + \Delta a_s = 4\times10^{-13}$ N/kg. Note that a tidal effect of this magnitude could be alternatively created while holding a pea (mass 1.5 g) some 0.5 m above one's head. In any case, an average tidal tension on a 80 kg person is about 1.6×10^{-11} N. Assuming average cross section area of human skeleton to be 1 cm^2 =

Another factor that one might wish to consider here is that the weight of the atmosphere overhead, and hence the atmospheric pressure on a human body, is reduced too. Such gravitational reactions of the atmosphere to the tidal forces are rather small,

$$\Delta P = 1.7\times10^{-7}\times1 \text{ atm} = 0.02 \text{ Pa}, \tag{16}$$

but there are larger tides in the atmosphere caused by a radiational energy input from the Sun. Analysis of tides in atmosphere is difficult due to contamination by broad-band noise of meteorological origin; nevertheless the tidal changes in the atmospheric pressure are found to be as high as ± 3 Pa, and a human body would stretch by some 10^{-9} m. One should keep in mind that such a pressure change could be alternatively achieved climbing up 0.2 meters, and that purely meteorological pressure variations can be as high as ± 3000 Pa.

Other Effects

What happens when the Moon is at the perigee, i.e. at the point closest to Earth? While it is true that the Moon's pull on Earth is at its maximum, this circumstance *per se* has no bearing on the tidal mechanism. It is the difference in the Moon's pulls, described by Eq.(4), that creates the tidal force. Since the eccentricity of the Moon's orbit is $\varepsilon = 0.0549$, the average distance d_m in Eq.(2) and Eq.(4) is replaced by the perigee distance of $d_p = d_m\times(1 - \varepsilon) =$ 363,000 km. This increases the Moon's differential pull of Eq.(4) by a factor of about $1/(1\text{-}0.0549)^3 = 1.18$, i.e.

$$\Delta a_{m,perigee} = 1.34\times10^{-7} \text{ g}, \tag{17}$$

and the tidal action of the Moon is about 18 % stronger. If the Moon's passage through the perigee coincides with spring tides, the combined tidal effect of the Sun and the Moon is approximately 12% stronger than that described by Eq.(14). Such perigean spring tides happen approximately every 6.5 months. Also, because of a three-body effects in the Sun-Earth-Moon system, the lunar perigee distance can be actually as small as 356,4000 km, and the Moon's tidal effects can be up to 25% stronger than those at the average distance [11].

Earth's orbit around the Sun is also elliptical, and the Earth passes through the perihelion (point closest to the Sun) around January 3. The eccentricity of Earth's orbit is only $\varepsilon =$ 0.017, and at the perihelion the Sun's differential pull of Eq.(2) increases by about $1/(1\text{-}0.017)^3 = 1.052$, i.e. by some 5%. Since the time of perigean spring tide is different every year, it will eventually coincide with Earth's passage through perihelion. This will produce the highest tides in the entire tidal cycle. If such unusually strong tides happen to coincide with meteorological tides caused by a winter storm with low atmospheric pressure and onshore winds, then coastal flooding is a definite possibility.

Length of a Day

As we pointed out above, for the Earth covered with an ideal, uniform ocean, the surface of the water would form a spheroid, whose long (major) axis would point in a direction determined by the position of the Moon (more about in the next Section). As the Earth and oceans rotate beneath the spheroid, continents crash into the water bulge during a high tide, piling water up. It is important to remember that the water of the ocean rotates together with the Earth – it is only the *shape* of the water that remains fixed in space. As seen from a geocentric reference frame that rotates with the Earth, the tidal bulges appear as waves that circulate around the globe crashing into continents, thus imparting an impulse to the continents. (Keep in mind that a traveling wave transports both momentum and energy in the direction of motion). These inelastic collisions between the tidal bulges and continents cause a gradual decrease in the Earth's spinning rate. As a result, days are getting longer, by about 1.6 milliseconds per century. While it does not seem much, over a period of 1500 years the accumulated time difference equals 2 hours. In other words, over the period of 1500 years the Earth has rotated through 30 degrees more than it would if the angular velocity in the past had today's value. This is beautifully verified by the solar eclipse of January 14, 484 A.D., that would have been observed in southern Spain, if the Earth had spun at a today's rate, but was in fact observed in Greece, 30 degrees of longitude eastward. [12]

On a quite different time scale, an extrapolation back some 370 million years shows that a day then was about 22 hours long, and there were about 400 such short days in a year. This agrees with results of analysis of Devonian corals, that reveal a stunning periodicity in growth rings – there are about 400 daily growth rings in each annual set of rings.

Future of the Earth - Moon System

Since neither the Earth nor the water is perfectly elastic, the rotation of the Earth causes a delay in response of the ocean to the tidal force, and the spheroid is not exactly aligned with the direction toward the Moon, c.f. Figure 2. As a result, the high tide appears later than the time of highest moon.

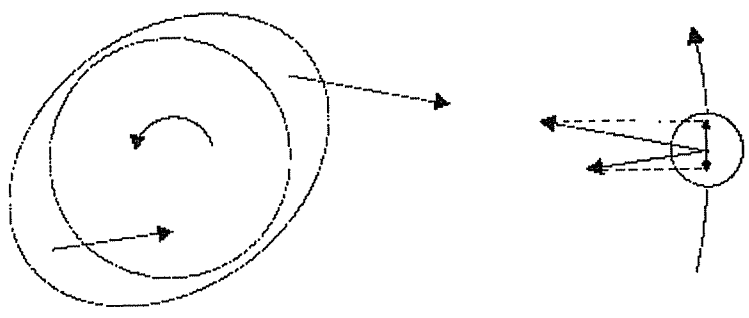

Figure 2. As the spinning Earth carries the water spheroid away, the spheroid is not exactly aligned with the direction toward the Moon. The bulge closer to the Moon exerts the gravitational force with larger component along the Moon's path than the force exerted by the more distant bulge. This results in a non-zero net force in the direction of the Moon's orbital motion, that causes the Moon to move into a higher orbit and hence away from the Earth.

This misalignment of the water spheroid causes the net gravitational pull exerted by the water bulges to have a small forward component along the direction of the Moon motion. As a result, the Moon moves into a higher orbit and hence away from the Earth, at the rate of some 4 meters per 100 years, and the orbital period of the Moon increases. (*This is similar*

1×10^{-4} m^2, the resulting tidal tensile stress is some 1.6×10^{-7} Pa. Further, taking the value of effective elastic modulus of human skeleton as $E = 1\times10^9$ N/m^2, we conclude that the strain is $\Delta H/H = 10^{-16}$, i.e. the expected change in person's height is million times smaller than a size of atom! So much for psychiatrist's "explanation" quoted in the introduction. On the other hand, a calculation of the strain due to a stress produced by the body's own weight yields a reasonable result $\Delta H/H = 1\times10^{-2}$.

A related and interesting problem came up as another question in "Ask Marilyn" column. *"We know that the rise and fall of the ocean tides are caused by the gravitational pull of the Moon as it revolves around Earth. Would this same gravitational pull affect the exact weight of a solid object?"* [10]

"Yes", answered Marilyn vos Savant, and this is, in principle, a correct answer. But how big is this effect? How much weight would you lose during a full Moon?

During the full Moon (as during the new Moon), a rock dropped at the point A would fall with acceleration whose component towards the center of Earth is g =9.81 m/s^2. For a 80 kg person, the weight at point A is $W_A = mg$ = 785 N or 176 lb. But at the points C and F, the person would fall towards the ground with acceleration

$$g' = g - \Delta g, \tag{12}$$

while at point L this acceleration would be

$$g' = g + \Delta g/2, \tag{13}$$

where

$$\Delta g = \Delta a_m + \Delta a_s = 1.65\times10^{-7}\ g. \tag{14}$$

Therefore the difference between downward accelerations at points C and L is $3/2 \times \Delta g = 2.4\times10^{-7}$ g, and an apparent weight of a 80 kg person at points C and F would be by the amount $\Delta W = 2.0\times10^{-4}$ N = 0.4×10^{-4} lb smaller than at point L. If the scale were calibrated in kilograms, the reading would be 20 mg (milligram) less. Hence during the spring tide your weight oscillates with a period of 12.5 hours, and with the full Moon overhead (or during the lunch time) you weigh some 2.5×10^{-5} percent less than some 6 hours earlier or later.

Is it enough of a weight difference to be noticed, and even influence humans? Suppose one wishes to achieve a similar reduction in weight simply by climbing up a vertical distance H above Earth's surface to reduce Earth's gravitational field by 2.5×10^{-7} g. We have

$$\frac{GM_e}{R^2} - \frac{GM_e}{(R+H)^2} = \frac{2GM_e H}{R^3} = g\frac{2H}{R} = 2.5\times10^{-7} g \tag{15}$$

Solving for H, we obtain H = 0.79 m, or about five steps up. (Note that this results is in fact identical to the value Δh obtained previously, c.f. discussion following Eq.(7)). Hence going to a bedroom upstairs results in a weight change that is about 5 times larger than that produced by the tidal mechanism during a spring tide.

to launching a satellite from a Newton's Mountain by firing it horizontally from a cannon with a speed slightly higher than one required for a perfectly circular orbit. As a result, the satellite begins to follow an elliptical orbit, initially rising above the surface of the Earth while slowing down. Likewise, the forward component of the tidal force keeps" re-launching" the Moon ever higher, resulting in the Moon's recession from the Earth along a tightly wound spiral orbit. See Ref. [13] for a discussion of forces along the spiral trajectory.) However, since the orbital period of the Moon increases at smaller rate than the length of the day does, both periods will eventually match. The Earth will be then tidally locked with the Moon, and the length of the day and the month will both be equal to some 50 present days, with the same side of Earth always facing the Moon. Note that the same side of the Moon already always faces the Earth, as the tidal action of the Earth on the Moon caused the Moon's original spin to slow down, and Moon became tidally locked with the Earth long time ago, in the sense that the Moon spins once on its axis for each revolution around the Earth.

Once the Earth becomes tidally locked with the Moon, the solar tides will tend to slow the Earth's rotation even more, so the day will be longer than the month and the Moon will rise in the West and set in the East. The water spheroid generated by the Sun will cause the high tide to appear earlier than the time of highest moon, a situation exactly opposite to that of Fig. 2. Then the tidal force of Earth on the Moon will pull the Moon into a lower orbit and eventually inside the Roche limit (18500 km), whereupon the Moon will disintegrate producing a ring around the Earth. [8]

Why is the water spheroid not aligned with the Moon?

A glance at Fig. 2 may suggest an explanation for the misalignment of the tidal spheroid and the Earth-Moon line and the resulting lag of a high tide with respect to the overhead position of the Moon. Due to the Earth's rotation, the speed of points on the Earth surface near the equator is about 1670 km/h. On the other hand, speed of very long waves on the surface of a relatively shallow water is determined by the Earth's gravity g and the depth of the ocean h, $v = \sqrt{gh}$. Assuming the average value for the oceans depth h = 4.0 km, one obtains v = 700 km/h. So a popular explanation claims that the traveling tidal wave simply can't move fast enough to remain directly under the Moon and ends up being carried away by Earth's rotation. Note that the speed of points along the Antarctic Circle at 66°33'39" S latitude is some 660 km/h, so the circulating tidal wave could actually keep up with the Earth-Moon line despite the rotation of the Earth.

Such an explanation ignores a crucial dynamic character of the periodic tide-rising forces that for a given location on the Earth has a period T = 12 h 25min. The observed tides are a steady-state response of the oscillating ocean to this external periodic driving force. For a simplest model for this response, imagine a hypothetical wide canal encircling the whole Earth along the equator. The surface of the water in the canal has its own period of natural oscillations T_0, that could be estimated as a time required for a circulating tidal wave to travel along half of the globe, T_0 = (20000 km)/(700 km/h) = 29 h. Clearly, the period of natural oscillations T_0 is much longer than the period of the driving force T, $T_0 >> T$, and the theory of forced oscillations tells us that while the water surface will be forced to oscillate with the period T of the driving force, the response will lag behind the driving force by ½ of the period T, i.e. by about 6 hours. This means that the resulting tidal bulges in the ocean will be *perpendicular* to the Earth-Moon line [13], certainly a counterintuitive result.

Interestingly, the southern ocean is the only place on Earth where a circulating wave can travel practically unimpeded by land. For a channel encircling the Earth along the 65^{th} parallel whose length is only 0.42 of the equatorial one, the period of natural oscillations of the water surface is some 12 h, which is close to the period of driving tidal force. The system is nearly resonant, and while the resulting forced oscillations will always have the period T, the lag of the response is now ¼ of the period, or about 3 h. Moving further south towards the Antarctic Circle one encounters progressively shorter canals for which the period of natural oscillations T_0 may become shorter than T. Under such conditions, the forced oscillations of the ocean still have the period T of the driving tidal force, but there is no time lag, i.e. one has a direct tide. Since the actual periods of natural oscillations of ocean in these locations may differ from our simple estimates, the value of latitude for which one would have a resonant response is difficult to determine theoretically. It is therefore interesting to note that the tidal lag in southern ocean along 65^{th} parallel is about 2 h, and further south one observes a direct tide.

Falling into a Black Hole

We have seen that the tidal force manifests itself as downward compressional force toward the center of the Earth at point L and a stretching force pulling upward away from the center of the Earth at points C and F. This results in a permanent tidal flattening of Earth's poles since compressional tidal forces of Moon and Sun always add at Earth's poles. This tidal flattening of the poles occurs independent of the rotation of the Earth. Likewise, tidal forces generated by all planets add to the flattening of the Sun's poles. Most of the stars in our galaxy contribute to the tidal force that helps to flatten the galaxy.

What would be a fate of a person who were to fall feet-first into a black hole? That person would not feel the force of gravity, because of being in a free fall (similar to a person in a freely falling elevator or astronaut in orbiting shuttle not feeling her own weight), but would do feel a devastating tidal forces pulling upwards on her head and downwards on her feet, and squeezing her waistline inward [14].

Acknowledgments

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The Lagrange Points

There are five equilibrium points to be found in the vicinity of two orbiting masses. They are called *Lagrange Points* in honour of the French-Italian mathematician Joseph Lagrange, who discovered them while studing the restricted three-body problem. The term "restricted" refers to the condition that two of the masses are very much heavier than the third. Today we know that the full three-body problem is chaotic, and so cannot be solved in closed form. Therefore, Lagrange had good reason to make some approximations. Moreover, there are many examples in our solar system that can be accurately described by the restricted three-body problem.

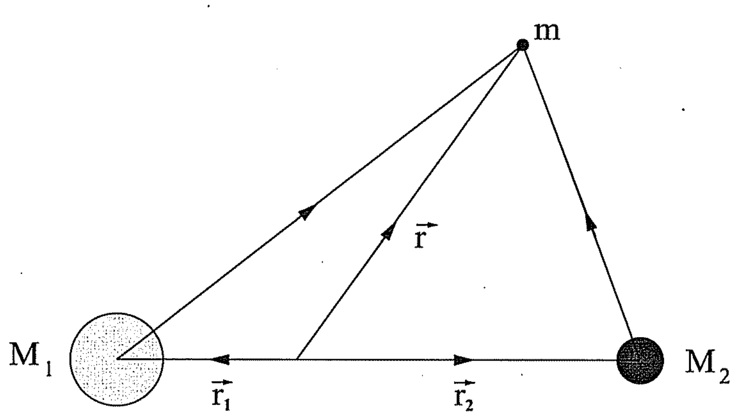


Figure 1: The restricted three-body problem

The procedure for finding the Lagrange points is fairly straightforward: We seek solutions to the equations of motion which maintain a constant separation between the three bodies. If M_1 and M_2 are the two masses, and $\vec{r_1}$ and $\vec{r_2}$ are their respective positions, then the total force exerted on a third mass m, at a position $\vec{r}$, will be

$$\vec{F} = -\frac{GM_1 m}{|\vec{r} - \vec{r_1}|^3}(\vec{r} - \vec{r_1}) - \frac{GM_2 m}{|\vec{r} - \vec{r_2}|^3}(\vec{r} - \vec{r_2}) \,. \tag{1}$$

The catch is that both $\vec{r_1}$ and $\vec{r_2}$ are functions of time since M_1 and M_2 are orbiting each other. Undaunted, one may proceed and insert the orbital solution for $\vec{r}_1(t)$ and $\vec{r}_2(t)$ (obtained by solving the two-body problem for M_1 and M_2) and look solutions to the equation of motion

$$\vec{F}(t) = m\frac{d^2\vec{r}(t)}{dt^2}\,, \tag{2}$$

that keep the relative positions of the three bodies fixed. It is these *stationary* solutions that are know as Lagrange points.

The easiest way to find the stationary solutions is to adopt a co-rotating frame of reference in which the two large masses hold fixed positions. The new frame of reference has its origin at the center of mass, and an angular frequency Ω given by Kepler's law:

$$\Omega^2 R^3 = G(M_1 + M_2)\,. \tag{3}$$

Here R is the distance between the two masses. The only drawback of using a non-inertial frame of reference is that we have to append various pseudo-forces to the equation of motion. The effective force in a frame rotating with angular velocity $\vec{\Omega}$ is related to the inertial force $\vec{F}$ according to the transformation

$$\vec{F}_\Omega = \vec{F} - 2m\left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right) - m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})\,. \tag{4}$$

The first correction is the coriolis force and the second is the centrifugal force. The effective force can be derived from the generalised potential

$$U_\Omega = U - \vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r})\,, \tag{5}$$

as the generalised gradient

$$\vec{F}_\Omega = -\nabla_{\vec{r}}U_\Omega + \frac{d}{dt}\left(\nabla_{\vec{v}}U_\Omega\right)\,. \tag{6}$$

The velocity dependent terms in the effective potential do not influence the positions of the equilibrium points, but the are crucial in determining the dynamical stability of motion about the equilibrium points. A plot of U_Ω with $\vec{v} = 0$, $M_1 = 10M_2 = 1$ and $R = 10$ is shown in Figure 2. The extrema of the generalised potential are labeled L1 through L5.

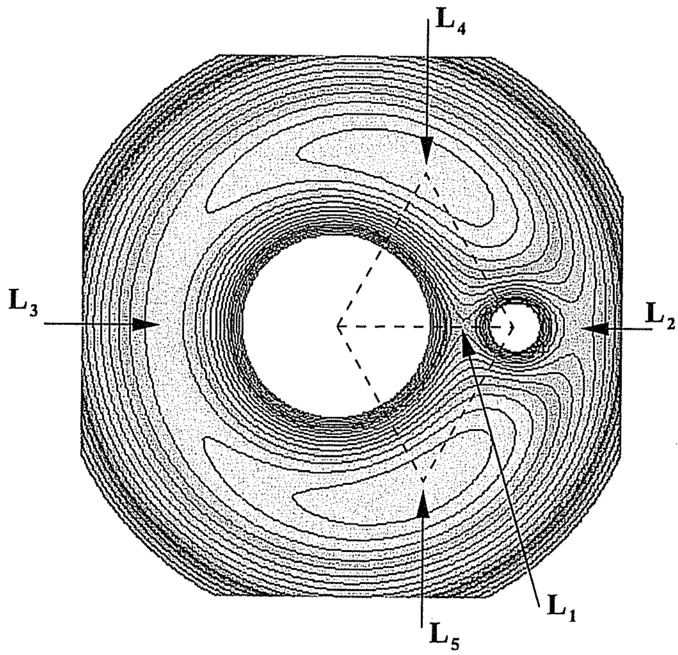


Figure 2: A countour plot of the generalised potential.

Choosing a set of cartesian coordinates originating from the center of mass with the z axis alined with the angular velocity, we have

$$\begin{aligned}\vec{\Omega} &= \Omega\,\hat{k}\\ \vec{r} &= x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}\\ \vec{r}_1 &= -\alpha R\,\hat{\imath}\\ \vec{r}_2 &= \beta R\,\hat{\imath}\end{aligned} \tag{7}$$

where

$$\alpha = \frac{M_2}{M_1+M_2}, \quad \beta = \frac{M_1}{M_1+M_2}. \tag{8}$$

To find the static equilibrium points we set the velocity $\vec{v} = d\vec{r}/dt$ to zero and seek solutions to the equation $\vec{F}_\Omega = \vec{0}$, where

$$\begin{aligned}\vec{F}_\Omega &= \Omega^2\left(x - \frac{\beta(x+\alpha R)R^3}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha(x-\beta R)R^3}{((x-\beta R)^2+y^2)^{3/2}}\right)\hat{\imath}\\ &\quad \Omega^2\left(y - \frac{\beta y R^3}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha y R^3}{((x-\beta R)^2+y^2)^{3/2}}\right)\hat{\jmath}.\end{aligned} \tag{9}$$

Here the mass m has been set equal to unity without loss of generality. The brute-force approach for finding the equilibrium points would be to set the magnitude of each force component to zero, and solve the resulting set of coupled, fourteenth order equations for x and y. A more promising approach is to think about the problem physically, and use the symmetries of the system to guide us to the answer.

Since the system is reflection-symmetric about the x-axis, the y component of the force must vanish along this line. Setting $y = 0$ and writing $x = R(u + \beta)$ (so that u measures the distance from M_2 in units of R), the condition for the force to vanish along the x-axis reduces to finding solutions to the three fifth-order equations

$$u^2((1-s_1)+3u+3u^2+u^3) = \alpha(s_0+2s_0u+(1+s_0-s_1)u^2+2u^3+u^4)\,, \quad (10)$$

where $s_0 = \text{sign}(u)$ and $s_1 = \text{sign}(u+1)$. The three cases we need to solve have (s_0, s_1) equal to $(-1, 1)$, $(1, 1)$ and $(-1, -1)$. The case $(1, -1)$ cannot occur. In each case there is one real root to the quintic equation, giving us the positions of the first three Lagrange points. We are unable to find closed-form solutions to equation (10) for general values of α, so instead we seek approximate solutions valid in the limit $\alpha \ll 1$. To lowest order in α, we find the first three Lagrange points to be positioned at

$$\begin{aligned}
L1: &\quad \left(R\left[1-\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right)\,, \\
L2: &\quad \left(R\left[1+\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right)\,, \\
L3: &\quad \left(-R\left[1+\frac{5}{12}\alpha\right], 0\right)\,.
\end{aligned} \quad (11)$$

For the earth-sun system $\alpha \approx 3 \times 10^{-6}$, $R = 1\,\text{AU} \approx 1.5 \times 10^8\,\text{km}$, and the first and second Lagrange points are located approximately 1.5 million kilometers from the earth. The third Lagrange point - home of the mythical planet X - orbits the sun just a fraction further out than the earth.

Identifying the remaining two Lagrange points requires a little more thought. We need to balance the centrifugal force, which acts in a direction radially outward from the center of mass, with the gravitational force exerted by the two masses. Clearly, force balance in the direction perpendicular to centrifugal force will only involve gravitational forces. This suggests that we should resolve the force into directions parallel and perpendicular to $\vec{r}$. The appropriate projection vectors are $x\hat{\imath} + y\hat{\jmath}$ and $y\hat{\imath} - x\hat{\jmath}$. The perpendicular projection yields

$$F_{\Omega}^{\perp} = \alpha\beta y\Omega^2 R^3 \left(\frac{1}{((x - R\beta)^2 + y^2)^{3/2}} - \frac{1}{((x + R\alpha)^2 + y^2)^{3/2}} \right) . \tag{12}$$

Setting $F_{\Omega}^{\perp} = 0$ and $y \neq 0$ tells us that the equilibrium points must be equidistant from the two masses. Using this fact, the parallel projection simplifies to read

$$F_{\Omega}^{\parallel} = \Omega^2 \frac{x^2 + y^2}{R} \left(\frac{1}{R^3} - \frac{1}{((x - R\beta)^2 + y^2)^{3/2}} \right) . \tag{13}$$

Demanding that the parallel component of the force vanish leads to the condition that the equilibrium points are at a distance R from each mass. In other words, L4 is situated at the vertex of an equilateral triangle, with the two masses forming the other vertices. L5 is obtained by a mirror reflection of L4 about the x-axis. Explicitly, the fourth and fifth Lagrange points have coordinates

$$\begin{aligned} L4: &\quad \left(\frac{R}{2} \left(\frac{M_1 - M_2}{M_1 + M_2} \right), \frac{\sqrt{3}}{2} R \right) , \\ L5: &\quad \left(\frac{R}{2} \left(\frac{M_1 - M_2}{M_1 + M_2} \right), -\frac{\sqrt{3}}{2} R \right) . \end{aligned} \tag{14}$$

Stability Analysis

Having established that the restricted three-body problem admits equilibrium points, our next task is to determine if they are stable. Usually it is enough to look at the shape of the effective potential and see if the equilibrium points occur at hills, valleys or saddles. However, this simple criterion fails when we have a velocity dependent potential. Instead, we must perform a linear stability analysis about each Lagrange point. This entails linearising the equation of motion about each equilibrium solution and solving for small departures from equilibrium. Writing

$$\begin{aligned} x &= x_i + \delta x\,, \quad v_x = \delta v_x\,, \\ x &= y_i + \delta y\,, \quad v_y = \delta v_y\,, \end{aligned} \tag{15}$$

where (x_i, y_i) is the position of the i-th Lagrange point, the linearised equations of motion become

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{d^2 U_\Omega}{dx^2} & \dfrac{d^2 U_\Omega}{dxdy} & 0 & 2\Omega \\ \dfrac{d^2 U_\Omega}{dydx} & \dfrac{d^2 U_\Omega}{dy^2} & -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix}\,. \tag{16}$$

Here the second derivatives of U_Ω are evaluated at $\vec{r} = (x_i, y_i)$.

Stability of L1 and L2

The stability of the first and second Lagrange points of the earth-sun system is an important consideration for some NASA missions. Currently the solar observatory SOHO is parked at L1, and NASA plans to send the Microwave Anisotropy Probe (MAP) out to L2. It has also be suggested that the Next Generation Space Telescope (NGST) should be positioned at L2.

The curvature of the effective potential near L1 and L2 reveals them to be saddle points:

$$\frac{d^2 U_\Omega}{dx^2} = \mp 9\Omega^2\,, \quad \frac{d^2 U_\Omega}{dy^2} = \pm 3\Omega^2\,, \quad \frac{d^2 U_\Omega}{dxdy} = \frac{d^2 U_\Omega}{dydx} = 0\,. \tag{17}$$

Solving for the eigenvalues of the linearised evolution matrix we find

$$\lambda_\pm = \pm\Omega\sqrt{1 + 2\sqrt{7}} \quad \text{and} \quad \sigma_\pm = \pm i\Omega\sqrt{2\sqrt{7} - 1}\,. \tag{18}$$

The presence of a positive, real root tells us that L1 and L2 are dynamically unstable. Small departures from equilibrium will grow exponentially with a e-folding time of

$$\tau = \frac{1}{\lambda_+} \approx \frac{2}{5\Omega}. \tag{19}$$

For the earth-sun system $\Omega = 2\pi\,\text{year}^{-1}$ and $\tau \approx 23$ days. In other words, a satellite parked at L1 or L2 will wander off after a few months unless course corrections are made.

Stability of L3

A popular theme in early science fiction stories was invasion by creatures from Planet X. The requirement that Planet X remain hidden behind the sun places it at the L3 point of the earth-sun system. Unfortunately for our would-be invaders, the L3 point is a weak saddle point of the effective potential with curvature

$$\frac{d^2 U_\Omega}{dx^2} = -3\Omega^2\,, \quad \frac{d^2 U_\Omega}{dy^2} = \frac{7M_2}{8M_1}\Omega^2\,, \quad \frac{d^2 U_\Omega}{dxdy} = \frac{d^2 U_\Omega}{dydx} = 0\,. \tag{20}$$

To leading order in M_2/M_1, the eigenvalues of the linearised evolution matrix are

$$\lambda_\pm = \pm\Omega\sqrt{\frac{3M_1}{8M_2}} \quad \text{and} \quad \sigma_\pm = \pm i\Omega\sqrt{7}\,. \tag{21}$$

The real, positive eigenvalue spells disaster for Planet X. Its orbit is exponentially unstable, with an e-folding time of roughly $\tau = 150$ years. While there can be no Planet X, the long e-folding time makes L3 a good place to park your invasion force while final preparations are made...

Stability of L4 and L5

The stability analysis around L4 and L5 yields something of a suprise. While these points correspond to local maxima of the generalised potential - which usually implies a state of unstable equilibrium - they are in fact stable. Their stability is due to the coriolis force. Initially a mass situated near L4 or L5 will tend to slide down the potential, but as it does so it picks up speed and the coriolis force kicks in, sending it into an orbit around the Lagrange point. The effect is analogous to how a hurricane forms on the surface of the earth: as air rushes into a low pressure system it begins to rotate because of the coriolis force and a stable vortex is formed. Explicitly, the curvature of the potential near L4 is given by

$$\frac{d^2U_\Omega}{dx^2} = \frac{3}{4}\Omega^2 \,, \quad \frac{d^2U_\Omega}{dy^2} = \frac{9}{4}\Omega^2 \,, \quad \frac{d^2U_\Omega}{dxdy} = \frac{d^2U_\Omega}{dydx} = \frac{3\sqrt{3}}{4}\kappa\Omega^2 \,, \tag{22}$$

where $\kappa = (M_1 - M_2)/(M_1 + M_2)$. The eigenvalues of the linearised evolution matrix are found to equal

$$\lambda_\pm = \pm i\,\frac{\Omega}{2}\sqrt{2 - \sqrt{27\kappa^2 - 23}}$$
$$\sigma_\pm = \pm i\,\frac{\Omega}{2}\sqrt{2 + \sqrt{27\kappa^2 - 23}}\,. \tag{23}$$

The L4 point will be stable if the eigenvalues are pure imaginary. This will be true if

$$\kappa^2 \geq \frac{23}{27} \quad \text{and} \quad \sqrt{27\kappa^2 - 23} \leq 2\,. \tag{24}$$

The second condition is always satisfied, while the first requires

$$M_1 \geq 25 M_2 \left(\frac{1 + \sqrt{1 - 4/625}}{2}\right)\,. \tag{25}$$

When the L4 and L5 points yield stable orbits they are refered to as Trojan points after the three Trojan asteroids, Agamemnon, Achilles and Hector, found at the L4 and L5 points of Jupiter's orbit. The mass ratios in the earth-sun and earth-moon system are easily large enough for their L4 and L5 points to be home to Trojan satellites, though none have been found.

Author

These notes were written by Neil J. Cornish with input from Jeremy Goodman.

Chapter 13
Oscillation

1. Mathematics

(a) Definition
(b) Free Oscillation
(c) Damped Oscillation
- (i) Overdamping
- (ii) Critical damping
- (iii) Underdamping

(d) Forced Oscillation.
- (i) Driven Oscillation of Pure Harmonic Oscillator
- (ii) Driven Oscillation of Real Harmonic Oscillator
- (iii) Average Power Dissipation

2. Simple Harmonic Oscillation

(a) Spring
- (i) Equation of Motion.

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$ a, b, c are constant

$b = 0$ free oscillation

$$F = -kx$$ $x = x - x_0$ equilibrium

$$m\frac{d^2x}{dt^2} = -kx$$

$(a = m;\ c = k,\ b = 0)$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$ $\omega^2 = \frac{k}{m} > 0$

$$= -\omega^2 x$$ $\omega^2 = \frac{k}{m} = \frac{c}{a}$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t$$

On the other hand

$$a\frac{d^2x}{dt} + cx = 0 \quad \Rightarrow \quad \frac{d^2x}{dt} = -\frac{c}{a}x$$

If $\frac{c}{a} < 0$, then $\omega^2 = -\frac{c}{a} > 0$

$$\frac{d^2x}{dt^2} = +\omega^2 x$$

$$\Rightarrow \quad x(t) = Ae^{\omega t} + Be^{-\omega t}$$

A, B are determined by the initial conditions.

Damped Oscillation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Ansatz $\quad x(t) = e^{-bt/2a} f(t)$

$$\frac{dx}{dt} = e^{-bt/2a}\frac{df}{dt} + e^{-bt/2a}\left(-\frac{b}{2a}\right)f$$

$$\frac{d^2x}{dt} = e^{-bt/2a}\frac{d^2f}{dt^2} + e^{-bt/2a}\left(-\frac{b}{2a}\right)\frac{df}{dt}$$

$$+ e^{-bt/2a}\left(-\frac{b}{2a}\right)\frac{df}{dt} + e^{-bt/2a}\left(-\frac{b}{2a}\right)^2 f$$

$$LHS = e^{-bt/2a} \times \Big[a\frac{d^2f}{dt^2} - \cancel{\frac{b}{2}\frac{df}{dt}} - \cancel{\frac{b}{2}\frac{df}{dt}} + \frac{b^2}{4a}f$$

$$+ \cancel{b\frac{df}{dt}} - \frac{b^2}{2a}f + cf\Big] = 0$$

$$\Rightarrow \quad a\frac{d^2f}{dt^2} = \frac{b^2-4ac}{4a}f$$

$$\Rightarrow \quad \frac{d^2f}{dt^2} = \frac{b^2-4ac}{4a^2}f$$

$$F = -kx - \text{"}b\text{"}\dot{x}$$

$$m\ddot{x} = -kx - \text{"}b\text{"}x'$$

$$\underset{\underset{a}{\downarrow}}{m}\ddot{x} + \underset{\underset{b}{\downarrow}}{\text{"}b\text{"}}x' + \underset{\underset{c}{\uparrow}}{k}x = 0 \qquad a, c > 0$$

Damped Oscillation (Overdamping)

$$\underbrace{\frac{b^2 - 4ac}{4a^2}}_{= \omega^2} > 0 \quad \Rightarrow \quad \omega < \frac{b}{2a}$$

$$\frac{d^2f}{dt^2} = \omega^2 t \quad \Rightarrow \quad f = Ae^{\omega t} + Be^{-\omega t}$$

$$\Rightarrow \quad x(t) = e^{-\frac{b}{2a}t}\left[Ae^{\omega t} + Be^{-\omega t}\right]$$

A, B are constants to be determined by initial condition

$\Rightarrow$ The motion of the displacement is always decreasing

Critical Damping

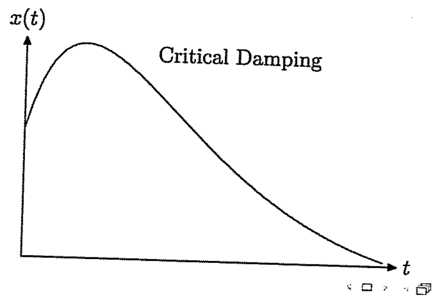
$b^2 - 4ac = 0 \Rightarrow b^2 = 4ac$

$$\Rightarrow \frac{d^2 f}{dt^2} = 0$$

$$\downarrow$$

$$f(t) = A + Bt$$

$$\Rightarrow \quad x(t) = e^{-bt/2a}(A + Bt)$$



Underdamping

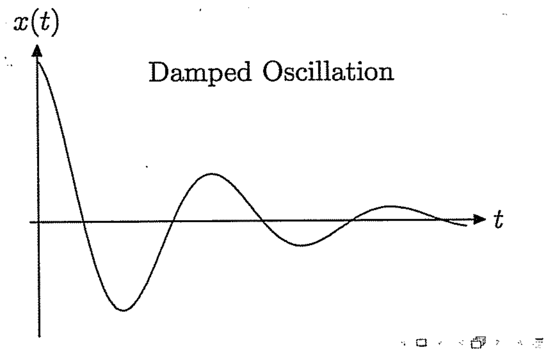
$$\frac{b^2-4ac}{4a^2} < 0$$

Let $\omega^2 = \dfrac{4ac-b^2}{4a^2} > 0$, then $\dfrac{d^2f}{dt^2} = -\omega^2 f$

$$f(t) = A\sin\omega t + B\cos\omega t$$

$$\Rightarrow \quad x(t) = e^{-bt/2a}(A\sin\omega t + B\cos\omega t)$$

A, B are constants to be determined by initial conditions



Inhomogeneous D.E.

The inhomogeneous term $F(t)$ represents an external driven force

$$a\ddot{x} + b\dot{x} + cx = F(t)\,.$$

We are mainly interested in a periodically driven force $F_0 \cos \omega t$ where ω is the external angular frequency which can varied over a range. The general solution is the homogeneous solution plus a particular solution

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} + p(t)\,,$$

so that

$$a\ddot{p} + b\dot{p} + cp = F(t)\,.$$

Driven Oscillation of Pure Harmonic Oscillator

The inhomogeneous D.E. is

$$\ddot{x} + \omega_0^2 x = (F_0/m)\cos\omega t\,.$$

(i) For $\omega \neq \omega_0$ we can take $p(t) = D\cos\omega t$. Then

$$-\omega^2 D\cos\omega t + \omega_0^2 D\cos\omega t = (F_0/m)\cos\omega t\,,$$

giving

$$D = \frac{F_0}{m(\omega_0^2 - \omega^2)}\,,$$

So the particular solution represents an "in phase motion" with the external driven periodic force.

The general solutions for

$$\ddot{x} + \omega_0^2 x = 0$$

$$\Rightarrow \quad x_g = A\sin\omega t + B\cos\omega t$$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t + \frac{F_0}{m(\omega_0^2-\omega^2)}$$

(ii) For $\omega = \omega_0$ we would take $p(t) = Dt \sin \omega t$,

$$(-\omega^2 Dt \sin \omega t + 2\omega D \cos \omega t) + \omega^2 Dt \sin \omega t = (F_0/m) \cos \omega t \,,$$

giving

$$D = \frac{F_0}{2m\omega} \,.$$

So the particular solution at resonance ($\omega = \omega_0$) gives a displacement growing with time, absorbing energy from the driven force. In order to attain this, part of the velocity must be in phase with the external force. So there is an abrupt phase change at resonance.

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So the particular solution at resonance ($\omega = \omega_0$) gives a displacement growing with time, absorbing energy from the driven force. In order to attain this, part of the velocity must be in phase with the external force. So there is an abrupt phase change at resonance.

$$p(t) = Dt \sin\omega t$$

$$\frac{dp}{dt} = Dt\,\omega\cos\omega t + D\sin\omega t$$

$$\frac{d^2p}{dt^2} = Dt(-\omega^2)\sin\omega t + D\omega\cos\omega t + D\omega\cos\omega t$$

$$\Rightarrow \quad (\cancel{-\omega^2 Dt\sin\omega t} + 2\omega D\cos\omega t) + \cancel{\omega^2 Dt\sin\omega t}$$

$$= \frac{F_0}{m}\cos\omega t$$

$$\Rightarrow \quad D = \frac{F_0}{2m\omega}$$

$$\Rightarrow \quad x(t) = A\sin\omega t + B\cos\omega t + \frac{F_0}{2m\omega}t\sin\omega t$$

Driven Oscillation of a Real Oscillator

$$F = -kx - \underset{\lambda}{\underset{\uparrow}{b}}v + F_0 \cos\omega t$$

$$m\frac{d^2x}{dt^2} + \underset{\lambda}{\underset{\updownarrow}{b}}\frac{dx}{dt} + kx = F_0 \cos\omega t$$

$$x(t) = \underset{\substack{\downarrow \\ \text{solution of the} \\ \text{homogeneous} \\ \text{equation}}}{x_g(t)} + \underset{\substack{\downarrow \\ \text{particular} \\ \text{solution}}}{p(t)}$$

$$m\frac{d^2p}{dt^2} + \lambda\frac{dp}{dt} + kp = F_0 \cos\omega t$$

$$m\ddot{p} + \overset{b}{\lambda}\dot{p} + kp - F_0 \cos\omega t = 0 \quad \Rightarrow \quad m\ddot{p} + \lambda\dot{p} + \underbrace{\frac{k}{m}}_{\omega_0^2} mp - F_0 \cos\omega t$$

$$p(t) = C \sin(\omega t + \delta)$$

$$\dot{p}(t) = C\omega\cos(\omega t + \delta)$$

$$\ddot{p}(t) = C(-\omega^2)\sin(\omega t + \delta)$$

$$(\omega_0^2 - \omega^2)\, mC \sin(\omega t + \delta) + \lambda\omega C \cos(\omega t + \beta) - F_0 \cos\omega t = 0 \qquad (A)$$

$$A \equiv \omega t + \delta, \quad B = \delta$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos\omega t = \cos\delta \cos(\omega t + \delta) + \sin\delta \sin(\omega t + \delta)$$

Substitute back into A

$$[(\omega_0^2 - \omega^2)C.. - F_0 \sin\delta]\sin(\omega t + \delta) + (\lambda\omega C - F_0\cos\delta)\cos(\omega t + \delta) = 0$$

$$\Rightarrow \quad \begin{aligned} (\omega_0^2 - \omega^2)Cm &= F_0 \sin\delta \qquad (B) \\ \lambda\omega C &= F_0\cos\delta \end{aligned}$$

$$(\omega_0^2 - \omega^2)C^2 m^2 = F_0^2 \sin^2\delta$$

$$\lambda^2\omega^2 C^2 = F_0^2\cos^2\delta$$

$$C^2[(\omega_0^2 - \omega^2)m^2 + \lambda^2\omega^2] = F_0^2$$

$$C = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)m^2 + \lambda^2\omega^2}}$$

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\omega_0^2 - \omega^2}{\lambda\omega}$$

The mathematical problem has been solved

$$x(t) = e^{-bt/2a} f(t) + p(t)$$

Time Average Steady-State Power Transferred into the Oscillator from the Deriving Force

$$\langle P \rangle = \frac{1}{T}\int_0^T \underbrace{P(t)}_{F(t)\,\underbrace{v(t)}_{\frac{dx}{dt}}} dt = \frac{1}{T}\int_0^T F_{DR}(t)\, v(t)\, dt$$

$$\left\{\begin{array}{l} T = \text{period} \\ \omega T = 2\pi \end{array}\right.$$

$$P = \frac{dW}{dt}$$

$$= \frac{1}{T}\int_0^T F_0 \cos\omega t \underbrace{P(t)}_{C\omega\cos(\omega t+\delta)} dt$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \qquad A = \omega t,\ B = \delta$$

$$\cos(\omega t+\delta) = \cos\omega t \cos\delta - \sin\omega t \sin\delta$$

$$F_0 \cos\delta = C\lambda\omega$$

$$F_0 \sin\delta = (\omega_0^2 - \omega^2) C m \qquad \Rightarrow F_0^2 = C^2[\lambda^2\omega^2 + m^2(\omega_0^2-\omega^2)^2]$$

$$= \frac{1}{T}\int_0^T F_0 \cos\omega t\, [C\omega \cos\omega t \cos\delta - C\omega \sin\omega t \sin\delta]\, dt$$

$$= \frac{1}{T} F_0 C\omega \cos\delta \int_0^T \cos^2\omega t\, dt - \frac{1}{T} F_0 \sin\delta \int_0^T \cos\omega t \sin\omega t\, dt$$

$$= \frac{F_0 C\omega\cos\delta}{2}$$

$$\int_0^T \frac{1}{2}(\ \sin 2\omega t)\, dt \quad = 0$$

$\downarrow$

$\sin\omega t \cos\omega t$

$$\cos^2\omega t - \sin^2\omega t = 2\cos^2\omega t - 1$$

$\shortparallel$

$\cos 2\omega t$

$$\frac{1}{T}\int_0^T \cos^2\omega t\, dt = \frac{1}{T}\int_0^T \left[\frac{1}{2}(1+ \cos 2\omega t\right] dt$$

$$\int_0^T \sin 2\omega t\, dt$$

$\omega T = 2\pi$

$\omega t = x$ $\quad t = 0 \quad x = 0$

$\quad t = T \quad x = \omega T = 2\pi$

$\omega dt = dx$

$dt = \frac{1}{\omega} dx$

$$= \frac{1}{\omega}\int_0^{2\pi} \sin x\, dx$$

$$= \frac{1}{\omega}\left[-\cos x\right]_0^{2\pi} = 0$$

$$\int_0^T \cos 2\omega t\, dt = 0$$ using the same argument

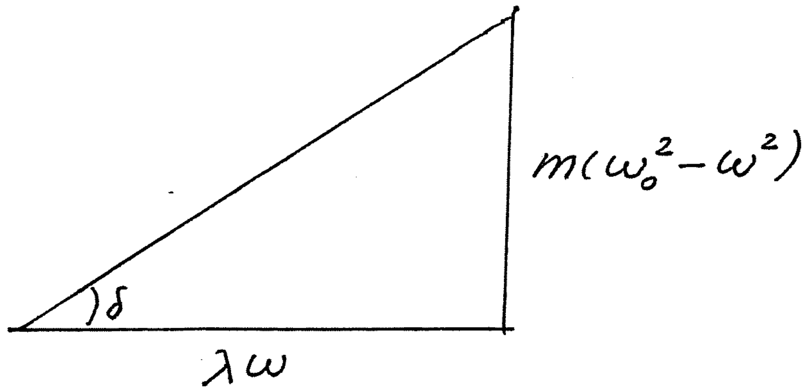
$$\Rightarrow \frac{1}{T}\int_0^T \cos^2\omega t\, dt = \frac{1}{2}$$

(See Figure in MISN-0-31 P. 5)

$$\frac{1}{T}\int_0^T \cos\omega t \sin\omega t\, dt = 0$$

Elimating the Phase Angle

$$\tan\delta = \frac{\omega_0^2 - \omega^2}{\lambda\omega}$$



$$\cos\delta = \frac{\lambda\omega}{\sqrt{\lambda^2\omega^2 + m^2(\omega^2 - \omega_0^2)^2}}$$

$$\langle P \rangle = \frac{F_0 \,\omega\, C \cos\delta}{2} \qquad C = \frac{F_0}{\sqrt{\lambda^2\omega^2 + m^2(\omega_0^2 - \omega^2)}}$$

$$= \frac{F_0\, \omega C \lambda\omega}{2\sqrt{\lambda^2\omega^2 + m^2(\omega_0^2 - \omega^2)}}$$

$$= \frac{F_0^2 \omega^2 \lambda}{2[m^2(\omega_0^2 - \omega^2)^2 + \lambda^2\omega^2]}$$

(one dimensional)

Forced Harmonic Oscillation of a Real Oscillator

$$F = -kx - \lambda v + F_0 \cos \omega t$$

$$\downarrow$$

$$ma$$

$$m\ddot{x} + \lambda \dot{x} + kx = F_0 \cos \omega t$$

Solution : $x(t) = x_g(t) + p(t)$

$x_g(t) \to e^{-bt/2a} f(t)$ $\qquad$ $p(t) \to C \sin(\omega t + \delta)$

Overdamping $\quad f(t) = Ae^{\omega t} + Be^{-\omega t}$

Critical $\quad f(t) = A + Bt$

Underdamping $\quad f(t) = A \sin \omega t + B \cos \omega t$

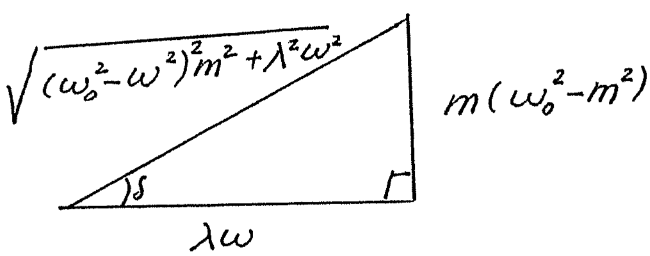
As t increase, the $p(t)$ dominates

$$x(t) \longrightarrow p(t) = C \sin(\omega t + \delta)$$

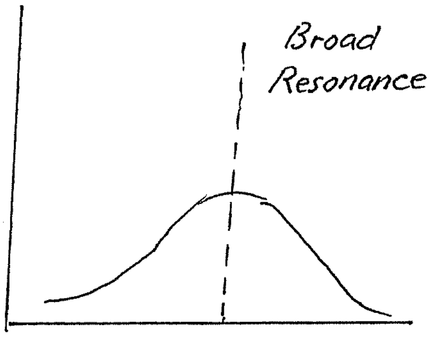
$p(t)$: steady state solution

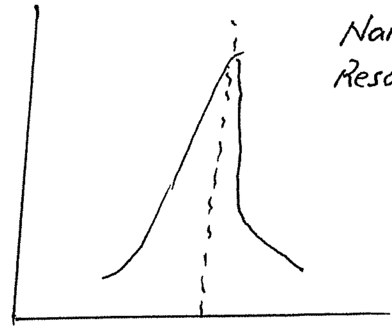
$$C = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)m^2 + \lambda^2 \omega^2}} \qquad \omega_0^2 = \frac{k}{m}$$

$$\tan \delta = \frac{\omega_0^2 - \omega^2}{\lambda \omega}$$



$\langle P \rangle$ ↓ function of ω



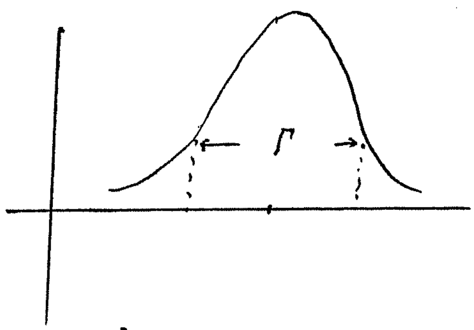


Maximum occur at $\omega = \omega_0$

$\omega_{1/2}$

Definition

$$\langle P \rangle_{\omega_{1/2}} = \frac{1}{2}\langle P \rangle_{\omega_0}$$



$$\frac{\cancel{F_0^2}\,\omega_{1/2}^2\,\cancel{x}}{m^2(\omega_0^2-\omega_{1/2}^2)^2+\lambda^2\omega_{1/2}^2} = \frac{1}{2}\,\frac{\cancel{F_0^2}\,\cancel{\omega_0^2}\,\cancel{x}}{\lambda^2\cancel{\omega_0^2}}$$

$$\Rightarrow \quad \omega_0^2 - \omega_{1/2}^2 = \pm \lambda\, \omega_{1/2}/m$$

$$\| $$

$$(\omega_0 + \omega_{1/2})(\omega_0 - \omega_{1/2}) = \pm\lambda\,\omega_{1/2}/m$$

$\omega_0 \sim \omega_{1/2}$ (narrow width approximation)

↓

$$\omega_0 + \omega_{1/2} \approx 2\omega_0$$

$$\Rightarrow \quad (\omega_0 - \omega_{1/2})\, 2\omega_0 \cong \pm\lambda\,\omega_0$$

$$\omega_{1/2} = \omega_0 \pm \underbrace{\lambda/m}_{\Gamma}$$

λ 大 $\rightarrow$ distribution wide

$$\langle P \rangle_\omega = \frac{F_0^2\,\Gamma/8m}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

$$\langle P \rangle_{\omega_0} = \frac{F_0^2}{2\Gamma m}$$

Examples of Simple Harmonic Motion

(1) Spring

(2) Particle Moving on a circle with radius A and constant angular velocity ω.

(3) A Single Pendulum

$$m\ell \frac{d^2\theta}{dt^2} = -mg \sin\theta$$

$$\frac{d^2\theta}{dt^2} = -(g/\ell) \sin\theta$$

For $\theta \ll 1$ $\quad \sin\theta \sim \theta$

$$\frac{d^2\theta}{dt^2} \cong -\underbrace{\left(\frac{g}{\ell}\right)}_{\omega^2} \theta$$

$$\omega = \sqrt{\frac{g}{\ell}}.$$

$$\sin\theta \simeq \theta - \frac{1}{3!}\theta^3 \Rightarrow \text{aharmonic}$$

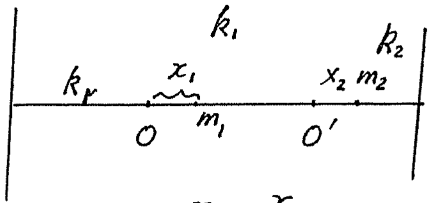
can be solved by perturbation method

Extension

Coupled oscillation

Parall $\qquad k = k_1 + k_2$

Series $\qquad \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$



$x_2 - x_1$

$m_1 \qquad -k_1 x_1$

$\qquad +k(x_2 - x_1)$

$m_2 \qquad -k(x_2 - x_1)$

$\qquad -k_2 x_2$

$$m_1 \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1$$

$$m_2 \frac{d^2 x_2}{dt} = -k_2 x_2 - k(x_2 - x_1)$$

$m_1 = m_2 = m \quad , \quad k_1 = k_2$

$$\begin{cases} m \dfrac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1 \\ m \dfrac{d^2 x_2}{dt^2} = -k_1 x_2 - k(x_2 - x_1) \end{cases}$$

coupled

$\oplus \qquad m \frac{d^2}{dt^2}\underline{(x_1 + x_2)} = -k_1 \underline{(x_1 + x_2)}$

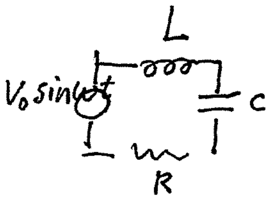
$\ominus \qquad m \frac{d^2}{dt^2}\underline{(x_1 - x_2)} = -(2k + k_1)\underline{(x_1 - x_2)}$

decoupled $\qquad\qquad$ 2 wide

normal mode

Two, Three dimensional problem

RLC Circuits (P. 926)



$$V_0 \sin\omega t - L\frac{d^2Q}{dt^2} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

$$F_0 \sin\omega t - m\frac{d^2x}{dt^2} - b\frac{dx}{dt} - kx = 0$$

对应

Q	x
$\frac{1}{C}$	k
R	b
L	m
$V_0 \sin\omega t$	$F_0 \sin\omega t$
$\frac{1}{\sqrt{LC}}$	$\sqrt{\frac{k}{m}}$

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第八章 振動

第一節 簡諧運動 (I)

簡介 在物理中，簡諧運動佔著一極重要的地位，我們在此節中將詳細地討論此種運動。我們並將解釋一個質點在其穩定平衡點之附近大多作簡諧運動。我們討論一定空間的情形。

基本觀念 若一質點之 $x(t)$ 可寫成

$$x(t) = A\sin(\omega t + \alpha) \qquad (1)$$

時則稱此質點在作簡諧運動[1,2,3]

由於 $x(t + \frac{2\pi}{\omega}) = x(t)$，因此它是一週期運動，其週期為 $T = \frac{2\pi}{\omega}$，其頻率 $\nu = \frac{\omega}{2\pi}$

在第(1)式中 A 是該質點之最大位移[4]，又稱為振幅，$\omega t + \alpha$ 則稱為相角，而在 $t=0$ 之相角 α 則稱為初相角。

對第(1)式微分，則可得

$$v(t) = A\omega\cos(\omega t + \alpha) \qquad (2)$$

再對第(2)式微分，則可得

$$a(t) = -A\omega^2\sin(\omega t + \alpha) \qquad (3)$$

比較第(1)式及第(3)式，我們得到

$$a(t) = -\omega^2 x \qquad (4)$$

因此，加速度之大小與位移成正比，但方向卻相反。

引入牛頓第二定律，我們得知質點所受之力為

$$F = ma = -m\omega^2 x = -kx \qquad (5)$$

此處 $k = m\omega^2$ 稱為力常數 (6)

我們可以從另外一個角度來討論此一問題，假設一質點所受之外力為

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$$F = -kx \tag{7}$$

則由牛頓第二定律得知

$$m\frac{d^2x}{dt^2} = -kx \tag{8}$$

此為一微分方程式。其普遍解為

$$x(t) = A'\genfrac{}{}{0pt}{}{\cos}{\sin}\omega t + B'\cos\omega t \tag{9}$$

此處 $\omega = \sqrt{\frac{k}{m}}$ (10)

是取決於所受之力常數 k 及其質量.

由簡單的代數計算,我們可以證明第(9)式可寫成第(11)式而 A, α 與 A', B' 間之關係是

$$A^2 = A'^2 + B'^2 \tag{11 a}$$

$$\tan\alpha = \frac{B'}{A'} \tag{11b}$$

A, α (或是 A', B') 是由該質點之初位移及初速所決定. 若在 $t=0$ 時 $x=x_0$, $v=v_0$ 則我們可以很容易的求得

$$A = \left[x_0^2 + \left(\frac{v_0}{\omega}\right)^2\right]^{1/2} \tag{12 a}$$

$$\alpha = \tan^{-1}\left(\frac{\omega x_0}{v_0}\right) \tag{12 b}$$

由上章之結果,我們得知該粒子之位能是

$$E_p = -\int(-kx)\,dx = \frac{1}{2}kx^2 + C \tag{13}$$

此處 C 為一積分。我們將取 $x=0$ 之位能為 0, 則 $C=0$. 因此

$$E_p = \frac{1}{2}kx^2 \tag{14}$$

質點之動能為

$$E_k = \frac{1}{2}mv^2 \tag{15}$$

利用第(1)，(2)，(6)，(14)及(15)式，我們得到該系統之總能量為

$$E = E_p + E_k = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} k A^2 \qquad (16)$$

因此是一常數。此即是能量守恆定律。

3. 討論：

(1) 簡諧運動中之"簡"是指此一運動為單頻率，"諧"是指其位移是一正弦曲線。

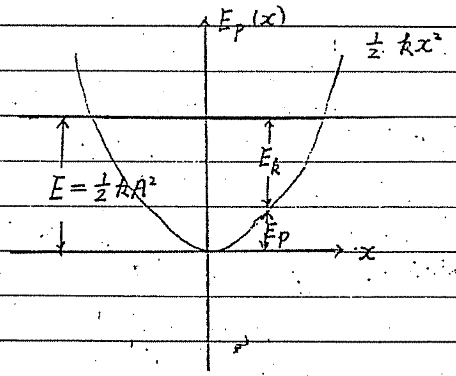
(2) 簡諧運動是週期運動之最簡單的例子。

(3) 我們對簡諧運動特別重視之理由 (一) 物理方面之理由，在物理中，簡諧運動時常發生。我們將在下節中說明在位移夠小時其運動均可以簡諧運動表示。

(二) 數學上之理由是源自 Fourier 定理，他在 1827 年證明任何週期為 T 之運動可由一組週期為 T, $\frac{T}{2}$, $\frac{T}{3}$, ⋯ 之簡諧運動所組成（當然各個振幅均需適當地選擇）。

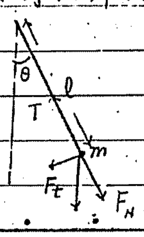
(4) 簡諧運動是局限於 $(-A, A)$ 之間。

(5) E_p, E_k 及 E 之間的關係可由下圖中看出。



4. 應用：

我們現在將要舉兩個簡諧運動的例子



(1) 單擺

$$F_t = -mg\sin\theta = ma_t$$

$$a_t = \underset{\uparrow\ \text{角加速度}}{l\alpha} = l\frac{d^2\theta}{dt^2}$$

$$-mg\sin\theta = l\frac{d^2\theta}{dt^2}$$

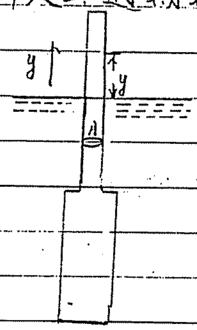
$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

當 θ 甚小時，$\sin\theta \simeq \theta$，上式即可寫成

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

與 $m\frac{d^2x}{dt^2} + kx = 0$ 相比較可得 $\frac{g}{\ell} \leftrightarrow \frac{k}{m}$ 因此此一簡諧運動之角速度 $\omega = \sqrt{\frac{g}{\ell}}$

也即是其週期為 $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$



若此水壓表是在比其平衡點高出 y 處則其向上之浮力比其在平衡處（所受之外力為零）時小 $\rho A g y$

ρ 為此液体之密度，A 是表上部之截面積，因此水壓力受一為 $\rho A g y$ 大小向下之淨力。由牛頓定律得

$$m\frac{d^2y}{dt^2} = -\rho A g y$$

因此該物体作簡諧運動，其角速度 $\omega = \sqrt{\frac{g\rho A}{m}}$，週期 $T = 2\pi\sqrt{\frac{m}{g\rho A}}$

習題

1. 一物体之質量為 1 Kg，在作簡諧運動其振幅為 (0.01 m) 其頻率為 2 sec^{-1} 在 $t=0$ 時該質點位於 $x=0$ 處。

(a) 求 $x(t)$ [D]

(b) 求在 $t = \frac{5}{8}T$ 時之位移，速度，動能及位能。 [B]

(c) 將位移及位能與時間之關係簡單地繪出。 [G]

(d) 將速度及動能與時間之關係簡單地繪出。 [I]

2. 一物体從事簡諧運動，其質量為 50 kg，力常數為 2 N/m，在 $t=0$ 時該物体位於離平衡點 0.5 m 處，而其速度為零

(a) 求該物体於 $t = 2.5\pi$ sec 時之速度 [C]

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(b) 求該物體之運動週期 [E]

(c) 求該物體於 $t = 5\pi$ sec 所受之力 [H]

3. 在一簡諧運動中 (a) 當位移為主振幅之一半時，動能及位能各佔總能量之百分比為何?[A]

(b) 當位移為振幅之百分之幾時 動能及位能相等? [F]

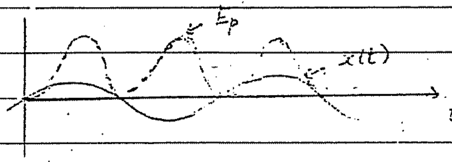
答案

[A] 動能 75%，位能 25% [B] -0.0071 m, -0.099 m/sec, 0.0039 J, 0.0040 J

[C] -0.1 m/sec [D] $x(t) = (0.01\ \text{m}) \sin(4\pi\ \text{sec}^{-1} \cdot t)$

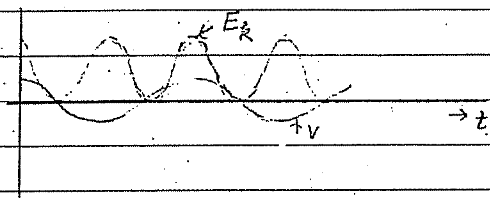
[E] 10π sec [F] 70.7%

[G]



[H] 1N

[I]



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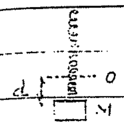
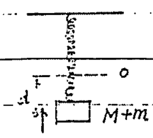
研究資料



$$Mg = kd$$

$$d = \frac{Mg}{k}$$ 離彈簧之平衡點之距離



子彈碰撞時，因不受外力，故動量守恒

$$m\underset{\text{入射子彈之速度}}{v} = (m+M)V_0$$

$V_0 = \dfrac{mv}{m+M}$ 方向為向上。碰撞中動能之變化 $= \frac{1}{2}(m+M)V_0^2 - \frac{1}{2}mv^2$

$$= -\frac{1}{2}\frac{M}{m+M}mv^2$$

碰撞所需之時間極短，故碰撞後之位置仍在 $y = d$ 處

牛頓第二定律可寫成

$$(m+M)\frac{d^2y}{dt^2} = (m+M)g - ky$$

解以上公式，$t=0$ 時 $V_0 = -\dfrac{mv}{M+m}$，$y_0 = d$.

$$(m+M)\frac{d^2y}{dt^2} = -k\left(y - \frac{m+M}{k}g\right)$$

令 $y' = y - \dfrac{m+M}{k}g$，則上式可寫成

$$(m+M)\frac{d^2y'}{dt^2} = -ky'$$

故 $y' = A\sin(\omega t + \alpha)$ $\omega = \sqrt{\dfrac{k}{m+M}}$

$$y = \frac{m+M}{k}g + A\sin(\omega t + \alpha)$$

$t=0$ 時 $y = d \Rightarrow d = \dfrac{m+M}{k}g + A\sin\alpha$

$t=0$ 時 $\left.\dfrac{dy}{dt}\right|_{t=0} = A\omega\cos\alpha = -\dfrac{mv}{m+M}$

$$A^2\sin^2\alpha = \left(\frac{Mg}{k} - \frac{mg}{k} - \frac{Mg}{k}\right)^2 = \frac{m^2g^2}{k^2}$$

$$A^2\cos^2\omega = \frac{m^2v^2}{(m+M)^2\omega^2} = \frac{m^2v^2}{k(m+M)}$$

$$A^2 = \frac{m^2g^2}{k^2} + \frac{m^2v^2}{k(m+M)} \Rightarrow A = \sqrt{\frac{m^2g^2}{k^2} + \frac{m^2v^2}{k(m+M)}}$$

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$$\cos\alpha = -\frac{mv}{(m+M)}\cdot\frac{1}{A\omega}$$

另外一種解 A 之方法是利用能量守恆定律

在碰撞後，此系統受之力均為保守力故能量守恆定律成立

在剛碰撞後 $E_p + E_k = -(m+M)gd + \frac{1}{2}kd^2 + \frac{1}{2}(m+M)v_0^2$

在轉向點 Y $E_p + E_k = -(m+M)gY + \frac{1}{2}kY^2$

$$-(m+M)gd + \frac{1}{2}kd^2 + \frac{1}{2}(m+M)v_0^2 = -(m+M)Y + \frac{1}{2}kY^2$$

由上式中可求出兩個轉向點 Y^+, Y^-

振動之中心点位於 $\frac{1}{2}(Y^+ + Y^-)$ 處

振幅 $= \frac{1}{2}(Y^+ - Y^-)$

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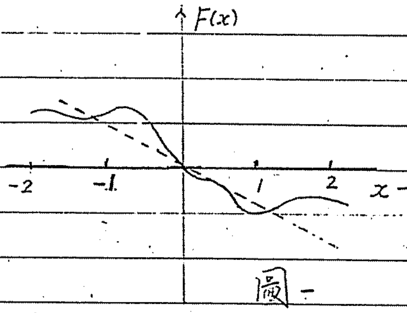
第二節 簡諧運動(II)

1. 簡介：在此節中我們將討論 (一) 當質點在其平衡點附近作小振動時，它通常均作簡諧運動 (二) 在簡諧運動中各物理量之平均值 (三) 以轉動向量圖來描述簡諧運動之方法 (四) 簡諧運動之相重疊。

2. 基本觀念

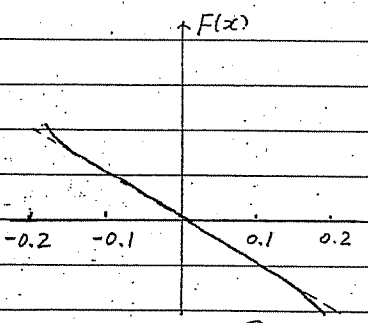
(一) 若一外力 $F(x)$ 作用於一質點上，當該質點位於平衡點，則其所受之外力知為 0。若我們取一坐標使 $x=0$ 點位於平衡點，則 $F(x)$ 可能如圖一中實線所示。



顯然地，$F(x) \doteq -kx$。

圖一

當我們將原點附近放大十倍，則情形就如圖二所示。



很明顯地，在 $-0.1 < x < 0.1$ 之間，$F = -kx$

因此在此區中，它從事簡諧運動。而力常數

$$k = -\frac{dF}{dx}\bigg|_{x=0} \qquad (1)$$

圖二

此一結果可由另一個方法來解釋。任一在 x_0 點附近分析函數 $f(x)$ 可利用泰勒展開法[2]寫成

$$f(x) = f(x_0) + \frac{1}{1!}\frac{df}{dx}\bigg|_{x=x_0}(x-x_0) + \frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{x=x_0}(x-x_0)^2 + \cdots \qquad (2)$$

將 $F(x)$ 用上式在其平衡點 $x_0=0$ 附近展開。

$$F(x) = \underset{0}{\underset{\parallel}{F(0)}} + \frac{dF}{dx}\bigg|_{x=x_0} x + \frac{1}{2!}\frac{d^2F}{dx^2}x^2 + \cdots$$

當 x 夠小時，$F(x) \simeq \frac{dF}{dx}\big|_{x=x_0} x = -kx$. (3)

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同樣的我們將位能函數 $E_p(x)$ 在平衡點 $(x_0=0)$ 附近用泰勒方法展開

$$E_p(x) = E_p(0) + \frac{dE_p(x)}{dx}\Big|_{x=0} x + \frac{1}{2!}\frac{d^2E_p(x)}{dx^2}\Big|_{x=0} x^2 + \frac{1}{3!}\frac{d^3E_p}{dx^3}\Big|_{x=0} x^3 + \cdots \quad (4)$$

在平衡點[3] $\frac{dE_p}{dx} = 0$, $\frac{d^2E_p(x)}{dx^2}\Big|_{x=0} \equiv k > 0$ 因此當 x 夠小時

$$E_p(x) - E_p(0) = \frac{1}{2} k x^2 \quad (5)$$

因此質點作簡諧運動。

(二) 若 $f(t) = f(t+T)$ 是一週期函數，則 $f(t)$ 之平均值

$$\langle f(t) \rangle = \frac{1}{T}\int_0^T f(t)\, dt \quad (6)$$

在簡諧運動中，

$$x(t) = A\sin(\omega t + \alpha), \quad v(t) = A\omega\cos(\omega t + \alpha) \quad (7)$$

$$E_p(x) = \frac{1}{2} k x^2 \Rightarrow E_p(t) = \frac{1}{2} k A^2 \sin^2(\omega t + \alpha) \quad (8)$$

$$E_k(x) = \frac{1}{2} m v^2 \Rightarrow E_k(t) = \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \alpha) \quad (9)$$

$$E = \frac{1}{2} k A^2, \quad T = \frac{2\pi}{\omega} \quad (10)$$

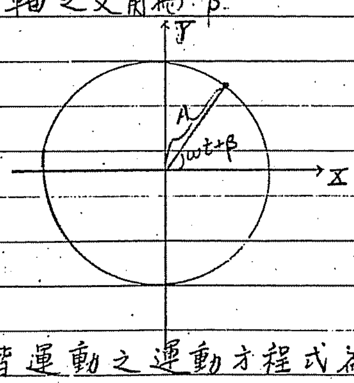
代入 (6) 式，可得[4]

$$\langle x(t) \rangle = \langle v(t) \rangle = 0 \quad (11)$$

$$\langle E_p \rangle = \langle E_k \rangle = \frac{1}{2}\langle E \rangle \quad (12)$$

[5]

(三) 若一質點在一半徑為 A 之圓周上作等速圓周運動，其角速度為 ω，其於 $t=0$ 時與 x 軸之夾角為 β. 則其於 x 軸之投影 $x(t) = A\cos(\omega t + \beta)$



圖三

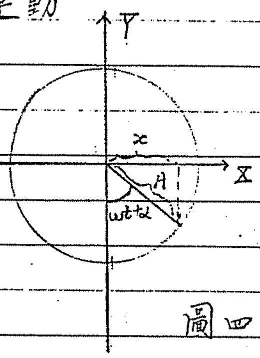
簡諧運動之運動方程式為 $x(t) = A\sin(\omega t + \alpha) = A\cos(\omega t + \alpha - \frac{\pi}{2})$ (13)

分類：	
編號：	3
總號：	

若是我們令 $\beta = \alpha - \frac{\pi}{2}$ 則此兩公式即完全相同。

若是我們對 $-y$ 軸沿反時鐘方向量之角為 α 則其對 x 軸之交角即為 $\alpha - \frac{\pi}{2}$

因此我們若以 $-y$ 軸為起點沿反時鐘量交角 α 則在半徑為 A 之圓周上之質點在 x 軸之投影即為簡諧運動 $x(t) = A\sin(\omega t + \alpha)$. 我們經常以這樣的一個轉動向量來描述簡諧運動



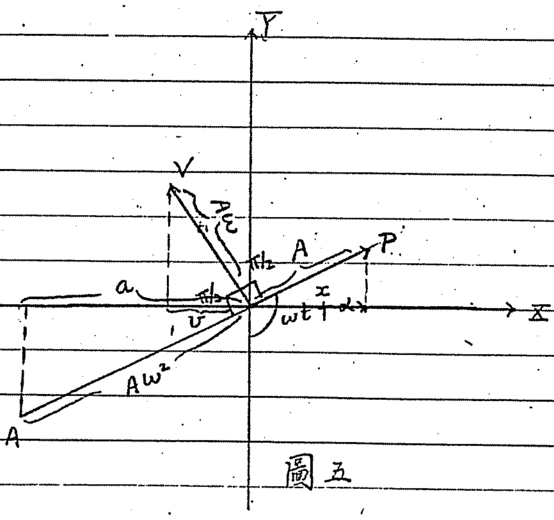
$x = A\sin(\omega t + \alpha)$

圖四

因為 $v(t) = A\omega\cos(\omega t + \alpha)$，所以代表它的轉動向量與 y 軸之交角比代表轉動向量位移與 y 軸之交角大 90°

其長度也變成 $A\omega$.

因為 $a(t) = -A\omega^2\sin(\omega t + \alpha) = -A\omega^2\cos(\omega t + \alpha + \frac{\pi}{2})$，所以代表它的轉動向量與 y 軸之交角比代表速度之轉動向量與 y 軸之交角大 $90°$。其大小變成 $A\omega^2$



圖五

由上圖，x, v 及 a 之關係可一目了然。

(四) 簡諧運動之相重疊

(1) 方向，頻率相同之兩簡諧運動相加

分類：	
編號：	4
總號：	

$$x_1 = A_1 \sin(\omega t + \alpha_1) \tag{14}$$

$$x_2 = A_2 \sin(\omega t + \alpha_2) \tag{15}$$

$$x = x_1 + x_2 \tag{16}$$

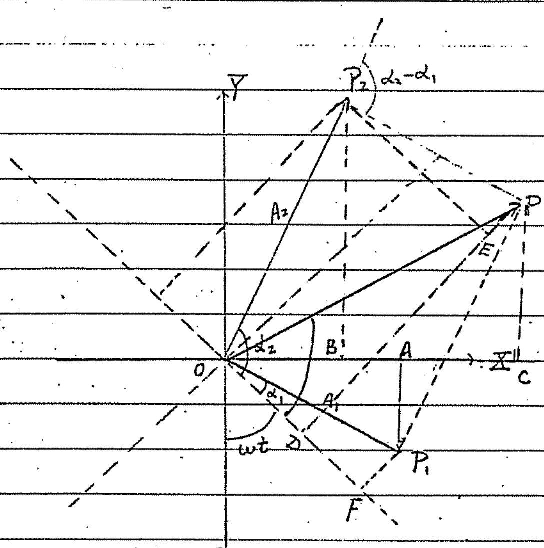
代數法：

$$\begin{aligned} x = x_1 + x_2 &= A_1 \sin(\omega t + \alpha_1) + A_2 \sin(\omega t + \alpha_2) \\ &= A_1 \sin\omega t \cos\alpha_1 + A_1 \cos\omega t \sin\alpha_1 + A_2 \sin\omega t \cos\alpha_2 + A_2 \cos\omega t \sin\alpha_2 \\ &= \underbrace{(A_1 \sin\alpha_1 + A_2 \sin\alpha_2)}_{A \sin\alpha} \cos\omega t + \underbrace{(A_1 \cos\alpha_1 + A_2 \cos\alpha_2)}_{A \cos\alpha} \sin\omega t \\ &= A \sin(\omega t + \alpha) \end{aligned} \tag{17}$$

A 与 α 是由 $A \sin\alpha = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$，及 $A\cos\alpha = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$

$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\alpha_2 - \alpha_1)$，$\tan\alpha = \dfrac{A_1 \sin\alpha_1 + A_2 \sin\alpha_2}{A_1 \cos\alpha_1 + A_2 \cos\alpha_2}$

圖解法



$\overrightarrow{OP_1}$ 是代表 x_1 簡諧運動之轉動向量

$\overrightarrow{OP_2}$ 是代表 x_2 簡諧運動之轉動向量

$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$

由圖中可看出 $|\overrightarrow{OP}| = A_1^2 + A_2^2 + 2A_1A_2 \cos(\alpha_1 - \alpha_2)$

$PD = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$

$OD = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$

因此 $\overrightarrow{OP}$ 与 $-y$ 軸之交角為 $\omega t + \alpha$。

結論 $\overrightarrow{OP}$ 是代表 x 之轉動向量 也即是 OP 在沿 x 軸之投影即為 $x(t)$.

分類：	
編號：	4A
總號：	

由向量的觀點來看

$\vec{OP_1} = OA\,\hat{i} + AP_1\,\hat{j}$ $\quad OA = x_1$

$\vec{OP_2} = OB\,\hat{i} + BP_2\,\hat{j}$ $\quad OB = x_2$

$\vec{OP} = \vec{OP_1} + \vec{OP_2} = (OA + OB)\,\hat{i} + (AP_1 + BP_2)\,\hat{j}$

因此 $\vec{OP}$ 在 x 軸之投影顯然的即是 $x = x_1 + x_2$

分類：
編號： 5
總號：

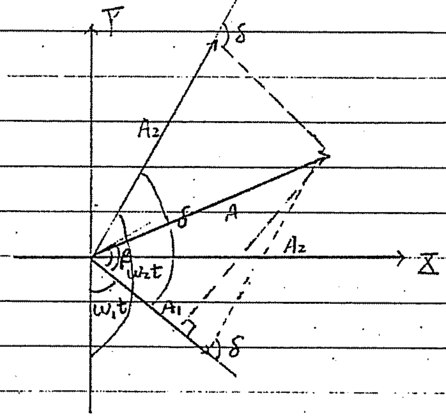
(2) 方向相同，頻率不同之兩簡諧運動相同

$$x_1(t) = A_1 \sin \omega_1 t \quad (18)$$

$$x_2(t) = A_2 \sin \omega_2 t \quad (19)$$

$$x(t) = x_1(t) + x_2(t) \quad (20)$$

我們現在利用圖解法來研究在時間 t 時的情形



現在 $\delta = \omega_2 t - \omega_1 t$ 是時間的函數

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\omega_1 - \omega_2)t}$$

所以振幅也是時間之函數，稱為拍

$$\tan\beta = \frac{A_2 \sin\delta}{A_1 + A_2 \cos\delta} \quad (21)$$

因此代表 x 之轉動向量与 -y 軸之交角 $= \omega_1 t + \beta = \omega_1 t + \tan^{-1}\dfrac{A_2 \sin(\omega_2 - \omega_1)t}{A_1 + A_2 \sin(\omega_2 - \omega_1)t}$

$$x(t) = A(t) \sin(\omega_1 + \beta(t))$$

因為 $A(t)$ 及 $\beta(t)$ 均為時間之函數，所以它並非簡諧運動

若 $A_1 = A_2$，則 (20) 式變成 $A = 2A_1 \cos[\frac{1}{2}(\omega_1 - \omega_2)t]$ (22)

(21) 式簡化為 $\tan\beta = \tan[\frac{1}{2}(\omega_1 - \omega_2)t]$ (23)

因此 $x(t) = 2A_1 \cos[\frac{1}{2}(\omega_1 - \omega_2)t] \cdot \sin[\frac{1}{2}(\omega_1 + \omega_2)t]$ (24)

(3) 互相垂直，頻率不同之兩簡諧運動之相加

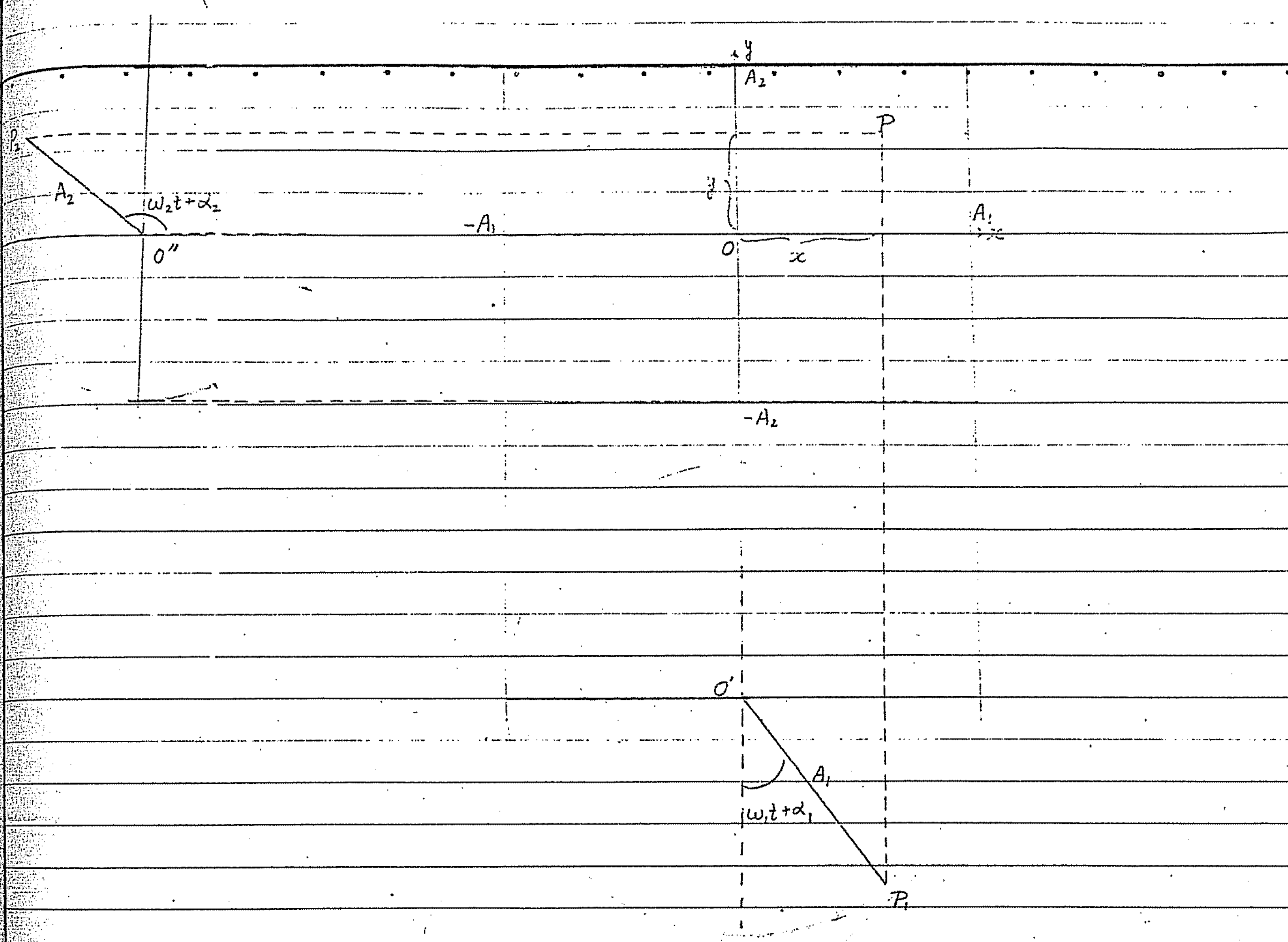
$$x = A_1 \sin(\omega_1 t + \alpha_1), \quad y = A_2 \sin(\omega_2 t + \alpha_2) \quad (25)$$

代數法：若 ω_1 / ω_2 是簡單分數，δ 是 2π 之簡單分數 (26)

由上兩式中將 t 消去，則得到 x-y 之關係，也即是在兩度空間中的軌跡。

圖解法。

分類：	
編號：	6
總號：	



圖解的方法如下 (一) 先畫兩圓，其半徑分別為 A_1 及 A_2 (二) $\overrightarrow{OP_1}$ 在 x 軸之投影為 $x(t)=A_1\sin(\omega_1 t+\alpha_1)$ (三) $\overrightarrow{O''P_2}$ 在 y 軸之投影為 $y(t)$，(四) 在以 O 為原點之 x-y 平面上 P 點即是該質點在 t 時之位置 (五) 以此法求在各時間的位置，則可得該質點在 x-y 平面上之軌跡。很顯然地，P 點之運動均局限以 $2A_1$ 及 $2A_2$ 為邊長的長方形中。若 ω_1 及 ω_2 不可共約，則質點 P 永不重複，若是經過夠長的時間後，其所經之途徑由物理的觀點而言，遍佈此一長方形。

若 ω_1/ω_2 是一簡單分數，同時 δ 是 2π 之簡單之分數時，則它的途徑在兩度空間中為一封閉曲線。其週期為 $T_1=\frac{2\pi}{\omega_1}$ 及 $T_2=\frac{2\pi}{\omega_2}$ 之最小公倍數。這個問題以個別討論為宜。我們將在應用一節中分別討論。

討論

分類：
編號： 7
總號：

(1) 我們所指的平衡點是穩定平衡點，因此在 $x=0$ 附近 $x>0$ 時 $F(x)<0$，在 $x=0$ 附近 $x<0$ 時 $F(x)>0$

(2) 參看任何微積分書。

(3) $x=x_0$ 為 $E_p(x)$ 函數之極小点，因此 $\frac{dE_p}{dx}\Big|_{x=x_0}=0$，$\frac{d^2E_p}{dx^2}>0$

(4) 這些結果可利用以下的公式很容易的得到。

(a) $\int_0^T \cos(\omega t+\alpha)\,dt=0$; $\int_0^T \sin(\omega t+\alpha)\,dt=0$ (27)

證明 令 $\omega t+\alpha=y$ $dt=\frac{1}{\omega}dy$, $t=0 \Rightarrow y=\alpha$; $t=T \Rightarrow y=\omega T+\alpha$
$=2\pi+\alpha$

則 $\int_0^T \cos(\omega t+\alpha)\,dt=\int_\alpha^{2\pi+\alpha}\frac{1}{\omega}\cos y\,dy$

$=\frac{1}{\omega}\sin y\Big|_\alpha^{2\pi+\alpha}=0$

利用同樣的方法可證明 $\int_0^T \sin(\omega t+\alpha)\,dt=0$ (28)

(b) $\frac{1}{T}\int_0^T \cos^2(\omega t+\alpha)\,dt=\frac{1}{2}$. $\frac{1}{T}\int_0^T \sin^2(\omega t+\alpha)\,dt=\frac{1}{2}$

證明 令 $\omega t+\alpha=y$, $dt=\frac{1}{\omega}dy$, $t=0\Rightarrow y=\alpha$; $t=T\Rightarrow y=\omega T+\alpha=2\pi+\alpha$

$\frac{1}{T}\int_0^T \cos^2(\omega t+\alpha)\,dt=\frac{1}{\omega T}\int_\alpha^{2\pi+\alpha}\cos^2 y\,dy$

$=\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha}\frac{1}{2}(1-\cos 2y)\,dy=\frac{1}{2}-\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha}\cos 2y\,dy$

$\int_\alpha^{2\pi+\alpha}\cos 2y\,dy=0$ (29)

令 $z=2y$ $dy=\frac{1}{2}dz$, $y=\alpha\Rightarrow z=2\alpha$; $y=2\pi+\alpha\Rightarrow z=4\pi+2\alpha$

$\int_\alpha^{2\pi+\alpha}\cos 2y\,dy=\frac{1}{2}\int_{2\alpha}^{4\pi+2\alpha}\cos z\,dz=\frac{1}{2}\left(\sin(4\pi+2\alpha)-\sin 2\alpha\right)=0$

因此 $\frac{1}{T}\int_0^T \cos^2(\omega t+\alpha)\,dt=\frac{1}{2}$ (30)

利用同樣的方法可證明 $\frac{1}{T}\int_0^T \sin^2(\omega t+\alpha)\,dt=\frac{1}{2}$

$\langle E_p\rangle=\frac{1}{2}kA^2\frac{1}{T}\int_0^T \sin^2(\omega t+\alpha)\,dt=\frac{1}{4}kA^2$ (31)

$\langle E_k\rangle=\frac{1}{2}mA^2\omega^2\frac{1}{T}\int_0^T \cos^2(\omega t+\alpha)\,dt=\frac{1}{4}kA^2$ (32)

(5) 此一結果為均功定理之一特例。在補充材料部分我們將證明均功定理。

分類：
編號： 8
總號：

應用

(a) 若 $F(x) = \frac{Ax}{x-a}$ 　$x<a$，$F(0)=0$，因此 $x=0$ 是平衡點。

在 $x=0$ 附近，$F(x)$ 可利用泰勒公式表開端

$$F(x) = -\frac{A}{a}x - \frac{A}{a^2}x^2 - \frac{A}{a^3}x^3 + \cdots \tag{33}$$

當 $x \ll a$ 時

$$F(x) = -\frac{A}{a}x \tag{34}$$

受此力質量為 m 之物体作簡諧運動，其頻率為 $\nu = \frac{1}{2\pi}\sqrt{\frac{A}{am}}$ (35)

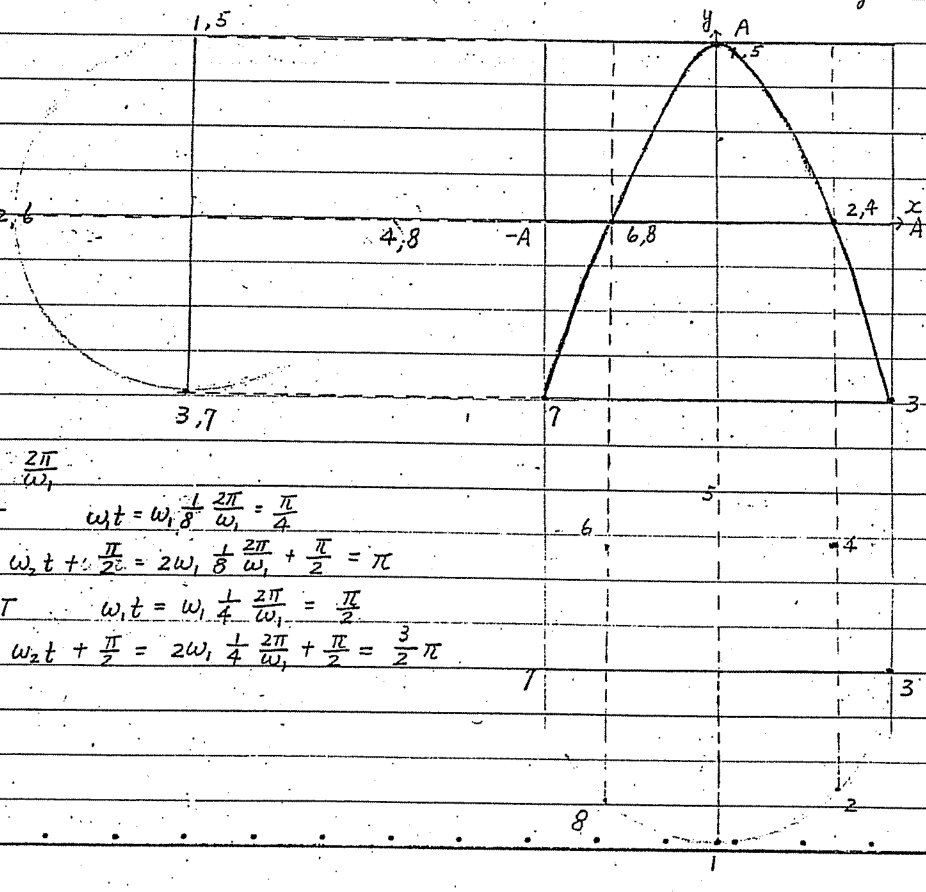
(b) 若 $x = A\sin\omega_1 t$ (36)

$$y = A\cdot\sin(2\omega_1 t + \tfrac{\pi}{2}) \tag{37}$$

消去 t 後可得利薩如圖形。

$$y = A\cos 2\omega_1 t = A[1-2\sin^2\omega_1 t] = A[1-2\frac{x^2}{A^2}] \tag{38}$$

在 x-y 平面上它描述拋物線的一段（$|x|<A_1$ 及 $|y|<A_2$ 之部分）



$T = \frac{2\pi}{\omega_1}$

$t = \frac{1}{8}T$ 　$\omega_1 t = \omega_1 \frac{1}{8}\frac{2\pi}{\omega_1} = \frac{\pi}{4}$

$\omega_2 t + \frac{\pi}{2} = 2\omega_1 \frac{1}{8}\frac{2\pi}{\omega_1} + \frac{\pi}{2} = \pi$

$t = \frac{1}{4}T$ 　$\omega_1 t = \omega_1 \frac{1}{4}\frac{2\pi}{\omega_1} = \frac{\pi}{2}$

$\omega_2 t + \frac{\pi}{2} = 2\omega_1 \frac{1}{4}\frac{2\pi}{\omega_1} + \frac{\pi}{2} = \frac{3}{2}\pi$

分類：	
編號：	9
總號：	

若 $x = A\cos\omega_1 t = A\sin(\omega_1 t + \frac{\pi}{2})$

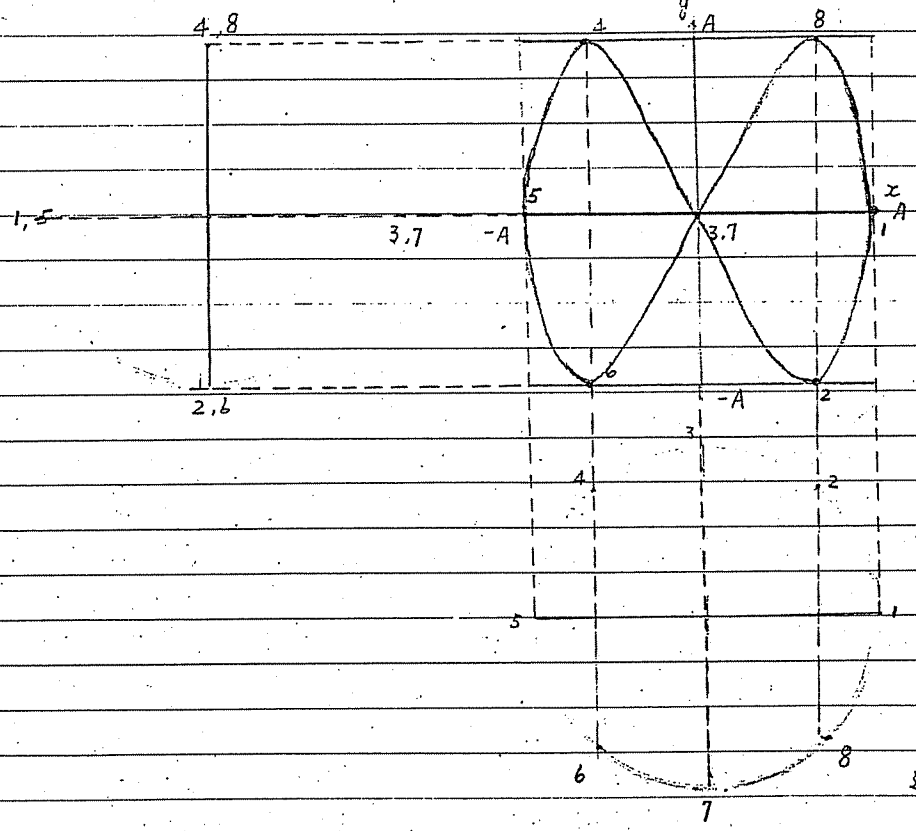
$y = A\cos(2\omega_1 t + \frac{\pi}{2}) = -A\sin 2\omega_1 t = A\sin(2\omega_1 t + \pi)$

代數法

$y = -A\sin 2\omega_1 t = 2A\sin\omega_1 t\cos\omega_1 t$

$y^2 = 4A^2\sin^2\omega_1 t\cos^2\omega_1 t = 4x^2(1-\frac{x^2}{A})$

圖解法



這些圖形稱為利薩如圖形

習題

(1) 兩原子分子中原子間之作用力可用以下之公式來約略描述

$$f(r) = A(r_0^6 - r^6)/r^{13}$$

此處 r 是兩個原子間之距離。A，r_0 是常數，因不同的分子而異

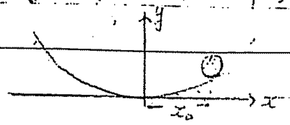
(a) 求靜力平衡點 [C]

(b) 導出作簡諧運動的條件 [J]

分類：
編號： 10
總號：

(c) 求在平衡點附近小振盪之頻率 [D]

(2) 一為質點之質量為 m，位於一平滑之拋物線形之表面 $y = \alpha x^2$



當球放開後 (a) 試證當 x 小時，它在 x 方向之運動為一簡諧運動 [E]

(b) 求此一振盪之頻率 [I]

(3) 繪出 $x(t) = A\sin(\omega t + \pi/2)$ 在 $t = \frac{T}{2}$ 時

(a) 代表位移之轉動向量 [A]

(b) 代表速度之轉動向量 [F]

(c) 代表加速度之轉動向量 [K]

(4) $x_1(t) = A\sin\omega t$

$x_2(t) = A\sin(2\omega t + \frac{\pi}{4})$

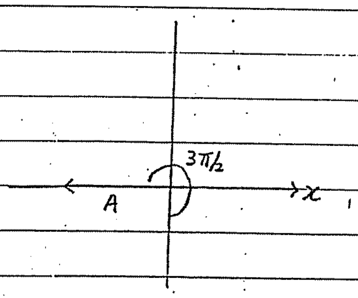
(a) 利用代數法 $x(t) = x_1(t) + x_2(t)$ 在 $t = \frac{\pi}{\omega}$ 時之值 [B]

(b) 利用圖解法求 $x(t) = x_1(t) + x_2 t$ 在 $t = \frac{\pi}{\omega}$ 時之值 [G]

(5) 利用圖解法求 $x = A\sin\omega_1 t$ 及 $y = A\sin(2\omega_1 t + \frac{\pi}{4})$ 所構成之利薩如圖形 [H]

答案

[A]



[B] $x = -\frac{1}{\sqrt{2}}A$

[C] $r = r_0$

[D] $\nu = \frac{1}{2\pi r_0^4}\sqrt{\frac{6A}{m}}$

[E] $\frac{dy}{dx} = 2\alpha x = \tan\theta \simeq \theta$

分類：	
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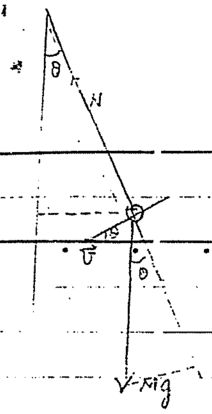
總號：	



沿線方向沒有運動 $|\vec{N}| = |Mg\cos\theta| \simeq Mg$

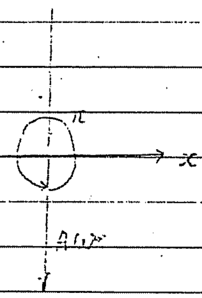
$$F_x = -N\sin\theta \simeq -Mg\theta$$

$$\frac{dy}{dx} = 2\alpha x = \tan\theta \simeq \theta$$

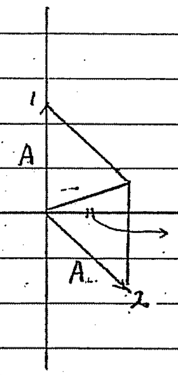
$$F_x \simeq -2Mg\alpha x$$

因此是一簡諧運動.

[F]

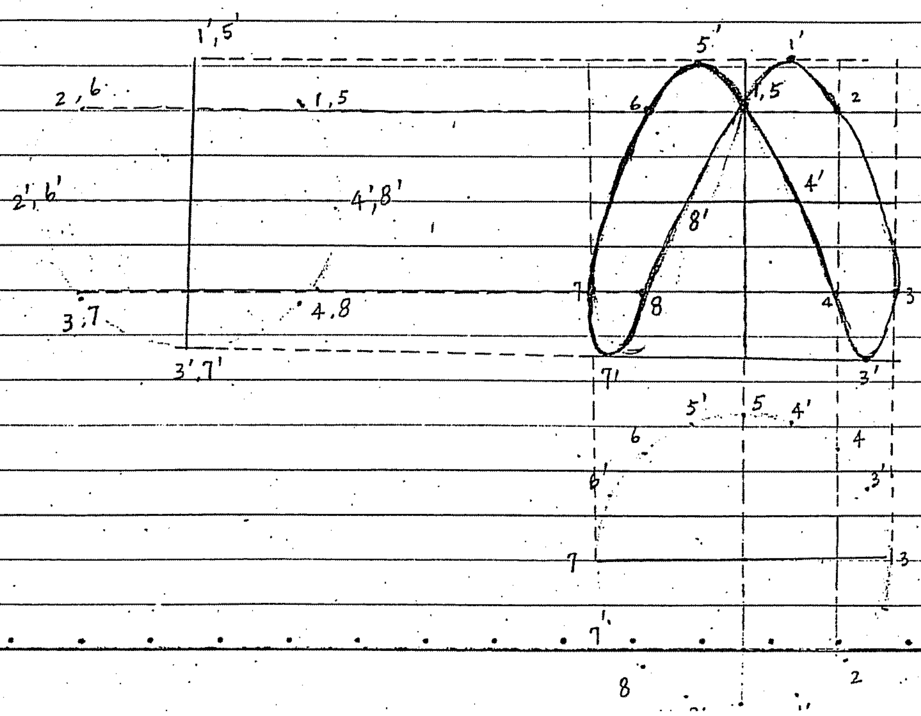


[G]



x 即是此段之長度, 顯然地 $x = \frac{A}{\sqrt{2}}$

[H]



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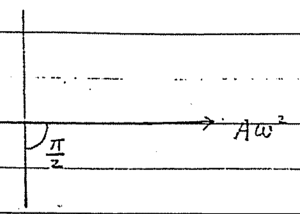
總號：	

[I] $\nu = \frac{1}{2\pi}\sqrt{2g\alpha}$

[J] $63 \frac{r-r_0}{r_0} \ll 6A$

[K]



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總號：

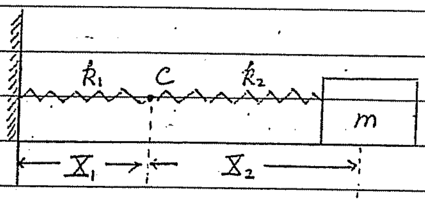
第三節 耦合振盪，非諧振動

簡介 在此節中我們將討論(一)當一些振盪器連接在一起的運動情況(二)傅立葉級數的性質(三)在上節中我們曾說明當一質點在其穩定平衡點附近作運動時它的運動與簡諧運動非常接近，當然實際上它的運動並非真正的簡諧運動，因此需要將簡諧運動方程式作少許的修正才能真正描述該質點的運動，我們將舉例來說明修正的方法。

基本觀念

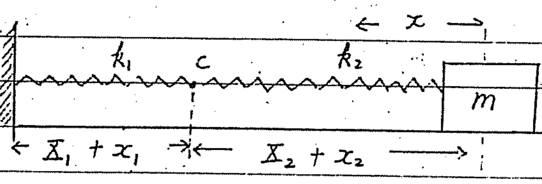
(一) 在很多情況下，我們會遇到一些耦合振盪，其運動狀況當然因連接的方法而異，但處理這些問題的方法卻大致相同。我們將在此處討論兩個最簡單的例子來說明基本的方法，在應用一節中再舉一個比較複雜的例子。

(1)



一質量 m 之物体与力常數為 k_1 及 k_2 之兩彈簧如圖所示（串联）

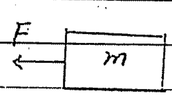
X_1, X_2 是兩彈簧之平衡長度



$$x = x_1 + x_2 \quad (1)$$

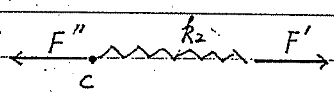
我們現在分別將在 m, 彈簧，及 c 點所受的力寫出

m.



F 是 k_2 作用於物体 m 之力

$$F = k_2 x_2 \quad (2)$$



F 之大小与 F' 相等，因為它們是作用力及反作用力。F'' 是 c 作用於 k_2 之力。由於我們假設彈簧之質量可略去不計，所以 F', F'' 之大小相等。

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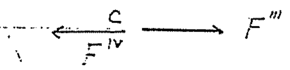


F''' 之大小與 F'' 之大小相同，因為它們是作用力與反作用力偶。由於 c 的質量假設為 0，所以 F^{IV} 之大小與 F'' 相等，F^{IV} 是 k_1 對 C 點之拉力。

$$F^{IV} = F = -k_1 x_1 \tag{3}$$

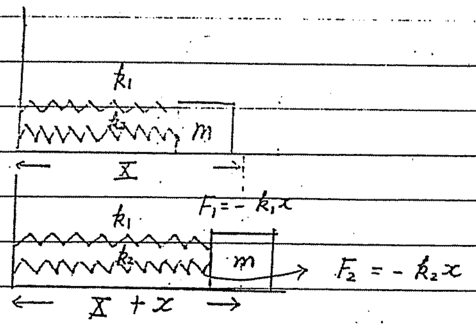
將 (1)，(2)，(3) 式連在一起即可得

$$x = x_1 + x_2 = -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -\frac{F}{k} \tag{4}$$

此處 $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2}$ [3] (5)

因此其振動週期 $T = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$ (6)

(2)



$$F_1 = -k_1 x \quad , \quad F_2 = -k_2 x \tag{7}$$

$F = F_1 + F_2 = -(k_1 + k_2)x = -kx \qquad k = k_1 + k_2$ [4] (8)

其振動週期為 $T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$ (9)

(二) 傅列葉定理[5] 任一週期為 $T = \frac{2\pi}{\omega}$ 之週期函數 $f(t)$ 可展開為

$$f(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \cdots + a_n \cos n\omega t + \cdots$$
$$+ b_1 \sin\omega t + b_2 \sin 2\omega t + \cdots + b_n \sin n\omega t + \cdots \tag{10}$$

以上的級數叫做傅列葉級數。ν 稱為基頻率，$2\nu, 3\nu, \cdots$ 則為諧頻率。

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$$a_0 = \frac{1}{T}\int_0^T f(t) \qquad (11a)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt \qquad (11b)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt \qquad (11c)$$

我們將在討論小節中證明以上的結果[5].

(三) 非諧振動.

我們首先將 $E_p(x)$ 在其穩定平衡點附近以泰勒級數展開

$$E_p(x) = E_p(x_0) + \frac{1}{2}k(x-x_0)^2 + \frac{1}{6}k'(x-x_0)^3 + \frac{1}{24}k''(x-x_0)^4 + \cdots \qquad (12)$$

$$k = \left.\frac{d^2E_p}{dx^2}\right|_{x=x_0}, \quad k' = \left.\frac{d^3E_p}{dx^3}\right|_{x=x_0}, \quad k'' = \left.\frac{d^4E_p}{dx^4}\right|_{x=x_0}, \cdots \qquad (13)$$

若 $x-x_0$ 夠小時[6], (12) 式第二項以後均可忽略時,則其運動是簡諧運動.若 $x-x_0$ 變大時則我們不可略去第二項以後之各項,則其運動是非諧運動。此時問題非常複雜. 但是,若是 $x-x_0$ 仍是不太時,在第二項以後只有極少的幾項需要考慮,而且後面這些項比第一,二項之貢獻小很多時,則我們可用微擾法來計算後面這些項所引起之修正.我們將在應用一節中舉一例來加以說明.

討論

1. c 點是接頭點,我們可以將它看成一質點,其質量趨近於 0.
2. k_1, k_2 分別是第一條及第二條彈簧之力常數,為了方便起見,我們也用它來代表第一及第二條彈簧
3. 當 $k_1 = k_2$ 時 $k = \frac{1}{2}k_1$, 此與我們的平常的想法相符. 當我們將 m 離之時 x_1, x_2 均只需為 $\frac{x}{2}$, 因此它對物体之拉力只有原來的一半.
4. 當 $k_1 = k_2$ 時 $k = 2k_1$, 此與我們的平常的想法相符。我們需要兩倍的力才能使質量相同之物体作同樣長度之位移.

5. 要導以上的公式我們需要首先證明以下幾個公式

(a) n, m 是整數時，$\int_0^T \cos n\omega t \cos m\omega t \, dt = \frac{1}{2} T \delta_{nm}$

證明 $\int_0^T \cos n\omega t \cos m\omega t \, dt$

$$= \frac{1}{2}\int_0^T \left[\cos \frac{2\pi(n+m)}{T} t + \cos \frac{2\pi(n-m)}{T} t \right] dt \tag{14}$$

$$= \begin{cases} \frac{1}{2} \dfrac{1}{\frac{2\pi(n+m)}{T}} \left[\sin \frac{2\pi(n+m)}{T} t \right]_0^T + \frac{1}{2} \dfrac{1}{\frac{2\pi(n+m)}{T}} \left[\sin \frac{2\pi(n-m)}{T} t \right]_0^T & n \neq m \\ \frac{1}{2} \dfrac{1}{\frac{4\pi n}{T}} \left[\sin \frac{4\pi n}{T} t \right]_0^T + \frac{1}{2}\int_0^T dt = \frac{1}{2} T & n = m \end{cases} \tag{15}$$

(b) n, m 是整數時，$\int_0^T \cos n\omega t \sin m\omega t \, dt = 0$，$\int_0^T \sin n\omega t \sin \omega t \, dt = \frac{1}{2} T \delta$ (16)

其證法與 (a) 式相同。

$\int_0^T f(t)\, dt = \int_0^T f(t) \cos(0\omega t)\, dt$

將 (10) 式代入上式 $\int_0^T f(t) \cos(0\omega t) dt = \int_0^T [a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n$

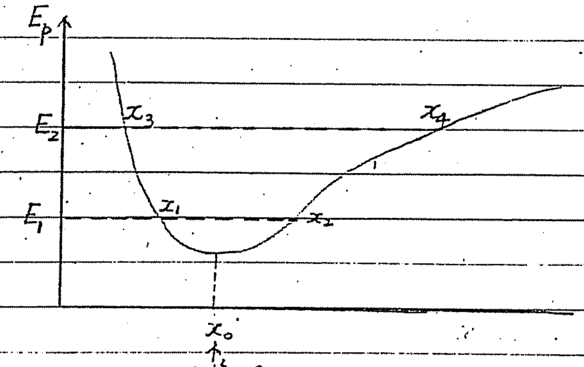
$\cdot \cos(0\omega t) dt$ (17)

利用公式 (a) 與 (b) 則上式變成 $\int_0^T f(t)\, dt = a_0 T$ 也即是 (11 a)

$\int_0^T f(t) \cos n\omega t \, dt = \frac{T}{2} a_n$ 也即是 (11b)

$\int_0^T f(t) \sin m\omega t \, dt = \frac{T}{2} b_n$ 也即是 (11c)

6. 在平衡點附近，非諧運動的位能函數大約如下圖所示



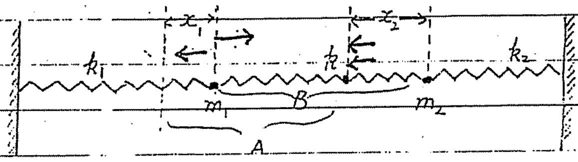
若能量較低時（如 $E=E_1$），則該質點的運動局限於 x_1 及 x_2 之間，此時其位能函數相當接近 $\frac{1}{2}k(x-x_0)^2$，所以其運動也接近簡諧運動。若能量較高時（如 $E=E$

則其運動可在 x_3 到 x_4 之間進行。顯然地，在區域中其位能函數為 $\frac{1}{2}k(x-x_0)^2$

相差也較多，所以其運動也與簡諧運動相差較多。同時也可由上圖中看出此種情形時 $\langle x\rangle > x_0$

應用

(一) 我們現在用一個例子來解釋耦合振盪中的一些觀念



有力常數為 k 之彈簧之長度增長了 $x_2 - x_1$

m_1 所受之力：k_1 向左拉大小是 $k_1 x_1$
k 向右拉大小是 $k(x_2 - x_1)$

m_2 所受之力 k 向左拉大小是 $k(x_2 - x_1)$
k_2 向左拉大小是 $k_2 x_2$

牛頓定律可寫成

$$m_1 \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1 \qquad (18)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2 x_2 - k(x_2 - x_1) \qquad (19)$$

我們將討論 $m_1 = m_2$ 及 $k_1 = k_2$ 的簡單的情形，則以上兩式可簡化為

$$m \frac{d^2 x_1}{dt^2} = k(x_2 - x_1) - k_1 x_1 \qquad (20)$$

$$m \frac{d^2 x_2}{dt^2} = -k_1 x_2 - k(x_2 - x_1) \qquad (21)$$

$$(20)+(21) \qquad m \frac{d^2}{dt^2}(x_1 + x_2) = -k_1(x_1 + x_2) \qquad (22)$$

$$x_1(t) + x_2(t) = A \sin(\omega t + \alpha) \qquad \omega = \sqrt{\frac{k_1}{m}} \qquad (23)$$

A, α 是取決於初位移及初速。

$$(20)-(21) \qquad m \frac{d^2}{dt^2}(x_1 - x_2) = 2k(x_2 - x_1) + k_1(x_2 - x_1)$$

$$= -(2k + k_1)(x_1 - x_2) \qquad (24)$$

$$x_1(t) - x_2(t) = A' \sin(\omega' t + \alpha') \qquad \omega' = \sqrt{\frac{k_1 + 2k}{m}} \qquad (25)$$

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A', α' 也是取決於初位移及初速。

因此 $x_1(t)+x_2(t)$ 及 $x_1(t)-x_2(t)$ 進行獨立之簡諧運動。它們稱為簡正振動。

由 (24) 及 (25) 式我們可得

$$x_1(t) = \frac{1}{2}\left[A\sin(\omega t+\alpha)+A'\sin(\omega' t+\alpha')\right] \tag{26}$$

$$x_2(t) = \frac{1}{2}\left[A\sin(\omega t+\alpha)-A'\sin(\omega' t+\alpha')\right] \tag{27}$$

$x_1(t)$, $x_2(t)$ 分別是兩個不同頻率的簡諧運動的和。這也是我們在第二節中討論簡諧運動的主要理由。

我們現在由能量的觀念來討論此一問題。為了方便起見我們令 $x_1=x_2=0$ 時之位能為 0.

$$E_p = \frac{1}{2}k_1x_1^2+\frac{1}{2}k(x_2-x_1)^2+\frac{1}{2}k_2x_2^2 \tag{28}$$

m_1 質點所受的力 $F_1 = -\frac{\partial E_p}{\partial x_1} = -k_1x_1+k(x_2-x_1)$ (29)

m_2 質點所受的力 $F_2 = -\frac{\partial E_p}{\partial x_2} = -k(x_2-x_1)-k_2x_2$ (30)

以上兩公式與 (18), (19) 完全一致。

此一系統之總能量 $E = E_k+E_p$

$$\begin{aligned}
&= \frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2+E_p\\
&= \frac{1}{2}mv_1^2+\frac{1}{2}(k_1+k)x_1^2 \quad ①\\
&\quad +\frac{1}{2}mv_2^2+\frac{1}{2}(k_2+k)x_2^2 \quad ②\\
&\quad -kx_1x_2 \quad ③
\end{aligned} \tag{31}$$

第①項代表 m_1 之能量 第②項代表 m_2 之能量 而第三項則代表耦合或作用能。

(二) 我們現在用一個簡單的例子來說明用微擾法來解非諧運動

$$E_p(x) = \frac{1}{2}kx^2-\frac{1}{4}ax^4 \tag{32}$$

此處我們假設第二項比第一項小得多

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$$m\frac{d^2x}{dt^2} = -kx + ax^3 \tag{33}$$

即是其運動方程式。

若是 $a=0$ 則 $x = A\sin(\omega_0 t+\alpha)$ 即是 (33) 式之解。 $\omega_0^2=\frac{k}{m}$；A,α 由初位移及初速決定

當 $a\neq 0$ 時，我們將假設

$$x = A'\sin(\omega t+\alpha') + B\sin[3(\omega t+\alpha')] \tag{34}$$

是較好的解。顯然地 $B \ll A$，而當 $a=0$ 時 B 亦為 0.

我們寫下 (34) 式之理由如下

(1) 若 $B=0$ 時右邊之結果 $-k\sin(\omega t+\alpha) + a(\sin(\omega t+\alpha))^3$. 由於

$[\sin(\omega t+\alpha)]^3 = \frac{3}{4}\sin(\omega t+\alpha) - \frac{1}{4}\sin[3(\omega t+\alpha)]$. 因此右邊之函數形式是

$\sin(\omega t+\alpha)$ 和 $\sin[3(\omega t+\alpha)]$ 的線性組合

(2) (34) 式對 t 之二度微分也具有以下之形式。

將 (34) 式代入 (33) 式

$$\text{左邊} = -mA\omega^2\sin(\omega t+\alpha) - 9mB\omega^2\sin[3(\omega t+\alpha)]$$

$$\text{右邊} = -k\{A\sin(\omega t+\alpha) + B\sin[3(\omega t+\alpha)]\}$$
$$+ a[A\sin(\omega t+\alpha)+B\sin[3(\omega t+\alpha)]]^3 \tag{35}$$

由於 a，B 比 k，A 小，所以在第二項中我們只需保留 $aA^3\sin(\omega t+\alpha)$

$$\text{右邊} \approx -k\{A\sin(\omega t+\alpha) + B\sin[3(\omega t+\alpha)]\} + aA^3[\sin(\omega t+\alpha)]^3$$

$$= -k\{A\sin(\omega t+\alpha) + B\sin[3(\omega t+\alpha)]\} + aA^3\{\frac{3}{4}\sin(\omega t+\alpha)$$
$$-\frac{1}{4}\sin[3(\omega t+\alpha)]\} \tag{36}$$

將左，右兩邊乘以 $\sin(\omega t+\alpha)$ 後對 t 由 0 至 T 積分得

$$-mA'\omega^2 = -kA + aA'^3\frac{3}{4}$$

因此 $\omega^2 = \omega_0^2 - \frac{3}{4}\frac{A}{m}A^2$ (37)

將左,右兩邊乘以 $\sin[3(\omega t+\alpha)]$ 後對 t 由 0 至 T 積分得

$$-9m\omega^2 B = -kB - \frac{1}{4}aA^3$$

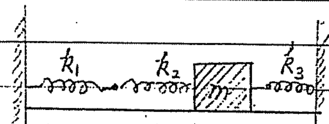
$$B\left[9\omega^2 - \omega_0^2\right] = \frac{1}{4}\frac{a}{m}A^3$$

因此 $B = \frac{1}{4}\frac{a}{m}A^3\frac{1}{(9\omega^2-\omega_0^2)}$ (38)

將由(37)式所得之 ω 及 (38)式所得之 B 代入(34)式,我們即得一對(33)式比

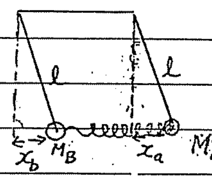
$x = A\sin(\omega t+\alpha)$ 較佳之解

習題



(1) 物体位於一平滑之桌面上,而彈簧連接的情形如圖所示。求物体振動之頻率。 [L]

(2) 兩個單擺之擺長均為 l,質量分別為 M_a, M_b 它們之間以力常數為 K 之彈簧相連如下圖所示。



(a) 證明當此單擺作小幅度振動時之運動方程式為

$$M_a\frac{d^2x_a}{dt^2} = -M_a\frac{g}{l}x_a + K(x_b - x_a)$$

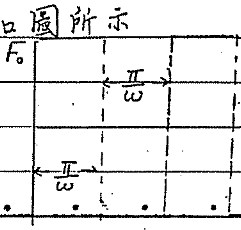
$$M_b\frac{d^2x_b}{dt^2} = -M_b\frac{g}{l}x_b - K(x_b - x_a)$$ [F]

(b) 證明 $x_1 \equiv (M_a x_a + M_b x_b)/(M_a+M_b)$ 及 $x_2 = x_a - x_b$ 是正坐標。 [A]

(c) 求 x_1 及 x_2 之振動頻率 [I]

(d) 若在 $t=0$ 時 $v_a = v_b = 0$, $x_a = A$, $x_b = 0$ 求 $x_a(t)$ 及 $x_b(t)$ [K]

(3) $F(t)$ 如圖所示



將 $F(t)$ 以傅列葉級數展開為

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega t) + \sum_{n=1}^{\infty} b_n\sin(n\omega t)$$

(a) 求 a_0 [B]

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(b) 求 a_n [G]

(c) 求 b_n [J]

(4) 一質點m在位能函數 $E_p(x)=\frac{1}{2}kx^2-\frac{1}{3}ax^3$ 之影響下運動

(a) 寫出該質點之運動方程式 [D]

(b) 設 $a \ll k$， $x = A\cos\omega t + B\cos 2\omega t + x_1$，後兩項為由非諧項所引起者.

以微擾法求 (i) ω [E]，(ii) x_1 [H]，及 (iii) B [C]

答案

[A] 將 F 結果的兩式相加除以 M_A+M_B

$$\frac{d^2}{dt^2}(M_a x_a + M_b x_b) = -(M_a x_a + M_b x_b)\frac{g}{l}$$

$$(M_a+M_b)\frac{d^2}{dt^2}\frac{M_a x_a + M_b x_b}{M_a+M_b} = -\left(\frac{g}{l}\right)(M_a+M_b)\frac{M_a x_a + M_b x_b}{M_a+M_b}$$

$$(M_a+M_b)\frac{d^2}{dt^2}x_1 = -\frac{g}{l}(M_a+M_b)\,x_1$$

將 F 結果的分別除以 M_a，M_b 後相減

$$\frac{d^2}{dt^2}(x_a-x_b) = -\frac{g}{l}(x_a-x_b) + \left(\frac{K}{M_a}+\frac{K}{M_b}\right)(x_b-x_a)$$

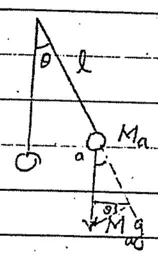
$$= -\left(\frac{g}{l}-\frac{K}{M_a}-\frac{K}{M_b}\right)(x_a-x_b)$$

因此 x_1，x_2 分別是正坐標.

[B] $a_0 = 0$ [C] $B = -\frac{aA^2}{6\omega_0^2 m}$，$\omega_0^2 = \frac{k}{m}$

[D] $m\frac{d^2x}{dt^2} = -kx + ax^2$ [E] $\omega = \omega_0 = \sqrt{\frac{k}{m}}$

[F]



M_a上之力： $F_x \sim -M_a g\sin\theta\cos\theta$

$\sim -M_a g\theta \simeq -M_a\frac{g}{l}x_a$

$(x \sim l\theta)$

彈簧壓縮了 $x_b - x_a$ ⇒ 在A點有一向右之力

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所以

$$M_a \frac{d^2x_a}{dt^2} = -M_a \frac{g}{l} x_a + K(x_b - x_a)$$

同理可得另一式。

[G] $a_n = 0$ [H] $x_1 = \frac{aA^2}{2m\omega_0^2}$ $\omega_0^2 = \frac{k}{m}$

[I] x_1 之振動角速度 $\omega_1 = 2\pi\nu_1 = \sqrt{\frac{g}{l}}$

x_2 〃 〃 〃 〃 〃 $\omega_2 = 2\pi\nu_2 = \sqrt{\frac{g}{l} - \frac{K}{M_a} - \frac{K}{M_b}}$

[J] $b_n = F_0 \frac{4}{\pi} \cdot \frac{1}{n}$

[K] $x_1 = \frac{M_a A}{M_a + M_b}$, $x_2 = A$, $v_1 = 0$, $v_2 = 0$

$x_1 = B\sin(\omega_1 t + \alpha_1)$ $v_1 = B\omega\cos(\omega_1 t + \alpha)$ $\omega_1 = \sqrt{\frac{g}{l}}$

$v_1 = 0 \Rightarrow \alpha_1 = \frac{\pi}{2}$

$x_1 = B\cos\omega_1 t$ $\frac{M_a A}{M_a + M_b} = B$

$$x_1 = \frac{M_a A}{M_a + M_b}\cos\omega_1 t$$

$x_2 = B'\sin(\omega_2 t + \alpha_1)$ $v_2 = B'\omega_2\cos(\omega_1 t + \alpha)$ $\omega_2 = \sqrt{\frac{g}{l} - \frac{K}{M_a} - \frac{K}{M_b}}$

$v_2 = 0 \Rightarrow \alpha_2 = \frac{\pi}{2}$

$x_2 = B'\cos\omega_2 t$ $A = B'$

$$x_2 = A\cos\omega_2 t$$

$$x_a = x_1 + \frac{M_b}{M_a + M_b} x_2 = \frac{M_a}{M_a + M_b} A\cos\omega_1 t + \frac{M_b}{M_a + M_b} A\cos\omega_2 t$$

$$x_b = x_1 + \frac{M_a}{M_a + M_b} x_2 = \frac{M_a A}{M_a + M_b}\cos\omega_1 t + \frac{M_a}{M_a + M_b} A\cos\omega_2 t.$$

[L] $\nu = \frac{1}{2\pi}\sqrt{\frac{k_1k_2 + k_3k_1 + k_2k_3}{m(k_1 + k_2)}}$

Oscillation

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解下列联立微分方程式的方法

$$\frac{d^2x_1}{dt^2} + \omega_1^2 x_1 = -k_1(x_1 - x_2)$$

$$\frac{d^2x_2}{dt^2} + \omega_2^2 x_2 = -k_2(x_2 - x_1) \qquad (1)$$

令 $x_1 = Ae^{i\lambda t}$, $x_2 = Be^{i\lambda t}$ (2)

代入(1)式得

$$A(\omega_1^2 - \lambda^2 + k_1) = k_1 B$$

$$B(\omega_2^2 - \lambda^2 + k_2) = k_2 A \qquad (3)$$

$$\frac{B}{A} = \frac{\omega_1^2 - \lambda^2 + k_1}{k_1} = \frac{k_2}{\omega_2^2 - \lambda^2 + k_2} \qquad (4)$$

$$\begin{vmatrix} \omega_1^2 - \lambda^2 + k_1 & -k_1 \\ -k_2 & \omega_2^2 - \lambda^2 + k_2 \end{vmatrix} = 0$$

令上式 λ^2 之解為 ω^2 及 ω'^2

所以 $x_1 = a\cos\omega t + b\sin\omega t + a'\cos\omega' t + b'\sin\omega' t$

$$x_2 = \gamma a\cos\omega t + \gamma b\sin\omega t + \gamma' a'\cos\omega' t + \gamma' b'\sin\omega' t \qquad (5)$$

$$\gamma = \frac{\omega_1^2 - \omega^2 + k_1}{k_1}, \quad \gamma' = \frac{\omega_1^2 - \omega'^2 + k_1}{k_1} \qquad (6)$$

是第(1)式之解，a, b, a', b' 是取决於在 $t=0$ 時之 x_1, x_2 及 $\frac{dx_1}{dt}$ 及 $\frac{dx_2}{dt}$

例如在 $t=0$ 時 $x_1 = c$, $x_2 = 0$, $\frac{dx_1}{dt} = 0$, $\frac{dx_2}{dt} = 0$ 則得

$$a + a' = c, \quad \gamma a + \gamma a' = 0$$

$$\omega b + \omega' b' = 0, \quad \gamma\omega b + \gamma'\omega' b' = 0 \qquad (7)$$

顯然以上公式之解為 $b = b' = 0$

$$a = \frac{\gamma'}{\gamma' - \gamma} c, \quad a' = \frac{\gamma}{\gamma - \gamma'} c \qquad (8)$$

因此 $x_1 = \frac{c}{\gamma' - \gamma}(\gamma'\cos\omega t - \gamma\cos\omega' t)$

$$x_2 = \frac{c}{\gamma' - \gamma}\gamma'\gamma(\cos\omega t - \cos\omega' t) \qquad (9)$$

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第四節 減幅振盪

簡介 由於通常有阻力的存在，而阻力的方向與運動的方向相反，使得現實振盪器之振幅不為常數，也就是它們作減幅振盪。

基本觀念

通常阻力來自摩擦力，其方向與運動方向相反而大小與速度的大小成正比。因此阻力可寫成

$$F = -\lambda v \tag{1}$$

一個受阻力之振盪器所受之外力為

$$F = -kx - \lambda v \tag{2}$$

牛頓定律可寫成

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0 \tag{3}$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad (\gamma = \frac{\lambda}{2m}) \tag{4}$$

我們假設 $x(t) = e^{-\gamma t}u(t)$ (5)

是第(3)式之解。 (6)

$$\frac{dx}{dt} = e^{-\gamma t}\frac{du}{dt} - \gamma e^{-\gamma t}u \tag{7}$$

$$\frac{d^2x}{dt^2} = e^{-\gamma t}\frac{d^2u}{dt^2} - \gamma e^{-\gamma t}\frac{du}{dt} + \gamma^2 e^{-\gamma t}u - \gamma e^{-\gamma t}\frac{du}{dx} \tag{8}$$

將(7),(8)代入(4)式得

$$e^{-\gamma t}\left(\frac{d^2u}{dt^2} + (\frac{k}{m} - \gamma^2)u\right) = 0 \tag{9}$$

要解的方程式變成

$$\frac{d^2u}{dt^2} + (\frac{k}{m} - \gamma^2)u = 0 \tag{10}$$

(1)當 $\frac{k}{m} - \gamma^2 > 0$ 時

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$$\frac{d^2u}{dt^2} = -\left(\frac{k}{m}-\gamma^2\right)u \tag{11}$$

即是簡諧運動之微分方程式。其解為

$$u(t) = A\sin(\omega t+\alpha) \tag{12}$$

也即是說

$$x(t) = Ae^{-\gamma t}\sin(\omega t+\alpha) = Ae^{-t/2\tau}\sin(\omega t+\alpha) \quad \tau\text{是鬆弛時間}\ \tau=\frac{1}{2\gamma} \tag{13}$$

$\gamma = \frac{\lambda}{2m}$, $\omega = \sqrt{\frac{k}{m}-\gamma^2}$ 是第(3)式之解。此處 A, α 是兩常數，由初位移及初速度決定。

(2) 當 $\frac{k}{m}-\gamma^2<0$ 時，第(10)式變成

$$\frac{d^2u}{dt^2} = \left(\gamma^2-\frac{k}{m}\right)u \tag{14}$$

其解為

$$u(t) = Ae^{\omega' t}+Be^{-\omega' t} \tag{15}$$

$\omega' = \sqrt{\gamma^2-\frac{k}{m}}$。也即是此時

$$x(t) = e^{-\gamma t}\left(Ae^{+\sqrt{\gamma^2-\frac{k}{m}}\,t}+Be^{-\sqrt{\gamma^2-\frac{k}{m}}\,t}\right) \tag{16}$$

為第(3)式之解。此處 A, B 是兩常數，由初位移及初速度決定。在此時，我們稱之為超阻尼振盪。

(3) 當 $\frac{k}{m}-\gamma^2=0$ 時，第(10)式變成

$$\frac{d^2u}{dt^2} = 0 \tag{17}$$

其解為

$$u(t) = A+Bt \tag{18}$$

也即是此時

$$x(t) = e^{-\gamma t}(A+Bt)$$

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為第(3)式之解，此處 A, B 也是常數，由初位移及初速度決定。此時我們稱之為臨界阻尼振盪。

討論

1. 這個公式只在速度較小時成立

2. 我們做此轉換的理由為 (a) 由日常生活的經驗中得知，當有阻力時，其振幅隨時間急速減小，這是物理的理由。(b) 經此轉換後，u 滿足之公式比 x 滿足的公式簡單，此點可由第(10)式中無一度微分可看出，這是數學的理由。

應用

我們在此將討論低阻尼時，能量耗損的情形。低阻尼即是 $\frac{k}{m} \gg \gamma^2$，此時 $\omega \approx \omega_0 = \sqrt{\frac{k}{m}}$

$\omega_0^2 \gg \gamma^2 = \frac{1}{4\tau^2}$，因此低阻尼的條件又可寫成 $\omega_0 \tau \gg 1$.

此時

$$x(t) \approx A e^{-t/2\tau} \sin(\omega_0 t + \alpha) \tag{19}$$

動能是 $$K.E = \frac{1}{2} m \dot{x}^2 \tag{20}$$

將 (19) 式微分後平方帶入 (20) 得

$$K.E. = \frac{1}{2} m \Big[\left(\frac{1}{2\tau}\right)^2 A^2 e^{-t/\tau} \sin^2 \omega_0 t + \omega_0^2 A^2 e^{-t/\tau} \cos^2 \omega_0 t - \left(\frac{\omega_0}{\tau}\right) A^2 e^{-t/\tau} \sin \omega_0 t \cos \omega_0 t \Big] \tag{21}$$

我們現在要算在 t 到 $t+T = \frac{2\pi}{\omega_0}$ 之間，動能之平均值

$$\langle K.E. \rangle_t = \frac{1}{T} \int_t^{t+T} (K.E)\, dt \tag{22}$$

因為 $\omega_0 \tau \gg 1$，所以在 t 到 $t+T$ 之間 $e^{-t/2\tau}$ 之變化很小，因此可以當它是常數而在積分時抽出來

$$\langle K.E. \rangle_t = \frac{1}{2} m \Big[\left(\frac{1}{2\tau}\right)^2 A^2 e^{-t/2\tau} \langle \sin^2 \omega_0 t \rangle + \omega_0^2 A^2 e^{-t/\tau} \langle \cos^2 \omega_0 t \rangle - \left(\frac{\omega_0}{\tau}\right) A_0^2 e^{-t/\tau} \langle \sin \omega_0 t \cos \omega_0 t \rangle.$$

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$$= \frac{1}{4} m \left[\left(\frac{1}{2\tau}\right)^2 + \omega_0^2 \right] A^2 e^{-t/\tau} \qquad (23)$$

$\left(\frac{1}{2\tau}\right)^2 \ll \omega_0^2$，所以上式可寫成

$$\langle K.E.\rangle_t = \frac{1}{4} m \omega_0^2 A^2 e^{-t/\tau} \qquad (24)$$

同理 $\langle P.E.\rangle_t = \frac{1}{2} \underbrace{m\omega_0^2}_{k} A^2 \langle e^{-t/\tau} \sin^2 \omega_0 t \rangle$

$$\simeq \frac{1}{4} m\omega_0^2 A^2 e^{-t/\tau} \qquad (25)$$

功率的消耗等於能量的減少率

$$\langle P\rangle_t = -\frac{d}{dt}\langle E\rangle_t \approx +\frac{1}{\tau}\left(\frac{1}{2} m\omega_0^2 A^2 e^{-t/\tau}\right) = +\frac{\langle E\rangle_t}{\tau} \qquad (26)$$

另外一種算法是

$$P = -\langle \underset{\substack{\uparrow \\ \text{阻力對振盪器所作之功率}}}{F_f} v \rangle_t = \left\langle -\frac{m}{\tau}\frac{dx}{dt}\frac{dx}{dt} \right\rangle \simeq \frac{\langle E\rangle_t}{\tau} \qquad (27)$$

另外一個常用的量叫做性質係數 Q 其定義為

$$Q = 2\pi \frac{\text{儲存之能量}}{\text{週期中能量之損耗}} = 2\pi \frac{E}{PT} = \frac{E}{P/\omega} \qquad (28)$$

由 (26), (28) 兩式可得當 $\omega_0 \tau \gg$ 時

$$Q \simeq \frac{E}{E/\omega\tau} \simeq \omega_0 \tau \qquad (29)$$

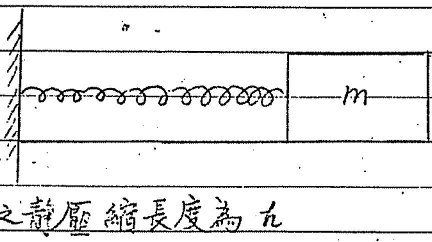
Q 越大則阻尼越小。

習題

1. 一物体之質量為 m 置於一桌面

上方一彈簧相連 同時我們也

知道有阻力存在，我們觀察到



以下的結果，(i) 當加以 mg 的力彈簧之靜壓縮長度為 h

(ii) 阻力在物体以 u 之速度進行時與 mg 相等

(a). 將運動方程式寫出來 [B]

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若 $u = 3\sqrt{gh}$

(b) 此一減幅振動之角頻率為何？ [C]

(c) 經多少時間後其能量是原來的 $\frac{1}{e}$？ [F]

(d) 此一振盪器之 Q 值為何？ [D]

(e) 若此振盪器原來是在靜止位置。一子彈在 $t=0$ 時打入質量而使振盪器開始運動。子彈之質量可略去不計，但其動量卻不能忽略。求 $x = Ae^{-\gamma t}\cos(\omega t - \delta)$ 中 δ 之值。 [A]

2. 若一人跳水後，跳板之振幅在 0.8 秒間由 14 cm 變成 8 cm 求跳板的 γ 之值 [E]

答案

[A] $\delta = \frac{\pi}{2}$

[B] $F = -\lambda v - kx$

$$kh = mg \Rightarrow k = mg/h$$

$$\lambda u = mg \Rightarrow \lambda = mg/u$$

$$m\frac{d^2x}{dt^2} = -\frac{mg}{u}\frac{dx}{dt} - \frac{mg}{h}x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{g}{u}\frac{dx}{dt} + \frac{g}{h}x = 0$$

[C] $\left(\frac{35g}{36h}\right)^{1/2}$　[D] $Q = 3$　[E] $\gamma = 0.7\ s^{-1}$

[F] $3\sqrt{\frac{h}{g}}$

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第三節 強迫振盪

簡介 在上一節中我們加一振盪的外力於此節中所討論的振盪器強迫它振盪，我們討論此一系統的性質

基本觀念

若外力 $F = F_0 \cos \omega_f t$，則該質點的運動方程式為

$$m \frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + kx = F_0 \cos \omega_f t \qquad (1)$$

利用上節所用之定義 $\gamma = \frac{\lambda}{2m}$， $\omega_0^2 = \frac{k}{m}$，上式可寫成

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_f t \qquad (2)$$

我們假設[2,3]

$$x(t) = A' \sin(\omega_f t - \alpha') \qquad (3)$$

將第(3)式代入第(2)式得

$$-A'\omega_f^2 \sin(\omega_f t - \alpha) + 2\gamma A'\omega \cos(\omega_f t - \alpha) + \omega_0^2 A' \sin(\omega_f t - \alpha) = \frac{F_0}{m} \cos \omega_f t \qquad (4)$$

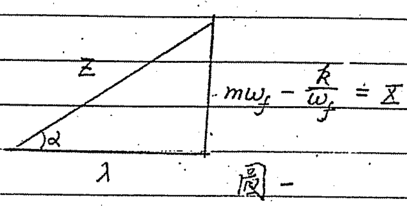
$$A'(\omega_0^2 - \omega_f^2)[\sin \omega_f t \cos\alpha - \cos \omega_f t \sin\alpha) + 2A'\gamma\omega_f [\cos \omega_f t \cos\alpha + \sin \omega_f t \sin\alpha]$$

$$= \frac{F_0}{m} \cos \omega_f t \qquad (5)$$

此式在 $\omega_f t = (n+\frac{1}{2})\pi$ 時應成立，因此

$$A'(\omega_0^2 - \omega_f^2) \cos\alpha + 2A'\gamma\omega_f \sin\alpha = 0 \qquad (6)$$

$$\tan\alpha = \frac{\omega_f^2 - \omega_0^2}{2\gamma\omega_f} = \frac{m\omega_f - \frac{k}{\omega_f}}{\lambda} \qquad (7)$$



圖一

定義[4] $m\omega_f - \frac{k}{\omega_f} = X$ (力抗)　　$\lambda = R$ (力阻)，

$$\underset{\text{機械阻抗}}{Z} = \sqrt{R^2 + X^2} = \sqrt{\lambda^2 + (m\omega_f - \frac{k}{\omega_f})^2} \qquad (8)$$

則 $\tan\alpha' = \frac{X}{R}$ 　　$(\alpha' = \tan^{-1}\frac{X}{R})$ (9)

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第(5)式在 $\omega_f t = n\pi$ 時也應成立，因此

$$-A'(\omega_0^2 - \omega_f^2)\sin\alpha + 2A'\gamma\omega_f\cos\alpha = \frac{F_0}{m} \quad (10)$$

$$A'\omega_f\left(m\omega_f - \frac{k}{\omega_f}\right)\sin\alpha + A'\omega_f\lambda\cos\alpha = F_0 \quad (11)$$

由圖一，可看出 $\left(m\omega_f - \frac{k}{\omega_f}\right) = Z\sin\alpha$ ， $\lambda = Z\cos\alpha$ (12)

代入(11)式可得

$$A'\omega_f Z = F_0 \quad \Rightarrow \quad A' = \frac{F_0}{\omega_f Z} \quad (13)$$

(9)，(13) 兩式完全決定了 $x(t)$。

將第(3)式對 t 微分則得[5]

$$v = \frac{F_0}{Z}\cos(\omega_f t - \alpha) = v_0\cos(\omega_f t - \alpha) \quad (14)$$

$$v_0 \equiv \frac{F_0}{Z} \quad (15)$$

$P = Fv$ =由外力轉移至振盪器之功率

$$= F_0\cos\omega_f t \; v_0\cos(\omega_f t - \alpha)$$

$$= \frac{F_0^2}{Z}\left(\cos^2\omega_f t\cos\alpha - \cos\omega_f t\sin\omega_f t\sin\alpha\right) \quad (16)$$

$$\langle P\rangle = \frac{F_0^2}{2Z}\cos\alpha = \frac{F_0^2}{2Z}\frac{R}{Z} = \frac{F_0^2 R}{2Z^2} \quad (17)$$

由上式顯然地可看出 $\langle P\rangle_{max}$ 對應於 Z 為最小時。由圖一中可看出這對應於 $\alpha = 0$ 時，也即是 $Z = R$ $\left(m\omega_f = \frac{k}{\omega_f} \Leftrightarrow \omega_f = \omega_0\right)$ 時，此時

$$\langle P\rangle_{max} = \frac{F_0^2}{2R} \quad (18)$$

討論

1. ω_0 是該振盪器的自然頻率。
2. 這個解在微分方程式中稱為特殊解。此一問題之完整解在 $\frac{k}{m} > \gamma^2$ 時是

$$x(t) = Ae^{-\gamma t}\sin(\omega t + \alpha) + A'\sin(\omega_f t - \alpha') \quad (19)$$

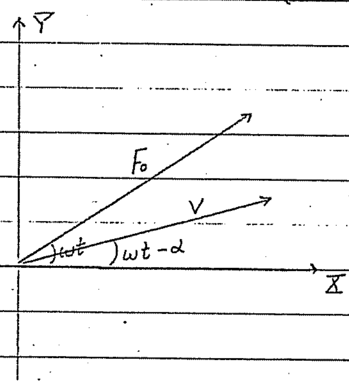
在第(19)式中 A', α' 是定值，由第(9)式及第(13)式所決定。A, α 是兩常數是取決於初位移及初速。

3. 在第(19)式中第一項的振幅隨時間的增加而減小，而第二項之振幅則與時間無關，因此當時間夠大時，則只剩第二項。第一項稱為暫時性的，以後我們將討論只剩第二項的情形。

4. 這些定義與在電磁學中之定義相似。

5. 以轉動向量來代表 F 及 v 之關係圖如下

這些轉動向量在 X 軸之投影即代表 F 及 v.



應用：

(一) 首先我們先將符號及公式整理一下

$$\gamma = \frac{\lambda}{2m} = \frac{1}{2\tau} \tag{20}$$

$$\omega_0^2 = \frac{k}{m} \tag{21}$$

$$R = \lambda = 2m\gamma = \frac{m}{\tau} \tag{22}$$

$$X = m\omega_f - \frac{k}{\omega_f} = m\omega_f \left[1 - \frac{k}{m\omega_f^2}\right] = m\omega_f \left(1 - \frac{\omega_0^2}{\omega_f^2}\right) \tag{23}$$

$$Z^2 = \lambda^2 + m^2\left(\omega_f - \frac{\omega_0^2}{\omega_f}\right)^2 = \frac{m^2}{\tau^2} + m^2\left(\omega_f - \frac{\omega_0^2}{\omega_f}\right)^2 \tag{24}$$

(二) 我們現在來討論共振問題

(a) 由第(13)式

$$A' = \frac{F_0}{\omega_f Z} =$$

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因此當 $\omega_f Z$ 為極小時，A' 為極大。

$$\omega_f Z = m\left[\left(\frac{\omega_f}{\tau}\right)^2 + (\omega_f - \omega_0^2)^2\right]^{1/2}$$

$\frac{d}{d\omega_f}\omega_f Z = 0$ 時，也即是 $\omega_f^2 = \omega_0^2 - \frac{1}{2\tau^2}$ 時，A' 為極大，稱為振幅共振。當 $\omega_0\tau \gg 1$ 時，共振在 $\omega_f \simeq \omega_0$ 處發生。

(b) 由第 (17) 式

$$\langle P\rangle = \frac{F_0 R}{2Z^2}$$

可以看出當 Z^2 為極小值時，由外力傳遞至振盪器上之功率為最大，稱為能量共振

$\frac{d}{d\omega_f} Z^2 = 0$，也即是 $\omega_f = \omega_0$ 時產生能量共振。

(c) 由第 (17) 式及第 (24) 式可得

$$\langle P\rangle|_{\omega_f} = \frac{F_0\tau}{2m\left[1 + \frac{(\omega_f - \omega_0)^2}{(\omega_f/\tau)^2}\right]} \qquad (25)$$

$$\langle P\rangle_{max} = \frac{F_0\tau}{2m}$$

定義 $\omega_\pm$ 為當 $\omega_f = \omega_\pm$ 時 $\langle P\rangle$ 為 $\frac{1}{2}\langle P\rangle_{max}$，$\omega_+ > \omega_0$，$\omega_- < \omega_0$

由 (25) 式中很容易的可以看出發生於 $\frac{(\omega_f^2 - \omega_0^2)^2}{(\omega_f/\tau)^2} = 1 \Rightarrow \frac{\omega_f^2 - \omega_0^2}{\omega_f/\tau} = \pm 1 \qquad (26)$

當 $\omega_f \simeq \omega_0$，第 (26) 式可寫成

$$2\omega_0(\omega_\pm - \omega_0) = \pm\,\omega_0/\tau$$

$$\omega_\pm = \omega_0 \pm \frac{2}{\tau} \qquad (27)$$

ω_+ 及 ω_- 之距離稱為此共振的半最高點之寬度 $(\Delta\omega)_{\frac{1}{2}}$

$$(\Delta\omega)_{\frac{1}{2}} = \frac{1}{\tau} = 2\gamma \qquad (28)$$

γ 小時尖峯較銳

由上節中我們發現當 $\omega_0\tau \gg 1$ 時

$$Q \simeq \omega_0\tau = \frac{\omega_0}{(\Delta\omega)_{1/2}}$$

因此 Q 與調諧的銳度有密切的關係。

(三) 在能量共振時，由外加力傳遞至振盪器的能量為最大，日常生活中最熟悉的例子

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編號：5
總號：

是收音機的選台。所有的電台都在使收音機產生強迫振動，但當我們選台時，即是使收音的電振盪器有一固定的自然頻率。當此頻率與電台之頻率一致時，其能量的吸收最大，因此我們就只聽到此一電台的信號。

很多系統在受一週期性之外力時，即進行強迫振盪，這些系統通常有其自然頻率。當外加力之頻率與此系統之自然頻率相符時，該系統吸收之能量為最大。因此我們可以變化外力之頻率來觀察何時該系統吸收之能量最大，由此來決定該系統之自然頻率。

	系統	外力	結果
(1)	分子中原子可以在其平衡點附近振動，通常形成一耦合振盪系統，具有其自然頻率	電場	找出分子之振動波譜
(2)	原子中之電子可看成具有自然頻率的振盪器	電場	
(3)	晶体如 NaCl 是由正負離子所組成的。當外加一週期之電場，正,負離子間產生振動	電場	找出離子間振動之自然頻率 ~ 5×10^{12} Hz

共振觀念之推廣：在某種情況下由一系統與另一系統間之能量傳遞最容易發生（我們現在並不要求此一系統可以用強迫振盪器來描述）。此一觀念在物理中有極重要之地位。

習題

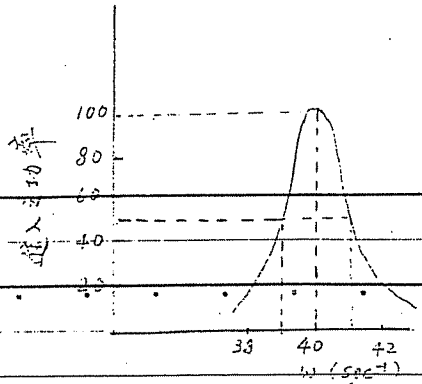
(1) 一物体之質量是 2kg 掛在一質量可略去不計的彈簧上。當物体掛在彈簧上時，彈簧伸長了 2.5 cm。彈簧的頂端在作簡諧運動其振幅為 1 mm。此一系統之 Q 值為 15

(a) 求此系統之 ω_0。 [H]

(b) 求強迫振盪在 $\omega=\omega_0$ 時之振幅 [B]

(2) 下圖中所示乃是某一系統受一 $F_0 \sin\omega t$ （F_0 = 常數，ω 是變數）之功率與 ω 之關係

分類：
編號：
總號：

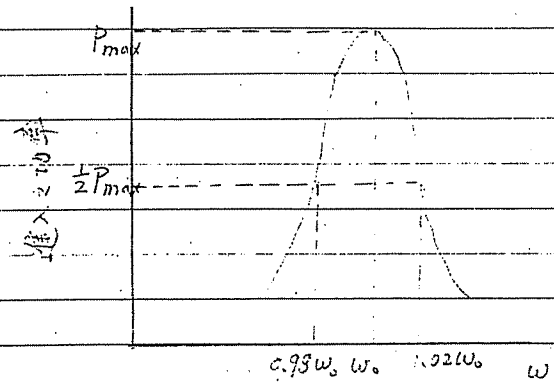


(a) 求此系統之 ω_0 值 [D]

(b) 求此系統之 Q 值 [F]

(c) 將外力關掉多久後經過多少自由振動（沒有外力）週期後，此一系統之能量為其最初值之 $\frac{1}{e^5}$? [J]

(3) 右圖所示的是由一質量及有阻力所組成系統，所組成系統之入射功率與策動頻率間之關係。



（策動力 $= F_0 \sin \omega t$，此處 F_0 為定值而 ω 是變數）

(a) 求 Q 值 [A]

(b) 若是去掉策動力則其能量按下列之公式減少

$$E = E_0 e^{-\gamma t}$$

γ 之值為何？ [G]

(c) 當策動力去掉後每一週期所損失之能量之分數為何？ [I]

(4) 一自由振動之力系統之角頻率為 ω_1，當此一系統受一策動力為 $F_0 \cos \omega t$（F_0 不變，ω 為變數）時其功率共振之角頻率之寬度為 $\omega_1/5$

(a) 進入功率最大時之角頻率為何？ [C]

(b) 此系統之Q值為何？ [K]

(c) 此系統之 λ 值為何？（寫成 m, k 之函數） [E]

答案：

[A]	$Q = 25^{-1}$	[B]	1.5 cm	[C]	$1.005\,\omega_1$
[D]	40 sec^{-1}	[E]	$\sim 0.2\sqrt{mk}$	[F]	Q = 20
[G]	$\gamma = 0.04\,\omega_0$	[H]	19.6 sec^{-1}	[I]	0.08π
[J]	16	[K]	$Q \cong 5$		

$\sin(kx - \omega t)$

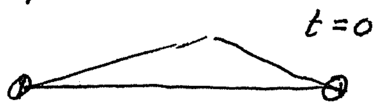
$= \sin kx \cos\omega t + \cos kx \sin\omega t$

Linear Combination.

趙凱華
李怡嚴

Mathematics

Wave Equation



$\underline{u(x,t)}$

Wave Equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = v^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$\left.\begin{aligned} u(x,0) &= u(x) \\ \left.\frac{\partial u}{\partial x}\right|_{t=0} &= v(x) \end{aligned}\right\} \text{initial condition}$$

Ansatz $\quad u(x,t) = \phi(t)\psi(x)$

$$\psi(x)\frac{d^2\phi}{dt^2} = \phi(t)\frac{d^2\psi}{dx^2}\cdot \underset{\substack{\uparrow \\ \text{constant}}}{v^2}$$

$$\underset{\substack{\downarrow \\ \text{function of } t}}{\frac{1}{\phi}\frac{d^2\phi}{dt^2}} = \underset{\substack{\downarrow \\ \text{function of } x}}{\frac{1}{\psi}\frac{d^2\psi}{dx^2}}\cdot v^2 = -\omega^2$$

$$\Rightarrow \quad \frac{d^2\phi}{dt^2} = -\omega^2\phi \quad \Rightarrow \quad \phi = C\sin\omega t + D\cos\omega t$$

$$\underset{\substack{\downarrow \\ \text{known}}}{\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{v^2}\psi} \quad \Rightarrow \quad \psi = E\sin\frac{\omega}{v}x + F\cos\frac{\omega}{v}x$$

The method is known as
seperation of variable

$$u(x,t) = [C\sin\omega t + D\cos\omega t]\cdot["E"\sin\frac{\omega}{v}x + "F"\cos\frac{\omega}{v}x]$$

The coefficients (constants) are to be dertermined by the initial condition and boundary conditions

$\psi(0, t) = 0$
u
$\psi(L, t) = 0$
u

$u(0, t) = 0 \Rightarrow$ "F" $= 0$

$u = (A \sin \omega t + B \cos \omega t) \sin \frac{\omega}{v} x$

C · "E"　　DE

$0 = u(L, t) = (A \sin \omega t + B \cos \omega t) \sin \frac{\omega}{v} L$

$\Rightarrow (\omega / v) L = n\pi$　　$n = 0, 1, 2, 3, \cdots$

$\omega_n = v \frac{n\pi}{L}$

$\psi_n(x, t) = u_n(x, t) = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin(\omega_n / v) x$

$u_n(x, t)$　　$n = 0, 1, \cdots$

$n = 1 \rightarrow$ first harmonic

$n = 2 \rightarrow$ second harmonic

satisfies the wave equation and the boundary condition

If $\{u_n(x, t)\}$ are solutions

then the general solution can be written as

constant

$$u(x, t) = \sum_n A_n' u_n(x, t)$$

$$= \sum_n A_n' (A_n \sin \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi}{L} x$$

$$= \sum (K_n' \cos \omega_n t + K_n'' \sin \omega_n t) \sin \frac{n\pi}{L} x$$

K_n', K_n'' are constants to be determined by initial conditions

$$u(x,0) = W(x) \quad \text{initial conditions}$$

$$\left(\frac{\partial u(x,t)}{\partial t}\right)_{t=0} = V(x)$$

$$u(x,0) = W(x) = \sum_n K_n' \sin \frac{n\pi}{L} x. \qquad (A)$$

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0 \quad \text{if } m \neq n$$

$$= \frac{L}{2} \quad \text{if } m = n.$$

Proof: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$A = \frac{n\pi}{L}, \quad B = \frac{m\pi}{L}$$

$$\frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right] dx$$

$$y \equiv \frac{\pi x}{L}$$

$$dy = \frac{\pi}{L} dx$$

$x = 0 \quad y = 0$

$x = L \quad y = \pi$

$$= \frac{1}{2}\frac{L}{\pi}\left[\int_0^\pi \cos \cancel{(n-m)} y \, dy + \int_0^\pi \cos(n+m) dy\right]$$

$\underline{n \neq m}$ $\quad n-m,\ n+m$ are all non-zero integrals

$\Rightarrow$ the integrals are all zero

if $n = m$ $\quad n-m = 0$, the first integral $= \pi$

$\Rightarrow$ thus the proof

$$\underline{\int_0^L} \underset{\underset{W(x)}{\shortparallel}}{u(x,0)} \sin\frac{m\pi}{L} \underline{dx} = \sum K_n' \underline{\int_0^L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx$$

$$= K_m'$$

$\downarrow$

have been found

Use the same technique to find K_m''

$$V(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum K_n'' \omega_n \sin\frac{n\pi}{L}x$$

$$\int_0^L V(x) \sin\frac{m\pi}{L}x = \int_0^L \left[\sum K_n'' \omega_n \sin\frac{n\pi x}{L}\right] \sin\frac{m\pi}{L}x \, dx$$

$$= K_m'' \omega_n \frac{L}{2}$$

$$\Rightarrow \quad K_m'' = \underbrace{\frac{2}{\omega_m L}}_{\text{known}} \int_0^L \cancel{W}(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[\sin\frac{n\pi}{L}x \int_0^L \frac{2}{L} W(x) \sin\frac{n\pi}{L} dx \right] \cos\omega_n t + \left[\sin\frac{n\pi}{L} x \int_0^L \frac{2}{\omega_n L} \overline{W}'(x) \sin\frac{n\pi}{L} dx \right] \sin\omega_n t \right.$$

$$\omega_n = v\frac{n\pi}{L}$$

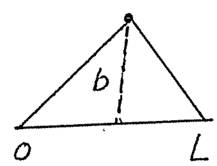
$$\sin \underbrace{\frac{n\pi}{L}x}_{A} \cdot \cos \underbrace{v\frac{n\pi}{L}t}_{B}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\frac{1}{2}\left[\sin \underbrace{\frac{n\pi}{L}(x+vt)}_{\text{function of } x+vt} + \cos \underbrace{\frac{n\pi}{L}(x-vt)}_{\text{function of } x-vt.}\right]$$

See P. 745 - P. 750 of 李怡嚴之"大學物理學



$$u(x) = \frac{2b}{L}x \qquad 0 < x < \frac{L}{2}$$

$$= \frac{2b}{L}(L-x) \qquad \frac{L}{2} < x < L$$

$$V(x) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$

Mathematics

Wave Equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \underset{\text{constant}}{v^2} \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \underline{\text{Linear}}$$

Seperation of variable

Ansatz : $u(x,t) = \underset{\phi(t)}{F(t)}\ \underset{\psi(x)}{G(x)}$

$$\frac{1}{\phi}\frac{d^2\phi}{dt^2} = \frac{1}{\psi}\frac{d^2\psi}{dx^2} v^2 = -\omega^2$$

$$\frac{d^2\phi}{dt^2} = -\omega^2\phi \quad \Rightarrow \quad \phi = C\sin\omega t + D\cos\omega t$$

$$\frac{d^2\psi}{dt^2} = -\frac{\omega^2}{v^2}\psi \quad \Rightarrow \quad \psi = C'\sin\frac{\omega}{v}x + D'\cos\frac{\omega}{v}x$$

$$u(x,t) = [C\sin\omega t + D\cos\omega t][C'\sin\frac{\omega}{v}x + D'\cos\frac{\omega}{v}x]$$

any ω can satisfy the wave equation

Linear $\Rightarrow$ any linear combination of above equation are solution of the wave equation

$u(0,t) = 0$ fixed end point

$u(L,t) = 0$

$D' = 0$, $\sin\omega t = 0$ $\boxed{\frac{\omega}{v}L = n\pi}$

$$\omega_n = \frac{v}{L}(n\pi)$$

only discrete values can satisfy the boundary condition

$$u_n(x,t) = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin(\omega_n/v)\, x$$

is a solution of the wave equation and the boundary condition

The general solution

$$u(x,t) = \sum_n (K_n' \cos \omega_n t + K_n'' \sin \omega t) \sin \frac{n\pi}{L} x$$

to be determined by the initial condition.

Initial Condition

$$u(x,0) = W(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=0} = \overline{W}(x)$$

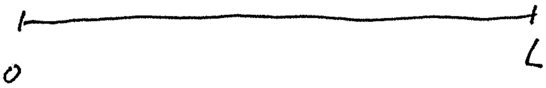
$\downarrow$

the solution of the problem

Comment

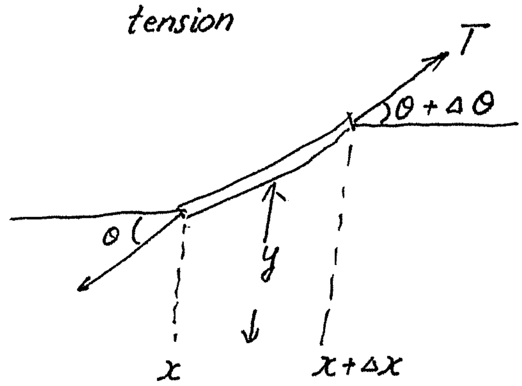
- Linear
- Boundary condition
- Initial condition
- Method of Seperation of Variable
- Fourier Series

Vibration String



μ, T ← tension

$y(x,t)$



Small

$$F_y = T\sin(\theta+\Delta\theta) - T\sin\theta$$

$$F_x = T\cos(\theta+\Delta\theta) - T\cos\theta$$

Transverse displacement y is small

$$F_y \simeq T\Delta\theta \quad ①$$

$$F_x \simeq 0$$

$$T\Delta\theta \simeq (\mu\Delta x)\, a_y \longrightarrow \text{Newton's second law}$$

$T\Delta\theta$ ↓ fixed t, $y(x)$

a_y ↓ fixed x, $y(t)$

The reason for partial derivatives

$$\tan\theta = \frac{\partial y}{\partial x}$$

$$\sec^2\theta\,\Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x \quad ②$$

$\sec^2\theta$ ↓ ① for small θ

$$a_y = \frac{\partial^2 y}{\partial t^2} \quad ③$$

$$\Rightarrow \quad T\frac{\partial^2 y}{\partial x^2}\Delta x = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$\downarrow$

combine ①, ②

$$\boxed{T\frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}}$$

$\parallel$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$$v = \sqrt{\frac{T}{\mu}}.$$

violen E string

$f = 640\ Hz$

$\omega \text{"}T\text{"} = 2\pi$

$\omega \frac{1}{f} = 2\pi$

$$\boxed{f = \frac{\omega}{2\pi}} \neq 640\ Hz$$

$$\omega = 2\pi \cdot 640\ Hz$$

$$\boxed{\omega_1 = v \,①\, \frac{\pi}{L}}$$

$$2\pi \cdot 640 = \sqrt{\frac{T}{\mu}}\ \frac{\pi}{L} \qquad \Rightarrow \sqrt{\frac{T}{\mu}}$$

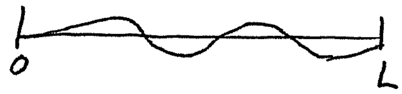
$$\mu = \frac{m}{L}$$

$L = 33\ cm$

$m = 0.125\ g$

$\Rightarrow \quad T \cong 68\ N$

Strected String



$\Rightarrow$ small oscillation in transverse direction

$\Rightarrow$ $\boxed{y(x,t)}$ described the transverse motion $\rightarrow$ wave function

Derivation of the Wave equation.

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} \qquad \boxed{v = \sqrt{\frac{T}{\mu}}}$$

This is the wave equation
Note it is linear.

Solution of the problem, mathematical problem

~~Calcula~~ Relation with Energy, Energy Density to go first.

Fundamental frequency of a string with both end fixed end at L

$$\boxed{\omega_n = v\frac{n\pi}{L}}$$

$\boxed{n=1}$ fundamental.

$$\boxed{\omega = \frac{v}{L}\pi}$$

application

$$= \frac{1}{L}\sqrt{\frac{T}{\mu}}\,\pi$$

Violen

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{L}\sqrt{\frac{T}{\mu}}\,\pi$$

$$= \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

$\downarrow$

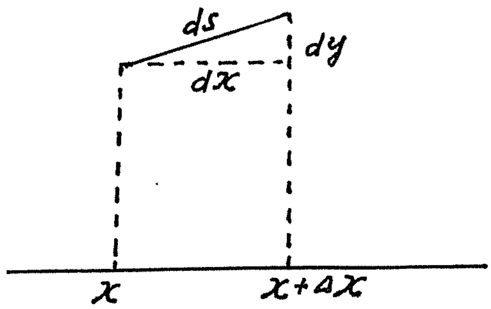
640 Hz

$L = 33$ cm

$\mu = \frac{m}{L}$ $\quad m = 0.125$ g

$\Rightarrow T \simeq 68$ N

The Energy in a Mechanical Wave



The mass of the small segment is $\mu\, dx$

The transverse velocity is $\frac{\partial y}{\partial t}$

$\Rightarrow$ for this segment the kinetic energy is $\frac{1}{2}\mu\, dx \left(\frac{\partial y}{\partial t}\right)^2$

$\Rightarrow$ kinetic energy per unit length $\equiv$ kinetic energy density

$$\frac{d(K.E)}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

The potential energy can be calculated by finding the amount by which the string, when deformed, is longer than when it is straight

The extension deformation . constant tension $\Rightarrow$ work done in the

$\Rightarrow$ for this segment potential energy $= T(ds - dx)$

$\boxed{y \leftrightarrow u}$

$$ds = ((dx)^2 + (dy)^2)^{1/2}$$

$$= dx\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}$$

$\frac{\partial y}{\partial x} \ll 1$

$$ds - dx \cong \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

potential energy $\simeq \frac{1}{2} T\left(\frac{\partial y}{\partial x}\right)^2 dx$

potential energy density $\equiv \frac{dE_p}{dx} \simeq \frac{1}{2} T\left(\frac{\partial y}{\partial x}\right)^2$

$\Rightarrow$ energy density $\cong \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2} T\left(\frac{\partial y}{\partial x}\right)^2$

Summary of Vibration String

1. Derivation of the Wave Equation

2. Find the wave function with the boundary condition + initial condition

$\downarrow$

linear Combination

$$(\sin kx \ , \ \cos kx) \cdot (\sin \omega t \ , \ \cos\omega t)$$

or

$$\sin kx \cos \omega t = \frac{1}{2}\left[\sin(kx+\omega t) + \sin(kx-\omega t)\right]$$

$$k = \frac{\omega}{v}$$

3. Given the wave function

$\downarrow$

calculate the physical interesting quantities

Need to calculate the energy of a segement

$$\boxed{\frac{K.E.}{\Delta x} = \frac{1}{2}\mu\left[\frac{\partial y}{\partial t}\right]^2}$$

clearly a function of x, t.

$$\frac{P.E.}{\Delta x} = \frac{1}{2} T\left(\frac{\partial y}{\partial x}\right)^2$$

$\tan\theta$

clearly a function of x, t

$\Rightarrow$ describe the energy distribution of the string vibration as a function of x, t.

Key of the wave phenomenon

- Distribuance

  $y(x,t)$ Elastic wave (Mechanical wave)

  Water wave
  Sound wave
  EM wave (Light)
  "Matter" wave

- Derivation of the wave equation

  Wave equation type.

Waves

Oscillations in space and time

$u(x,t)$

Transverse stretched string

Longitude sound wave

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

Note linear

Pulses of arbitrary shape

$$\underset{\updownarrow \atop u}{\cancel{z}}(x,t) = f(\underbrace{x \pm vt}_{z})$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial z}(\pm v)$$

$$\frac{\partial^2 u}{\partial t^2} = (\pm v)^2 \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}$$

$\Rightarrow$ the wave equation is satisfied.

$\cancel{A}^{c} \sin(kx - \omega t)$

$+ (\cancel{\sin})(kx + \omega t)$

$= \underline{2\sin(kx)\cos(\omega t)}$

$\cancel{\cos}$ $\left\{ \begin{array}{l} \sin A \cos B \\ = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \end{array} \right.$

$=$

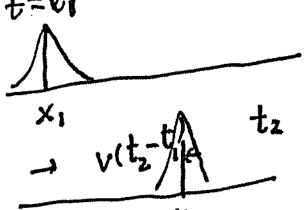
The same height $\quad x_2 - vt_2 = x_1 - vt_1 = z$

$x_2 = x_1 + v(t_2 - t_1)$

$f(z)$ the same $\downarrow$ the same height.

moving with velocity

$t = t_1$



x_1

$\rightarrow v(t_2 - t_1)$ $\quad t_2$

x_2

$f(x - vt)$ going to the right with velocity v

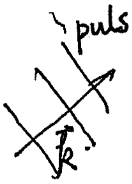
$f(x + vt)$ going to the left with velocity v

$f(z)$ function can be ~~Progress~~ found at $t = 0$

$f(x - vt)$

travelling wave

pulse



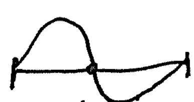
harmonic pulses

$$z(x,t) = z_0 \cos(k(x \pm vt) + \phi)$$

$$\boxed{k = \frac{\omega}{v}}$$

$\boxed{f(x)\,g(t)}$

$A \sin k$



it is

$$\underline{z(x,t)} = A\sin(kx - \omega t)$$

$$= A \sin kx \cos \omega t - A \cos kx \sin \omega t$$

Standing wave

$A \sin(kx - \omega t)$

$+ A \sin(kx + \omega t)$

$=$

$$P = \vec{F} \cdot v_{tra} = -T \frac{\partial z}{\partial x}\frac{\partial z}{\partial t}$$

$$= \mu v \omega^2 A^2 \cos^2(kx - \omega t)$$

notice the difference

$v \neq v_{trans}$

$v = \sqrt{\frac{T}{\mu}}$

$\tan\theta \sim \theta \sim \frac{\Delta y}{\Delta x}$

$T \sin\theta - T \sin\theta$

$T \Delta\theta = \vec{F}$

$\boxed{\frac{\omega}{v} = k}$ → useful definition

wave ~~numbe~~

This is an example of the Mechanical Wave

one dimensional

linear

General Discussion

Wave — $\boxed{\text{Oscillation}}$ in space, time

what to describe

origin — Mechanical Wave
- String
- Elastic rod
- Sound

Ocean Wave
⋮

EM Wave

Matter Wave

transverse $y(x,t)$

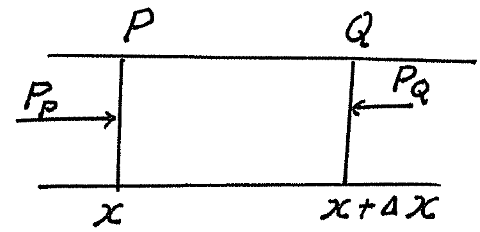
longitudinal $\xi(x,t)$ → sound wave

Wave equation

All partial differential equation involved ~~distrib~~

disturbance in space - time

Sound Wave



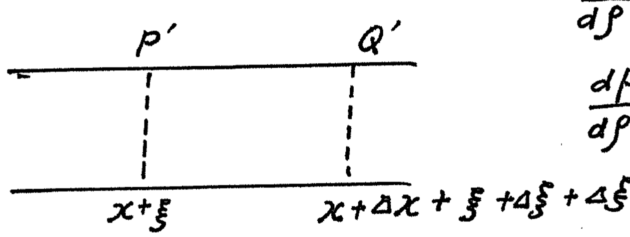
$$\Delta F = \underset{\text{area}}{A}\,[-p(x+\Delta x) + p(x)]$$

$$= -A \frac{\partial p}{\partial x} \Delta x$$

$$\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x} A \Delta x$$

$$\Rightarrow \rho \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x} \simeq -\frac{dp}{d\rho}\frac{\partial \rho}{\partial x}$$

$$= \frac{dp}{d\rho}\rho_0 \frac{\partial^2 \xi}{\partial x^2}$$



$$\rho(\Delta x + \Delta \xi)A = \rho_0 \Delta x A$$ continuity equation

$$\rho = \rho_0 \frac{1}{1+\frac{\Delta \xi}{\Delta x}}$$

$$\simeq \rho_0 \left[1 - \frac{\partial \xi}{\partial x}\right]$$

$$\frac{\partial \rho}{\partial x} = -\rho_0 \frac{\partial^2 \xi}{\partial x^2}$$

$$\downarrow$$

$$\frac{\partial^2 \xi}{\partial t^2} = \left(\frac{\partial p}{\partial \rho}\right)\frac{\partial^2 \xi}{\partial x^2}$$

Adiabatic expansion of gas

$$pV^\gamma = C \qquad p = \frac{C}{m^\gamma}\rho^\gamma = k\rho^\gamma$$

$$\frac{dp}{d\rho} = \gamma k \rho^{\gamma-1} = \frac{\gamma k p}{\rho}\frac{\rho^\gamma}{p} = \frac{\gamma k p}{\rho}\frac{1}{k} = \frac{\gamma p}{\rho}$$

$$\Rightarrow \frac{\partial^2 \xi}{\partial t^2} = \frac{\gamma p}{\rho}\frac{\partial^2 \xi}{\partial x^2}$$

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$p = 1$ atm

$\gamma = \frac{C_p}{C_V} = 1.41$

$\rho = \frac{28}{22.4}$ Kg/m^3

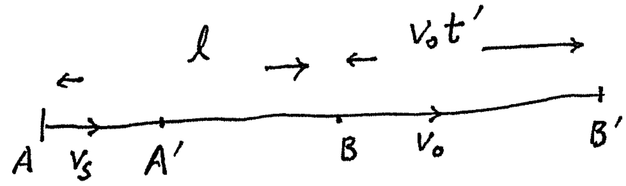
$\Rightarrow \quad v \sim 331$ m/sec

longitudinal wave

Doppler effect

V_o = velocity of the observer
observer relative to the medium

V_s = velocity of the source
source relative to the medium.



$AB = l$

$\boxed{t=0}$ $\begin{cases} \text{source at } A \\ \text{observer at } B \end{cases}$

Emit a wave
$\downarrow$
reach the observer at t

observer $V_o t$

$Vt = \boxed{l + V_o t}$

$$\boxed{t = \frac{l}{V - V_o}}$$

the first signal arrive at the observer

$t = \tau$ source is at A'
$\downarrow$
received by B'

reach the receiver at t'

$(l - V_s \tau) + V_o t'$
actual distance travelled

$\boxed{V(t' - \tau)} = (l - V_s \tau) + V_o t'$

$$\boxed{t' = \frac{l + (V - V_s)\tau}{V - V_o}} \Leftarrow (V - V_o)t' = l + V\tau - V_s\tau$$

$Vt' - V\tau$
$= l - V_s\tau + V_o t'$

$0 \to \tau$ $\quad \nu\tau$ number of pulse
$\tau' = t' - t$ $\quad \parallel$
$\nu'\tau'$ number of pulse receved.

$$\nu' = \frac{\tau}{\tau'} \nu =$$

emitted frequency

$$\tau' = t' - t = \frac{l + (V - V_s)\tau}{V - V_0} - \frac{l}{V - V_0}$$

$$= \frac{(V - V_s)\tau}{(V - V_0)}$$

$$\boxed{\nu' = \frac{\tau}{\tau'} \nu}$$

observed by the observer — emitted by the source

pitch of moving ve

$$= \frac{\cancel{\tau}}{\frac{(V - V_s)\cancel{\tau}}{V - V_0}} \nu = \frac{V - V_0}{V - V_s} \nu$$

V_s f

This is equation 14-58

The other

Use of the Doppler effect

sonic bome

$$\boxed{V = V_s}$$

~~shock~~ waves

Cerenkov radiation.

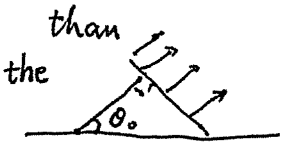
Light in vacuum c

Light in medium $v = \frac{c}{n}$ $\boxed{n > 1}$

if charged particle moving through a medium faster than

differential counter → vary the n

threshold counter



$$\cos\theta_0 = \frac{v}{v_s}$$

v → velocity in the medium

v_s → source

v could be the velocity of the electron ~~med~~

Chapter 15

Wave function and Wave Equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$$

<u>Linear</u> $\quad \xi_1 , \xi_2$ are solution

$a\xi_1 + b\xi_2$ is also a solution

$$\frac{\partial^2 \xi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi_1}{\partial t^2}$$

$$\frac{\partial^2 \xi_2}{\partial x^2} = \frac{1}{v}\frac{\partial^2 \xi_2}{\partial t^2}$$

$$\frac{\partial^2}{\partial x^2}(a\xi_1 + b\xi_2) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(a\xi_1 + b\xi_2)$$

$$a\left[\frac{\partial^2 \xi_1}{\partial x^2}\right] + b\underline{\underline{\frac{\partial^2 \xi_2}{\partial x^2}}} = a\frac{1}{v^2}\frac{\partial^2}{\partial t}\xi_1 + \underline{\underline{b\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\xi_2}}$$

This wave equation is <u>linear</u>

See Problem 15 - 59

$$\frac{\partial \psi_1}{\partial t} = 2\psi_1\frac{\partial \psi_1}{\partial t} + 3\psi_1$$

$$\frac{\partial \psi_2}{\partial t} = 2\psi_2\frac{\partial \psi_2}{\partial t} + 3\psi_2$$

$$\frac{\partial}{\partial t}(a\psi_1 + b\psi_2) \overset{?}{=} 2(a\psi_1 + b\psi_2)\frac{\partial}{\partial t}(a\psi_1 + b\psi_2) + 3(a\psi_1 + b\psi_2)$$

no!

We shall only discuss the wave equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} \quad (y)$$

$y(x - vt)$ is a travelling wave moving toward right

$y(x + vt)$ " " " " " " left

Chapter 14

$\downarrow$

harmonic waves

$\downarrow$

idealization

Superposition and interference

$\downarrow$

understand more complex waves in terms of simple harmonic waves

Different ways waves can be combined

Superposition Principle

Wave equations

Coherence

$\downarrow$

An interference pattern occurs only when the waves have a definite, stable relation between their frequencies and phase

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = A\sin(kx - \omega t) = A\sin k\left(x - \frac{\omega}{k}t\right)$$

$$= A\sin k(x - vt)$$

$$v = \frac{\omega}{k}.$$

Harmonic wave

$$y(x,t) = A\sin(kx-\omega t) = A\sin k(x - \underbrace{\tfrac{\omega}{k}}_{= v} t)$$

$$A\cos(kx-\omega t)$$

Sinusodial wave

$$y = \underbrace{A}_{\text{amplitude}} \cos(\underbrace{\omega t - kx}_{\theta} + \underbrace{\phi}_{\text{phase}})$$

$$y_1 = A_1\cos(\theta+\phi_1)\ , \quad y_2 = A_2\cos(\theta+\phi_2)$$

addition of similar Sinusodial Waves

Addition of Similar Sinusodial Waves

$$y = \underset{\text{amplitude}}{\underset{\downarrow}{A}} \cos(\omega t - kx + \underset{\text{phase}}{\underset{\uparrow}{\phi}})$$

$$y(x,t) = y_1 + y_2 = A_1 \cos(\theta + \phi_1) + A_2 \cos(\theta + \phi_2)$$

$$\underline{\theta = \omega t - kt}$$

$$\cos(\theta \pm \phi) = \cos\theta \cos\phi \mp \sin\theta \sin\phi$$

$$y = (A_1 \cos\phi_1 + A_2 \cos\phi_2) \cos\theta - (A_1 \sin\phi_1 + A_2 \sin\phi_2) \sin\theta$$

$$y = C \cos(\underbrace{\omega t - kx}_{\theta} + \delta)$$

$$= C \cos\delta \cos\theta - C \sin\delta \sin\theta$$

$$C \cos\delta = A_1 \cos\phi_1 + A_2 \cos\phi_2$$

$$C \sin\delta = A_1 \sin\phi_1 + A_2 \sin\phi_2$$

$$\Rightarrow \quad C^2 = (A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2$$

$$\tan\delta = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}$$

$\Rightarrow$ Any two sinusoidal function superimpose into yet another sinusoidal function, different from the original components, only in amplitude and phase

$$C^2 = A_1^2 (\sin^2\phi_1 + \cos^2\phi_1) + A_2^2 (\sin^2\phi_2 + \cos^2\phi_2) + 2A_1A_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$$

$$= A_1^2 + A_2^2 + \underbrace{2A_1A_2 \cos(\phi_1 - \phi_2)}_{\text{interference}}$$

note the importance of phase difference

$$\phi_1 = \phi_2 \Rightarrow C^2 = A_1^2 + A_2^2 + 2A_1A_2 \rightarrow \text{maximum}$$

$$\phi_1 - \phi_2 = \pi \qquad C^2 = A_1^2 + A_2^2 - 2A_1A_2 \rightarrow \text{minimum.}$$

Generalize to any number of components

$$y = \sum_i y_i = \sum_i A_i \cos(\omega t - kx + \phi_i)$$

$$y = C \cos(\omega t - kx + \delta)$$

$$C^2 = \left(\sum_i A_i \cos\phi_i\right)^2 + \left(\sum_{i=1}^{n} A_i \sin\phi_i\right)^2$$

$$\tan\delta = \frac{\sum_i A_i \sin\phi_i}{\sum_i A_i \cos\phi_i}$$

Any number of sinusoidal waves in superposition constitute still another sinusoidal wave

The Energy of Two Waves

The intensity of any sinusoidal wave is proportional to the square of the amplitude

Two such wave (different only in amplitude and phase

$$C^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_1 - \phi_2)$$

Average intensity over any number of complete cycles

$$I = I_1 + I_2 + \underbrace{2\sqrt{I_1 I_2} \cos(\phi_1 - \phi_2)}_{\text{interference}}$$

$\phi_1 - \phi_2 = \frac{\pi}{2}$ waves appear independent of one another

$$I = I_1 + I_2$$

If $A_1 = A_2$

$$I = 2I_1 [1 + \cos(\phi_1 - \phi_2)]$$

$\phi_1 - \phi_2 = 0$ $4I_1$

$\phi_1 - \phi_2 = \pi$ 0

appearance or disapparence of the wave energy

↓

Young's interference

↓

coherence

Interference of two oppositely travelling waves

$$y = y_1 + y_2 = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

$$= 2A\cos kx \cos \omega t$$

fixed points of zero distance

$$kx_n = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \cdots$$

$$f_n(x) = A_n \cos(k_n x + \phi_n)$$

$$f_n(x) = f_n(x + 2L) \quad \text{periodic.}$$

$$k_n(2L) = 2n\pi$$

$$\Rightarrow \quad k_n = \frac{n\pi}{L}$$

Fourier Series.

$$F(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x + \phi_n)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Fourier's theorem

$$\frac{n\pi x}{L} = n\theta$$

$$\theta = \frac{\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_x^{x+2L} F(x)\,dx$$

$$a_n = \frac{1}{L}\int_x^{x+2L} F(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_x^{x+} F(x)\sin\frac{n\pi x}{L}\,dx$$

$$\int_\theta^{\theta+2\pi} \sin n\theta \sin m\theta \, d\theta = \pi\delta_{n,m}$$

$$\int_\theta^{\theta+2\pi} \sin n\theta \cos m\theta \, d\theta = 0$$

$$\int_\theta^{\theta+2\pi} \cos n\theta \cos m\theta \, d\theta = \pi\delta_{m,n}$$

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

Define $\quad k_1 = \frac{\pi}{L}$

$$\cos nk_1 x = \frac{1}{2}\left(e^{ink_1 x} + e^{-ink_1 x} \right)$$

$$\sin nk_1 x = \frac{i}{2}\left(e^{-ink_1 x} - e^{ink_1 x} \right)$$

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_1 x}$$

$$c_0 = a_0$$

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2}$$

$$c_n = \frac{1}{2L} \int_x^{x+2L} F(x) e^{-ink_1 x} dx$$

$$\Rightarrow \quad F(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{ink_1 x} \int_{-L}^{L} F(x') e^{-ink_1 x'} dx'$$

$$k_1 \to \frac{\pi}{L}$$

$$F(x) = \lim_{L\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} k_1 \int_{-L}^{L} F(x') e^{-ink_1 (x'-x)} dx'$$

$$\Delta k = k_{n+1} - k_n = \frac{\pi}{L} = k_1$$

$$k_n = nk_1$$

$$\lim_{L\to\infty} nk_1 = \lim_{L\to\infty} k_n = k$$

$$\sum_{n=-\infty}^{\infty} \Delta k \, G(k_n)$$

$$F(x) = \lim_{L\to\infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta k \int_{-L}^{L} F(x') e^{-ik_n (x'-x)} dx'$$

$$= \underbrace{\frac{1}{2\pi}}_{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} dk \Big[\underbrace{\int_{-\infty}^{\infty} F(x') e^{-ikx'} dx'}_{= G(k)} \Big] e^{ikx}$$

$$\Rightarrow \quad F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ikx} dk$$

Fourier integral

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-ikx} dx$$

$$\int_{-\infty}^{\infty} e^{ikx}\, e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha}$$

$$\parallel$$

$$\int_{-\infty}^{\infty} \cos kx\, e^{-\alpha x^2} dx + i \underbrace{\int_{-\infty}^{\infty} \sin kx\, e^{-\alpha x^2} dx}_{0}$$

even - odd
argument

$$I_1 = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$I_2^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy$$

$$= \int_{-\infty}^{+\infty} e^{-\alpha(x^2+y^2)} dx\, dy$$

$$= \int_0^{\infty} r\, dr \int_0^{2\pi} d\theta\, e^{-\alpha r^2} r\, dr$$

$$= 2\pi \int_0^{\infty} e^{-\alpha r^2} r\, dr$$

$$z = \alpha r^2$$

$$dz = 2r dr \cdot \alpha$$

$$= 2\pi \int_0^{\infty} \frac{1}{2} e^{-z} \frac{dz}{\alpha}$$

$$= \frac{\pi}{\alpha} \int_0^{\infty} e^{-z} dz$$

$$= \frac{\pi}{\alpha} \left[- e^{-z} \right]_0^{\infty} = \frac{\pi}{\alpha}$$

$$I_1^2 = \frac{\pi}{\alpha}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$I_2 = \int_{-\infty}^{\infty} \cos kx \; e^{-\alpha x^2} dx$$

$$\cos kx = 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \cdots$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi}\frac{d}{d\alpha} \alpha^{-\frac{1}{2}}$$

$$\shortparallel$$

$$-\int_{-\infty}^{\infty} e^{-\alpha x^2} x^2 dx = \sqrt{\pi}\left(-\tfrac{1}{2}\right) \alpha^{-3/2}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-\alpha x^2} x^2 dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

Take derivative with respect to α again

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} x^4 dx = \frac{3}{2^2 \alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$I_2 = \sqrt{\frac{\pi}{\alpha}} \left[1 - \frac{k^2}{4\alpha} + \frac{3k^4}{\alpha^2 4! 2^2} - \frac{3\cdot 5 k^6}{\alpha^3 6! 2^3} + \cdots \right]$$

$$= \sqrt{\frac{\pi}{\alpha}} \left[1 - \frac{k^2}{4\alpha} + \left(\frac{k^2}{4\alpha}\right)^2 \Big/ 2! - \left(\frac{k^2}{4\alpha}\right)^3 \Big/ 3! + \cdots \right]$$

$$\downarrow$$

$$e^{-k^2/4\alpha}$$

$$F(x) = e^{-\alpha x^2} \qquad\qquad \sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha}$$

$$e^{-\tilde{k}_1^2/4\alpha} = e^{-1}$$

$$\tilde{k}_1^2 / 4\alpha = 1$$

$$\boxed{\tilde{k}_1 = 2\sqrt{\alpha}}$$

$$\Delta k = 4\sqrt{\alpha}$$

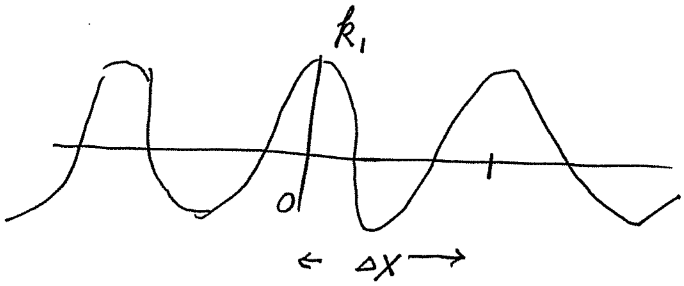
$$e^{-\alpha \tilde{x}^2} = e^{-1}$$

$$\tilde{x}^2 = \frac{1}{\alpha}$$

$$\Delta x = \frac{2}{\sqrt{\alpha}}$$

$$\Delta k \Delta x \sim O(1)$$

uncertainty principle



$x > \Delta x$ want the wave to "vanish" or small

The sinuodinal wave with k_1 outside must be destructively interfered by a sinuoidal wave with wave number k_2 such that the phase difference

$$k_1 \Delta x - k_2 \Delta x \sim \pi$$

$$\boxed{(\Delta k)(\Delta x) \sim \pi}$$

uncertainty priniple

Δx smaller

Δk must grow

$\boxed{p \sim \hbar k}$ in matter wave

$\boxed{\Delta p \Delta x \gtrsim h}$

Heisenberg's uncertainty principle.

The solution of the problems for Chapter 14 - 16 is on line

Final Jan 10 7:00 pm

Chapter 11 – 16

Thursday

$$F(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk$$

$$F(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x,0)\, e^{-ikx}\, dx.$$

Energy associated with the wave

$$\int_{x}^{x+2L} [F(x)]^2 \, dx$$

$$= \int_{x}^{x+2L} \left[\left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] \left[a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \right] dx$$

$$\sim a_0^2 (2L) + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) L$$

↓

Fourier Energy theorem.

Advantages of use Fourier Series Expansion.

Problem 46 of Chapter 15

Applicable to any periodic function.
(even with finite discontinunity)

Similar to Taylor expansion

Addition of two different k values

$$v = \frac{\omega}{k}$$

So far we have consider the case v = constant

k vary

ω, v may either one or both

$$y_1 = A e^{i[(k+dk)x - (\omega + d\omega)t]}$$

$$y_2 = A e^{i[(k-dk)x - (\omega - d\omega)t}$$

$$y_1 = y_1 + y_2 = A e^{i(kx - \omega t)} \underbrace{\left[e^{i(dkx - d\omega t} + e^{-i(dkx - d\omega t}\right]}_{2\cos(dkx - d\omega t}$$

Real of y

$$y = 2A \cos(dkx - d\omega t) \cos(kx - \omega t)$$

waves propagating with group velocity $\frac{d\omega}{dk}$

moving with phase velocity v_p

No dispersive medium

$$v = \frac{\omega}{k} \Rightarrow \omega = ck$$

$$\frac{d\omega}{dk} = v$$

The group velocity is the same as the phase velocity

Dispersive medium

$$v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk}$$

$$= v + k \underbrace{\frac{dv}{dk}}_{<0}$$

v is a function of $k = \frac{2\pi}{\lambda}$

Example different color light propagating with different velocity

$$v_g < v_p$$

The energy carried by the wave would propagate at the group velocity

11'

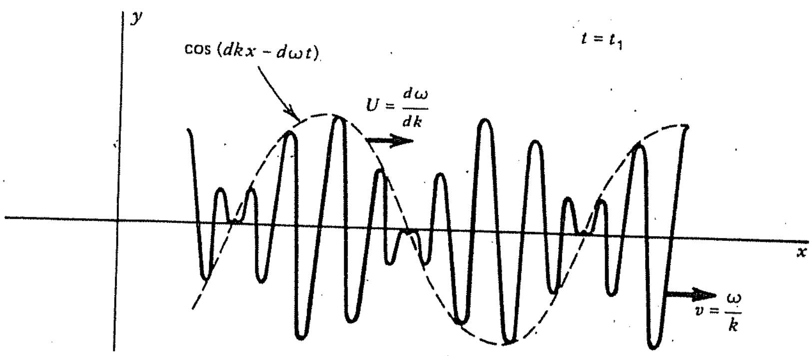


Figure 3.18. The sum of two waves of slightly differing angular frequency and wave number.

Gaussian wave packet

$$A(k) = \sigma\, e^{-(k-k_0)^2\sigma^2/2}$$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(k-k)^2\sigma^2/2}\, e^{i(kx-\omega t)}\, dk$$

$$\omega(k) \simeq \underbrace{\omega_0 + \left(\frac{d\omega}{dk}\right)_{k_0}(k-k_0)}_{\text{keep}} + \frac{1}{2}\frac{d^2\omega}{dk^2}(k-k_0)^2 + \cdots$$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}}\, e^{i(k_0x-\omega t)} \int_{-\infty}^{\infty} e^{-(k-k_0^2)\sigma^2-(k_0-k)(x-\upsilon t)}\, dk$$

Let $a^2 = \sigma^2/2$ $\qquad \upsilon = \left(\frac{d\omega}{dk}\right)_{k_0}$

$b = -i(x-\upsilon t)$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}}\, e^{i(k_0x-\omega_0 t)}\, \frac{e^{b^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= e^{i(k_0x-\omega_0 t)}\, e^{-(x-\upsilon t)^2/2\sigma^2}$$

center of the Gaussian is moving with group velocity $\frac{d\omega}{dk}$