縱波
聲波 will be discussed in Chapters 14 and 15
電磁波 will be discussed after electromagnetism.

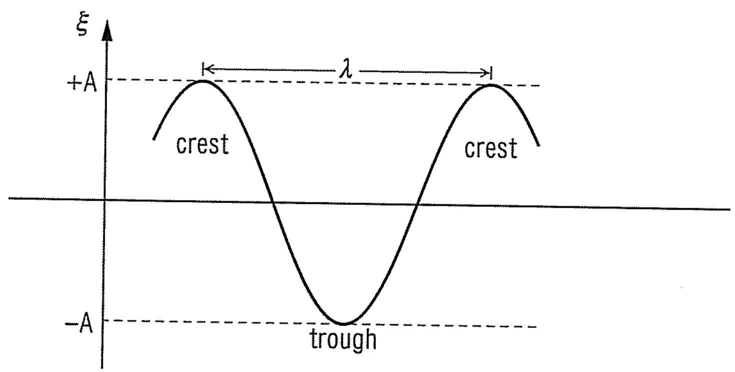
Wave Equation and its Solutions

- Waves → oscillations in space and time
  - $y(x, t)$
  - Transverse or longitudinal waves
  - Traveling or standing waves
- Solutions to wave equation
  - Pulses of arbitrary shape → $y(x, t) = f(x \pm vt)$
  - Harmonic pulses → $y(x, t) = y_0 \cos(k(x \pm vt) + \phi)$
  - Separable solutions → $y(x, t) = f(x) \cos(\omega t + \phi)$

PHYSNET

MISN-0-201

THE WAVE EQUATION AND ITS SOLUTIONS



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THE WAVE EQUATION AND ITS SOLUTIONS

by
William C. Lane
Michigan State University

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Input Skills:

1. Vocabulary: displacement, velocity (MISN-0-7); frequency, angular frequency, period (MISN-0-25).
2. Take the derivative of $\sin(kx+\omega t)$ with respect to t while treating x as a constant whose derivative is zero; alternatively with respect to x while treating t as a constant ((MISN-0-1).

Output Skills (Knowledge):

K1. Vocabulary: amplitude, wavelength, wave number, phase, phase constant, wave function, wave speed, wave equation, harmonic function, sinusoidal wave, traveling wave, boundary conditions, field.

K2. State the one-dimensional wave equation and its general solution.

K3. Take the partial derivative of $\sin(kx + \omega t)$ with respect to either x or t.

Output Skills (Rule Application):

R1. Given a wave function for a one-dimensional traveling wave, verify that it satisfies the wave equation.

Output Skills (Problem Solving):

S1. Given a sufficient number of parameters associated with a sinusoidal wave, write down the mathematical description of the traveling wave.

S2. Determine the unknown parameters of a one-dimensional sinusoidal wave, given its displacement as a function of either: (i) position at two different times; or (ii) time at two different positions.

S3. Determine the unknown parameters of a one-dimensional sinusoidal wave, given the wave function and its first derivative with respect to time at $x = 0$ and $t = 0$.

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New authors, reviewers and field testers are welcome.

THE WAVE EQUATION AND ITS SOLUTIONS

by

William C. Lane
Michigan State University

1. Overview

Waves and vibrations in mechanical systems constitute one of the most important areas of study in all of physics. Evidence of the existence of these phenomena can be observed for almost any kind of physical system. The propagation of sound and light, ocean waves, earthquakes, the transmission of signals from the brain are a few examples. In this unit we introduce the descriptors of waves and their motions: periodicity, amplitude, propagation speed, etc. We relate these to symbols in the differential form of the wave equation and in its formal solutions. We also relate these descriptors to the properties of some simple physical systems.

2. The Wave Function

2a. Graphical and Mathematical Representation. Waves are represented mathematically by a wave function that may be expressed as a graph or a formal function. This function describes the disturbance made by the wave at various times as it propagates in space. Because the wave function depends on both position and time, when we wish to draw a graph of the wave we usually keep one variable fixed. For example, Fig. 1

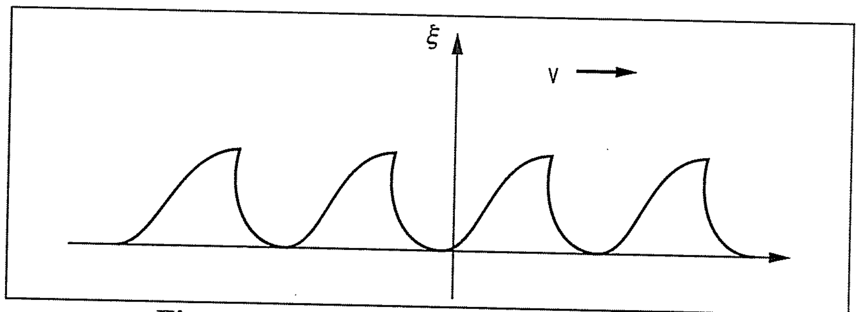


Figure 1. A "snapshot" of a water wave showing the wave profile at a given instant of time. The vertical axis indicates the displacement of water from its average level.

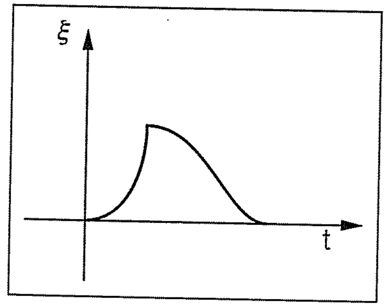


Figure 2. Vertical displacement of a floating object as a water wave passes.

shows the profile of a series of water waves at a given instant of time. You can think of this as a "snapshot" of the wave. Alternatively we can examine the time variation of the wave at a specific point x. If we look at a piece of driftwood as the waves pass, it will bob up and down, executing a periodic motion. The vertical displacement of the driftwood may also be represented graphically, as illustrated in Fig. 2. The exact mathematical form of the wave function, $\xi = f(x,t)$, depends on the type of wave that is being considered.

2b. Traveling Waves. One particular form of wave function, $\xi = f(x - vt)$, corresponds to the "traveling wave," the most common type of wave we encounter. A traveling wave consists of a periodic series of oscillations of some quantity that travel, or "propagate," through space with a speed characteristic of the wave and the medium through which it travels. The expression $\xi = f(x - vt)$ represents a one-dimensional traveling wave propagating in the positive x-direction with speed v. If the oscillations in the medium are simple harmonic oscillations, the functional form of the wave function is also harmonic. It is important to convince yourself that when $f(x)$ is a function representing some curve then the same function f, but with (x) replaced by $(x - a)$, represents a curve with the same shape but shifted along the positive x-axis by an amount "a" (see Fig. 3). Similarly $f(x + a)$ represents that curve shifted in the negative x-direction by an amount "a."

Try it with some simple, easy to compute function.[1] Understanding this is fundamental to understanding the mathematical description of wave motion. Replacing "a" by something linear in time, $a = vt$, gives you a curve that propagates either to the right or the left depending on the sign of v.

[1] For example, $f = ax$ or $f = C \sin bx$.

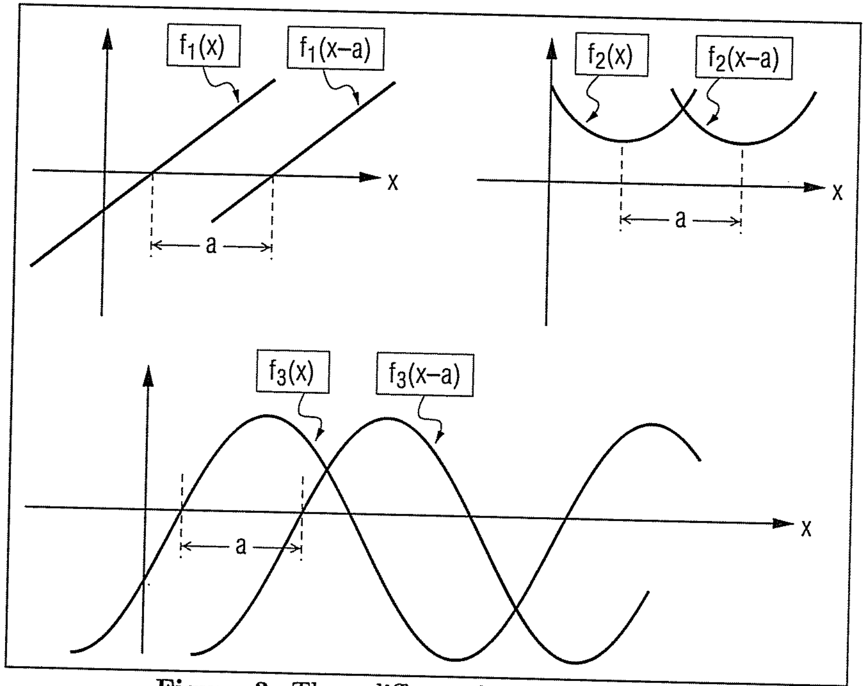


Figure 3. Three different functions shifted to the right by an amount "a": the accompanying replacement of (x) by $(x-a)$ is a general property of all functions.

3. Description of the Wave Motion

3a. The Harmonic Wave Function. A harmonic wave function is sinusoidal in functional form and for a one-dimensional wave may be expressed as either:

$$\xi = A \sin\left[\frac{2\pi}{\lambda}(x \pm vt) + \phi_0\right], \tag{1}$$

or

$$\xi = A \cos\left[\frac{2\pi}{\lambda}(x \pm vt) + \phi_0\right]. \tag{2}$$

This means that the profile of the wave at a particular time is a sine or cosine function, as shown in Fig. 4, and that at a particular point in space, the wave produces a simple harmonic oscillation in some quantity ξ, as indicated in Fig. 5. The symbols in Eqs. (1) and (2) are explained in the remaining paragraphs of this section. The parameters A and ϕ_0 are

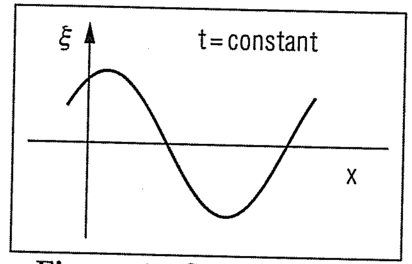


Figure 4. Snapshot of a harmonic wave function at some specified time t.

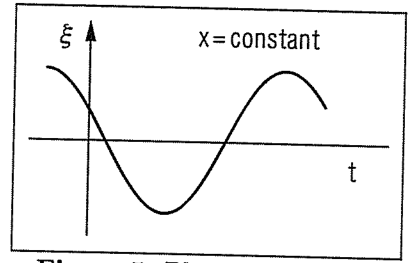


Figure 5. Plot of the same wave as in Fig. 4, but at a specified point x.

constants determined by the initial displacement and initial velocity of ξ at some point in space.

3b. Phase and Phase Constant. The argument of a harmonic function is called the "phase" of the wave, ϕ. For a one-dimensional traveling wave:

$$\phi(x,t) = \frac{2\pi}{\lambda}(x \pm vt) + \phi_0\,. \tag{3}$$

The phase describes the part of a complete wave oscillation that is occurring at a given place and time. The constant ϕ_0 is called the "phase constant" and is the value of ϕ at $x = 0$, $t = 0$. The units of the phase are radians when 2π occurs in Eq. (3). To express ϕ in degrees, replace 2π radians with its equivalent, 360°.

3c. Amplitude, Wavelength, and Wave Number. The maximum value that $\sin\phi$ or $\cos\phi$ may take is ± 1, so the maximum wave disturbance ξ is:

$$\xi_{\rm max} = \pm A\,. \tag{4}$$

This maximum value of ξ is called the "amplitude" of the wave. The point where $\xi = +A$ is typically called the "crest" of the wave and the point where $\xi = -A$ is called the "trough" of the wave.[2] The distance from crest to crest (or trough to trough) is called the "wavelength," the distance between points on the wave which have the same phase at the same instant of time. Figure 6 illustrates these wave dimensions for a sinusoidal wave. A useful expression involving the wavelength is the definition of a quantity

[2]Notice that the designation of a wave maximum as a crest or a trough is somewhat arbitrary since the maxima at $\xi = +A$ are identical to the maxima at $\xi = -A$.

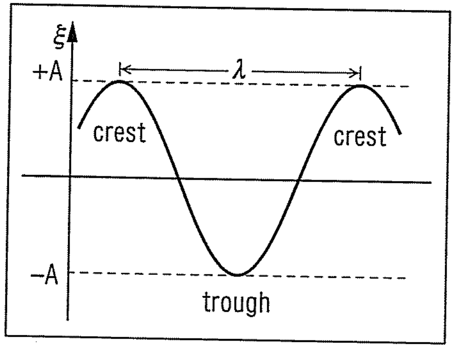


Figure 6. Amplitude and wavelength of a harmonic wave.

called the wave number, k, of a wave,

$$k = \frac{2\pi}{\lambda}. \tag{5}$$

Since wavelength has units of length and 2π radians is a dimensionless quantity, k has units of inverse length, usually m^{-1} or cm^{-1}. Using the wave number symbol, a one-dimensional sinusoidal wave function may be expressed as

$$\xi = A \sin\left[k(x \pm vt) + \phi_0\right]. \tag{6}$$

3d. Phase: Period, Frequency, Angular Frequency. There are a number of ways of writing the phase of a wave, depending on whether one uses period, frequency, or angular frequency. For a particle at a fixed point, undergoing simple harmonic motion, we can write the phase of its motion as

$$\phi = \left(\frac{2\pi}{T}\right) t + \phi_0, \tag{7}$$

where T is the period of the motion, and ϕ_0 is the initial phase. Comparing Eqs. (7) and (3), we see that the phase of a one-dimensional harmonic wave may be written as

$$\phi = 2\pi\left(\frac{x}{\lambda} \pm \frac{t}{T}\right) + \phi_0. \tag{8}$$

Using Eq. (8), the phase of a wave is easy to interpret: a change in position by one wavelength or a change in time by one period results in a change in phase of 2π radians (360°). The phase of a harmonic wave may also be expressed in terms of frequency or angular frequency. Using the relation between period and frequency

$$\nu = \frac{1}{T}, \tag{9}$$

the phase may be written as

$$\phi = 2\pi\left(\frac{x}{\lambda} \pm \nu t\right) + \phi_0, \tag{10}$$

or using the relation between frequency and angular frequency

$$\omega = 2\pi\nu \tag{11}$$

and the definition of wave number, Eq. (5), the phase becomes

$$\phi = kx \pm \omega t + \phi_0. \tag{12}$$

3e. Relations Among Traveling-Wave Descriptors. By comparing Eq. (3) and (8), a relation between the wave speed v, the wavelength λ, and the period T, may be determined to be:

$$v = \frac{\lambda}{T}. \tag{13}$$

This expression may be transformed into several equivalent forms by using the definition of frequency and angular frequency:

$$v = \lambda\nu, \tag{14}$$

and

$$v = \frac{\omega}{k}. \tag{15}$$

Using these relations between wave descriptors and their definitions, you should be able to transform between the several forms of the wave function we have encountered so far. These relations and the various forms of the harmonic wave function are summarized in Table 1. The variables used in Table 1 are listed in Table 2, although you should note that: (1) any part of a wave could be used in place of the word "crests"; and (2) the descriptions are only meant as reminders of the more complete descriptions given throughout the text.

Table 1. Useful wave relations and various one-dimensional harmonic wave functions. Remember that cosine functions may also be used as harmonic wave functions.

Wave Relations	One-Dimensional Wave Functions
$v = \frac{\lambda}{T}$	$\xi = A\sin\left[\frac{2\pi}{\lambda}(x \pm vt) + \phi_0\right]$
$v = \lambda\nu$	$\xi = A\sin\left[k\,(x \pm vt) + \phi_0\right]$
$v = \frac{\omega}{k}$	$\xi = A\sin\left[2\pi\left(\frac{x}{\lambda} \pm \frac{t}{T}\right) + \phi_0\right]$
$k = \frac{2\pi}{\lambda}, \nu = \frac{1}{T}$	$\xi = A\sin\left[2\pi\left(\frac{x}{\lambda} \pm \nu t\right) + \phi_0\right]$
$\omega = 2\pi\nu$	$\xi = A\sin\left[kx \pm \omega t + \phi_0\right]$

Table 2. Variables used in Table 1.

Variable	Brief Description
λ	wavelength: distance between successive crests at one time
T	period: time between successive crests at one place
ξ	wave function: the size of the wave at any time and place
A	amplitude: maximum value of the wave function
v	speed of each crest
t	the time at which the wave function is being described
ϕ_0	phase constant: the wave's phase at time zero, place zero
k	wave number: number of waves per unit length at one time
x	the place at which the wave function is being described
ω	angular frequency: 2π times the frequency
ν	frequency: rate at which crests go by at one place

4. The Equation of Wave Motion

4a. One-Dimensional Equation of Wave Motion . By applying Newton's second law and some forms of Hooke's law to the deformation ξ in an elastic medium, a differential equation of motion for ξ may be

derived.[3] If ξ is a one-dimensional traveling wave in the elastic medium, the differential equation of motion is found to be:

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2}\,. \qquad (16)$$

This partial differential equation[4] is called the "characteristic equation" of wave motion in one dimension. If, by using Newton's second law, you find for a physical system that the equation of motion is of this form, then you know that wave motion can result. From this differential equation you can read the propagation speed of any wave that obeys it.

4b. Forms of the General Solution. Since the wave equation is a second order linear partial differential equation, the general solution of the wave equation consists of a linear combination of two linearly independent harmonic functions:

$$\xi(x,t) = f_1(x \pm vt) + f_2(x \pm vt)\,. \qquad (17)$$

You should be able to verify Eq. (17) as a solution to Eq. (16), the wave equation, by direct substitution.

If the signs of the "vt" terms are the same in f_1 as in f_2, Eq. (17) represents a superposition of two waves traveling in the same direction.

If the signs of "vt" terms are opposite for the two functions in Eq. (17), we have the superposition of two waves traveling in opposite directions. With appropriate choices for boundary conditions, this particular solution to the wave equation is called a "standing wave" and it is a very important phenomenon in physics. It is treated elsewhere.[5]

4c. Restriction to Harmonic Waves. We can restrict Eq. (17) to a description of single-frequency harmonic waves by making one term a sine function and the other a cosine function (these functions are linearly independent):

$$\xi(x,t) = A\,\sin[k(x - vt)] + B\,\cos[k(x - vt)]\,. \qquad (18)$$

The two amplitudes A and B complete the description of particular waves. That is, specifying values for them picks out a specific case from all possible waves with the specified frequency and velocity already specified in

[3] For several examples of this derivation, see "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).

[4] For the meaning of "partial differential equation" and "partial derivative," as used in this module, see this module's Appendix.

[5] See "Standing Waves" (MISN-0-232) and "Standing Waves in Sheets of Materials" (MISN-0-233).

Eq. (18). For example, if the values of ξ and its first derivative with respect to time are known at some point in space and at some instant of time for the process at hand (usually at $x = 0$ and $t = 0$), then those values can be used to set A and B. As an alternative to Eq. (18), we can express ξ as a single sine or cosine function (more useful in certain situations):

$$\xi(x,t) = \xi_0 \sin[k(x - vt) + \phi_0]\,, \tag{19}$$

where Eqs. (18) and (19) are connected by:

$$\xi_0 = \left(A^2 + B^2\right)^{1/2} \text{ and } \phi_0 = \tan\left(\frac{B}{A}\right). \tag{20}$$

Either way we have two constants that must be established for any particular application.[6]

Acknowledgments

I would like to thank J. Kovacs, O. McHarris, and P. Signell for their contributions to an earlier version of this module. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **amplitude**: the maximum value of a wave function.
- **boundary conditions**: the values of the wave function and its first time-derivative, at some point in space, at some instant of time.
- **field**: a quantity that has a value at each point in space.
- **harmonic function**: a sinusoidal function; for example, $\sin(kx \pm \omega t)$ or $\cos(kx \pm \omega t)$.
- **partial derivative**: a derivative of an expression where only one quantity in the expression can vary.

[6] The requirement of two constants may not surprise you since the solution of a second order differential equation requires two sequential integrations, thus introducing two constants of integration. These two constants are called "boundary conditions" or "initial conditions," but both of those terms are misleading since the values used are not necessarily on any physical periphery nor are they necessarily values for the boundaries of the time interval being studied.

- **phase**: the argument of a harmonic wave function. The phase of a wave specifies what part of a complete oscillation or cycle the wave is producing at a given point in space and at a given time.
- **phase constant**: the value of the phase of a wave at time $t = 0$ at the origin of the relevant coordinate system.
- **sinusoidal wave**: a wave whose spatial profile at any given time is a sine function and which produces simple harmonic oscillations of the wave quantity ξ at any given point through which the wave passes.
- **traveling wave**: a periodic series of oscillations that propagate through space with a speed characteristic of the wave and the medium through which it travels.
- **wave equation**: a differential equation of motion whose solutions are mathematical representations of waves.
- **wave function**: a mathematical representation of a wave, and the solution to the wave equation.
- **wave number**: a quantity inversely proportional to the wavelength of a wave; symbolized by k. Note: $k = 2\pi/\lambda$.
- **wave speed**: the speed with which a wave propagates through space.
- **wavelength**: the distance in space between successive points of equal phase on a wave; symbolized by λ.

Partial Derivatives

a. Particles and Ordinary Derivatives. A "particle" is usually specified in part by its position, say x. The position of the particle usually changes with time so we write $x(t)$. Then the *rate* of change of the particle's position with respect to time is written $dx(t)/dt$, and that is its x-component of velocity. It is an example of the use of the ordinary derivative.

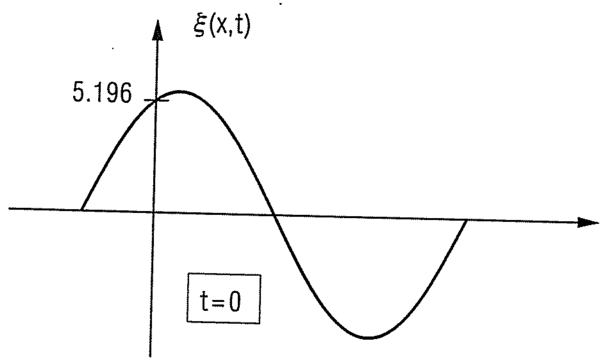
b. Fields: Temperature as an Example. A "field" is specified by its value at each space point (x) at each time (t). For example, the temperature of the air in this room can be written as $T(x,t)$: at a particular time t it has a value (in degrees) at each space point x as one moves along some straight line across the room. As time changes, the value at each

$T = 2 \times 10^{-3}\,\mathrm{s}$

$\nu = 500\,\mathrm{Hz}$

$\lambda = 0.66\,\mathrm{m}$

c. $\xi(x,t)$ at $t = 0$:



3. A wave described at two times:

   a. $A = 1.3\,\mathrm{m}$ *Help: [S-5]*

   b. $\lambda = 5.24\,\mathrm{m}$ *Help: [S-6]*

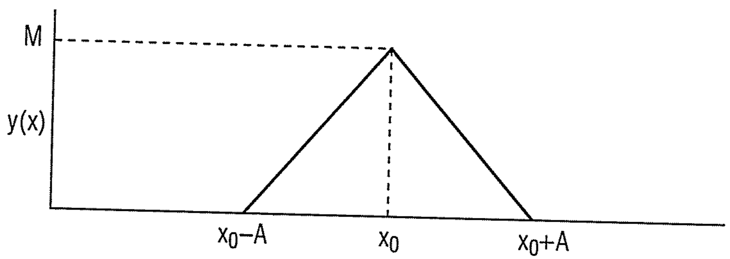
   c. $\nu = 0.50\,\mathrm{Hz}$ *Help: [S-4]*

   d. $v = 2.62\,\mathrm{m\,s^{-1}}$ *Help: [S-7]*

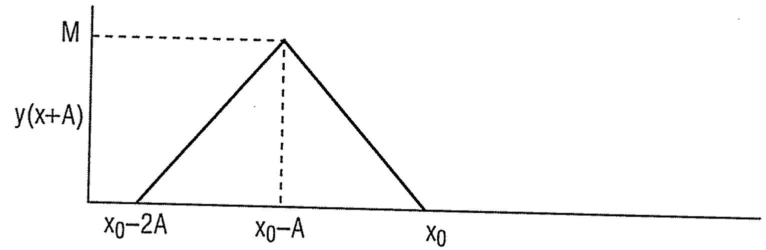
   e. $-\hat{x}$ direction *Help: [S-8]*

4. $y(x_0)$ is the maximum value of $y(x)$.

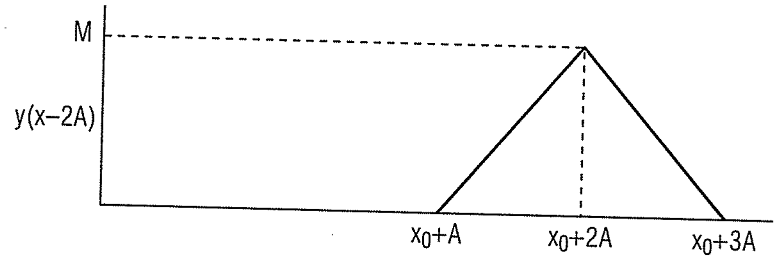
   a. $y(x)$ has its maximum value when $x = x_0$:



   b. $y(x + A)$ has its maximum value when $x + A = x_0$:



   c. $y(x - 2A)$ has its maximum value when $x - 2A = x_0$:



5. $k = \omega/v$; $kv = \omega$; $(\omega/v) = (2\pi/\lambda)$; $\omega = 2\pi\nu$; $\nu = 1/T$.

6. Determining the functional form.

   a. $\xi = (0.5\,\mathrm{m}) \sin\left[\left(\pi\,\mathrm{m}^{-1}\right)x - \left(4\pi\,\mathrm{s}^{-1}\right)t + \phi_0\right]$,

   or

   $\xi = (0.5\,\mathrm{m}) \cos\left[\left(\pi\,\mathrm{m}^{-1}\right)x - \left(4\pi\,\mathrm{s}^{-1}\right)t + \phi_0\right]$.

   b. $\phi_0 = 156.4^\circ$ if the sine function is used;
   $\phi_0 = 66.4^\circ$ if the cosine function is used. *Help: [S-9]*

7. A wave specified at two times:

   a. $A = 0.02\,\mathrm{cm}$; $\nu = 1.5\,/\mathrm{s}$; $\lambda = 8.0\,\mathrm{cm}$ *Help: [S-10]*

   b. The wave moves in the negative x direction with speed $v = 12\,\mathrm{cm/s}$.

8. Only (c) satisfies this equation.

9. $F(x,t) = x + (10\,\mathrm{Hz})\lambda t$

10. Displacements at two points:

    a. $1.5 \times 10^{-4}\,\mathrm{m}$

space point along the line changes. Note that there are an uncountable infinity of points along the line, at each of which the temperature can be specified at any one time. This is in contrast to a "particle," for which there is only one position at any one time.

c. A Mental Exercise. In your mind, go through the process of plotting an ordinary two-dimensional graph showing the temperature at all points x for a fixed time t. That is, plot $T(x)$ for a single time t, a "snapshot" of the temperature field. Now imagine plotting a separate graph showing the temperature at a single point (that is, at a single x value), as a function of time. Think about how the measurements would be made in each case.

d. Fields and Partial Derivatives: an Example. If we want to know the rate of change of temperature, T, with position along a line, x, *at a fixed time* t, we must take the derivative of T with respect to x while holding t fixed. This process of holding one variable fixed while taking the derivative with respect to another variable is called "taking a partial derivative." It is written with ∂ symbols replacing the usual d symbols in the derivative. Here are some examples of taking first and second partial derivatives of a particular function $f(x,t)$:

$$f = (x+vt)^2$$
$$\partial f/\partial x = 2(x+vt) \qquad \partial f/\partial t = 2v(x+vt)$$
$$\partial^2 f/\partial x^2 = 2 \qquad \partial^2 f/\partial t^2 = 2v^2$$

PROBLEM SUPPLEMENT

Note: Problems 8, 9, and 10 also occur in this module's *Model Exam.*

1. Given the function:

$$\xi(x,t) = A_1 \cos k_1(x+vt) - A_2 \sin k_2(x-vt)\,.$$

Determine whether this a solution to the wave equation, *Help: [S-1]*

$$v^2 \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2}\,.$$

2. Let $\xi(x,t) = A\sin(\omega t + kx + \phi_0)$ where ϕ_0 is the phase constant. This wave is traveling in the negative x-direction at the speed of sound in air, 330 m/s.

   a. Determine whether $\xi(x,t)$ satisfies the wave equation quoted in Problem 1.

   b. If $A = 6.0$ cm, $\xi(0,0) = 5.196$ cm and

$$\dot{\xi}(0,0) \equiv \left.\frac{\partial \xi}{\partial t}\right|_{\substack{x=0\\t=0}} = +94.2\,\text{m/s}\,,$$

   find ϕ_0, ω, k, T, ν, and λ.

   c. Sketch $\xi(x)$ at $t=0$.

3. A certain one-dimensional wave is observed at a certain instant of time to be described by:

$$\xi(x,t_1) = (1.3\,\text{m})\sin[(1.2\,\text{m}^{-1})x + 16\pi]$$

and 12 seconds later by:

$$\xi(x,t_1 + 12\,\text{s}) = (1.3\,\text{m})\sin[(1.2\,\text{m}^{-1})x + 28\pi].$$

Determine this wave's: (a) amplitude; (b) wavelength; (c) frequency (in hertz); (d) speed; and (e) direction of travel.

4. A function $y(x)$ consists of: (1) a straight line that increases from its value of zero at position $(x_0 - A)$ to its maximum value of M at position x_0; then (2) another straight line from this maximum value of M to be the value zero at position $(x_0 + A)$. Everywhere else, $y(x)$ is zero. Thus the function is a triangle in the x-y plane joining points $(x_0 - A, 0)$, (x_0, M) and $(x_0 + A, 0)$.

   a. Sketch the function $y(x)$.

   b. Sketch the function $y(x + A)$.

   c. Sketch the function $y(x - 2A)$.

5. Show that $\xi = \xi_0 \sin(kx - \omega t)$ may be written in the alternative forms:

   a. $\xi = \xi_0 \sin[k(x - vt)]$

   b. $\xi = \xi_0 \sin\left[\omega\left(\frac{x}{v} - t\right)\right]$

   c. $\xi = \xi_0 \sin\left[2\pi\left(\frac{x}{\lambda} - \nu t\right)\right]$

   d. $\xi = \xi_0 \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$

6. A one-dimensional sinusoidal wave, of wavelength 2 m, travels along the x-axis (in the positive x-direction). If its amplitude is 0.5 m and it has a period of $T = 0.5\,\mathrm{s}$:

   a. Write down an appropriate wave function to represent this wave.

   b. If the displacement of the wave is 0.2 m at $x = 0$, $t = 0$, and $\dot{\xi}(0,0) = +5.76\,\mathrm{m/s}$, find the phase constant.

7. A one-dimensional sinusoidal wave moves along the x-axis. The displacement at two points, $x_1 = 0$ and $x_2 = 2.0\,\mathrm{cm}$, is observed as a function of time:

$$\begin{aligned} \xi(x_1, t) &= (0.02\,\mathrm{cm}) \sin[(3\pi\,\mathrm{s}^{-1})t] \\ \xi(x_2, t) &= (0.02\,\mathrm{cm}) \sin[(3\pi\,\mathrm{s}^{-1})t + \frac{\pi}{2}] \end{aligned}$$

   a. What are the amplitude, frequency, and wavelength of this wave? *Help: [S-10]*

   b. In which direction and with what speed does the wave travel?

8. Verify whether or not each of the following functions is a solution to the one-dimensional wave equation:

   a. $\xi = \xi_0 \cos(\pi t)$

   b. $\xi = \xi_0 \sin M(x + 4vt)$ where M is a constant

   c. $\xi = Y(x - vt)$

9. A one-dimensional sinusoidal wave is traveling along the x-axis in the negative x-direction. It can be represented by:

$$\xi(x,t) = A\cos[\frac{2\pi}{\lambda}F(x,t)].$$

   The wave's frequency is 10 Hz and its wavelength is λ. Write down an appropriate function $F(x,t)$ which gives this wave the properties listed above.

10. The displacements at two points in space are observed as a wave $\xi(x,t)$ passes by. At the points $x_1 = 0.5\,\mathrm{m}$ and $x_2 = 2.5\,\mathrm{m}$ the displacement from equilibrium is observed as a function of time. These are found to be: $\xi(0.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m}) \sin[(6\pi\,\mathrm{s}^{-1})t]$
    $\xi(2.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m}) \sin[(6\pi\,\mathrm{s}^{-1})t + 2\pi/3]$

   a. What is the amplitude of this wave?

   b. What is the frequency of this wave in hertz?

   c. What is the wavelength?

   d. What is the speed with which this wave travels?

   e. What way is the wave traveling?

   f. What is the time rate of displacement at the point x_1 at times $t = 0$ and $t = 0.25\,\mathrm{s}$?

Brief Answers:

1. Yes *Help: [S-1]*

2. A traveling wave:

   a. Yes, if $\omega^2 = v^2 k^2$ (see Problem 1).

   b. $\phi_0 = \pi/3\,\mathrm{radians} = 60^\circ$ *Help: [S-3]*

   $\omega = 3.14 \times 10^3\,\mathrm{s}^{-1}$ *Help: [S-2]*

   $k = 9.5\,\mathrm{m}^{-1}$

b. 3 Hz

c. 6 m

d. 18 m/s.

e. Toward the negative x-direction

f. $9\pi \times 10^{-4}$ m/s and zero respectively.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 1)*

Taking the appropriate partial derivatives,

$$\begin{aligned}
\frac{\partial \xi}{\partial t} &= -k_1 v A_1 \sin\left[k_1(x+vt)\right] + k_2 v A_2 \cos\left[k_2(x-vt)\right] \\
\Rightarrow \frac{\partial^2 \xi}{\partial t^2} &= -k_1^2 v^2 A_1 \cos\left[k_1(x+vt)\right] + k_2^2 v^2 A_2 \sin\left[k_2(x-vt)\right] \\
\frac{\partial \xi}{\partial x} &= -k_1 A_1 \sin\left[k_1(x+vt)\right] - k_2 A_2 \cos\left[k_2(x-vt)\right] \\
\Rightarrow \frac{\partial^2 \xi}{\partial x^2} &= -k_1^2 A_1 \cos\left[k_1(x+vt)\right] + k_2^2 A_2 \sin\left[k_2(x-vt)\right]
\end{aligned}$$

Comparing the two equations marked $\Rightarrow$, it is clear that v^2 times the lower one equals the upper one. Thus ξ is a solution to the wave equation. However, if the velocity v in the two terms had not been the same, ξ would not have been a solution. Note that the first term represents a wave of amplitude A_1 and wavelength $2\pi/k_1$ traveling to the left, while the second term represents a wave of amplitude A_2 and wavelength $2\pi/k_2$ traveling to the right. These two waves are traveling through the same space points at the same time.

S-2 *(from PS-Problem 2b)*

$$\omega = \frac{\dot{\xi}(0,0)}{A\cos\phi_0} = \frac{94.2\,\mathrm{m/s}}{(6.0\,\mathrm{cm})(1/2)}$$

S-3 *(from PS-Problem 2b)*

$\xi(0,0) = A\sin(0+0+\phi_0) = A\sin\phi_0 = 5.196\,\mathrm{cm}$

$\sin\phi_0 = \dfrac{\xi(0,0)}{A} = \dfrac{5.196}{6.0} = 0.866,$

so: $\phi_0 = \sin^{-1}(0.866) = 60°$ ($\pi/3$ radians) or $120°$ ($2\pi/3$ radians)

To choose the correct value of ϕ_0, information from $\dot{\xi}(0,0)$ must be used:

$\dot{\xi}(x,t) \equiv \partial\xi/\partial t = \omega A\cos(kx+\omega t+\phi_0)$

$\dot{\xi}(0,0) = \omega A\cos\phi_0 = +94.2\,\mathrm{m/s}$

$\cos\pi/3 = +1/2$, $\cos(2\pi/3) = -1/2$.

Since $\dot{\xi}(0,0)$, ω, and A are all positive, $\cos\phi_0$ must be positive as well, so $2\pi/3$ is rejected as a possible value for ϕ_0, i.e. $\phi_0 = \pi/3$.

S-4 *(from PS-Problem 3c)*

Rule: $\Delta(\text{phase}) = 2\pi\left[\dfrac{\Delta x}{\lambda} \pm \dfrac{\Delta t}{T}\right]$.

For this case,

phase #1= $(1.2\,\mathrm{m}^{-1})x + 16\pi$

phase #2= $(1.2\,\mathrm{m}^{-1})x + 28\pi$

$\Delta(\text{phase}) = 12\pi$

but $\Delta x = 0$, hence: $2\pi\left(\dfrac{\Delta t}{T}\right) = 12\pi$, and $\Delta t = 12\,\mathrm{s}$.

S-5 *(from PS-Problem 3a)*

Just look at either $\xi(x,t_1)$ or $\xi(x,t_2)$.

S-6 *(from PS-Problem 3b)*

See Problem 1. $\lambda = 2\pi/(1.2\,\mathrm{m}^{-1})$.

S-7 *(from PS-Problem 3d)*

Compare to the general form of the wave equation to get:

$kv\Delta t = \Delta\phi \Rightarrow v = 12\pi/(1.2\,\mathrm{m}^{-1}\ 12\,\mathrm{s})$

S-8 *(from PS-Problem 3e)*

Read and understand this module's *text*.

S-9 *(from PS-Problem 6b)*

Note 1: $-203.6°$ is just as good an answer as $156.4°$.

Note 2: Electronic calculators face an ambiguity in giving an answer for an inverse trigonometric function. For example, $\sin(5.74°) = \sin(174.26°) = 0.1$. Therefore $\sin^{-1}(0.1)$ could be either $5.74°$ or $174.26°$: both are valid mathematical answers to the inverse sine problem, taken in isolation. You must decide which is the right answer by examining other aspects of the problem at hand.

S-10 *(from PS-Problem 7)*

Compare to the general form of the wave equation to get:

$(2\pi/\lambda)\Delta x = \Delta\phi \Rightarrow \lambda = 2\pi(2.0\,\mathrm{cm})/(\pi/2)$

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet.*
2. Verify whether or not each of the following functions is a solution to the one-dimensional wave equation:

   a. $\xi = \xi_0 \cos(\pi t)$

   b. $\xi = \xi_0 \sin M(x + 4vt)$ where M is a constant

   c. $\xi = Y(x - vt)$

3. A one-dimensional sinusoidal wave is traveling along the x-axis in the negative x-direction. It can be represented by:

$$\xi(x,t) = A\cos\left[\frac{2\pi}{\lambda}F(x,t)\right].$$

   The wave's frequency is 10 Hz and its wavelength is λ. Write down an appropriate function $F(x,t)$ which gives this wave the properties listed above.

4. The displacements $\xi(x,t)$ at points in space are observed as a wave passes by. At the points $x_1 = 0.5\,\text{m}$ and $x_2 = 2.5\,\text{m}$ the displacements from equilibrium, ξ, are found to be (as functions of time):

$$\xi(0.5\,\text{m}, t) = (1.5 \times 10^{-4}\,\text{m})\sin[(6\pi\,\text{s}^{-1})t]$$
$$\xi(2.5\,\text{m}, t) = (1.5 \times 10^{-4}\,\text{m})\sin[(6\pi\,\text{s}^{-1})t + 2\pi/3]$$

   a. What is the amplitude of this wave?

   b. What is the frequency of this wave in hertz?

   c. What is the wavelength?

   d. What is the speed with which this wave travels?

   e. What direction is the wave traveling?

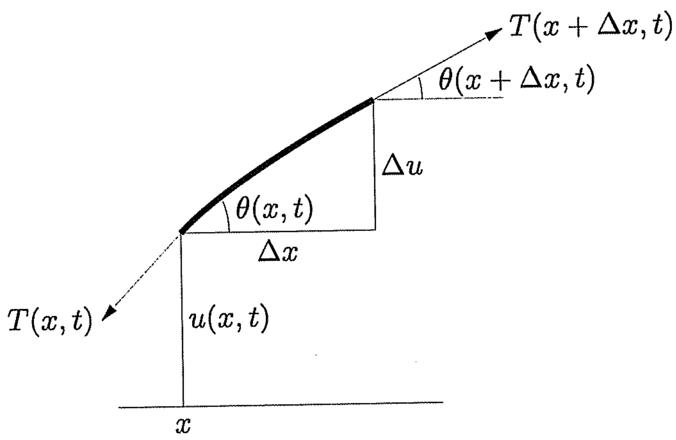
   f. What is the time-rate of displacement at the point x_1 at times $t = 0$ and $t = 0.25\,\text{s}$?

Brief Answers:

1. See this module's *text.*
2. See this module's *Problem Supplement*, problem 8.
3. See this module's *Problem Supplement*, problem 9.
4. See this module's *Problem Supplement*, problem 10.

Derivation of the Wave Equation

In these notes we apply Newton's law to an elastic string, concluding that small amplitude transverse vibrations of the string obey the wave equation. Consider a tiny element of the string.



The basic notation is

$u(x,t) =$ vertical displacement of the string from the x axis at position x and time t

$\theta(x,t) =$ angle between the string and a horizontal line at position x and time t

$T(x,t) =$ tension in the string at position x and time t

$\rho(x) =$ mass density of the string at position x

The forces acting on the tiny element of string are

(a) tension pulling to the right, which has magnitude $T(x+\Delta x,t)$ and acts at an angle $\theta(x+\Delta x,t)$ above horizontal

(b) tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle $\theta(x,t)$ below horizontal and, possibly,

(c) various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by $F(x,t)\Delta x$ the net magnitude of the external force acting on the element of string.

The mass of the element of string is essentially $\rho(x)\sqrt{\Delta x^2+\Delta u^2}$ so the vertical component of Newton's law says that

$$\rho(x)\sqrt{\Delta x^2+\Delta u^2}\tfrac{\partial^2 u}{\partial t^2}(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + F(x,t)\Delta x$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

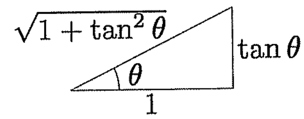
$$\begin{aligned} \rho(x)\sqrt{1+\left(\tfrac{\partial u}{\partial x}\right)^2}\,\tfrac{\partial^2 u}{\partial t^2}(x,t) &= \tfrac{\partial}{\partial x}\big[T(x,t)\sin\theta(x,t)\big] + F(x,t) \\ &= \tfrac{\partial T}{\partial x}(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\,\tfrac{\partial \theta}{\partial x}(x,t) + F(x,t) \end{aligned} \tag{1}$$

We can dispose of all the θ's by observing from the figure that

$$\tan\theta(x,t) = \lim_{\Delta x\to 0} \tfrac{\Delta u}{\Delta x} = \tfrac{\partial u}{\partial x}(x,t)$$

which implies, using the figure on the right below, that

$$\sin\theta(x,t) = \frac{\tfrac{\partial u}{\partial x}(x,t)}{\sqrt{1+\left(\tfrac{\partial u}{\partial x}(x,t)\right)^2}} \qquad \cos\theta(x,t) = \frac{1}{\sqrt{1+\left(\tfrac{\partial u}{\partial x}(x,t)\right)^2}}$$



$$\theta(x,t) = \tan^{-1}\tfrac{\partial u}{\partial x}(x,t) \qquad \tfrac{\partial \theta}{\partial x}(x,t) = \frac{\tfrac{\partial^2 u}{\partial x^2}(x,t)}{1+\left(\tfrac{\partial u}{\partial x}(x,t)\right)^2}$$

Substituting these formulae into (1) give a horrendous mess. However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that $|\theta(x,t)| \ll 1$ for all x and t. This implies that $|\tan\theta(x,t)| \ll 1$, hence that $\left|\tfrac{\partial u}{\partial x}(x,t)\right| \ll 1$ and hence that

$$\sqrt{1+\left(\tfrac{\partial u}{\partial x}\right)^2} \approx 1 \qquad \sin\theta(x,t) \approx \tfrac{\partial u}{\partial x}(x,t) \qquad \cos\theta(x,t) \approx 1 \qquad \tfrac{\partial \theta}{\partial x}(x,t) \approx \tfrac{\partial^2 u}{\partial x^2}(x,t) \tag{2}$$

Substituting these into equation (1) give

$$\rho(x)\tfrac{\partial^2 u}{\partial t^2}(x,t) = \tfrac{\partial T}{\partial x}(x,t)\tfrac{\partial u}{\partial x}(x,t) + T(x,t)\,\tfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{3}$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns u and T.

Fortunately there is a second equation lurking in the background, that we haven't used. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. Our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0$$

Dividing by Δx and taking the limit as Δx tends to zero gives

$$\frac{\partial}{\partial x}\big[T(x,t)\cos\theta(x,t)\big] = 0$$

For small amplitude vibrations, $\cos\theta$ is very close to one and $\frac{\partial T}{\partial x}(x,t)$ is very close to zero. In other words T is a function of t only, which is determined by how hard you are pulling on the ends of the string at time t. So for small, transverse vibrations, (3) simplifies further to

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) = T(t)\,\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{4}$$

In the event that the string density ρ is a constant, independent of x, the string tension $T(t)$ is a constant independent of t (in other words you are not continually playing with the tuning pegs) and there are no external forces F we end up with

$$\boxed{\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\frac{\partial^2 u}{\partial x^2}(x,t)}$$

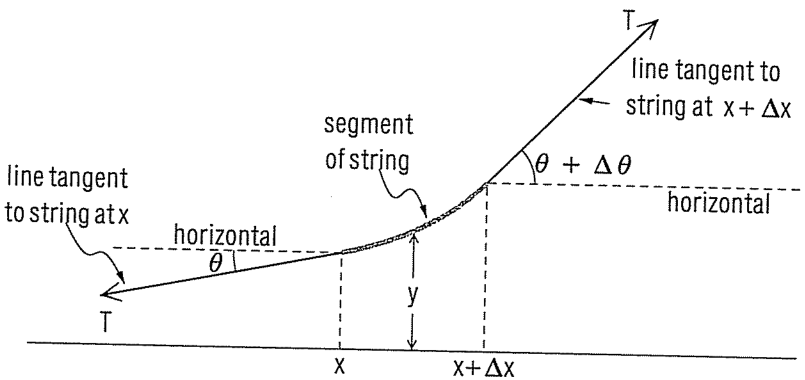
where

$$c = \sqrt{\frac{T}{\rho}}$$

PHYSNET

MISN-0-202

SOUND WAVES AND SMALL TRANSVERSE WAVES ON A STRING



Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

SOUND WAVES AND SMALL TRANSVERSE WAVES ON A STRING

by

J. S. Kovacs and O. McHarris

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Title: **Sound Waves and Small Transverse Waves on a String**

Author: J. Kovacs and Orilla McHarris, Lansing Community College

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Length: 1 hr; 20 pages

Input Skills:

1. Vocabulary: wavelength, amplitude, wave number, wave speed, traveling wave, wave equation (MISN-0-201); Hooke's law (MISN-0-26).
2. State the one-dimensional differential wave equation and its traveling wave solution (MISN-0-201).

Output Skills (Knowledge):

K1. Vocabulary: sound wave, longitudinal wave, transverse wave, compressions, rarefactions, bulk modulus (of elasticity), Young's modulus, stress, strain.

K2. Starting with Newton's second law, derive the expression relating the net force on an element of mass of a stretched string to the transverse acceleration of that string. Comparing the resultant expression to the one-dimensional wave equation, find the expression for the speed of a transverse wave in a stretched string.

K3. Determine the speed of the waves (in terms of the properties of the medium), given the differential wave equations describing: (i) transverse waves in a stretched string; (ii) longitudinal compressional waves in a solid or a gas.

Output Skills (Rule Application):

R1. For a given harmonic (sinusoidal) disturbance write down the equation representing the waveform and calculate the wavelength and frequency of the wave for: (i) transverse waves in a stretched string; and (ii) longitudinal compressional waves in a solid or a gas.

R2. Given a transverse harmonic disturbance, determine the displacement, speed, and acceleration of the waveform at any time in the wave cycle.

SOUND WAVES AND SMALL TRANSVERSE WAVES ON A STRING

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J. S. Kovacs and O. McHarris

1. Overview

Many physical systems are described by the wave equation and its solutions. By applying Newton's second law, the wave equations are found for transverse waves on a string, and longitudinal waves on a rod or in a gas. The speed of each type of wave is then found by inspection of the wave equation.

2. Introduction

2a. The Wave Equation and its Solutions. The solutions of the equation:

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{1}$$

are waves of displacement ξ traveling at speed v in the positive and negative x-direction.[1]

2b. Finding the Wave Equation for a Physical System. Let us look at appropriate physical systems, apply Newton's second law, and see that a wave equation results. For a physical displacement ξ we will use Newton's second law in the form:

$$F = ma = m\frac{\partial^2 \xi}{\partial t^2}.$$

Then whenever the net force F on the system under study is proportional to the displacement's spatial "bending function," $\partial^2\xi/\partial x^2$, we will have the wave equation, Eq. (1). The solution of this equation is the equation of motion of the system. The resulting motion should be that of a traveling wave and the velocity of this wave can be determined just by inspecting the differential equation. As examples of this procedure we will examine a transverse wave on a stretched string, a longitudinal wave in a rod of solid material, and a longitudinal wave in a gas. In each of these cases we will derive the correct differential equation for the system and then determine the wave velocity by comparison to Eq. (1).

[1]See "The Wave Equation and Its Solutions" (MISN-0-201).

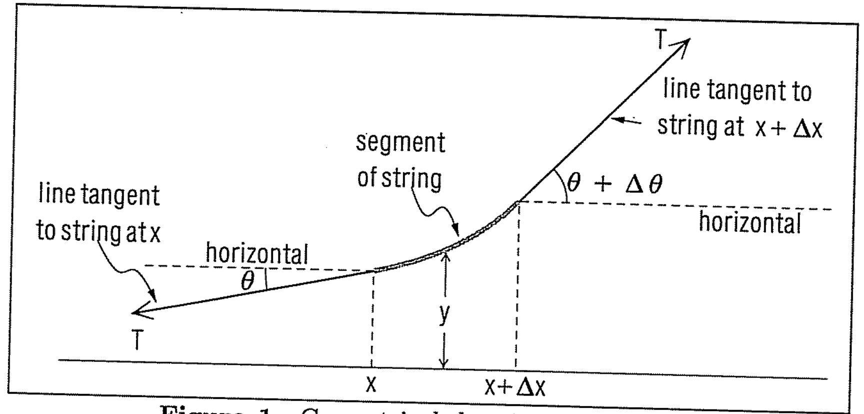

Figure 1. Geometrical descriptors for the displacement y of a small segment Δx of a stretched string.

3. Small Transverse Waves on a String

3a. Geometrical Descriptions. Above is shown a very short length Δx of a string that is vibrating transversely (in the figure, up and down). The string is stretched along the x-axis and the wave propagates along it in the x direction. However, the individual particles of the string move, parallel to the y-axis, at right angles to the direction of the wave's motion. The equilibrium position of the string is at $y = 0$, and the displacement y is assumed to be small, as are the angles θ and $\Delta\theta$.[2]

3b. Net Force on a Segment. Neglecting the force of gravity, the two forces on the segment Δx are the tension T in the string on the right hand end (the tangent at that point) pulling the segment up and to the right and the tension T in the string on the left hand end pulling it down and to the left. Due to the shape of the wave, the tangents to the two ends of the segment are at different angles to the x-axis. At the right hand end of the segment:

$$F_y = T\sin(\theta + \Delta\theta) \qquad \text{and} \qquad F_x = T\cos(\theta + \Delta\theta)\,,$$

while at the left hand end:

$$F'_y = -T\,\sin\theta \qquad \text{and} \qquad F'_x = -T\cos\theta\,.$$

[2]For small displacements the restoring force on the string varies linearly with displacement and hence produces simple harmonic motion. This makes the motion of the string easily soluble mathematically.

Since both θ and $\Delta\theta$ are small, $\cos\theta$ and $\cos(\theta+\Delta\theta)$ are both essentially equal to 1 and there is negligible net force in the x-direction.[3] For small θ, $\sin\theta$ on the other hand is essentially equal to θ, and the net force in the y-direction is:[4]

$$F_y = T\,\Delta\theta\,.$$

3c. Applying Newton's Second Law. If the linear density (mass per unit length) of a stretched string is μ, a segment with length Δx has a mass $\Delta m = \mu\Delta x$ and a transverse acceleration $a = \partial^2 y/\partial t^2$ (see Fig. 1). Notice that for "a" we must use partial derivatives of y with respect to t since the displacement y is a function of both x and t. Application of Newton's second law then gives us (with F from the previous section):

$$T\,\Delta\theta = \mu\,\Delta x \partial^2 y/\partial t^2\,.$$

This implies:

$$T\frac{\Delta\theta}{\Delta x} = \mu\frac{\partial^2 y}{\partial t^2}\,. \qquad (2)$$

We can now let Δx and $\Delta\theta$ shrink until $\Delta\theta/\Delta x$ becomes $\partial\theta/\partial x$ so Eq. (2) becomes:

$$T\frac{\partial\theta}{\partial x} = \mu\frac{\partial^2 y}{\partial t^2}\,. \qquad (3)$$

3d. Getting the Wave Equation. For a wave equation, Eq. (1), we need $\partial^2 y/\partial x^2$ rather than the $\partial\theta/\partial x$ of Eq. (3). Let us therefore rewrite $\partial\theta/\partial x$ in terms of y and x. These quantities are related by:

$$\tan\theta = \text{ slope of curve } = \partial y/\partial x\,.$$

Differentiating with respect to x gives us:

$$\frac{\partial}{\partial x}\tan\theta = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right),$$

$$\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial x^2},$$

[3] $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$; For small θ, the θ^2 and higher terms can be neglected.

[4] $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

where we now have the second derivative we need. Remembering that $\cos\theta \approx 1$ for small θ, we have finally:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}\,. \qquad (4)$$

This is the wave equation for any small traveling wave on a stretched string. Comparing to Eq. (1) we see that the speed of the wave is:

$$v = \sqrt{T/\mu}\,. \qquad (5)$$

3e. Physical Solutions. The wave equation contains symbols whose values must come from the physical problem at hand. The wave's speed comes from the properties of the medium in which the wave propagates, as illustrated in Eq. (5) with tension and mass-per-unit-length for a string. The wave's shape, whether sinusoidal or something else, and its frequency, depend as well on the driving force (on the manner in which the string is made to keep vibrating) and on the damping properties of the string.

4. Longitudinal Waves on a Rod

4a. Overview. Applying Hooke's law and Newton's second law to longitudinal compression in a solid rod, we can derive the equation for acoustic waves in the rod. Such a compression occurs when, for example, a rod is struck on one end, displacing the individual particles of the rod in the direction of the rod's length and causing a displacement wave to travel down the rod in the same direction as the motion of the particles.

4b. Stress, Strain, Young's Modulus. In general, when a rod is subjected to a force in the direction of its length and acting over its cross section—when, for example, it is vertical and holds up the roadbed of a bridge or some other weight suspended from its end—Hooke's Law applies. This law states, in general, that in an elastic medium:[5]

$$\frac{\text{stress}}{\text{strain}} = \text{constant}\,, \qquad (6)$$

where stress is the applied force per unit cross sectional area and strain is the rod's fractional deformation due to the stress.

$$\text{stress } = F/A\,.$$

[5] Our rod will remain "elastic" as long as it is not stretched too much in proportion to its length.

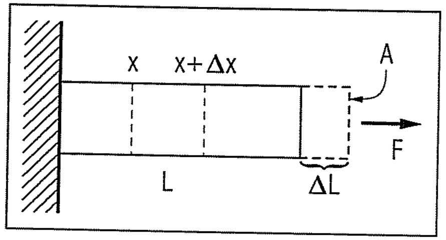

Figure 2. Geometrical descriptions for a rod undergoing longitudinal compression and/or extension.

In the case of a rod, the stress causes a change in the length of the rod. The strain is thus the fractional change in length. Writing L as the original length of the rod and ΔL is its elongation due to the stress on it, the strain is $\Delta L/L$. For a rod, the "constant" of Eq. (6) is called Young's modulus and is designated by the letter Y; it is a property of the material of which the rod is made. Thus for elastic longitudinal deformations of our rod:

$$Y = (F/A)/(\Delta L/L)\,,$$

and hence:

$$F = YA\,\Delta L/L\,.$$

4c. Force on a Segment: Static Case. In order to derive a wave equation for a rod, let us analyze what happens to a small segment of the rod. Consider a segment with unstretched length Δx and cross section A, where A is also the cross section for the rod as a whole (see Fig. 2). In a static situation, such as that of a spring pulling on the end of the rod, the force is constant along the rod, and the segment of length Δx is stretched by its proportionate amount $\Delta\xi$ such that the $\Delta\xi$'s summed over all the Δx's of the rod equals the total elongation ΔL. In this case, Hooke's law for the segment gives us:

$$F = YA\,\Delta\xi/\Delta x\,.$$

We now let Δx shrink until that expression becomes:

$$F = YA\frac{d\xi}{dx}\,. \tag{7}$$

4d. Force on a Segment: Dynamic Case. When a wave is traveling along the rod, the situation is not static and the forces at the two ends of any particular segment are not equal. The acceleration of the segment

at any particular time t is due to the net force on it at that time:

$$\begin{aligned} dF_{\text{net}} &= F(x+dx) - F(x) \\ &= \frac{dF}{dx}\,dx \\ &= YA\frac{d^2\xi}{dx^2}\,dx. \qquad \text{(fixed time)}\,. \end{aligned}$$

We must remember what $\xi(x,t)$ is, though. It is the displacement, from equilibrium, of the individual particles in the rod as the wave travels along the rod. As such, it is a function of both x and t. Thus to get rid of the "fixed time" label in Eq. (8) we can use partial derivatives:

$$dF = YA\frac{\partial\xi}{\partial x}\,dx\,.$$

4e. Applying Newton's Second Law. We apply Newton's Second Law to a small segment of a rod that has volume density ρ and cross-sectional area A. The small segment has mass dm and length dx:

$$dm = \rho\,dV = \rho A\,dx\,.$$

The acceleration of the rod particles at x and t is:

$$a(x,t) = \frac{\partial^2\xi}{\partial t^2}\,.$$

Then Newton's second law, for the segment of length dx at x and t, gives us:

$$YA\frac{\partial^2\xi}{\partial x^2}\,dx = \rho A\,dx\frac{\partial^2\xi}{\partial t^2}\,,$$

which can be written:

$$\frac{\partial^2\xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2\xi}{\partial x^2}\,. \tag{8}$$

This is the wave equation so by comparison to Eq. (1) speed for a longitudinal wave on a rod is:

$$v = (Y/\rho)^{1/2}\,. \tag{9}$$

5. Longitudinal Waves In a Gas

5a. Stress, Strain, Bulk Modulus of Elasticity. Longitudinal waves in a gas, such as sound waves through the air, obey Hooke's law:

$$\frac{\text{stress}}{\text{strain}} = \text{constant}. \qquad (10)$$

This equation also applies to a long thin rod where a stress causes a one-dimensional strain (a change in length). Actually, a more thorough analysis of that case shows that the rod's cross section decreases slightly as the length increases: the rod's volume stays approximately constant. A gas does not act the same way at all since it is compressible and its volume changes with pressure. Here the appropriate constant in Eq. (10) is the gas's bulk modulus of elasticity, K, usually defined by:[6]

$$K = -\frac{dP}{dV}V.$$

5b. The Wave Equation. The equation of motion for a volume element of a gas is mathematically more complicated than that for a segment of a rod. However, in advanced texts it is shown that the wave equation for a longitudinal wave in a gas is like that for a longitudinal wave in a rod, except that K replaces Y and the mass density is specifically designated as that at equilibrium:

$$\frac{\partial^2\xi}{\partial t^2} = \frac{K}{\rho_0}\frac{\partial^2\xi}{\partial x^2}. \qquad (11)$$

For a gas, then, the wave speed is

$$v = \sqrt{K/\rho_0} \qquad (12)$$

5c. Pressure, Density, and Heat Capacity Ratio. Using the definition of K, we can write the wave velocity in terms of gas properties that we have seen before. The first gas-related equation you are likely to recall is the equation of state for an ideal gas,[7] $PV = nRT$. This

[6] Some authors write K in terms of density rather than volume.

[7] "For an ideal gas, the simplest model of intermolecular forces is assumed: there are no interactions between the molecules unless their centers happen to coincide, in which case they bounce off one another like hard spheres, and the molecules are assumed to be point masses. For such a model, the equation of state, that is, the relation between the measurable quantities (pressure, volume, temperature), can be derived in a straightforward way once temperature and average pressure are defined." (Quoted from *Temperature And Pressure of an Ideal Gas: The Equation Of State*, MISN-0-157.)

makes it look as if we could solve for P, differentiate with respect to V, and end up with a value for K. Historically, this was the first approach. However, it gives the wrong value for the wave speed and the reason is this: it implicitly assumes that the temperature T is constant. That is, it assumes that as the wave travels through the gas, heating it in the wave-peak compressions and cooling it in the wave-trough rarefactions, the gas is able to instantaneously exchange heat with its surroundings and stay at the same temperature. Actually, a wave usually travels through a gas so rapidly that there is no time for heat transfer. Thus the appropriate relationship is the one for adiabatic conditions,

$$PV^\gamma = \text{constant},$$

where γ is the ratio of heat capacities:

$$\gamma = \frac{C_p}{C_v}.$$

Differentiating this equation gives us:

$$dP\,V^\gamma + \gamma P V^{\gamma-1}\,dV = 0.$$

Then:

$$V\frac{dP}{dV} = -\gamma P,$$

and hence:

$$K = \gamma P.$$

Thus the wave speed in a gas becomes

$$v = (\gamma P/\rho_0)^{1/2}. \qquad (13)$$

5d. Dependence on Temperature. Starting from the expression for wave speed as a function of pressure, Eq. (13), we can use the equation of state of an ideal gas to find the speed as a function of temperature. That is, we can substitute

$$PV = nRT$$

into Eq. (13) and get:

$$v = \left(\frac{\gamma nRT}{V\rho_0}\right)^{1/2} = \left(\frac{\gamma nRT}{m}\right)^{1/2}.$$

We can put this in a form useful for determining sound speeds in different gases by writing it in terms of $M = m/n$, the mass of one mole of the gas:

$$v = \left(\frac{\gamma RT}{M}\right)^{1/2} . \qquad (14)$$

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **bulk modulus (of elasticity)**: the ratio of the increase in pressure to the decrease in the fractional volume of a fluid.
- **compressions**: the portions of the phase of a longitudinal acoustical wave when the pressure of the medium is higher than the equilibrium pressure.
- **longitudinal wave**: a wave in which the wave displacement from equilibrium is parallel to the wave velocity (example: sound waves).
- **rarefactions**: the portions of the phase of a longitudinal acoustical wave when the pressure of the medium is lower than the equilibrium pressure.
- **sound wave**: a longitudinal acoustical wave traveling through an elastic medium.
- **strain**: the fractional deformation of the dimension or dimensions of a material subjected to a stress. If only one dimension is relevant, the strain is given by the ratio of the change in length to the undeformed length. If the entire material volume is altered, the strain is given by the ratio of the change in volume to the undeformed volume.
- **stress**: applied force per unit cross-sectional area. For a fluid this is the same as pressure.

- **transverse wave**: a wave in which the wave displacement from equilibrium is perpendicular to the wave velocity (example: water waves).
- **Young's modulus**: the ratio of stress to strain for a deformation of length.

PROBLEM SUPPLEMENT

$$\frac{T}{\mu}\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2} \qquad \frac{Y}{\rho}\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2} \qquad \frac{\gamma P}{\rho_0}\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2}$$

Note 1. Work the following problems *in order*, completely finishing each one successfully before going on to the next.

Note 2: Get appropriate velocity formulas by simply looking at the wave equations shown above.

Note 3: Some of the Answers have references to *help* sequences in this module's *Special Assistance Supplement.*

1. A guitar string of length 2.00 ft has a mass of 0.700 grams. When mounted on a guitar the string is placed under a tension of 20.0 N. Determine the speed of transverse waves traveling in the string when plucked.

2. A stretched steel wire under a tension of 1.50×10^4 N is attached at one end to an oscillator with a period of 4.00×10^{-4} sec. Given that the diameter of the wire is 0.3572 cm and that the density of steel is $7.850 \times 10^3\ \mathrm{kg/m^3}$, determine the speed of propagation, frequency, and wavelength of transverse waves traveling along the wire. Write the equation for the waveform of the wave.

3. In Westerns, the "Indians" frequently detected an approaching train by placing their ears to the railroad track and listening to the transmitted sound of the train wheels in contact with the track. Calculate the speed of sound in the steel track, treating the problem as compressional waves in a simple rod. Determine the speed of sound in air at 75 °F (23.9 °C). How do the two speeds compare?

   Steel:
   $Y = 1.95 \times 10^{11}\ \mathrm{N/m^2}$
   $\rho = 7.850 \times 10^3\ \mathrm{kg/m^3}$

   Air:
   $\gamma = 1.4$
   $M = 29.8$ grams/mole

   $R = 8.31$ J/(K mole)

4. A copper and an aluminum rod, each of cross section $0.75\ \mathrm{cm^2}$, are welded together end-on-end to form one continuous length of metal rod. Longitudinal waves are excited in the copper rod by a vibrating source

with a frequency of 675 Hz. If the wavelength of longitudinal waves in the copper portion of the rod is 5.25 m, determine the wavelength of the waves in the aluminum.

   Copper:
   $Y = 11.2 \times 10^{10}\ \mathrm{N/m^2}$
   $\rho = 8.93\ \mathrm{grams/cm^3}$

   Aluminum:
   $Y = 6.9 \times 10^{10}\ \mathrm{N/m^2}$
   $\rho = 2.7\ \mathrm{grams/cm^3}$

5. Scuba divers must use exotic mixtures of gases instead of air when diving to great depths in order to avoid nitrogen narcossis and oxygen poisoning. Such breathing mixtures typically consist mainly of helium as an "inert" gas, and a small amount of oxygen. However, when communicating with the surface, the divers' voices sound high-pitched and squeaky because sound travels faster in helium than in air (which is mostly nitrogen). Calculate the frequency of "middle C" in an environment of pure helium assuming the note is 256 Hz in air. Note that the wavelength is the same in both media since it depends only on the geometry of the voice box producing the sound. Also, the temperature is assumed to be the same in both environments.

   Helium:
   $\gamma = 5/4$
   $M = 4.00$ grams/mole

   Air:
   $\gamma = 7/5$
   $M = 29.8$ grams/mole

6. A stereo speaker cone is attached to a long tube in such a way as to generate longitudinal waves traveling down the air-filled tube. The speaker cone is made to oscillate harmonically as described by the equation:

   $$y = (0.010\,\mathrm{cm})\sin\left[(8.0 \times 10^2\,\pi\,\mathrm{sec}^{-1})\,t\right]$$

   where y is the horizontal displacement of the speaker surface as a function of time. Assuming the equilibrium pressure of the air is atmospheric pressure, determine the speed of waves travelling in the tube and write the equation for the waveform (for air, $\gamma = 1.4$, $\rho_0 = 1.33 \times 10^{-3}\ \mathrm{grams/cm^3}$, 1 atmosphere $= 1.01 \times 10^5\ \mathrm{N/m^2}$).

7. The density of Aluminum is $2.7 \times 10^3\ \mathrm{kg/m^3}$ and $Y_{Al} = 0.70 \times 10^{11}\ \mathrm{N/m^2}$. A thinly-drawn aluminum wire of length 10.00 m and cross-sectional area of $5.0\ \mathrm{mm^2}$ is held under a tension of 2.0×10^2 N.

   a. Compare the velocity of propagation of transverse mechanical waves and longitudinal sound waves in this wire.

   b. Which propagates faster?

c. Can both velocities be made the same?

8. The molecular mass of oxygen is 16 times the molecular mass of hydrogen, while the molecular mass of krypton (a noble gaseous element) is 42 times the molecular mass of hydrogen. Compare the speed of sound in oxygen and krypton at a given temperature. Oxygen is diatomic while krypton is monatomic so the relevant γ's are: $\gamma_{O_2} = 1.4, \gamma_{Kr} = 1.6$.

Brief Answers:

1. $v = 132\,\text{m/s}$. *Help: [S-2]* *Help: [S-6]*
2. $v = 437\,\text{m/s}$; *Help: [S-1]* *Help: [S-2]* *Help: [S-5]* *Help: [S-6]*
   $\nu = 2.5 \times 10^3\,\text{Hz}$;
   $\lambda = 0.175\,\text{m}$;
   $y = y_0 \sin[(35.9\,\text{m}^{-1}\,x \pm (1.57 \times 10^4\,\text{s}^{-1})\,t + \theta_0]$
3. $v_{\text{steel}} = 4.98 \times 10^3\,\text{m/s}$; *Help: [S-2]* *Help: [S-6]*
   $v_{\text{air}} = 3.40 \times 10^2\,\text{m/s}$ *Help: [S-4]* *Help: [S-6]*
   The speed of sound is 14.4 times larger in steel than in air.
4. $\lambda = 7.5\,\text{m}$ *Help: [S-2]* *Help: [S-7]*
5. $\nu_{\text{helium}} = 662\,\text{Hz}$; nearly 2 octaves higher. *Help: [S-2]*
6. $v = 327\,\text{m/s}$; *Help: [S-2]* *Help: [S-6]*
   $\xi = 0.010\,\text{cm} \sin[(7.7\,\text{m}^{-1}\,x - (8.00 \times 10^2\,\pi\,\text{s}^{-1})\,t + \theta_0]$
7. a. $v_{\text{mechanical}} = 121\,\text{m/s}$, $v_{\text{sound}} = 5.09 \times 10^3\,\text{m/s}$.

   b. sound.

   c. A tension of $3.5 \times 10^5\,\text{N}$ would equalize the speeds. The tensile strength of aluminum is less than that so the aluminum would pull apart.
8. $v_{\text{O}} = 1.52\,v_{\text{Kr}}$. [NOT 1.07, NOT 1.62] *Help: [S-3]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 2)*

The problem statement says **transverse** waves in a **wire**, not *longitudinal* waves in a *rod* or a *gas*. Use the right wave equation.

S-2 *(from PS-Problems 1, 2, 3, 4, 5, 6)*

Non-MKS units (e.g., "feet," "grams," "cm,") must be converted to MKS units.

S-3 *(from PS-Problem 8)*

$M_O/M_{Kr} = (M_O/M_H)/(M_{Kr}/M_H)$ which is *given* to you as 16/42.

S-4 *(from PS-Problem 3)*

$\mu = 1.148 \times 10^{-3}\,\text{kg/m}$.

S-5 *(from PS-Problem 2)*

$\mu = 0.0787\,\text{kg/m}$. If I buy 1 m of the wire I get 0.0787 kg of wire. If I buy 2 m of the wire I get 0.1574 kg of wire. That is what "linear density" μ is all about.

S-6 *(from PS-Problem 3)*

Kindly read Note 2 at the head of the *Problem Supplement* and do what it says to do.

S-7 *(from PS-Problem 4)*

$v_{al} = 5.06 \times 10^3\,\text{m/s}$. Think about what happens physically at the interface between the two metals. What quantity is preserved across the interface? Wavelength? Velocity? Frequency?

MODEL EXAM

$$\frac{T}{\mu}\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} \qquad \frac{Y}{\rho}\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} \qquad \frac{\gamma P}{\rho_0}\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2}$$

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.

2. The density of Aluminum is $2.7 \times 10^3\,\text{kg/m}^3$ and $Y_{Al} = 0.70 \times 10^{11}\,\text{N/m}^2$. A thinly-drawn aluminum wire of length 10.00 m and cross-sectional area of $5.0\,\text{mm}^2$ is held under a tension of $2.0 \times 10^2\,\text{N}$.

   a. Compare the velocity of propagation of transverse mechanical waves and longitudinal sound waves in this wire.

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3. The molecular mass of oxygen is 16 times the molecular mass of hydrogen, while the molecular mass of krypton (a noble gaseous element) is 42 times the molecular mass of hydrogen. Compare the speed of sound in oxygen and krypton at a given temperature. [Note that oxygen is diatomic while krypton is monatomic and $\gamma_{O_2} = 1.4, \gamma_{Kr} = 1.6$.]

Brief Answers:

1. See this module's *text.*

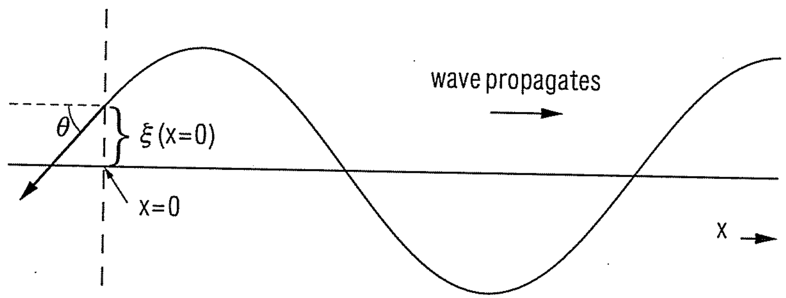
2. See this module's *Problem Supplement*, problem 7.

3. See this module's *Problem Supplement*, problem 8.

PHYSNET

MISN-0-203

INTENSITY AND ENERGY IN SOUND WAVES



Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

INTENSITY AND ENERGY IN SOUND WAVES

by
William C. Lane, J. Kovacs and O. McHarris,
Michigan State University and Lansing Community College

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Title: **Intensity and Energy in Sound Waves**

Author: William C. Lane, J. Kovacs and O. McHarris, Michigan State University

Version: 2/1/2000 Evaluation: Stage 0

Length: 1 hr; 20 pages

Input Skills:

1. Vocabulary: compressions, rarefactions, longitudinal wave, transverse wave, sound wave (MISN-0-202); energy, power (MISN-0-20).
2. For transverse waves in a stretched string and longitudinal compressional waves in a solid, relate the wave speed and the force on the individual particles of the medium to the physical properties of the material transmitting the wave (MISN-0-202).

Output Skills (Knowledge):

K1. Vocabulary: decibel, plane wave, spherical wave, wave intensity, wave vector, wave front.

K2. State the expression for the intensity of a plane wave and the power transmitted across a given cross-sectional area of the medium through which the wave travels.

Output Skills (Rule Application):

R1. For intensity calculations associated with plane and spherical waves, use the unit of intensity level, the decibel.

Output Skills (Problem Solving):

S1. For an acoustic plane wave of given amplitude and frequency, traveling through an elastic medium of given mass density and elasticity, calculate the wave's intensity, energy density, and rate at which it propagates energy.

S2. Given the intensity of a spherical wave at one radial distance from the wave source, calculate the wave's intensity at any other radial distance.

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1. Introduction

In a previous module we have seen that sound (acoustic) waves are the result of periodic disturbances of an elastic medium.[1] These disturbances may be represented by variations in the local pressure of a region in the medium (compressions and rarefactions) or by the displacement of portions of the medium from their equilibrium position. However, this displacement is in the form of a simple harmonic oscillation, whether it be a transverse displacement as with waves on a stretched string or a longitudinal displacement as with sound waves in a gas. Either way there is no net movement of matter. It would seem appropriate then to ask what it is that is "traveling" when a traveling wave passes through a medium.

2. One-Dimensional Elastic Waves

2a. Energy is What is Propagated. When a one-dimensional traveling wave propagates through an elastic medium, the wave carries energy in the direction that the wave travels. This energy is the kinetic and potential energy of the deformation of the elastic medium. The total mechanical energy of an infinitesimal mass element dm in the elastic medium is

$$dE = \frac{1}{2}\omega^2 \xi_0^2 \, dm \,, \tag{1}$$

where ω is the angular frequency of the wave and ξ_0 is the displacement amplitude of the wave.[2] Depending on the geometry of the medium this energy may be expressed as one of several energy densities. For a stretched wire of linear mass density μ, the mass element dm may be expressed as:

$$dm = \mu \, dx \,. \tag{2}$$

[1] See "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).
[2] For a detailed derivation of this relation, see the Appendix.

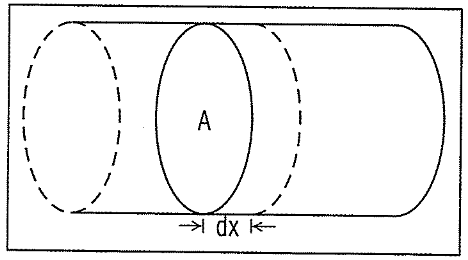


Figure 1. An infinitesimal volume element of an elastic medium of constant cross-sectional area.

Substituting this expression for dm into Eq. (1) and dividing by dx,[3] we may define a linear energy density E_ℓ:

$$E_\ell = \frac{dE}{dx} = \frac{1}{2}\mu\omega^2\xi_0^2 \,. \tag{3}$$

Similarly, for energy propagating through a three-dimensional elastic medium of constant cross-sectional area, such as a solid rod or a column of gas, a volume energy density E_v may be defined as

$$E_v = \frac{dE}{dV} = \frac{1}{2}\rho\omega^2\xi_0^2 \,, \tag{4}$$

where ρ is the volume density of the elastic medium.

2b. Power is Required to Maintain a Train of Waves. For a continuous series of sinusoidal pulses, called a "wave train," to be maintained in an elastic medium, the energy that propagates in the medium must be supplied by some external agent. The rate at which this energy is supplied is the "power" of the wave source, defined as

$$P = \frac{dE}{dt} \,. \tag{5}$$

For a transverse wavetrain propagating on a stretched wire, we may use the chain-rule of differential calculus to express Eq. (5) as

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = E_\ell v \,, \tag{6}$$

or

$$P = \frac{1}{2}\mu\omega^2\xi_0^2 v \,, \tag{7}$$

where $v = (T/\mu)^{1/2}$ is the speed with which energy propagates along the wire, the wave speed. A similar expression may be derived for a wave

[3] See "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).

traveling through a three-dimensional medium of constant cross-sectional area. If an infinitesimal volume element dV is chosen with cross-sectional area A and thickness dx, where the x-direction is chosen parallel to the direction of wave propagation, then $dV = A\,dx$. Thus

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = \frac{dE}{dV}\frac{dx}{dt}A,$$

so:

$$P = E_v\,A\,v\,, \quad (8)$$

or

$$P = \frac{1}{2}\rho\omega^2\xi_0^2 A v\,, \quad (9)$$

where $v = (Y/\rho)^{1/2}$ for longitudinal waves on a solid rod or $v = (K/\rho_0)^{1/2}$ for longitudinal waves in a column of gas.

2c. Definition of Wave Intensity. For cases of a one-dimensional wave traveling in a three-dimensional medium of constant cross-sectional area, an important quantity called the "wave intensity" may be defined as the power per unit cross-sectional area, I:

$$I = \frac{P}{A}. \quad (10)$$

Substituting Eq. (9) into Eq. (10), the intensity of a one-dimensional wave propagating in an elastic medium of constant cross-sectional area may be written as:

$$I = \frac{1}{2}\rho\omega^2\xi_0^2 v\,, \quad (11)$$

which is constant for waves of given amplitude and frequency. Intensity is the physical quantity which, for a sound wave, roughly corresponds to the "loudness" or "softness" of the sound. Since the MKS unit power is the watt (W), the MKS unit of intensity is the watt per square meter (W/m 2).

2d. The Intensity Level of Sound. Because the range of intensities of audible sounds is so wide, sound intensities are often expressed using a logarithmic scale, referred to as the "intensity level." Intensity levels are determined using the following equation:

$$I(\text{db}) = 10\log\left(\frac{I}{I_{\text{ref}}}\right), \quad (12)$$

where I is the intensity of interest, I_{ref} is a reference intensity, and $I(\text{db})$ is the intensity level of the intensity I in units of "decibels," abbreviated

"db." For sound waves in air, I_{ref} has arbitrarily been chosen to be 10^{-12} W/m^2. A list of typical sounds and their corresponding intensities and intensity levels are shown in Table 1.

Table 1. The Intensities of Some Sounds.

SOUND	INTENSITY (W/m^2)	RELATIVE INTENSITY (db)
Threshold of hearing	10^{-12}	0
Rustling leaves	10^{-10}	20
Talking (at 3 ft.)	10^{-8}	40
Noisy Office or Store	10^{-6}	60
Elevated train	10^{-4}	80
Subway car	10^{-2}	100
Threshold of pain	1	120

3. Plane Waves

3a. Definition of a Plane Wave. A one-dimensional wave for which the wave disturbance is distributed uniformly over a planar surface (either finite or infinite), is called a "plane wave." If the surface is of finite extent, the plane wave is said to be "collimated." The wave is still one-dimensional as long as it travels in a single direction. Figure 2a is an illustration of how we visualize a plane wave as a series of parallel plane surfaces, called "wave fronts" moving in the direction indicated. A wave

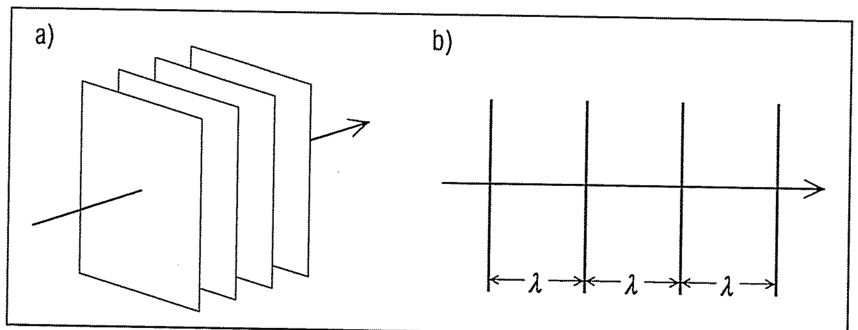


Figure 2. An illustration of a plane wave: a) oblique view; b) cross-sectional view.

front is a surface over which the wave disturbance has the same phase, i.e. for a plane wave traveling in the x-direction:

$$\phi = kx - \omega t + \phi_0 = \text{ constant}. \tag{13}$$

Figure 2b shows a cross-sectional view of the wave fronts of a plane wave. We typically draw such a sketch with the wavefronts one wavelength apart, so that they are all crests, or all troughs, or any other given wave disturbance.

3b. The Wave Vector. In order to describe a plane wave propagating in any direction, we will define a quantity called the "wave vector." The wave vector, symbolized by $\vec{k}$, is a quantity whose magnitude is the wave number, k, of the wave, and whose direction is the direction of propagation of the wave.[4] Using the wave vector we may define the phase of a plane wave as:

$$\phi = \vec{k} \cdot \vec{r} - \omega t + \phi_0\,, \tag{14}$$

where $\vec{r}$ is the position vector of a point on a particular wave front with respect to a specified coordinate system. Thus the wave function for a plane wave may be written as:

$$\xi = \xi_0 \sin(\vec{k} \cdot \vec{r} - \omega t + \phi_0)\,. \tag{15}$$

3c. The Intensity of a Plane Wave. The intensity of a plane wave is defined as the power propagated per unit area of wave front. Since the power is distributed uniformly over the wave front, the intensity is constant for plane waves of given amplitude, frequency, and wave speed. Thus one-dimensional waves travelling in an elastic medium of constant cross-sectional area are examples of collimated plane waves, and the relations derived for the energy density and the intensity of these waves are applicable to plane waves in general. Note that if the wave front of the plane wave is of infinite extent, the total power propagating across its surface is also infinite, although the energy density and intensity are not. However, for systems of physical interest, only a finite portion of the wave front impinges on a system capable of detecting the power propagated, so the power detected is finite. Furthermore, the concept of a plane wave front of infinite area is usually an idealized approximation.

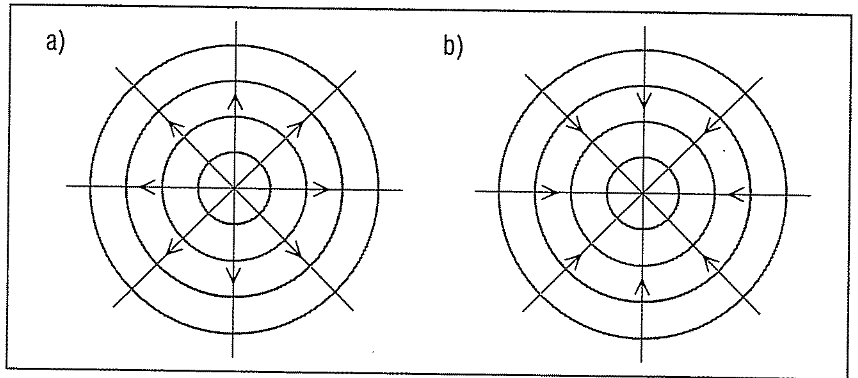


Figure 3. (a) A spherical wave propagating outward; (b) a spherical wave propagating inward.

4. Spherical Waves

4a. Wave Front of a Spherical Wave. Another type of wave frequently encountered is the "spherical wave." In contrast to a plane wave, the wave fronts of a spherical wave are concentric spherical surfaces. The surfaces travel radially outward or inward, depending on the sign of the frequency term in the phase of the wave function (see Fig. 3).

4b. Intensity of a Spherical Wave. The intensity of a spherical wave is defined as the power propagated per unit area of wave front, just as for a plane wave. However, the power and energy are distributed over a spherical wave front of area $4\pi r^2$. Since each wave front is expanding radially outward (or contracting inward) as the wave propagates, the intensity varies as r^{-2} for a spherical wave. Assuming the power output of the wave source, P_0, is a constant, the intensity of the spherical wave is given by

$$I = \frac{P_0}{4\pi r^2}\,. \tag{16}$$

If the intensity is known at a specific radial distance r_0, then since $P_0 = I_0 4\pi r_0^2$, the intensity at any other radial distance r may be expressed as:

$$I = I_0 \left(\frac{r_0}{r}\right)^2\,. \tag{17}$$

4c. The Wave Function of a Spherical Wave. By solving the wave equation for a wave source of spherical symmetry, the wave function for

[4] Note: $k = \omega/v$.

a spherical wave can be shown to be:

$$\xi(r,t) = \xi_0 \left(\frac{r_0}{r}\right) \sin(kr - \omega t + \phi_0)\,, \tag{18}$$

where ξ_0 and ϕ_0 are determined from the boundary conditions.[5] If we consider the entire coefficient of the sine function to be the "amplitude," then we see that the amplitude of a spherical wave decays as r^{-1}. Since the wave intensity is proportional to the square of the wave amplitude, for a spherical wave the intensity is:

$$I = \beta \xi_0^2 \left(\frac{r_0}{r}\right)^2 . \tag{19}$$

This expression is equivalent to Eq. (17) if $I_0 = \beta\xi_0^2$. The constant of proportionality, β, depends on the specific medium through which the wave travels and the physical nature of ξ_0.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **decibel:** a unit of intensity on a logarithmic scale of "intensity level," abbreviated as "db." An additive increase in the intensity level of 10 db implies a multiplicative increase in the actual intensity by a factor of 10.
- **plane wave:** a one-dimensional wave traveling in a direction defined by the wave vector $\vec{k}$, whose surfaces of equal phase are parallel planes of finite or infinite extent.
- **spherical wave:** a three-dimensional wave emanating from a wave source of spherical symmetry, whose surfaces of equal phase are concentric spheres.

[5]Note that the actual value of ϕ_0 will depend on whether a sine or a cosine function is used.

- **wave intensity:** the power propagated by the wave per unit area perpendicular to the propagation direction; the units of intensity are W/m^2.
- **wave front:** a continuous surface of wave disturbances of the same phase, such as a crest or a trough.
- **wave vector:** a vector whose magnitude is the wave number k and whose direction is the direction of wave propagation.

Energy Density of a 1D Elastic Wave

Only For Those Interested. To derive the energy density of a one-dimensional elastic wave traveling in an elastic medium of constant cross-sectional area, consider an infinitesimal volume element of the medium. The total energy in this element is:

$$dE = dE_K + dE_P \tag{20}$$

where dE_K is the kinetic energy due to the motion of the infinitesimal volume element and dE_P is the potential energy of the element's displacement from equilibrium. If the infinitesimal volume element has mass dm, the kinetic energy may be represented as

$$dE_K = \frac{1}{2}\dot{\xi}^2\, dm \tag{21}$$

where $\dot{\xi} = \partial\xi/\partial t$, the speed of the medium's deformation. For a one-dimensional sinusoidal wave, ξ may be represented as:

$$\xi(x,t) = \xi_0 \sin(kx - \omega t + \phi_0) \tag{22}$$

so

$$\dot{\xi} = \pm\omega\xi_0 \cos(kx - \omega t + \phi_0). \tag{23}$$

The potential energy of the mass element may be represented as:

$$dE_P = \frac{1}{2}\xi^2\, dk \tag{24}$$

where dk is the elastic constant of the restoring force acting on the infinitesimal mass element. This force constant may be identified by applying Newton's second law to the mass element. The force on the mass element is given by:

$$dF = dm\frac{\partial^2\xi}{\partial t^2} = -\omega^2 dm\xi_0 \sin(kx - \omega t + \phi_0) = -\omega^2 dm\xi. \tag{25}$$

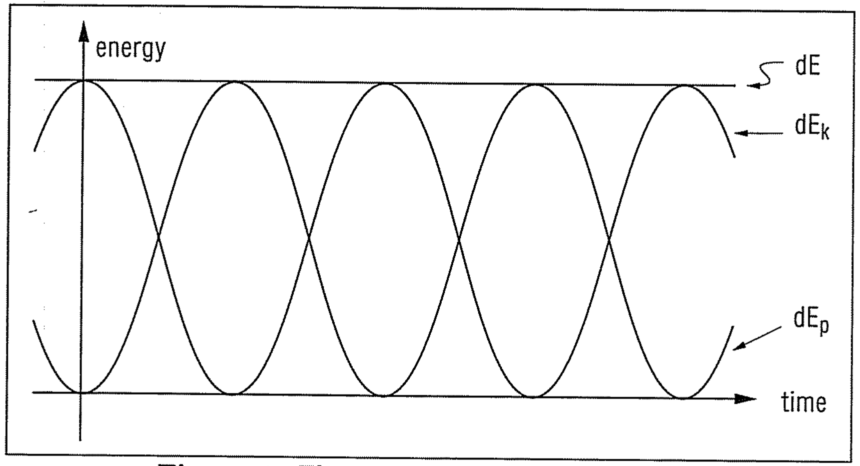


Figure 4. The energy of a mass element dm in an elastic medium, as a function of time, as a one-dimensional wave passes by.

This result is a form of Hooke's law, so the force constant must be

$$dk = \omega^2 dm. \tag{26}$$

Therefore the potential energy of the mass element is:

$$dE_P = \frac{1}{2}\xi^2\omega^2\, dm = \frac{1}{2}dm\omega^2\xi_0^2 \sin^2(kx - \omega t + \phi_0), \tag{27}$$

and the kinetic energy of the mass element is:

$$dE_K = \frac{1}{2}dm\omega^2\xi_0^2 \cos^2(kx - \omega t + \phi_0). \tag{28}$$

Since $\sin^2\theta + \cos^2\theta = 1$, the total energy of the mass element is:

$$dE = \frac{1}{2}\omega^2\xi_0^2 dm. \tag{29}$$

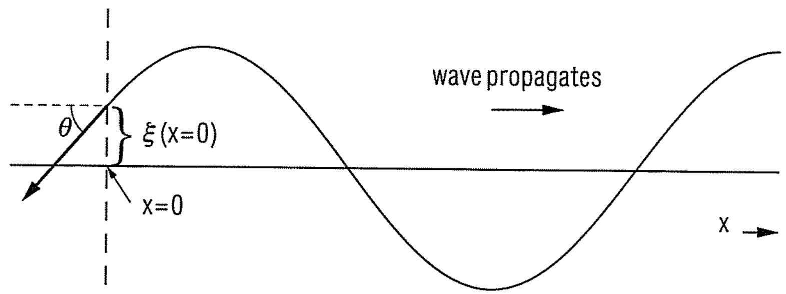
The results of Eqs. (27) - (29) are illustrated in Fig. 4. You can see that at a given point in the medium the energy oscillates between kinetic and potential energy; yet the total energy remains constant, dependent only on the mass of the infinitesimal element, the angular frequency of the harmonic oscillation, and the amplitude of the wave displacement.

PROBLEM SUPPLEMENT

$$Y(\text{iron}) = 2.06 \times 10^{11}\,\text{N/m}^2$$
$$\rho(\text{iron}) = 7.86 \times 10^3\,\text{kg/m}^3$$

Problems 4 and 5 also occur on this module's *Model Exam.*

1.



Consider a string of length L, mass M (mass per unit length $\mu = M/L$) under tension T. Suppose the left end of the string is being moved up and down (transversely) such that a sinusoidal wave, of angular frequency ω and wave number k, travels away from this end along the string.

a. What is the component of the applied force in the transverse direction at any particular time, written in terms of the tension T in the string and the angle θ the string makes with the string axis at that time? The energy of motion of the particles on the string is all transverse so that's the direction of the component of the force which provides this energy.

b. For small displacements ξ the angle θ will always be small such that $\sin\theta \approx \tan\theta \approx \theta$ is a good approximation. Therefore, express your answer to (a) in terms of the slope of the curve traced by the string at the point where the driving force acts.

c. Write down the functional form of $\xi(x,t)$ for a sinusoidal wave propagating to the right.

d. Substitute this into your answer to (b). This tells you the time dependence of the transverse force required to generate a traveling sine wave to the right along the string.

e. If this force displaces the point at the end of the string (at $x = 0$) by an amount $d\xi$, how much work is done by the agent exerting this force?

f. Determine the amount of work done by this driving force during one complete period of motion.

g. Find the average rate at which this work is done using the fact that, for transverse waves on a stretched string, $v = \sqrt{T/\mu}$. Express this rate in terms of the mass density of the string.

2. Longitudinal sound waves propagate down a length of iron rod of cross-sectional area $30\,\mathrm{cm}^2$. The average sound energy density distributed throughout the volume of this iron rod is $4\,\mathrm{J/m}^3$.

a. How is this energy density distributed between kinetic energy of oscillation (of the atoms of the rod) and potential energy?

b. What is the speed of sound along this rod?

c. What is the wave intensity along the rod?

d. What power must be supplied to one end of this rod to maintain this energy density in the rod?

3. The intensity of a spherical sound wave emanating from a point source is observed to be $4 \times 10^{-8}\,\mathrm{W/m}^2$ at a distance of one meter from the source.

a. Find the intensity in decibels of the sound that reaches a point 10 meters from the source.

b. Find the total power supplied by the source.

c. What is the total energy per second crossing the sphere of radius 1 meter with the source at the center?

d. Repeat (c) for the sphere of 10 meters.

e. What is the intensity in decibels 100 meters from the source?

4. The differential equation satisfied by transverse waves along a stretched string of mass M, length L, under tension T is, when the string is aligned parallel to the x-direction:

$$\frac{\partial^2\xi(x,t)}{\partial t^2} = \frac{TL}{M}\frac{\partial^2\xi(x,t)}{\partial x^2}.$$

The string's length is 10 meters, it has a mass of 5 grams, and it is under a tension of 30 newtons. A sinusoidal source at one end of the

string sends sinusoidal waves of wavelength 2 meters down the length of the string. The power supplied by the driving oscillator is 3 watts.

a. What is the speed with which waves propagate along this string?

b. What is the amplitude of the transverse waves propagating along the string?

5. A point source of sound emits 50,000 joules of sound energy every 20 seconds. At a distance 100 meters from the source, what is the intensity of the sound (in decibels), if no energy is lost in the intervening space?

Brief Answers:

1. a. $F_y = -T\sin\theta$ *Help: [S-3]*
   b. $F_y = -T(\partial\xi/\partial x)$ at $x = 0$
   c. $\xi(x,t) = \xi_0 \sin(kx - \omega t + \phi_0)$
   d. $F_y = -kT\xi_0\cos(\omega t - \phi_0)$
   e. $dW = F_y d\xi$
   f. $W = \omega T\xi_0^2\pi/v$ *Help: [S-1]*
   g. $P = \omega^2 T\xi_0^2/(2v)$ *Help: [S-2]*
2. a. Evenly distributed between kinetic and potential.
   b. $v = 5119\,\mathrm{m/s}$
   c. $I = 2.05 \times 10^4\,\mathrm{W/m}^2$
   d. 61.4 watts
3. a. 26 db
   b. 5.03×10^{-7} watts
   c. 5.03×10^{-7} watts
   d. 5.03×10^{-7} watts
   e. 6 db
4. a. $v = 245\,\mathrm{m/s}$.
   b. $\xi_0 = 9$ millimeters.
5. 103 db.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-problem 1f)*

$$\begin{aligned} W_{\text{one cycle}} &= \oint F_y \, d\xi \\ &= kT\xi_0 \oint \cos\omega t d(\xi_0 \sin\omega t) \\ &= k\omega T\xi_0^2 \int_0^{2\pi/\omega} \cos^2(\omega t)\, dt \\ &= kT\xi_0^2 \int_0^{2\pi} \cos^2(\omega t)\, d(\omega t) \\ &= kT\xi_0^2 \pi \\ &= \frac{\omega}{v} T\xi_0^2 \pi \end{aligned}$$

S-2 *(from PS-problem 1g)*

$\Delta t_{\text{one cycle}} = 2\pi/\omega$

S-3 *(from PS-problem 1a)*

See Module 202, Sect. 3b.

MODEL EXAM

$$I_{ref} = 10^{-12}\,\text{W}/m^2$$

1. See Output Skills K1-K2 in this module's *ID Sheet*. One or more of these skills may be on the actual exam.

2. The differential equation satisfied by transverse waves along a stretched string of mass M, length L, under tension T is, when the string is aligned parallel to the x-direction:

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{TL}{M}\frac{\partial^2 \xi(x,t)}{\partial x^2}.$$

   The string's length is 10 meters, it has a mass of 5 grams, and it is under a tension of 30 newtons. A sinusoidal source at one end of the string sends sinusoidal waves of wavelength 2 meters down the length of the string. The power supplied by the driving oscillator is 3 watts.

   a. What is the speed with which waves propagate along this string?

   b. What is the amplitude of the transverse waves propagating along the string?

3. A point source of sound emits 50,000 joules of sound energy every 20 seconds. At a distance 100 meters from the source, what is the intensity of the sound (in decibels), if no energy is lost in the intervening space?

Brief Answers:

1. See this module's *text*.
2. See Problem 4 in this module's *Problem Supplement*.
3. See Problem 5 in this module's *Problem Supplement*.

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第二十章 波的運動

第一節 波動及波動方程式

1. 簡介 在我們在本節中我們將簡單的介紹波動的一些特徵及描述波動的波動方程式。

2. 基本觀念

波動之定義 任何以一固定速度進行之擾動

波動的描述 為了簡單起見，我們先討論一度空間的情形

$\phi(x,t)$ 是一場，ϕ 是一物理量

若 $\phi(x,t=0) = f(x)$，而在 $\phi(x,t) = f(x-vt)$

則我們稱此為一向 x 軸方向以速度 v 進行之波

在 t 時在 $x = x_0 + vt$ 位置 ϕ 之值與在 $t=0$ 時 x_0 處 ϕ 值相同

因此 $\phi(x,t)$ 是一沿 x 軸方向以速度 v 進行之波。

同理 $\phi(x,t) = f(x+v_0 t)$ 是一沿 $-x$ 軸方向以速度 v 進行的波

重疊原理 若兩個獨立之波在空間進行，其在空間中任一點之總擾動仍是由此兩個波動所引起之擾動之和

若兩個以相反方向進行之波 $f_1(x-vt)$ 及 $f_2(x+vt)$ 則其所形成的總擾動為 $\phi(x,t) = f_1(x-vt) + f_2(x+vt)$

若 ϕ 代表的是物質中粒子的位移，而此位移之方向與波動進行之方向相垂直則此波稱為橫波。

若 ϕ 是物質中粒子之位移，而此位移之方向與波動進行之方向同向則此波為縱波

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諧和波

$f(x-vt) = A\sin k(x-vt) = \phi(x,t)$

$\phi(x+\frac{2\pi}{k}, t) = A\sin k(x+\frac{2\pi}{k} - vt) = \phi(x,t)$

$\lambda = \frac{2\pi}{k}$

↳波長 波數

$\phi(x,t) = A\sin(kx - kvt)$

$\phi(x, t+T) = A\sin(kx - kv(t+T)) = \phi(x,t)$

↳週期

$kvT = 2\pi$, $kv = \omega = \frac{2\pi}{T} = 2\pi\nu$

角頻率　　頻率

$\frac{2\pi}{\lambda}v = 2\pi\nu \Rightarrow \lambda\nu = v$

$\phi(x,t) = f(x-vt) = f(u)$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial f}{\partial u}$

$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\frac{\partial f}{\partial u} = -v\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial t} = v^2\frac{\partial^2 f}{\partial u^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$

所以得 $v^2\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial t^2}$

或 $\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$ 此一公式稱為波動方程式

x_1 , x_2

$\phi(x_1,t) = 1.5\times10^{-4}\sin 6\pi t$ at $x = x_1 = 0.5$ cm

$\phi(x_2,t) = 1.5\times10^{-4}\sin(6\pi t + \frac{2\pi}{3})$ at $x = x_2 = 2.5$ m.

$A\sin k(x-vt)$ $A = 1.5\times10^{-4}$

$kx - \omega t = 6\pi t$ (0.5) $\omega = 6\pi = 2\pi\nu$ ($\nu = 3$)

$k(2.5) - \omega t = 6\pi t + \frac{2\pi}{3}$ v negative

$2k = \frac{2\pi}{3}$ $k = \frac{\pi}{3}$ $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/3} = 6$

$\lambda\nu = v = 18$ m/sec.

第　章 干涉與繞射

第一節 兩個同步源之干涉

1. 簡介. 我們在此節中將討論同調及干涉兩個基本觀念

2. 基本觀念

干涉是波動中極重要的現象. 在討論干涉現象時, 我們需要利用波動之下列兩項特性

(1) 波的重疊性

(2) 諧和波之平均強度與其振幅之平方成正比

我們現在舉一簡單的例子來說明.

若 $u_1(x,t) = A_0 \sin(kx - \omega t)$ (1)

及 $u_2(x,t) = A_0 \sin(kx - \omega t - \delta)$ (2)

則利用重疊原理, 則此兩波所形成之結果

$$u(x,t) = A_0 [\sin(kx - \omega t) + \sin(kx - \omega t - \delta)]$$

$$= 2A_0 \cos\frac{\delta}{2} \sin(kx - \omega t - \frac{\delta}{2}) \quad (3)$$

仍是一諧和波. 此一波之振幅

$$A = 2A_0 \cos\frac{\delta}{2} \quad (4)$$

由於諧和波之強度與其振幅成正比. 因此重疊後波之強度

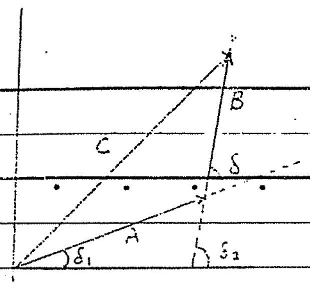
$$I = 4\cos^2\frac{\delta}{2} I_0 = I_0 + I_0 + 2I_0 \cos\delta \quad (5)$$

此處 I_0 是 u_1, u_2 波之強度.

若 $u_1(x,t) = A \sin(kx - \omega t + \delta_1)$ ✓ (6)

$u_2(x,t) = B \sin(kx - \omega t + \delta_2)$ ✓ (7)

則我們可用相量 (Phasor) 的方法來討論此一問題

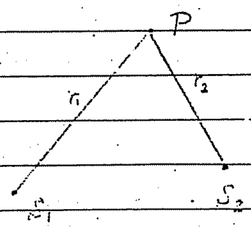


$$C^2 = A^2 + B^2 + 2AB\cos\delta \qquad (8)$$

因此

$$I_C = I_A + I_B + 2\sqrt{I_A I_B}\,\cos\delta \qquad (9)$$

現在我們討論兩個源產生之波在一點 P 之和



若由 S_1 所產生之波在 P 點可寫成

$$\xi_1 = A_1 \sin(\omega t - k r_1) \qquad (10)$$

由 S_2 所產生之波在 P 點可寫成

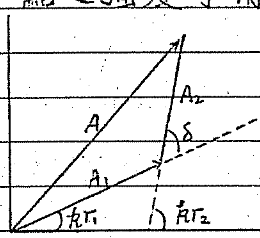
$$\xi_2 = A_2 \sin(\omega t - k r_2) \qquad (11)$$

此處 r_1 及 r_2 分別是 S_1P 及 S_2P 之距離

此時此兩波之相差

$$\delta = k(r_1 - r_2) = \frac{2\pi}{\lambda}(r_1 - r_2) \qquad (12)$$

在 P 點之強度可用相量的方法求得



$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\delta \qquad (13)$$

若 $\cos\delta = +1$，則 A^2 為極大，稱為相長干涉，其條件為

$\delta = 2n\pi$　　n 是整數

也即是 $\frac{2\pi}{\lambda}(r_1 - r_2) = 2n\pi$，$(r_1 - r_2) = n\lambda$ [1]　　(14)

定義雙曲面以 S_1, S_2 為焦點

若 $\cos\delta = -1$ 則 A^2 為極小，稱為相消干涉，其條件為

$\delta = (2n+1)\pi$

也即是 $\frac{2\pi}{\lambda}(r_1 - r_2) = (2n+1)\pi$，$(r_1 - r_2) = (n + \frac{1}{2})\pi$　　(15)

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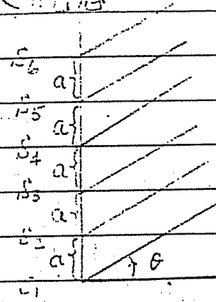
第二節 多個同步源產生之干涉

1. 簡介：我們在此節中討論用相量之方法來處理多個同步源產生之干涉

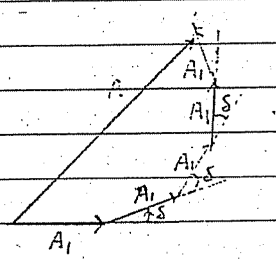
2. 基本觀念



每個光線間所行路徑之差 $r_1 - r_2 = a\sin\theta$，因此每個光源間之相差為

$$\delta = \frac{2\pi}{\lambda} a\sin\theta$$

若有 N 個源則利用相量我們可得到以下之結果



(1) 若 $\delta = 2n\pi$，n 是整數，也即是 $\frac{2\pi}{\lambda} a\sin\theta = 2n\pi$，$a\sin\theta = n\lambda$

則每個相量均是沿著同一方向，因此其結果為一極大值，稱為主要極大點。Principal Maximum

$$A = NA_1$$

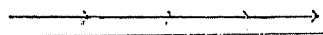
$$I = N^2 I_1$$

此處 N 是源的數目。

(2) 當 $N\delta = 2n'\pi$ $n' = 1, 2, \cdots N-1, N+1, N+2, \cdots$ 時則其圖形變成封閉之多邊形，此時 $A = 0$ 因此 $I = 0$ 為一極小值。

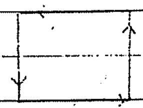
我們現在以 $N = 4$ 為例來說明此一結果。

(i) $n' = 0$ $\delta = 0$ 是一極大點



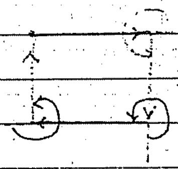
(ii) $n' = 1$ $\delta = \frac{\pi}{2}$ 是一極小點

$4\delta = 2\pi$



(iii) $n' = 2$ $\delta = \pi$ 是一極小點

(iv) $n' = 3$ $\delta = \frac{3\pi}{2}$ 是一極小點



(v) $n' = 4$ $\delta = 2\pi$，此時 $n = 1$ 是一極大值

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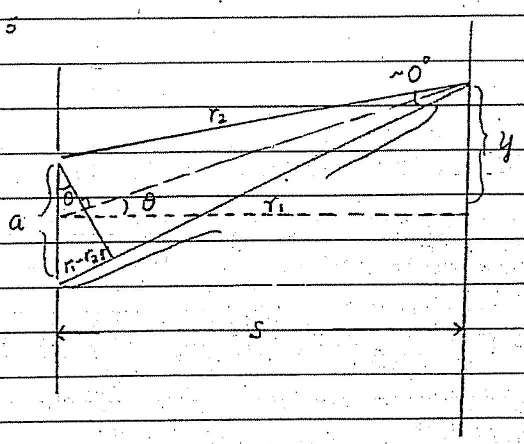
同調：若兩波源以相同之頻率振動並保持一定相差者稱為同調。

我們以上的討論需同調源時方成立。楊氏双狹縫實驗即是產生同調源最簡單之方法。[4,5]



$$r_1 - r_2 = a \sin\theta$$
$$\simeq a\theta$$
$$\simeq \frac{ay}{s}$$

相長干涉 $\frac{ay}{s} = n\lambda$

$$\delta = k(r_1 - r_2)$$

因此其強度 $I = 4I_0 \cos^2 \frac{k(r_1 - r_2)}{2} = 4I_0 \cos^2\left(\frac{\pi a}{\lambda}\sin\theta\right)$

$$= 4I_0 \cos^2 \frac{ya\pi}{s\lambda}$$

由干涉圖案可量度 λ

3. 討論

(1) 此處 A_1, A_2 可以是位置的函數

(2) $r_1 - r_2$ = 常數定義雙曲線面以 S_1, S_2 為焦點。

(3) 此處重要的是相位差

(4) 當 s 大時，θ 是指觀察的方向

(5) 當以下的情形發生時則以上的公式需要修正

(i) 光含不止單一頻率

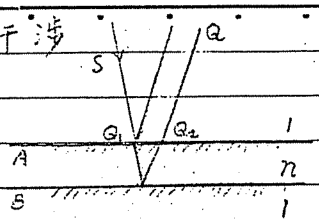
(ii) 光源並不同調

(iii) 狹縫不在比 λ 小很多時

(iv) 光屏離光源不太遠時

應用

(1) 薄膜之干涉



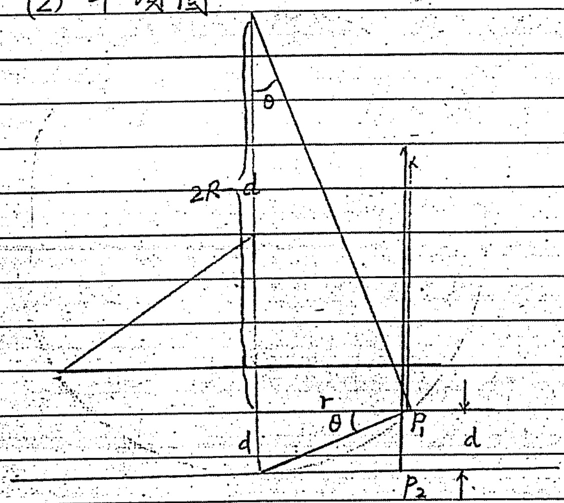
$n>1$ 因此在第 A 面反射時，它會產生 π 的相差

SQ_1，SQ_2 間之相差 $\delta = \frac{2\pi}{\lambda'} \cdot 2d - \pi = \frac{2\pi}{\lambda/n} \cdot 2d - \pi = \frac{4\pi n d}{\lambda} - \pi$

因此當 $\frac{4\pi n d}{\lambda} - \pi = 0, 2\pi, 4\pi, \cdots$ 時是相長干涉

$= -\pi, \pi, 3\pi, \cdots$ 時是相消干涉

(2) 牛頓圈



$\delta = \frac{2\pi}{\lambda} 2d + \pi$

因此 $\frac{4\pi d}{\lambda} + \pi = \pi, 3\pi, 5\pi, \cdots$ 時即是極小值，也即是

$d = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \cdots$

我們可將它寫成 $d = \frac{N\lambda}{2}$ $N = 0, 1, 2, \cdots$

$\tan\theta = \frac{d}{r} = \frac{r}{2R-d}$

$r^2 = 2Rd - d^2 \simeq 2Rd = 2R\frac{N\lambda}{2}$ $(d \ll R)$

也即是 $r = \sqrt{NR\lambda}$ 時會產生相消干涉。

分類：
編號：
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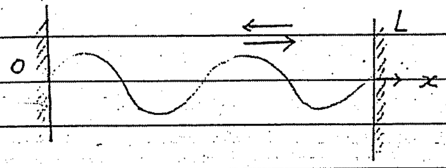
第三節 一度空間之駐波

1. 簡介 我們在此節中將討論一度空間之駐波及其性質

2. 基本觀念



$$\xi(x,t) = A\sin(\omega t + kx) + A'\sin(\omega t - kx) \tag{1}$$

邊界條件 $\xi(0,t)=0$ 此一條件在任何時間均需成立。也即是在任何時間 t

$$(A + A')\sin\omega t = 0 \tag{2}$$

均成立。所以 $A = -A'$ (3)

將第(3)式代入(1)式得

$$\xi(x,t) = A[\sin(\omega t + kx) - \sin(\omega t - kx)] = 2A\sin kx\cos\omega t \tag{4}$$

此一波稱為駐波。在一固定 x 點，它是時間 t 之諧和波，其振幅是位置之函數。

$\frac{2\pi}{\lambda}x = n\lambda$

當 $kx = n\pi$ 時其振幅為零稱為節，也即是在 $x = \frac{1}{2}n\lambda$ 時，它是波節。

相隣波節之距離為 $\frac{1}{2}\lambda$

在此問題中 $\lambda = \frac{v}{\nu}$ 是一任意值

當在 $x=L$ 處，該波也恒為零時，則 $x=L$ 必須是波節，也即是

$$kL = n\pi \tag{5}$$

因此 $\lambda = \frac{2L}{n}$，n 是整數 (6)

它的頻率必須為 $\nu = n\frac{\sqrt{\frac{T}{m}}}{2L} = n\frac{1}{2L}\sqrt{\frac{T}{m}}$ (我們以彈性繩為例) (7)

$= n\nu_1$ (8)

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$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$ 稱為基本頻率

我們現在從波動方程式來討論駐波

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$$

ξ 是 x, t 之函數. 以上討論的駐波有一特性即是 $\xi(x,t)$ 可寫成一 x 函數與一 t 函數之乘積.

若我們假設 $\xi = f(x)\sin\omega t$、代入波動方程式得

$$\frac{d^2 f(x)}{dx^2} = -\frac{\omega^2}{v^2} f(x)$$

令 $k = \frac{\omega}{v}$

則 $\frac{d^2 f}{dx^2} + k^2 f = 0$

其解為 $f = A\sin kx + B\cos kx$

也即是 $\xi = (A\sin kx + B\cos kx)\sin\omega t$

A 與 B 取決於邊界條件

這是討論駐波的出發點

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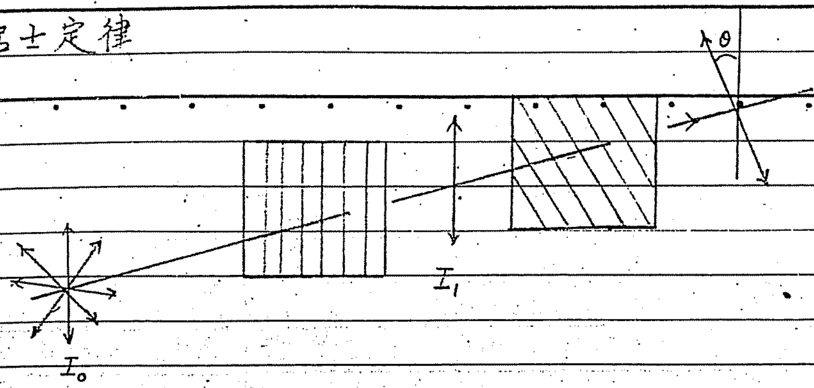
總號：

馬呂士定律



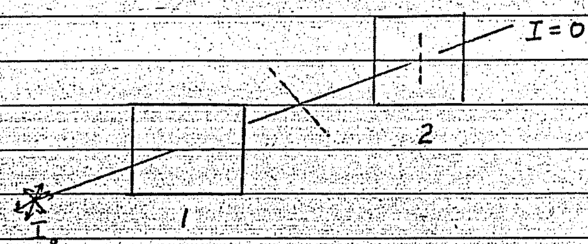
若入射光是無偏極化，則

穿過第一個分析片後，它變成沿垂直方向線偏極，其強度為 $\frac{1}{2}I_0$

穿過第二個後它變成沿與垂直方向成 θ° 線偏極之電磁波其強度為

$\frac{1}{2}I_0 \cos^2\theta$

若 $\theta = 90°$ 則其最終之強度為 0



但若在 1，2 分析片之間加一與 1，2 成 45° 之分析片則

經第 1 片後之強度為 $\frac{1}{2}I_0$

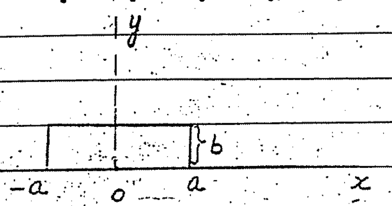
經第 3 片後之強度為 $\frac{1}{2}I_0 \cos^2 45° = \frac{1}{4}I_0$

經第 2 片後之強度為 $\frac{1}{4}I_0 \cos^2 45° = \frac{1}{8}I_0$

Wave Equation

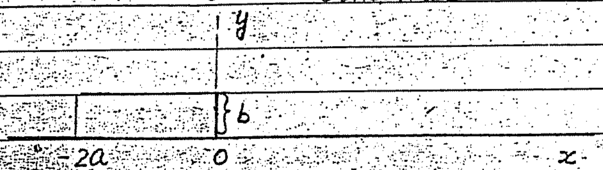
$y = f(x) = b$ for $-a < x < a$

$\quad\;\; = 0$ otherwise



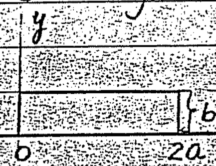
$y = f(x+a) = b$ for $-a < x+a < a$

$\quad\quad\quad\;\; = 0$ otherwise



→ curve has been shifted along the x axis to the right by amount a without deformation

$y = f(x-a) = b$ for $-a < x-a < a$



→ curve has been shifted along the x axis to the left by amount a without deformation

$a = vt$

$y = f(x - vt)$ represents a curve moving to the right with velocity v
$\quad\quad v$ ‖ phase velocity

$y = f(x + vt)$ represents a curve moving to the left with velocity v

Claim $\quad v^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2} \qquad u = x \pm vt \qquad \dfrac{\partial u}{\partial t} = v$

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = \frac{df}{du}(\pm v)$$

$$\frac{\partial^2 y}{\partial t^2} = (\pm v)\frac{d^2 f}{du^2}\frac{\partial u}{\partial t} = (\pm v)(\pm v)\frac{d^2 f}{du^2} = v^2 \frac{d^2 f}{du^2}$$

Similarly $\quad \dfrac{\partial^2 y}{\partial x^2} = \dfrac{d^2 f}{du^2}$

$$\Rightarrow \quad v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \quad \leftarrow \text{wave equation}$$

Claim $\quad y = A \sin k(x - vt)$ is a solution of the wave equation

$$\frac{\partial y}{\partial x} = Ak \cos k(x - vt) \qquad \frac{\partial^2 y}{\partial x^2} = Ak^2(-)\sin k(x - vt)$$

$$v^2 \frac{\partial^2 y}{\partial x^2} = -Ak^2 v^2 \sin k(x - vt)$$

$$\frac{\partial y}{\partial t} = A(kv)\cos k(x - vt) \qquad \frac{\partial^2 y}{\partial t^2} = A(-kv)(-kv)(-)\sin k(x - vt)$$

$$\frac{\partial^2 y}{\partial t^2} = -Ak^2 v^2 \sin k(x - vt)$$

Use the same method, one can show

$y = B \sin k(x + vt)$ is also a solution of the wave equation

Look at this harmonic wave

$$y(x,t) = A \sin k(x - vt) \qquad kx = \text{dimensionless}$$

Claim

$$x' = x + \frac{2\pi}{k}$$

$$y(x',t) = A \sin k\left(x + \frac{2\pi}{k} - vt\right) = A \sin k(x - vt)$$

$$= y(x,t)$$

$$y\left(x + \frac{2\pi}{k}, t\right) = y(x,t) \quad ✓$$

$$\lambda = \frac{2\pi}{k} = \text{"space period"} = \text{wavelength}$$

$$k = \frac{2\pi}{\lambda} = \text{wavelength}$$

$$y = A \sin \frac{2\pi}{\lambda}(x - vt)$$

$$y = A \sin(kx - \underbrace{kvt}_{\omega})$$

$$\omega = \frac{2\pi}{\lambda} v = 2\pi\nu$$

ω: angular frequency

$$P = \frac{2\pi}{\omega} = \frac{1}{\nu} \qquad \omega = \frac{2\pi}{P}$$

P: period

$$y(x,t) = y(x, t + P)$$

$$y(x, t+P) = A \sin(kx - \omega(t+P))$$

$$= A \sin\left(kx - \omega t - \frac{2\pi}{P} P\right)$$

$$= A \sin(kx - \omega t) = y(x,t) \quad ✓$$

$$\lambda = \frac{v}{\nu} = vP$$

λ: distance advanced by the wave motion in one period

Table 23-1

$y = A \sin k(x \pm vt)$ $\qquad$ $y = A \sin \omega\left(t \pm \frac{x}{v}\right)$

$y = A \sin(kx \pm \omega t)$ $\qquad$ $y = A \sin(\omega t \pm kx)$

$y = A \sin 2\pi\left(\frac{x}{\lambda} \pm \frac{t}{P}\right)$ $\qquad$ $y = A \sin 2\pi\left(\frac{t}{P} \pm \frac{x}{\lambda}\right)$

$$P = \frac{2\pi}{\omega}, \quad \lambda = \frac{2\pi}{k}, \quad k = \frac{\omega}{v}$$

$y = y_0 \sin k(x \pm vt)$

$\lambda = \frac{2\pi}{k}$

↑ wavength

k = wave number

$kv = \omega$ = angular frequency

$\omega = 2\pi\nu$ (↑ frequency)

$P = \frac{1}{\nu} = \frac{2\pi}{\omega}$ (↑ period)

y_0 = amplitude

Examples.

(i) $y(x,t) = A_1 \cos k_1(x+vt) + A_2 \sin k_2(x-vt)$ is a solution of $v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

Proof. By direct differentiation

$\frac{\partial y}{\partial t} = -k_1 v A_1 \sin k_1(x+vt) + k_2 v A_2 \cos k_2(x-vt)$

$\Rightarrow \frac{\partial^2 y}{\partial t^2} = -k_1^2 v^2 A \cos k_1(x+vt) + k_2^2 v^2 A \sin k_2(x-vt)$

$\frac{\partial y}{\partial x} = -k_1 A_1 \sin k_1(x+vt) - k_2 A_2 \cos k_2(x-vt)$

$\Rightarrow \frac{\partial^2 y}{\partial x^2} = -k_1^2 A \cos k_1(x+vt) + k_2^2 A_2 \sin k_2(x-vt)$

Comparing the two equations ($\Rightarrow$), it is clear that v^2 times the lower one equals the upper one. However, if the velocity v in the two terms had not been the same, the sum of these terms would not have been a solution. Note that the first term represents a wave (of amplitude A_1 and wavelength $\frac{2\pi}{k_1}$) traveling to the left and the second represents a wave (of amplitude A_2 and wavelength $\frac{2\pi}{k_2}$) traveling to the right. These are traveling through the same medium at the same time.

(ii) Two points on a traveling wave are observed as the wave passes by. At each point x_1 and x_2 the displacement from equilibrium is observed as a function of time. This is found to be (ξ is in meters)

$\xi(x_1, t) = 1.5\times10^{-4} \sin 6\pi t$ at $x = x_1 = 0.5\,m$

$\xi(x_2, t) = 1.5\times10^{-4} \sin(6\pi t + \frac{2\pi}{3})$ at $x = x_2 = 2.5\,m$

(a) amplitude $= 1.5\times10^{-4}\,m$

(b) ν: the coefficient of $t \Rightarrow kv = 6\pi = \omega = 2\pi\nu$

$\nu = 3$ Hertz

(c) $k(x_2 - x_1) = \frac{2\pi}{3}$

$\lambda = \frac{2\pi}{k} = 3(x_2 - x_1) = 6\,m$

(d) $k = \frac{2\pi}{6}$

$\frac{2\pi}{6} v = 6\pi$

$v = 18$ m/sec = speed with which this wave propagates.

(e) The relative sign between the coefficient of x and t is positive.
⇒ the wave is travelling toward the left.

(f) $\frac{\partial \xi}{\partial t}\Big|_{\text{at } x_1 \text{ time } t} = 9\pi \times 10^{-4} \cos 6\pi t$

At $t = 0$ $\quad \frac{\partial \xi}{\partial t}\Big|_{\text{at } x_1,\ t=0} = 9\pi \times 10^{-4}$ m/sec

At $t = 0.25$ sec $\quad \frac{\partial \xi}{\partial t}\Big|_{\text{at } x_1,\ t=0.25\text{ sec}} = 9\pi \times 10^{-4} \cdot \underbrace{\cos 6\pi \times \frac{1}{4}}_{\cos \frac{3\pi}{2} = 0}$

$= 0$

Elasticity of Solids

No solid is perfectly rigid

When several external forces act on a solid at rest and the resultant of these forces is zero ⇒ the body remains at rest

The solid size, or both will, however, be altered by the external force
⇒ the body will be deformed

When a particle is dispaced by a small amount from an equilibrium position ⇒ the particle is subject to a linear restoring force

$$F = -kx$$

Similarily for small deformation
⇒ the size of the deformation is proportional to the deformation force

A solid which springs back to its undeformed configuration when external forces are removed ⇒ perfectly elastic

A solid deformed beyond its elastic limit by a larger external forces
⇒ acquires a permanent set

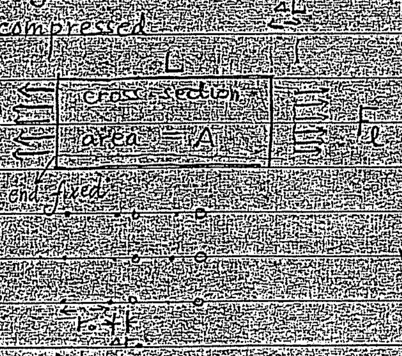
k ⇒ force constant
⇒ elastic modulus
Young's modulus Y
Shear modulus G
Bulk modulus B

Hook's law

$$\text{Elastic modulus} = \frac{\text{stress}}{\text{strain}}$$

stress ⇒ unit of force / area
strain ⇒ dimensionless number describing relative deformation

(1) Young's modulus ⇒ elastic properties of a solid which is streched or compressed



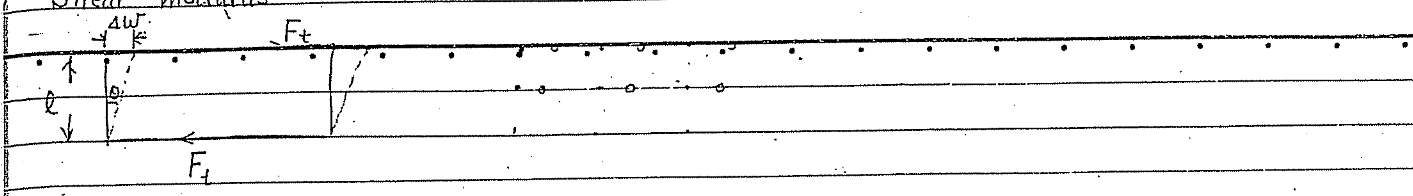
$$\left\{\begin{array}{l} L = n r_0 \\ \Delta L = n \Delta r \end{array}\right.$$

$$\text{stress} = \frac{F_\ell}{A}$$

$$\text{strain} = \frac{\Delta L}{L}$$

$$Y = \frac{F_\ell / A}{\Delta L / L}$$

Shear modulus



$\sigma_t = \frac{F_t}{A}$ → stress

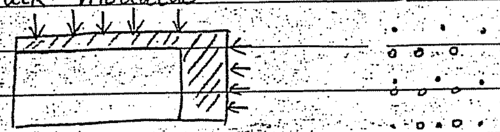
area over which the shear force is applied

$\tan\theta = \frac{\Delta W}{\ell}$ strain

$\simeq \theta$

$G = \frac{F_t / A}{\Delta W / \ell}$

Bulk modulus



$P = \frac{F_n}{A}$ stress

F_n = normal force

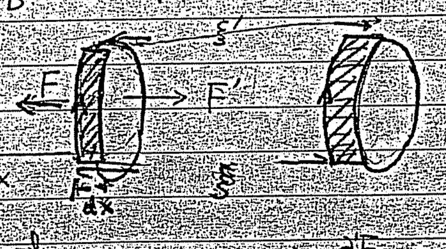
A = surface area where the normal force is applied.

Strain $\frac{\Delta V}{V}$

$B = -\frac{P}{\Delta V / V}$ = Bulk modulus

$\frac{1}{B}$ = compressibility

② Elastic waves in a solid rod



A → ξ

A' → ξ'

$L = dx$

$dx + (\xi' - \xi) = dx + d\xi$

$\Delta L = d\xi$

$L = dx$ $\xi' - \xi = \Delta L$

longitudinal

$F' - F = dF = \frac{\partial F}{\partial x} dx$

due to pull from the right of the rod

$Y = \frac{F_\ell / A}{\Delta L / L} = \frac{F_\ell / A}{\partial \xi / dx}$

mass = $\rho A\, dx$

$F_\ell = YA \frac{\partial \xi}{\partial x}$

acceleration = $\frac{\partial^2 \xi}{\partial t^2}$

$\rho A\, dx \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial F}{\partial x} dx$ $\frac{\partial F_\ell}{\partial x} = YA \frac{\partial^2 \xi}{\partial x^2}$

$\rho A \frac{\partial^2 \xi}{\partial t^2} = YA \frac{\partial^2 \xi}{\partial x^2}$ $v^2 \Rightarrow \frac{Y}{\rho}$, $\Rightarrow v = \sqrt{\frac{Y}{\rho}}$

$$F = YA \frac{\partial \xi}{\partial x}$$

$$\frac{\partial F}{\partial t} = YA \frac{\partial^2 \xi}{\partial x \partial t}$$

$$\frac{\partial^2 F}{\partial t^2} = YA \frac{\partial^3 \xi}{\partial x \partial t^2}$$

$$= YA \frac{\partial}{\partial x} \frac{\partial^2 \xi}{\partial t^2}$$

$$= YA \frac{\partial}{\partial x} \frac{Y}{\rho} \frac{\partial^2 \xi}{\partial x^2}$$

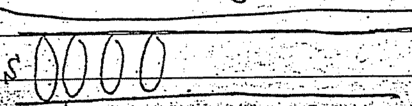
$$= YA \frac{Y}{\rho} \frac{\partial^2}{\partial x^2} \frac{\partial \xi}{\partial x}$$

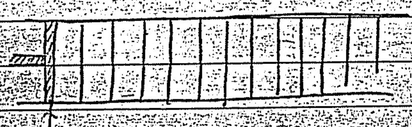
$$= \frac{Y}{\rho} \frac{\partial^2 F}{\partial x^2} \qquad v \equiv \sqrt{\frac{Y}{\rho}}$$

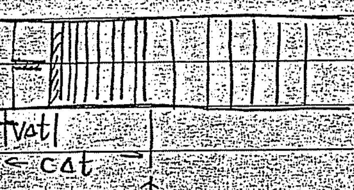
force also satisfied a wave equation with the same v

ξ, F $\parallel$ direction of the propagation
$\Rightarrow$ longitudinal wave.

Pressure wave in a gas column. → compression







$F_x \Delta t = \Delta(mv)$ $\qquad \underbrace{\rho c \Delta t S}_{\text{mass}}$

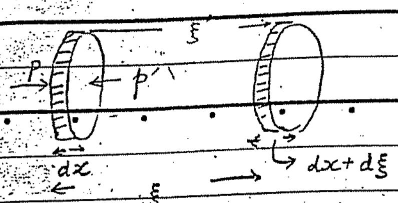
$\Delta v = 0 \rightarrow v$

$F_x \Delta t = \rho c v \Delta t S$ $\qquad \Delta p = \rho c v$

$\therefore F_x = \rho c v S$ $\qquad V = cS\Delta t$

$\Delta V = -vS\Delta t$

$$\frac{\Delta P}{-\Delta V / V} = B \Rightarrow B = \rho c^2 \Rightarrow c = \sqrt{\frac{B}{\rho}}$$



Compressi

Pressure wave in a gas column

p_0, ρ_0 => equilibrium pressure and density of the gas

Volume change ⇒ density will change

$\rho_0 A\, dx = \rho A\,(dx + d\xi)$ ⇒ mass conservation

$$\rho_0 = \rho\left(1 + \frac{d\xi}{dx}\right)$$

$$\Rightarrow \rho = \frac{\rho_0}{1+\frac{d\xi}{dx}} \simeq \rho_0\left(1 - \frac{\partial \xi}{\partial x}\right) \qquad \rho - \rho_0 = -\rho_0 \frac{\partial \xi}{\partial x} \quad ①$$

PV^{γ} = constant $\qquad \rho \propto \frac{1}{V}$ ②

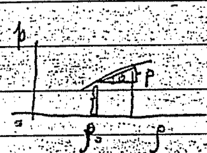
⇒ p is function of ρ

$p = p(\rho)$

Near $\rho = \rho_0$ we can make a Taylor series expansion

$$p = p_0 + (\rho - \rho_0)\left(\frac{dp}{d\rho}\right)_0 + \frac{1}{2}(\rho-\rho_0)^2\left(\frac{d^2p}{d\rho^2}\right)_0$$

$$= p_0 + (\rho - \rho_0)\left(\frac{dp}{d\rho}\right)_0 \quad ③ \quad \Longrightarrow$$



$p - p_0 \sim (\rho - \rho_0)\frac{d}{d}$

$\frac{p - p_0}{\rho - \rho_0} = \tan\theta = \frac{\partial}{\partial}$

$$B = -\frac{dp}{\frac{dV}{V}} \qquad V = \frac{C}{\rho} \qquad dV = \frac{-C}{\rho^2}\, d\rho$$

$$\frac{dV}{V} = \frac{-\frac{C}{\rho^2}\, d\rho}{\frac{C}{\rho}} = -\frac{d\rho}{\rho} \quad ④$$

$$B = \frac{dp}{\frac{d\rho}{\rho}} \Rightarrow \left(\frac{dp}{d\rho}\right)_0 = B\frac{1}{\rho_0} \quad ⑤$$

$$p = p_0 + B\frac{\rho - \rho_0}{\rho_0}$$

$$= p_0 - B\frac{\partial \xi}{\partial x} \checkmark \qquad \underbrace{p - p_0}_{\partial p} = -B\frac{\partial \xi}{\partial x}$$

$$\frac{\partial p}{\partial x} = -B\frac{\partial^2 \xi}{\partial x^2} \checkmark$$

Newton's law

$$-A\, \underset{\text{force}}{dp} = \underbrace{\rho_0 A\, dx}_{\text{mass}}\ \underset{\text{acceleration}}{\frac{\partial^2 \xi}{\partial t^2}}$$

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial^2 \xi}{\partial t^2} \checkmark$$

$$B\frac{\partial^2 \xi}{\partial x^2} = \rho_0 \frac{\partial^2 \xi}{\partial t^2}$$

$$\rightarrow \frac{\partial^2 \xi}{\partial t^2} = \frac{B}{\rho_0}\frac{\partial^2 \xi}{\partial x^2}$$

$$\rightarrow v^2 = \frac{B}{\rho_0} \quad ⑥ \qquad v = \sqrt{\frac{B}{\rho}}$$

$$B = \rho\left(\frac{dp}{d\rho}\right)_0 \checkmark$$

Adiabatic processes

$pV^{\gamma} = c$

$$\Rightarrow \quad p = \frac{c}{V^{\gamma}} = c'\rho^{\gamma}$$

$$\frac{dp}{d\rho} = c'\gamma\rho^{\gamma-1}$$

$$B = \rho_0\left(\frac{dp}{d\rho}\right)_{\rho=\rho_0} = \gamma c'\rho_0^{\gamma} = \gamma p_0$$

$$\therefore \quad v = \sqrt{\frac{\gamma p_0}{\rho_0}} \qquad (7)$$

$PV = NRT \qquad \rho = \frac{M}{V}$

(8)

$$P\frac{M}{\rho} = NRT$$

$$\frac{P}{\rho} = \frac{N}{M}RT \qquad m = \frac{M}{N} = \text{mass of one mole of gas}$$

$$= \frac{RT}{m}$$

$$v = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\frac{\gamma RT}{m}} = \alpha\sqrt{T}$$

$$\alpha = \sqrt{\frac{\gamma R}{M}}$$

At $T = 0°c$, $v_{\text{sound in air}} = 331.45 \ m/sec$

$$\Rightarrow \quad \alpha = 20.055 \ m/sec \cdot (°K)^{1/2}$$

$v = 20.055\sqrt{T} \ m/sec \qquad T$ measured in $°K$

{oscillation takes place quickly
{thermal conductivity of the gas is low } adiabatic

plane independent of r

$2\pi r l$ area for a cylinder

$I = \frac{P}{A}$

$I = \text{const.}$

$I \propto \frac{1}{r}$ for cylindrical wave

$I \propto \frac{1}{r^2}$ for spherical wave

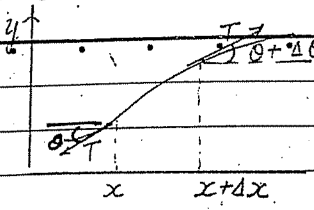
Wave front

direction wave moves → ray

surface on which all points have the same phase of oscillation.

(string)

Transverse waves in a string



$F_y = T_1 \sin(\theta+\Delta\theta) - T_2 \sin\theta$

$F_x = T_1 \cos(\theta+\Delta\theta) - T_2 \cos\theta$ → 0 to first order in $\Delta\theta$ ($T_1 = T_2$)

Short segment (of length Δx) of a string (of mass per unit length μ)

Vibrate transversely, displacement from equilibrium is y

($y=0$ is the equilibrium position)

Neglect the force of gravity

$F_y = T\sin(\theta+\Delta\theta) - T\sin\theta$

$F_x = T\cos(\theta+\Delta\theta) - T\cos\theta$

To first in θ and $\Delta\theta$ $\qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}$, $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\Rightarrow F_y = T\Delta\theta$

$F_x = 0$

$\underline{T\Delta\theta} = (\mu\Delta x)\, a_y = F_y$ $\qquad$ Newton's law

$$T\frac{\Delta\theta}{\Delta x} = \mu\frac{\partial^2 y}{\partial t^2}$$

$\frac{\Delta\theta}{\Delta x} \Rightarrow \frac{\partial\theta}{\partial x}$

$$\frac{\partial y}{\partial x} = \tan\theta$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\tan\theta = \sec^2\theta\,\frac{\partial\theta}{\partial x} \xrightarrow[\text{to first order in }\theta]{} \frac{\partial\theta}{\partial x}$$

$$\boxed{T\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

⇒ satisfy the wave equation with $v^2 = \frac{T}{\mu}$

...t propagates in wave motion? Intensity – definition

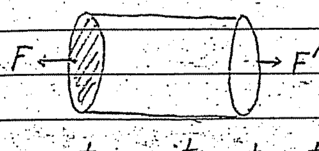
Physical condition ⇒ propagates

Not matter that propagates, but the state of motion of matter

Usually, we describe a dynamical condition in terms of momentum and energy

⇒ in wave motion, energy and momentum are transferred or propagated

Example: longitudinal elastic waves propagating along a rod



Power = force · velocity

Power going out from the right $= F' \frac{d\xi}{dt}$

Power coming in from the left $= F \frac{d\xi}{dt}$

Power transmits to the right from the left 這一段有問題

$= (F' - F) \frac{d\xi}{dt}$ $F > F'$

$= -|\Delta F| \frac{d\xi}{dt}$ $-F \frac{d\xi}{dt}$

$\xi = \xi_0 \sin(kx - vt)$

$\frac{d\xi}{dt} = -\omega \xi_0 \cos(kx - \omega t)$

$\Delta F = YA \frac{\Delta \xi}{\Delta x} \rightarrow YAk\xi_0 \cos(kx - \omega t)$

$\omega = kv \Rightarrow k = \frac{\omega}{v}$

$v = \sqrt{Y/\rho} \Rightarrow v^2 = \frac{Y}{\rho} \Rightarrow Y = \rho v^2$

$\frac{dE}{dt} = YA\omega k \xi_0^2 \cos^2(kx - \omega t)$

$= \rho v^2 A \omega \frac{\omega}{v} \xi_0^2 \cos^2(kx - \omega t)$

$= vA \left[\rho \omega^2 \xi_0^2 \cos^2(kx - \omega t)\right]$

$\left(\frac{dE}{dt}\right)_{ave} = vA \underbrace{\left[\frac{1}{2}\rho\omega^2\xi_0^2\right]}_{\tilde{E}}$ fixed x

↑ energy density

$\frac{dE}{dt} = vA\tilde{E}$

I intensity = energy which flows per unit time across a unit area ⊥ to the direction of propagation

$\frac{dE}{dt} = IA \Rightarrow I = v\tilde{E}$

$B = 10 \log \frac{I}{I_0}$

↑ db = decibel

I_0 is a reference intensity $= 10^{-12}$ W/m²

Waves in two and three dimensions

$\xi = f(x - vt)$

At given time t, ξ take the same value at all points having the same x.

$\Rightarrow$ plane wave propagating // to the X-axis

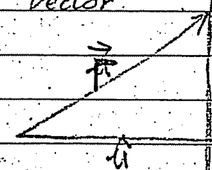
~~$\xi = f(\hat{u}\cdot\vec{r} - vt)$, ξ take the same value at all points having the same $\hat{u}\cdot\vec{r}$~~ (constant)

~~$\Rightarrow$ defines a plane $\perp$ $\hat{u}$ $\qquad \xi = \xi_0 \sin k(\hat{u}\cdot\vec{r} - vt)$ sinusoidal plane wav~~

Define $\vec{k} = k\hat{u}$ = propagation vector = wave number vector

{ length = $k = \frac{2\pi}{\lambda}$ $\qquad$ $kv = \omega$

{ pointing in the direction of propagation



$\xi = \xi_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$

$= \xi_0 \sin(k_x x + k_y y + k_z z - \omega t)$

$kv = \omega$

$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2/v^2$

The wave equation becomes

$$\frac{\partial^2\xi}{\partial t^2} = v^2\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) \checkmark$$

Wave front = wave surface = a surface passing through all points of the medium reached by the wave motion at the same time

Other kinds of waves that propagate in several directions.

cylindrical and spherical waves

If a disturbance that originates at a certain point propagates with the same velocity in all directions (the medium is isotropic)

$\Rightarrow$ spherical waves ✓ $\qquad$ (the same direction)

Sometimes the velocity of propagation is not the same in all directions

$\Rightarrow$ the medium is anisotropic $\Rightarrow$ the waves are not spherical

Spherical waves

Example pressure waves in a homogeneous, isotropic fluid

$p - p_0 = \frac{1}{r} f(r - vt)$

(p_0: equilibrium pressure)

$v = \sqrt{\frac{B}{\rho_0}}$

$\frac{1}{r}$ $\qquad$ wave surface $\propto r^2$

$\left(\frac{dE}{dt}\right)_{ave}$ constant $\Rightarrow$ intensity must $\propto \frac{1}{r^2}$

intensity $\propto |\text{amplitude}|^2$

$\Rightarrow$ amplitude $\propto \frac{1}{r}$

Furthermore, one can show $\frac{1}{r} f(r - vt)$ is a solution of the wave-equation

$$\frac{\partial^2\xi}{\partial t^2} = v^2\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right)$$

Wave motion

Displacement ξ

Velocity of propagation v

$$\xi(x,t) = f_1(x - vt) + f_2(x + vt)$$

ξ, F oriented along the direction of propagation ⇒ the wave motion is longitudinal

ξ is perpendicular to the direction of propagation ⇒ " " " " transverse

Transverse wave: Let us take the direction of propagation to be along X-axis

⇒ there are many directions of displacement ⊥ to the X-axis

⇒ ξ may be consider as a vector with components along the Y and Z-axes

While the disturbance propagates, the direction of ξ may change from point to point ⇒ twisting of the string

If all the displacement is always in the same direction

⇒ linearly polarized

If ξ is constant length but changes in direction ⇒ circularly polarized

Transverse wave elastic waves in a bar ⇒ produce longitudinal and transverse wave

Torsional wave: torque applied at free end of a rod

Surface waves in a liquid:

Possible relevant variables: gravity, surface tension, and density

$$v = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\mathcal{T}}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda}}$$

$\mathcal{T}$ → surface tension $\quad \frac{\text{dyne}}{\text{cm}}$

depth >> wavelength

Note: the velocity of propagation depends on the wavelength

[$\nu = v/\lambda$ ⇒ the velocity of propagation depends on the frequency]

If the velocity of propagation of a wave motion depends on the wavelength or the frequency ⇒ there is dispersion

If a wave motion resulting from the superposition of several harmonic waves of different frequencies impinge on a dispersive medium, the wave is distorted, since each of its component waves propagates with a different velocity

Dispersion is a very important phenomenon present in several types of wave propagation

⇒ example: prism

Group velocity

Pulse, velocity of the pulse

Example: $\xi = \xi_0 \sin(kx - \omega t) + \xi_0 \sin(k'x - \omega' t)$

$$= \xi_0 [\sin(kx - \omega t) + \sin(k'x - \omega' t)]$$

$$= 2\xi_0 \cos\tfrac{1}{2}[(k'-k)x - (\omega'-\omega)t] \sin\tfrac{1}{2}[(k+k')x - (\omega'+\omega)t]$$

$k \approx k'$ ⇒ $k \approx \frac{1}{2}(k'+k)$

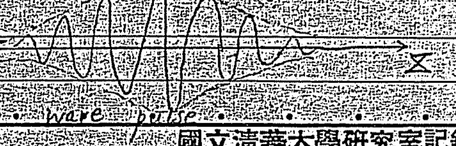
$\omega \approx \omega'$ ⇒ $\omega \approx \frac{1}{2}(\omega'+\omega)$

$$\xi = \underbrace{2\xi_0 \cos\tfrac{1}{2}[(k'-k)x - (\omega'-\omega)t]}_{A} \sin(kx - \omega t)$$

$A = 2\xi_0 \cos\frac{1}{2}[(k'-k)x - (\omega'-\omega)t]$

↑ modulating amplitude

↳ corresponds to a wave motion propagated with velocity v_g



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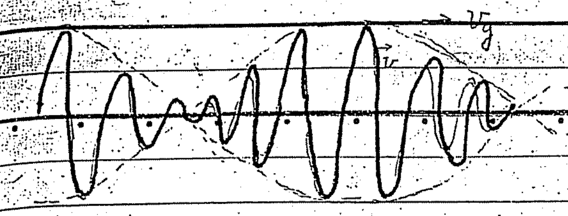


$$v_g = \frac{\omega' - \omega}{k' - k} \rightarrow \frac{d\omega}{dk} \qquad v = \frac{\omega}{k}$$

$$v_g = \frac{d\omega}{dk}$$
$$= \frac{d(vk)}{dk} = v + k\frac{dv}{dk}$$

v_g = group velocity
v = phase velocity
Maximum of the pulse propagates with the group velocity v_g
$\Rightarrow$ in dispersive medium, the signal velocity is the group velocity
For the case of surface wave in the long wavelength limit

$$v = \sqrt{\frac{g\lambda}{2\pi}} \qquad \lambda = \frac{2\pi}{k}$$

$$= \sqrt{\frac{g\frac{2\pi}{k}}{2\pi}} = \sqrt{g}\, k^{-\frac{1}{2}}$$

$$v_g = v + k\frac{dv}{dk} \qquad \frac{dv}{dk} = \sqrt{g}\left(-\frac{1}{2}\right)k^{-3/2}$$

$$= -\frac{1}{2}\sqrt{g}\, k^{-\frac{1}{2}} k^{-1}$$

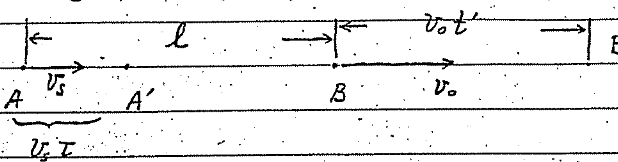
$$= -\frac{1}{2}\frac{v}{k}$$

$$\therefore \quad v_g = v + k\left(-\frac{1}{2}\frac{v}{k}\right) = \frac{1}{2}v$$

v independent of $k \Rightarrow v_g = v$

Doppler effect.

Source of the wave and the observer are in relative motion with respect to the medium in which the waves propagates, the frequency of the wave observed is different from the frequency of the source ⇒ Doppler effect.



v_o = velocity of the observer relative to the medium

v_s = velocity of the source relative to the medium

t=0 source at Ⓐ   AB = l
observer at Ⓑ

source emits a wave that reaches the observer at time t

In that time the observer has moved a distance $v_o t$

Total distance is $l + v_o t$

v = velocity of propagation in the medium

$$vt = l + v_o t$$

$$t = \frac{l}{v - v_o}$$

At time τ the source is at A', the wave reaches the observer at time t'

point of emission A'

point of receiving B'

distance travelled = $(l - v_s\tau) + v_o t'$

actually travelling time of the wave = $t' - \tau$

$$v(t' - \tau) = (l - v_s\tau) + v_o t'$$

$$vt' - v\tau = l - v_s\tau + v_o t'$$

$$(v - v_o)t' = l + (v - v_s)\tau$$

$$t' = \frac{l + (v - v_s)\tau}{v - v_o}$$

Source time interval τ number of wave emitted = $\nu\tau$

Observer time interval $\tau' = t' - t$ number of " received = $\nu'\tau'$

$$\nu'\tau' = \nu\tau \qquad \nu' = \frac{\tau}{\tau'}\nu$$

$$\tau' = t' - t = \frac{l + (v - v_s)\tau}{v - v_o} - \frac{l}{v - v_o} = \frac{(v - v_s)\tau}{v - v_o}$$

$$\nu' = \frac{\tau}{\tau'}\nu = \frac{v - v_o}{v - v_s}\nu$$

ν = frequency emitted by the source

ν' = " " observed by the observer

$$\nu' = \frac{v - v_o}{v - v_s}\,\nu$$

$$= \frac{1 - v_o/v}{1 - v_s/v}\,\nu$$

$$\cong \left(1 - \frac{v_o}{v}\right)\left(1 + \frac{v_s}{v}\right)\nu$$ here we assume $v_s \ll v$, $v_o \ll v$

$$= \left[1 - \frac{v_o}{v} + \frac{v_s}{v} - \frac{v_o v_s}{v^2}\right]\nu$$

$$\simeq \left[1 - \frac{v_o - v_s}{v}\right]\nu = \left(1 - \frac{v_{os}}{v}\right)\nu \qquad v_{os} = v_o - v_s$$

$$\nu' = \left(1 - \frac{v_{os}}{v}\right)\nu$$

$v_{os} = v_o - v_s$

$v_{os} > 0$ observer is moving away from the source
ν' decreases, i.e., $\nu' < \nu$

$v_{os} < 0$ observer is approaching the source
ν' increases $\nu' > \nu$

$\Rightarrow v_o = 0$, $v_s > v$ $\Rightarrow$ Mach wave $\Leftrightarrow$ shock wave
must be problem

Example:

source at rest observer at rest

$\nu = 600$ c/sec

(i) How many "wavelets" (complete cycles of the traveling wave) does this source emit in time T?
Take $T = 5$ sec
$\Rightarrow$ 5 sec × 600 c/sec = 3000

(ii) Over what distance in space is this wave-train spread out if the speed of sound is 340 m/sec?
$\Rightarrow$ vt = 340 m/sec × 5 sec = 1700 m

(iii) What is the distance between adjacent "crests" of this wave
$\Rightarrow \lambda = \frac{1700\,m}{3000} = 0.567\,m$

$\lambda = \frac{v}{\nu} = \frac{340\ m/sec}{600\ c/sec} = \frac{3.4}{6}\,m = 0.567\,m$

(iv) With what speed does this wave-train move by the observer?
v = 340 m/sec

(v) What length of wavetrain moves by the observer in 1 sec?
340 m/sec · 1 sec = 340 m

(vi) How many wavelets are contained in this wavetrain (of 1 second duration)?
340 m / 0.567 m = 600

(vii) How many wavelets move by the observer in 1 sec?
600

(viii) What frequency of the sound does then the observer detect?

600 hertz

Now suppose the source moves toward the observer at a speed of 40 m/sec while the observer moves toward the source, at 60 m/sec

(i) The source emits sound waves as it moves. In 5 seconds how many wavelets has it emitted?

5 second × 600 c/sec = 3000

(ii) How far from the original source location has the front of the wavetrain gone?

5 sec × 340 m/sec = 1700 m

(iii) How far from the original source location was the last of the 5 second wavetrain emitted?

5 sec × 40 m/sec = 200 m

(iv) From (ii), (iii), over what distance in the air is this wavetrain extended?

1500 m

(v) To an observer fixed with respect to the air what is the distance between the adjacent wavelets?

$$\frac{1500\,m}{3000} = 0.5\,m$$

So what wavelength does he think the sound wave has?

0.5 m

(vi) Compare this with the wavelength of the source were at rest

$\lambda_{at\ rest} = 0.567\ m$

$\lambda_{now} = 0.5\ m$

(vii) How fast does the wavetrain go by the moving observer?

$v = 340\ m/sec + 60\ m/sec = 400\ m/sec$

(viii) How many wavelets go by him in 1 sec?

[400 m/sec] / 0.5 m = 800

(ix) What frequency does observer then detect?

800 hertz

(x)

$$\nu' = \frac{v - v_o}{v - v_s}\,\nu = \frac{340\ m/sec + 60\ m/sec}{340\ m/sec - 40\ m/sec}\cdot 600\ \text{Hertz}$$

↑ eq. 23.50

$$= \frac{4}{3}\ 600\ \text{hertz} = 800\ \text{hertz}$$, the same answer

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Spherical Wave

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$\psi(x, y, z, t)$

Now if the problem has spherical symmetry $\Rightarrow \psi(x,y,z,t) \rightarrow \psi(r,t)$

We want to show $\psi = \frac{1}{r} f(r - vt)$ is a solution of the wave equation.

Proof $\quad r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial \psi(r,t)}{\partial x} = \frac{\partial \psi(r,t)}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 \psi(r,t)}{\partial x^2} = \frac{\partial \psi(r,t)}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\frac{\partial}{\partial x}\frac{\partial \psi(r,t)}{\partial r}$$

$$= \frac{\partial \psi(r,t)}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial \psi(r,t)}{\partial r}$$

$$= \frac{\partial^2 \psi(r,t)}{\partial^2 r}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial \psi(r,t)}{\partial r}\frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} + x\frac{\partial}{\partial x}\frac{1}{r}$$

$$= \frac{1}{r} + x\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{1}{r}\right)$$

$$= \frac{1}{r} + x\frac{x}{r}\left(-\frac{1}{r^2}\right)$$

$$= \frac{1}{r}\left[1 - \frac{x^2}{r^2}\right]$$

$$\frac{\partial^2 \psi(r,t)}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2 \psi(r,t)}{\partial^2 r} + \frac{1}{r}\left[1 - \frac{x^2}{r^2}\right]\frac{\partial \psi(r,t)}{\partial r}$$

With the same methed, we obtain

$$\frac{\partial^2 \psi(r,t)}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 \psi(r,t)}{\partial^2 r} + \frac{1}{r}\left[1 - \frac{y^2}{r^2}\right]\frac{\partial \psi(r,t)}{\partial r}$$

and

$$\frac{\partial^2 \psi(r,t)}{\partial z^2} = \frac{z^2}{r^2}\frac{\partial^2 \psi(r,t)}{\partial r} + \frac{1}{r}\left[1 - \frac{z^2}{r^2}\right]\frac{\partial \psi(r,t)}{\partial r}$$

The three dimensional wave equation becomes

$$\underbrace{\frac{\partial^2 \psi(r,t)}{\partial^2 r} + \frac{2}{r}\frac{\partial \psi(r,t)}{\partial r}}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r,t))} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r,t)) = \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\frac{\partial \psi(r,t)}{\partial r} + \psi(r,t)\right]\right\}$$

$$= \frac{1}{r}\left[r\frac{\partial^2 \psi(r,t)}{\partial r^2} + \frac{\partial \psi(r,t)}{\partial r}\cdot 1 + \frac{\partial \psi(r,t)}{\partial r}\right]$$

$$= \frac{\partial^2 \psi(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial \psi(r,t)}{\partial r}$$

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$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r,t)) = \frac{1}{v^2}\frac{\partial^2\psi(r,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2}{\partial r^2}(r\psi(r,t)) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi(r,t)) = 0$$

$r\psi(r,t) = f(r-vt)$ is a solution of the problem

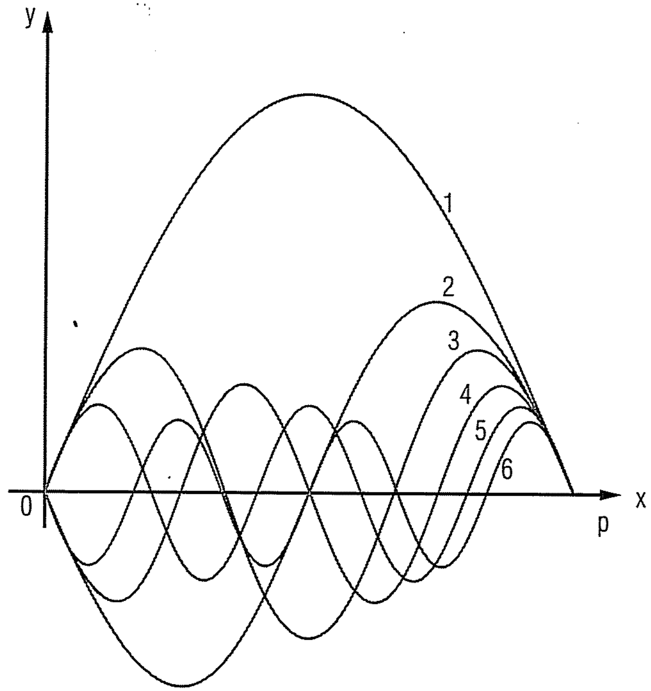
$$\Downarrow$$

$$\psi(r,t) = \frac{1}{r}f(r-vt)$$

PHYSNET

MISN-0-50

FOURIER SERIES AND INTEGRALS



Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

FOURIER SERIES AND INTEGRALS

by

E. H. Carlson, Michigan State University

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Title: **Fourier Series and Integrals**

Author: E. H. Carlson, Department of Physics, Michigan State University

Version: 2/1/2000 Evaluation: Stage 0

Length: 1 hr; 24 pages

Input Skills:

1. Compute definite and indefinite integrals of simple functions, including sine and cosine functions (MISN-0-1).
2. Understand the definite integral as an area (MISN-0-1).
3. Be familiar with the possibility of expansion of a function in a power series (MISN-0-4).

Output Skills (Knowledge):

K2. State sufficient conditions for the existence of the Fourier transform of a function.

Output Skills (Rule Application):

R1. Estimate the sizes of the Fourier coefficients by inspection of $f(x)$, considering its overlap with sine and cosine functions and noting discontinuities, cusps, peaks, wiggles in $f(x)$ of size ℓ, and symmetry.

R2. Compute the sine and cosine Fourier transform of a given $f(x)$.

R3. Sketch, by inspection, the Fourier transform of a given $f(x)$.

FOURIER SERIES AND INTEGRALS

by

E. H. Carlson, Michigan State University

1. Introduction

Suppose you have a function $f(x)$ defined in an interval

$$-L/2 < x < L/2$$

on the x-axis, as in Fig. 1.

You are probably familiar with the notion that, if $f(x)$ is sufficiently well behaved, you can expand it in a power series:

$$f(x) = a_0 + a_1 x + a_2 x + \dots,$$

a Taylor Series. It is also possible to expand it in a series of sine and cosine functions:

$$f(x) = a_0 + a_1 \cos\left(2\pi\frac{x}{L}\right) + a_2 \cos\left(2\pi\frac{2x}{L}\right) + \dots +$$
$$b_1 \sin\left(2\pi\frac{x}{L}\right) + a_2 \sin\left(2\pi\frac{2x}{L}\right) + \dots,$$

a Fourier Series. Here are the advantages of using a Fourier expansion:

1. Many problems, such as those involving waves and oscillations, are particularly simple when expressed this way. That is because they generally have some periodicity, some interval of x over which $f(x)$ repeats.

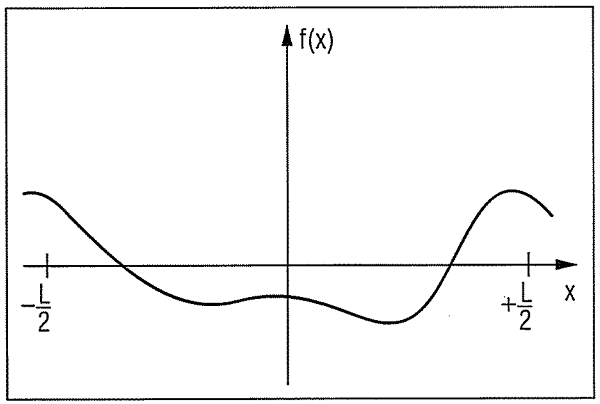

Figure 1. Some function which we wish to represent with an appropriate series.

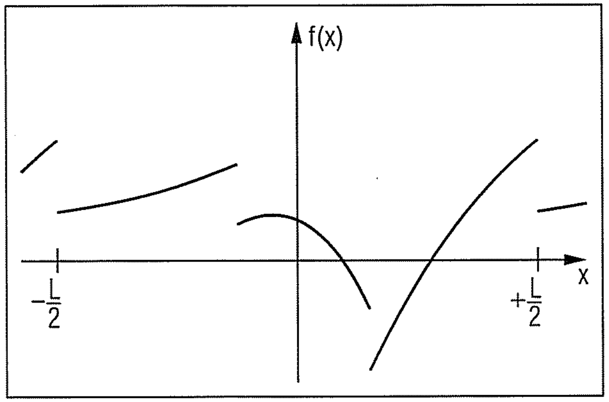

Figure 2. A function $f(x)$ having three discontinuities, counting the two at $(\pm L/2)$ as one (they are the same).

2. The criteria that $f(x)$ must satisfy, in order that the series converge, are not very stringent. It is sufficient that, in the interval $-L/2 < x < L/2$, $f(x)$ is finite and has a finite number of maxima and minima. It may even have a finite number of discontinuities. These are called Dirichlet's conditions.

3. If $f(x)$ is not periodic in x, we can still use the general idea by letting $L \to \infty$, thereby obtaining the Fourier Integral representation:

$$f(x) = \int_{-\infty}^{\infty} [A(k)\cos(kx) + B(k)\sin(kx)]\, dk \qquad (1)$$

2. The Fourier Series

2a. The Coefficient Equations. We will discuss the relationship between $f(x)$ and the coefficients a_k, b_k, leaving derivations and proofs to a mathematics text. The coefficients are defined by the integrals:

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, dx$$
$$a_k = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(2\pi k x/L)\, dx \qquad (2)$$
$$b_k = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin(2\pi k x/L)\, dx.$$

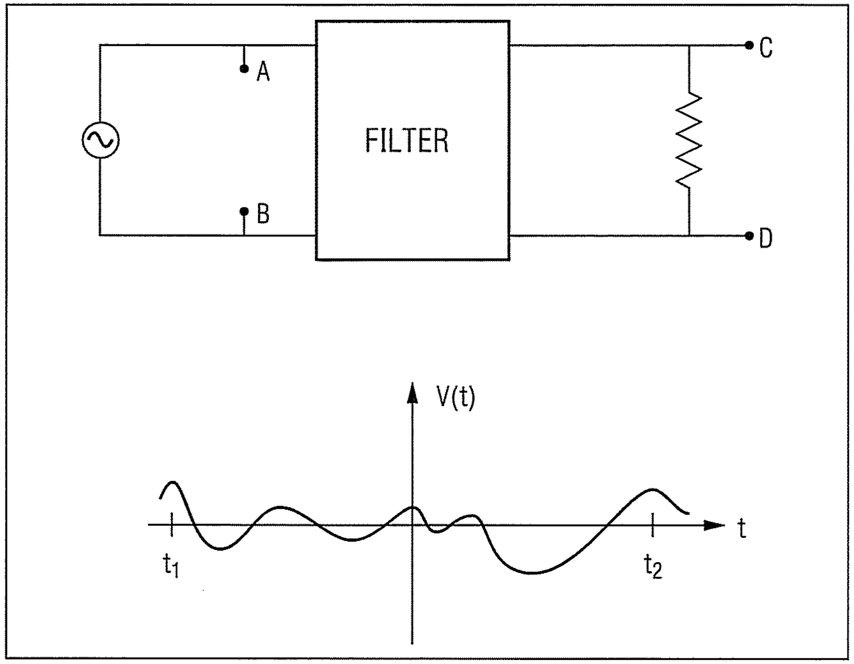

Figure 3. A voltage wave that is input to a filter.

2b. An Example. Fourier Series are useful not only as a computational tool, for which use Eqns. (2) must be evaluated, but even more so as a conceptual tool which simplifies the description of $f(x)$ for many applications. An example is shown in Fig. 3. Here the generator produces a repetitive time-varying wave form whose voltage across points A, B is $V(t)$. The voltage across points C, D will be most clearly expressed when the Fourier Series for $V(t)$ is known. Write:

$$V(t) = V_0 \sum_{k=1}^{\infty} (a_k \cos \omega_0 kt + b_k \sin \omega_0 kt).$$

If the filter passes only frequencies above some frequency ω_1 then only voltage Fourier components for which $k > \omega_1/\omega_0 0$ will appear at points C, D. One will usually begin the analysis of such a physical problem by inspecting $f(x)$ and seeing which coefficients a_k, b_k are large and which are small or zero. This gives insight into the solution, guidance in calculating the series coefficients, and a check on possible gross errors in the solution.

2c. Partial Sum and Formal Definition. Suppose we keep only the first $2n + 1$ terms in the Fourier Series, as we certainly would do in any numerical calculation of the coefficients. This defines the "partial sum" ϕ:

$$\phi_n(x) = a_0 + \sum_{k=1}^{n} [a_n \cos(2\pi kx/L) + b_n \sin(2\pi kx/L)]. \qquad (3)$$

The series obtained as $n \to \infty$ defines the Fourier Series if a_k and b_k are calculated using the Fourier coefficient equations given below.

3d. Non-Periodic but Localized Functions. For many physical situations $f(x)$ will be "localized," i.e., $f(x) \to 0$ as $x \to \pm\infty$. For example, $f(x)$ may represent a pulse or a wave train of finite dimensions. Then one can simply pick an interval L so large that it contains essentially all of $f(x)$, i.e., so that $f(x) \approx 0$ for $|x| > L/2$. Then $f(x)$ can be represented by a Fourier Series inside the interval but not outside the interval. Outside the interval we abandon the Fourier Series and simply set $f(x)$ equal to zero.

▷ Show that the resulting Fourier Series will not correctly represent $f(x)$ outside the interval.

2e. Estimating the Coefficients. Here is how professionals estimate the coefficients to see which are important, which are marginal, and which are negligible:

1. a_0 is just the average value of $f(x)$ over the interval.

2. Each a_k, b_k is proportional to the "overlap" of the corresponding cosine or sine function with $f(x)$. That is, when $f(x)$ and $\cos 2\pi kx/L$ are large and positive in the same places, negative or zero in the same places etc., then a_k will be large and positive. (What shape must $f(x)$ have for a_k to be large and negative?) The "overlap" idea is central to our method for roughly evaluating the integrals of Eqns. (2), and some particular cases will be discussed in the next few numbered remarks.

3. Low values of k contribute (through a_k and b_k) to the overall, broad outline of $f(x)$, while smaller scale structures (wiggles, peaks, etc.) that occupy a length $\ell < L$ on the x-axis require contributions from sine and cosine functions whose wavelength λ is near ℓ in size ($\lambda = L/k$).

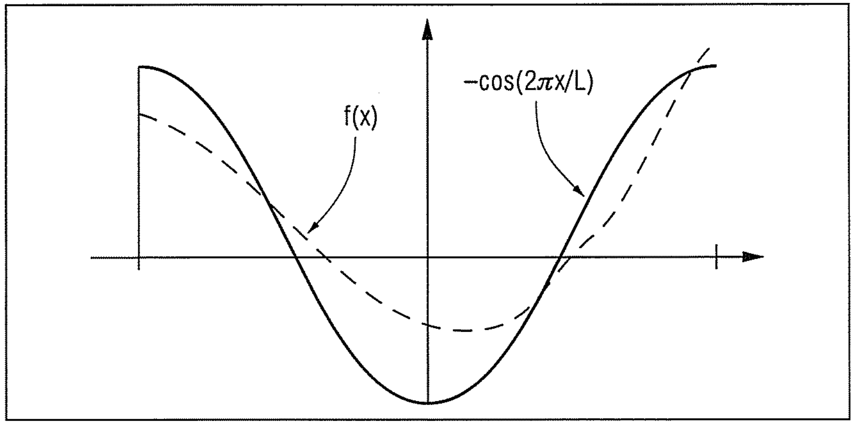

Figure 4. The overlap is large between $f(x)$ and $\cos(2\pi x/L)$. What about the overlap of $f(x)$ with $\sin(2\pi x/L)$?

4. If the size of the smallest structure in $f(x)$ is ℓ, then a_k, b_k fall off in size rapidly as k becomes much larger than L/k. The reason can be seen from Fig. 6.

5. The polynomials ϕ_n are periodic in x, so $\phi_n(x \pm mL) = \phi_n(x)$, where m is an integer. Thus $f(x)$ (which we did not necessarily define outside of $-L/2 < x < L/2$) is treated as being periodic. This may introduce a discontinuity or cusp at the points $x = +L/2$.

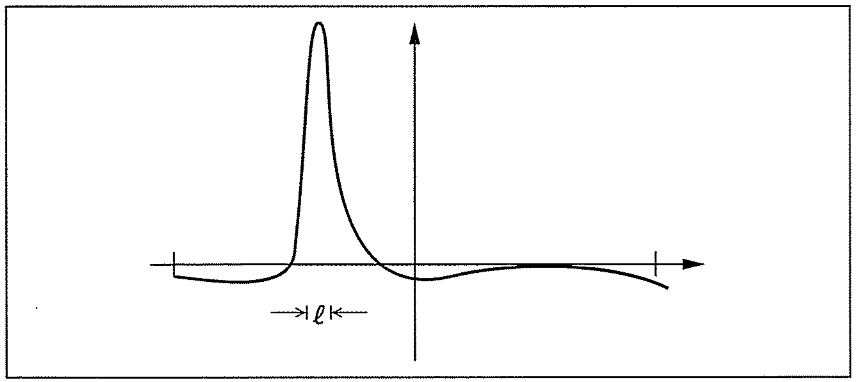

Figure 5. Coefficients a_k, b_k with $k = L/\ell$ will be relatively large.

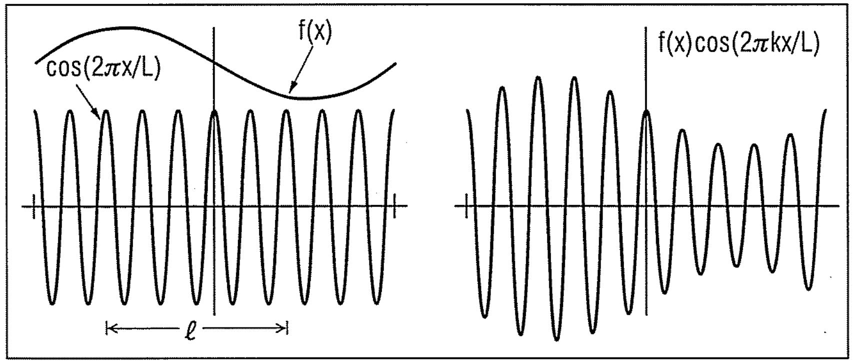

Figure 6. The integral defining a_k has many positive lobes that are nearly cancelled by the adjacent negative one of nearly the same size; so the whole integral is small.

6. Discontinuities in $f(x)$ are "structure" whose characteristic dimension ℓ is zero. They introduce the requirement that $a_k \propto 1/k$, $b_k \propto 1/k$ as $k \to \infty$, and likewise, "cusps" (where df/dx is discontinuous) give a_k, $b_k \propto 1/k^2$.

7. The symmetry of $f(x)$ may simplify the series. A function $f(x)$ is called "even" if $f(x) = f(-x)$ and "odd" if $f(x) = -f(-x)$. (What are the symmetries of $\sin x$, of $\cos x$?) If $f(x)$ is even, only a_k's will be non zero, if $f(x)$ is odd, only b_k's will be non zero. (Why?)

8. The partial sum $\phi_n(x)$ is a least squares fit to $f(x)$. The error in representing $f(x)$ by $\phi_n(x)$ is $e_n(x) = f(x) - \phi_n(x)$. Of all the possible methods of choosing the coefficients a_k, b_k, that given by

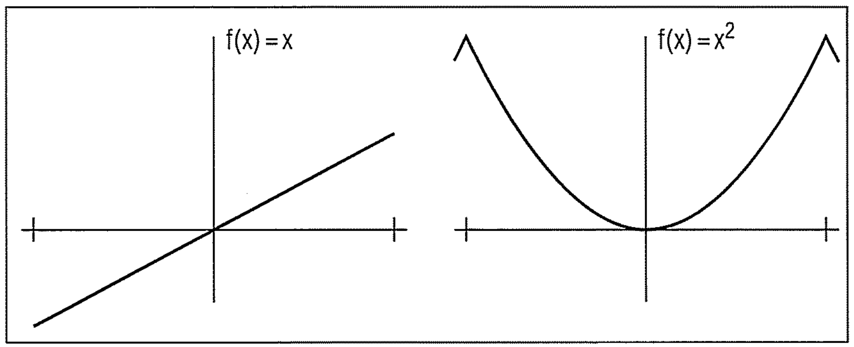

Figure 7.

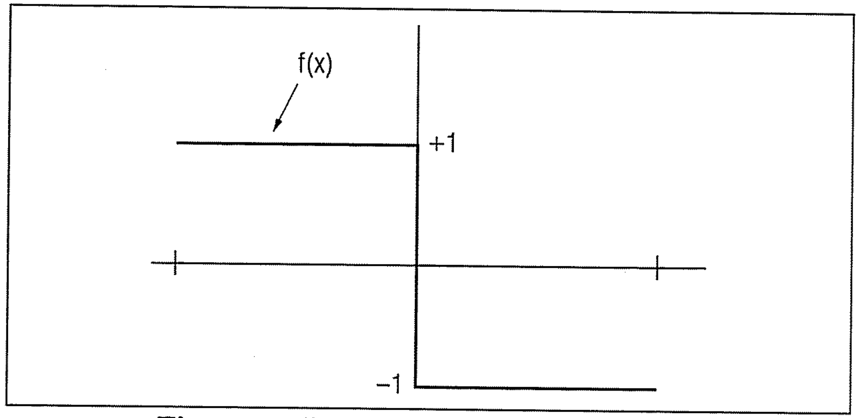

Figure 8. Fourier transform of a step function. $f(x) = -4/\pi \sum_{k=1,3,5,\ldots} (1/k)\sin(2\pi kx/L)$. Can you explain why the k-even terms are missing? Hint: sketch-in $\sin(4\pi x/L)$. Explain why there is a minus sign in front.

Eqns. (2) minimizes the integral $I = \int |e_n(x)|^2\, dx$.

9. From the fact that Eqns. (2) do not contain n, we see that when we approximate $f(x)$ by $\phi_n(x)$ and determine the coefficients a_k, b_k, $k < n$, and then decide to make a better approximation $\phi_p(x)$, $p > n$, the coefficients a_k, b_k for $k < n$ already determined will not change.

10. A single wave at wavelength $\lambda = L/\ell$ cannot, of course, form the peak. There must be many other waves of wave lengths near $\lambda =$

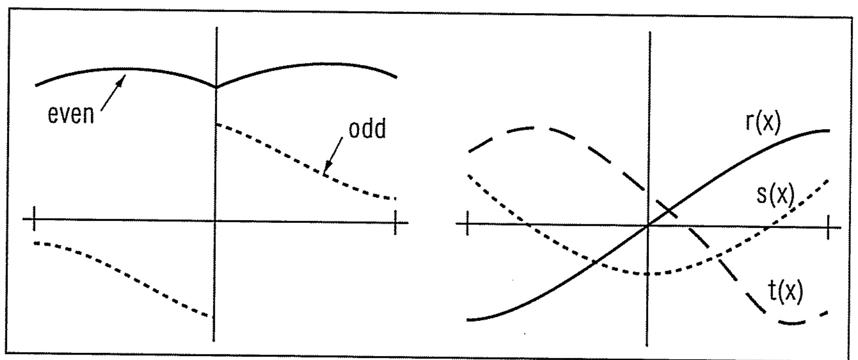

Figure 9. Which of r, s, t is even, odd, neither? Can one function be both even and odd?

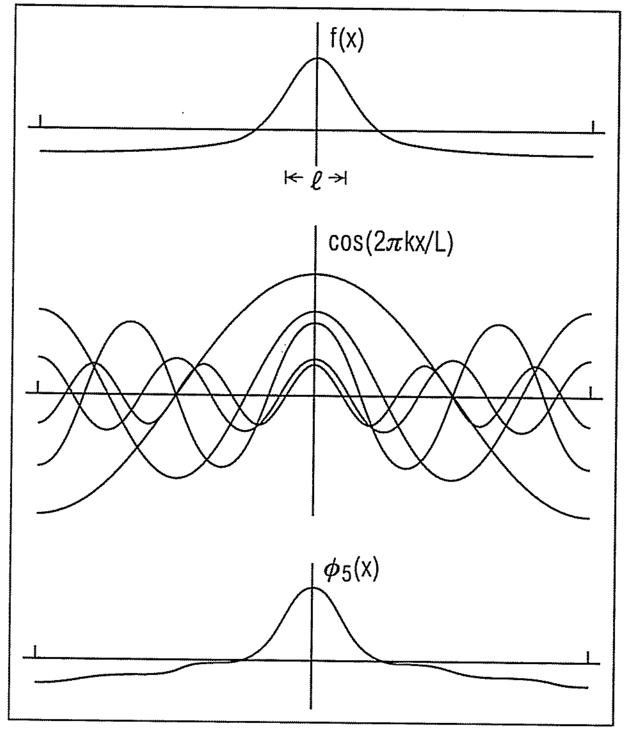

Figure 10. The cosine functions are plotted with 4-fold vertical exaggeration for clarity.

L/ℓ which add to each other at the position of the peak and cancel each other elsewhere.

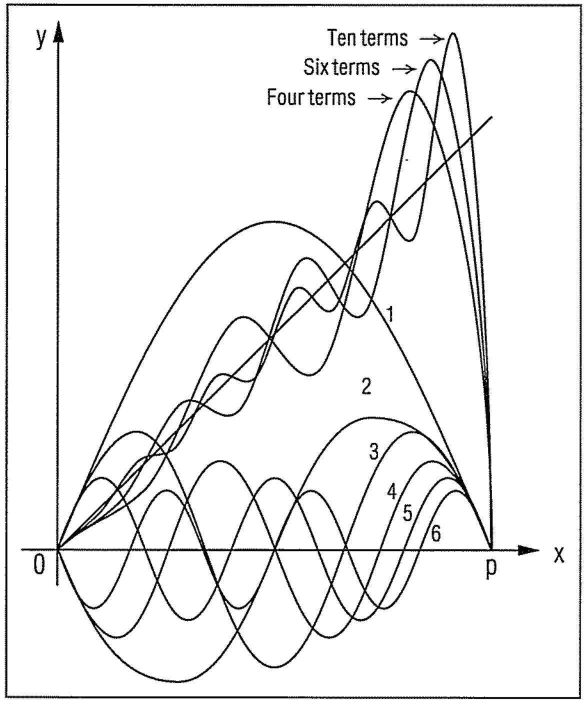

Figure 11. Representation of $y(x) = x$, $-\pi \leq \pi$, by $\phi_n(x)$ for $n = 4$, 6, 10 (only the region where $x > 0$ is shown). Here $b_k = (2/k) \cdot (-1)^k$, $a_k = 0$.

3. The Fourier Integral

3a. Series vs. Integral. If a function $f(x)$ is not periodic or is not restricted to a finite interval of x, the function cannot be expanded in a Fourier Series and one must turn to the Fourier Integral. This is equivalent to letting the period or the localization interval go to infinity so the sum in the Fourier Series becomes an integral.

3b. Transition: Series to Integral. Here we will make the transition from the Fourier Series to the Fourier Integral.

For simplicity of derivation, let $f(x)$ be an even function. Assume a periodicity or locality of length L and suppose we have already obtained a set of Fourier Series coefficients a_k where $k = 0$, 1, 2, ... We can write the arguments of the sine and cosine functions as:

$$2\pi\frac{kx}{L} = 2\pi\frac{x}{L/k} = 2\pi\frac{x}{\lambda_k}$$

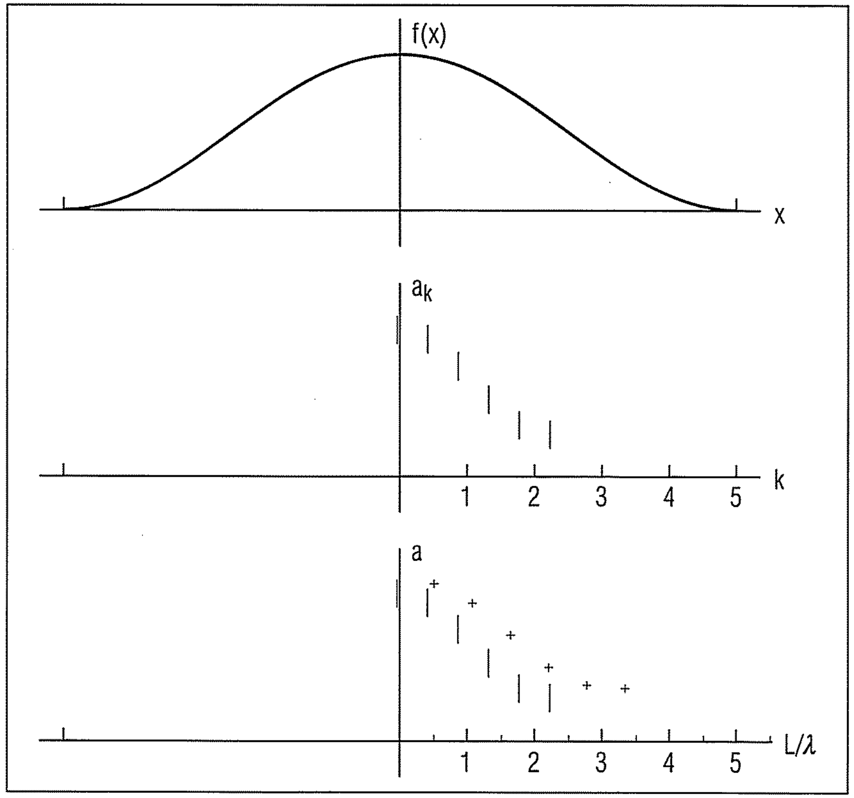

Figure 12. Fourier coefficients added by going to the larger interval are marked with crosses.

where $\lambda_k = L/k$ is obviously the wavelength of the k^{th} wave in the series. Note that exactly k complete waves fit into the periodicity distance L.

Now suppose we need to make the the interval twice as large, so the interval is $-L \leq x \leq L$, and we recalculate the a_k's. We will be using the wavelengths $\lambda_k = 2L/k = 2L$, L, $2L/3$, $L/2$, ... so all the coefficients we calculated before will still be here but with different k subscripts. That means we can use the original graph and not have to rename anything if we label things by their *wavelength*, instead of by the k integer. Writing the abscissa as $\lambda = L/k$ instead of k, we have Fig. 11.

As we let the interval L grow larger and larger, the scale of the graph will change and the points labeled by integer values of k will get ever more closely spaced. As $L \to \infty$ the points go toward becoming a continuum.

3c. The Continuum Case. In the continuum case we want to describe the continuum by some parameter that looks like our series-case $k = L/\lambda$ but which does not involve L. It is traditional to use the quantity called the "wave number" k defined by:

$$k = 2\pi/\lambda.$$

The Fourier coefficients a_k, b_k, with integer k, now become the "Fourier amplitudes" $A(k)$, $B(k)$, with a continuous dimensional k.

▷ What are k's dimensions?

Skipping further details on the transition from a sum to an integral, we write the equations equivalent to Eq. (3) and Eq. (2).

If $f(x)$ obeys Dirichlet's conditions in every finite interval, no matter how large, and if, in addition,

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

is finite, then

$$f(x) = \int_{-\infty}^{\infty} [A(k)\cos(kx) + B(k)\sin(kx)]\, dk \qquad (4)$$

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos(kx)\, dx$$
$$B(k) = \int_{-\infty}^{\infty} f(x)\sin(kx)\, dx. \qquad (5)$$

Note that k can take on negative values. This feature will be important when traveling waves, rather than standing waves, are used. However, for our purposes, $A(k)$ is even and $B(k)$ is odd and we will only plot the portion of each that has k positive.

3d. Eyeballing the Amplitudes. Just as for Fourier Series, the Fourier analysis of a function $f(x)$ into waves of various amplitudes and wavelengths can clarify its physical properties, and it is often sufficient to get a rough idea of the shape of $A(k)$ and $B(k)$ by "eyeballing" the function $f(x)$. Most of the ideas presented in the discussion of Fourier Series are still valid, including symmetry, overlap, structure size verses wavelength, etc.

Let us consider the examples in Fig. 13. For small k (long wavelength) $cos(kx) \approx 1$ and so near the origin, $A(k)$ is constant and equal to the area

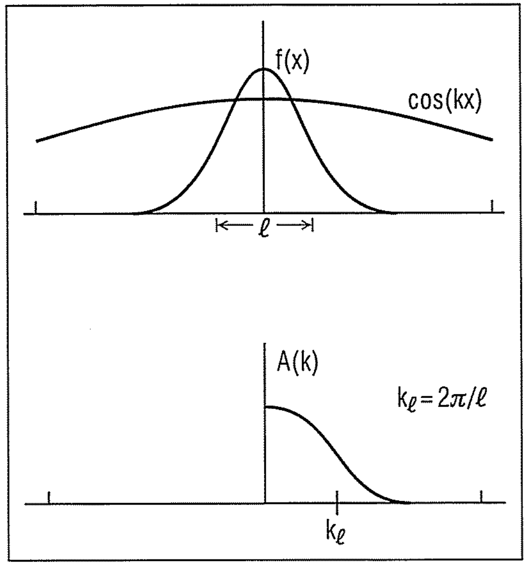

Figure 13.

under $f(x)$ divided by 2π. For k large [R smaller than any structure in $f(x)$], $A(k)$ approaches zero from cancellation of the + and − lobes of the integral (see Fig. 6). The region in which $A(k)$ drops off rapidly is near $\lambda = 2\pi/k\ell \approx \ell$, the size of the major structure in $f(x)$.

▷ For the examples in Figs. 14-16, see if you can justify exactly the form of $A(k)$ and $B(k)$.

The most elegant and useful form of the Fourier integral comes when we use the notation of complex numbers. Then

$$e^{ikx} = \cos(kx) + i\sin(kx), \qquad i = \sqrt{-1},$$

and we write for the real (or complex) valued function $f(x)$ of the real variable x:

$$f(x) = \int_{-\infty}^{\infty} G(k)e^{ikx}\, dk \qquad (6)$$

where

$$G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dk \qquad (7)$$

is the *Fourier transform of* $f(x)$ and is generally a complex valued function. If $f(x)$ is real, then $G(k)$ is related to the Fourier amplitudes of Eq. (5) by:

$$A(k) = ReG(k)$$

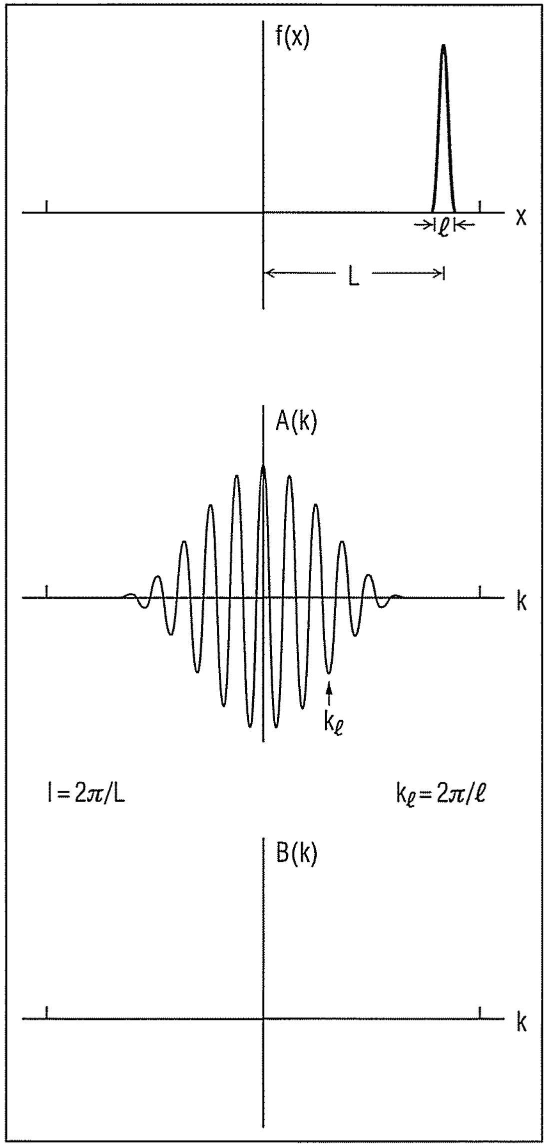

Figure 14.

$$B(k) = -ImG(k).$$

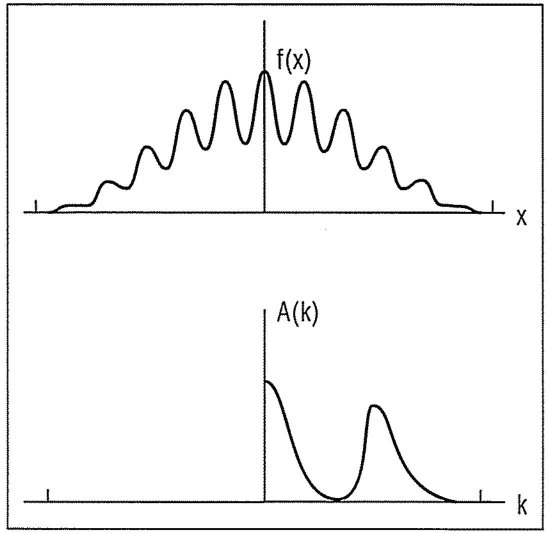

Figure 15.

Acknowledgments

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A. Some Indefinite Integrals

$$\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$\int x^2 \sin ax\, dx = \frac{2x}{a^2} \sin ax - \frac{x^2a^2 - 2}{a^3} \cos ax$$

$$\int x^3 \sin ax\, dx = \frac{3x^2a^2 - 6}{a^4} \sin ax - \frac{a^2x^3 - 6x}{a^3} \cos ax$$

$$\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos ax\, dx = \frac{2x}{a^2} \cos ax + \frac{a^2x^2 - 2}{a^3} \sin ax$$

$$\int x^3 \cos ax\, dx = \frac{3x^2a^2 - 6}{a^4} \cos ax + \frac{a^2x^3 - 6x}{a^3} \sin ax$$

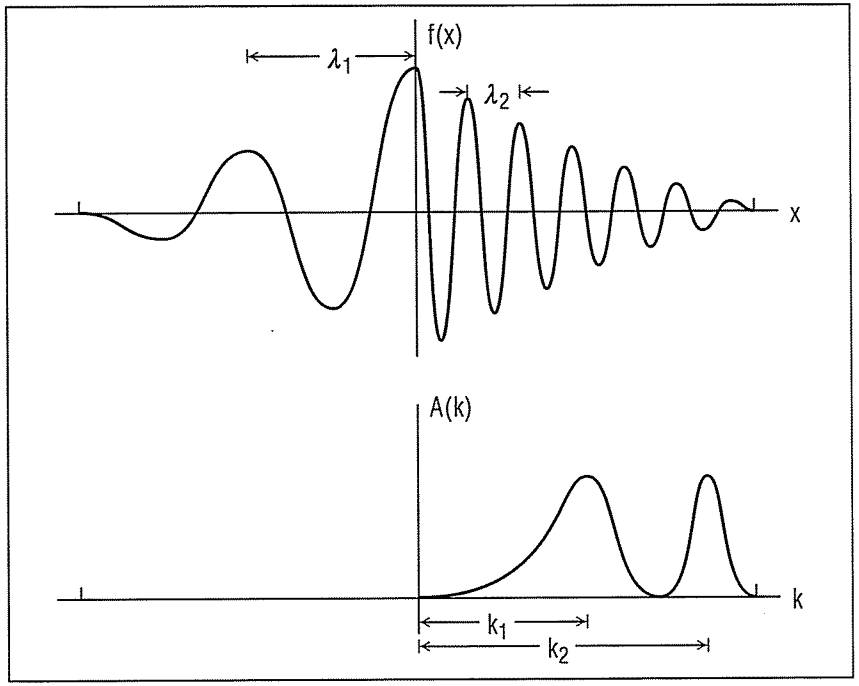


Figure 16.

$\int e^{ax} \sin px\, dx = e^{ax} \left(\dfrac{a \sin px - p \cos px}{a^2 + p^2} \right)$

$\int e^{ax} \cos px\, dx = e^{ax} \left(\dfrac{a \cos px + p \sin px}{a^2 + p^2} \right)$

B. A Definite Integral

$\int_0^\infty e^{-a^2/x^2} \cos bx\, dx = \dfrac{\sqrt{\pi}}{2a} e^{(-b^2/4a^2)}, \qquad (ab \neq 0)$

PROBLEM SUPPLEMENT

1. For each function listed below, sketch the function, apply symmetry conditions to see if any set of coefficients are zero, consider structure and overall shape to predict which coefficients may be large, consider the rate at which coefficients approach zero as $k \to \infty$ and then use Eqs. (2) to evaluate the coefficients. Each function is defined in $-\pi \leq x \leq \pi$.

   (a) $f(x) = 1$ for $|x| \leq \pi/2$; $= 0$ elsewhere

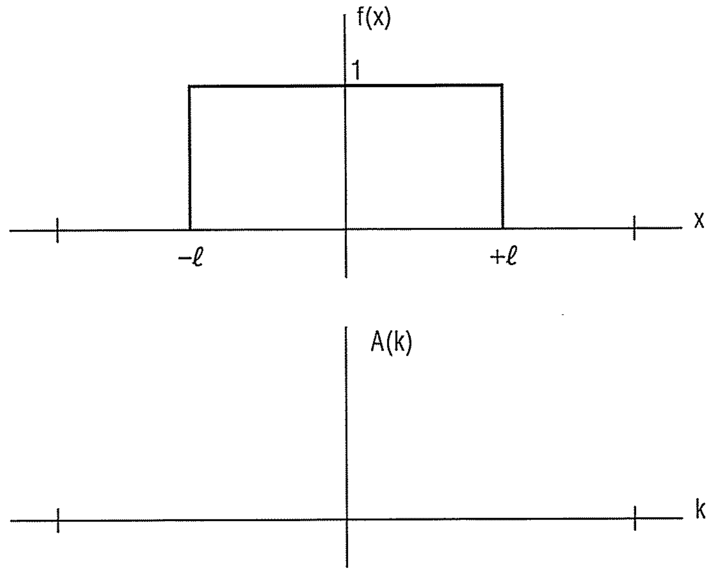
   (b) $f(x) = x$ for $|x| \leq \pi/2$; $= 0$ elsewhere

   (c) $f(x) = x$ for $0 \leq x \leq \pi$; $= 0$ elsewhere

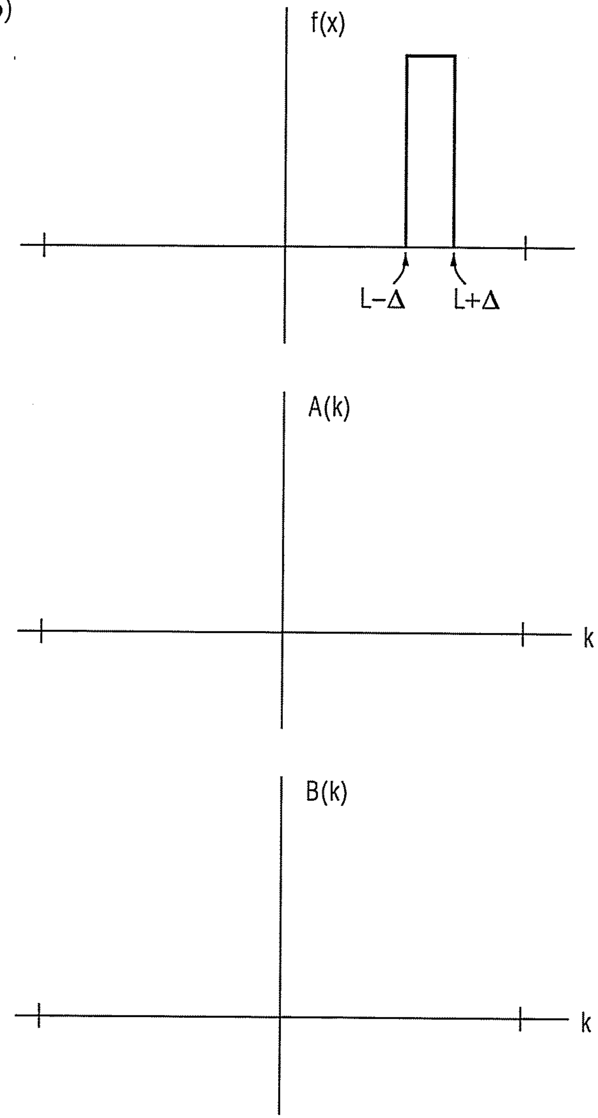
   (d) $f(x) = x^2$ in the interval.

2. For each function below, sketch $A(k)$, $B(k)$ by inspection of $f(x)$, then compute $A(k)$ and $B(k)$ and compare.

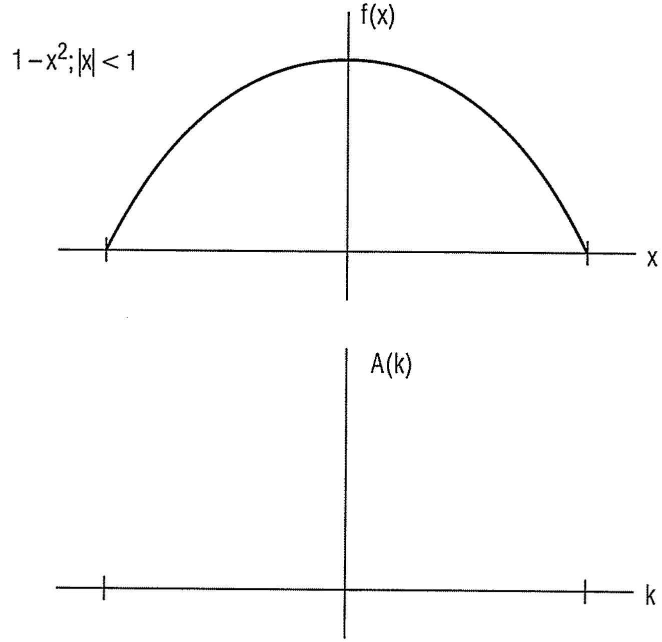
   (a)



(b)



(c)



WAVE MOTION

10

10.1 Introduction

Man has always been fascinated by the rhythmic undulations of the surface of the sea. The complex breakers which regularly approach our shores have been observed and studied for thousands of years, and yet our understanding of ocean waves is even now far from complete. Ocean waves may originate very far from shore and may travel at high speed toward their final destiny, but during their long journey they neither transport the debris in the ocean nor do they move any water over large distances. This statement contains the very essence of wave motion, for a wave is a disturbance of a medium, a disturbance which itself moves from place to place but which does not carry the medium with it as it goes. The wave is capable of transporting *energy* over large distances but in doing so it does not transport any *matter*.

The world we live in is full of waves. Without the electromagnetic waves that light the sky and warm our planet, our world would be ice cold and devoid of life. Without the waves we detect as sound, we could not hear the music of Beethoven or Brahms nor could we communicate by speech. Without waves, we would have no radios, no television sets, nor most of our modern technology. Without waves, we could never have gone to the moon, or even have been aware of its existence. Wave phenomena are one of the most basic aspects of physical reality and are fundamental to any description of matter on an atomic or sub-atomic scale.

Not all waves work for the benefit of man. Tidal waves (*tsunamis*) have devastated coastal cities and caused great loss of life in the past and will probably continue to do so in the future. Shock waves from supersonic aircraft cause a sonic boom which can shatter glass and damage the structure of buildings. Overdoses of energy carried by waves in the form of radiation can be harmful to life.

Physicists have studied waves at the macroscopic and microscopic levels, and these studies have proved to be of enormous value to civilization.

10.2 Types of Waves

Let us begin our study of wave motion by discussing the waves that are somewhat familiar to us. We will first consider waves that move in a physical medium such as water, air, or a string. Later, in a subsequent chapter, we will take up the very important special case of electromagnetic waves, which can travel, or *propagate*, in empty space or vacuum.

If someone were to ask you to describe in words the concept of a wave, you would probably have some difficulty doing this. Perhaps the best you could do would be to say that a wave is a *self-propagating disturbance* which travels in a medium and which owes its existence to a source that initially created the disturbance. Your questioner is likely to be confused by your choice of words, and perhaps the best thing to do is to take him into a laboratory and show him some examples of waves. This is likely to be the soundest way to approach the subject, and therefore this is the procedure we will adopt. After a number of examples have been discussed, the principal features of wave motion will become apparent.

One of the simplest examples of wave motion can be demonstrated by using a long rope tied to a post. If the rope is taut and the free end is jiggled, a disturbance propagates along the rope by traveling toward the post. Figure 10.1 schematically illustrates a number of snapshots of the rope taken at equal time intervals. This type of wave, which moves in a one-dimensional system, is called a *one-dimensional wave*. The rope is a one-dimensional medium since each element of it may be located by means of a single coordinate x when the medium is in equilibrium.[1] The surface of water is two-dimensional, and, therefore, surface waves in a fluid are called two-dimensional waves.

[1] In this example, since the motion of the rope that is initiated by the passage of the wave is in the y-direction, this is no longer strictly true when the equilibrium of the system is disturbed by the propagation of the wave. Now, the coordinate x locates the part of the system we are discussing, while a displacement along the y-direction describes the magnitude of the disturbance.

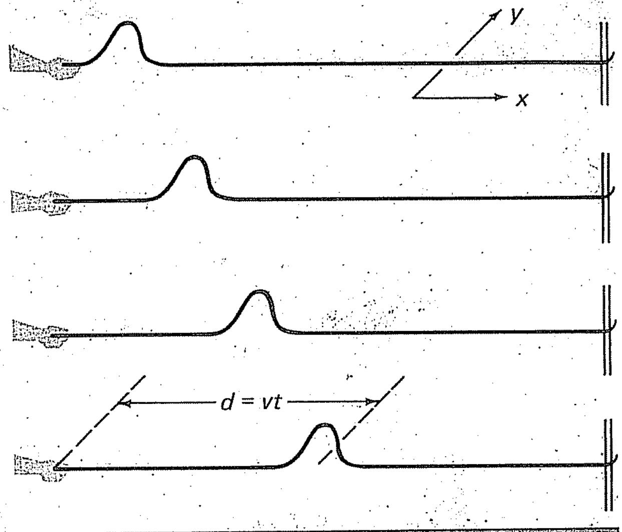


FIGURE 10.1. Propagation of a wave in the form of a transverse pulse along a rope.

As the wave of Fig. 10.1 progresses toward the post, those parts of the rope which are moving experience a transverse displacement along the y-direction. This transverse displacement will vary as a function of time and also as a function of the distance x along the rope. We must, therefore, regard the displacement $y(x, t)$ as a function of two variables, x and t. The displacements of various elements of the rope in this type of wave motion are always perpendicular, or transverse, to the direction of the wave propagation We say that we have a *transverse wave* whenever particle velocity and wave velocity are perpendicular in this manner. Note that in this wave motion there is no net movement of any part of the rope in the direction of the wave.

There are many possible ways of initiating disturbances of different shapes on the rope of Fig. 10.1. All of these produce transverse waves, and all of these waves move at the same speed, provided the tension in the rope is the same in each instance.

Not all waves are transverse. Another class, known as *longitudinal waves*, are present when the displacements of particles undergoing disturbance are *parallel* to the direction of wave propagation. An example of this type of wave is illustrated in Fig 10.2. The left end of a spring which is 1 a horizontal position on a frictionless table top is given a sudden momentary displacement by quickly pushing it to the right (causing a compression) and then pulling it back to the left (creating a rarefaction). This disturbance then propagates to the right with a speed governed by the characteristics of the spring. If we focus our attention on a small segment of the spring which is in motion, we notice that its motion is either *parallel or antiparallel* to the direction of motion of the wave. This wave is another example of a one-dimensional

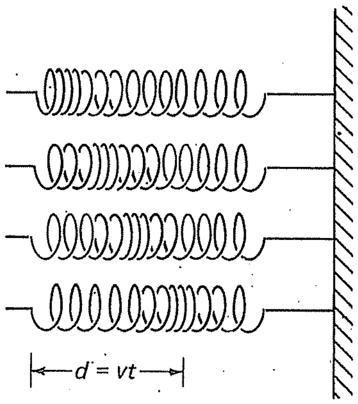
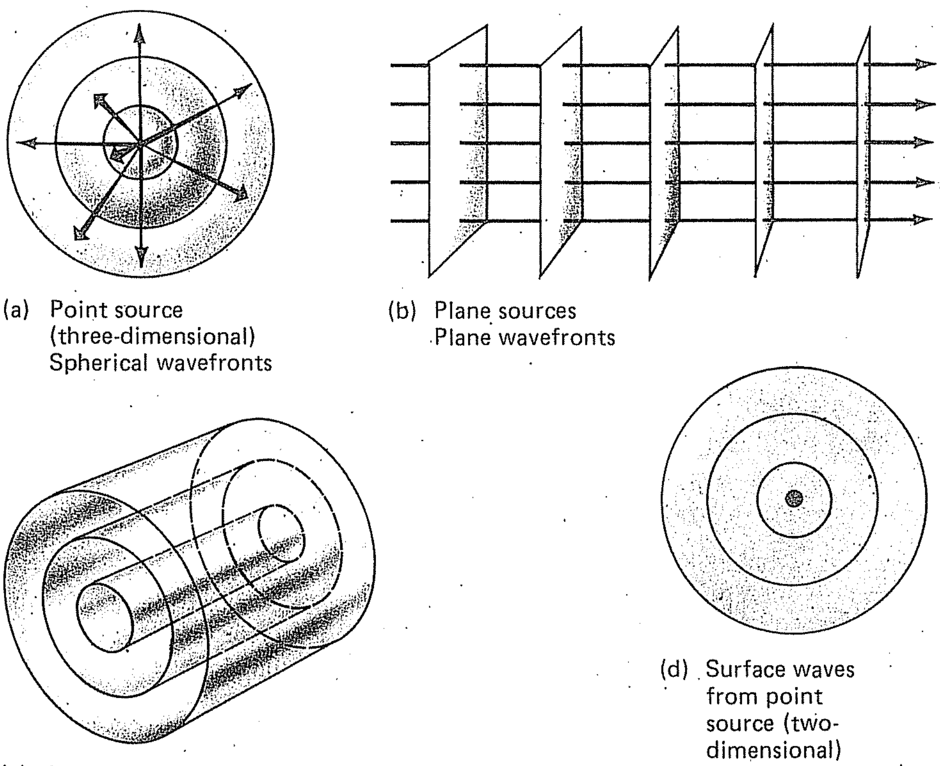


FIGURE 10.2. Propagation of a wave in the form of a longitudinal pulse along a spring.

wave. The displacements can again be characterized by means of a variable, u, which depends on the single variable x and also on t. In this case, however, it is important to realize that the displacement $u(x, t)$ is *parallel* to the x-axis. Any such wave is referred to as a longitudinal wave.

An essential characteristic of one-dimensional wave motion is that the initial disturbance moves with constant speed. Photographs of the medium taken at equal intervals of time all look alike in the sense that the disturbance maintains its initial shape but merely moves to a different location. The speed and direction of motion of the disturbance define a velocity vector known as the *wave velocity*. In the next section, some of the qualitative ideas discussed above will be made more quantitative.

We have mentioned two examples of waves, both of which are one-dimensional. There are also examples of waves which move in two dimensions and in three dimensions. An example of the former is provided by the surface waves which spread on a pond after a stone has been tossed into the water. An example of the latter is provided by sound waves coming from a train whistle. The type of wave created and transmitted depends on the medium that carries the wave as well as on the geometry of the source that produced the wave. For example, a point source such as a whistle will send out sound waves in all directions. At a given time, any displacement of the medium that exists at any point on a sphere, centered at the source, is the same at any other point on this sphere. In this case we say that *the wavefronts are spherical.* The stone falling into a pond sends out circular surface waves which are disturbances with *circular wavefronts.* Waves with plane and cylindrical wavefronts are also very commonly encountered. In Fig. 10.3, a number of



(a) Point source (three-dimensional) Spherical wavefronts

(b) Plane sources Plane wavefronts

(c) Line source Cylindrical wavefronts

(d) Surface waves from point source (two-dimensional) Circular wavefronts

FIGURE 10.3. Spherical (a), plane (b), and cylindrical (c) wave fronts. (d) Circular waves on the surface of a liquid, generated by a point source.

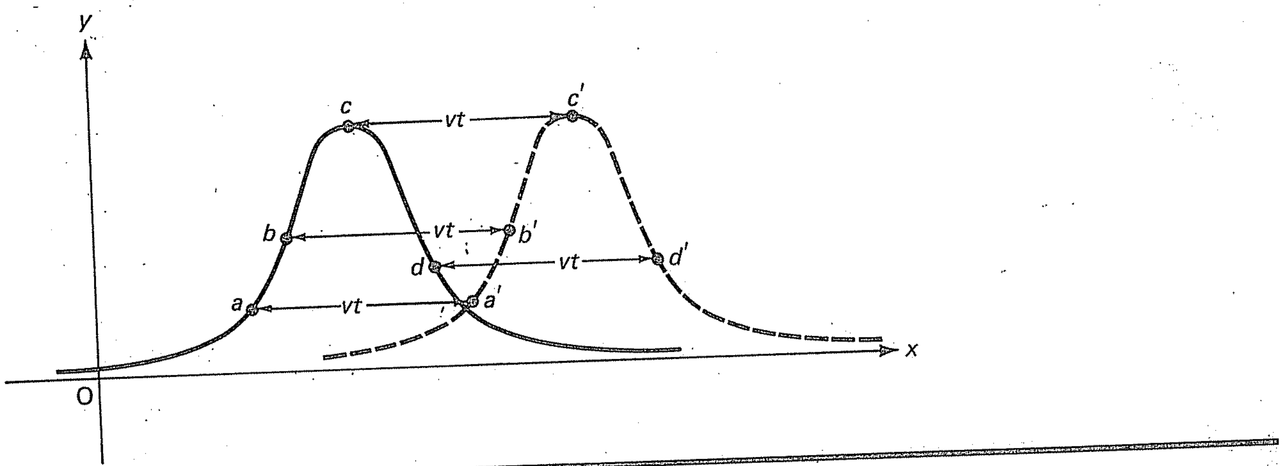


FIGURE 10.4. Constant velocity associated with different points in a disturbance propagating as a wave.

different waves are illustrated, with several wavefronts for each case. These wavefronts have the property that the magnitude of the disturbance of material particles (displacement from equilibrium) is everywhere the same on a given wavefront. Furthermore, the wave always propagates along a direction perpendicular to the wavefront at any point.

10.3 Mathematics of One-Dimensional Waves

There are two considerations which must be incorporated into the mathematical description of one-dimensional waves. One of these is the constant speed of propagation of the initial disturbance, while the second is the maintenance of the original form of the disturbance.

The physics can best be described with reference to the rope of Fig. 10.1. At $t = 0$, the rope has experienced an initial disturbance. This initial disturbance at $t = 0$ may then be characterized by means of the equation

$$y(x, 0) = F(x) \tag{10.3.1}$$

which gives the transverse displacement of each point x on the rope at the initial time. How do we now determine the dependence of y on t for all subsequent times for each value of x, subject to the considerations mentioned above? In order for the disturbance to maintain its initial shape, it is essential that after a time interval t has elapsed, each and every displacement y should have moved to a *new location* some constant distance d from the original location. The constancy of speed is guaranteed by requiring that $d = vt$, where the constant v is called the *wave velocity*, or the *phase velocity*. These considerations are demonstrated in Fig. 10.4. The conditions mentioned above are fulfilled if the time dependence of the disturbance is given by the equation[2]

$$\boxed{y(x, t) = F(x - vt)} \tag{10.3.2}$$

If we set $t = 0$, this describes the initial disturbance as given by (10.3.1), but it also describes the situation at any later time t. Equation (10.3.2) gives a one-dimensional traveling wave moving to the right. For a wave traveling to the left, (10.3.2) would be modified by changing the sign in front of the velocity v. We shall see later that the magnitude of the phase velocity is in all instances determined by the physical properties of the medium in which the wave propagates.

The most important feature of (10.3.2) is the occurrence of the combination $x - vt$ as the *argument* of the function F. For example, if the maximum of the disturbance defined by the function $F(x - vt)$ should occur where the argument of the function is zero, it is clear that the location of that point as a function of time will be defined by the condition that $x - vt = 0$ or $x = vt$; in other words, the point of the disturbance moves to the right, along the direction of propagation, with constant wave velocity v. The same general remarks clearly apply to all other points defined by $F(x - vt)$, and it is evident, therefore, that writing the argument of such a function in the form $x - vt$ (rather than simply as x) causes the function to "travel" in the x-direction with velocity v just the way a wave would! The function F itself must be determined from initial conditions; and although *any* function F leads to a wave motion in the mathematical sense, in practice,

[2] In this equation, the quantity $x - vt$ represents the *argument* of the function F described (for $t = 0$) in Eq. (10.3.1). It is decidedly *not* a factor that multiplies F!

physical considerations will limit the range of possibilities, as the following example demonstrates.

EXAMPLE 10.1

Which of the following mathematical expressions can represent one-dimensional waves that are physically reasonable?

a. $y(x, t) = y_0 e^{-\lambda(x-vt)^2}$ (10.3.3)

b. $y(x, t) = \beta(x + vt)^4$ (10.3.4)

c. $y(x, t) = y_0 e^{-\lambda x^2(1+t/t_0)}$ (10.3.5)

d. $y(x, t) = y_0 \sin k(x - vt)$ (10.3.6)

Let us first consider which of these expressions represents a wave motion in one dimension. We recall that a one-dimensional traveling wave must be in all cases a function of $x - vt$ or of $x + vt$. The functions given by **(a)**, **(b)**, and **(d)**, therefore, meet the mathematical criteria, but **(c)** does not. The "wave" given by **(b)** is not physically sensible since it predicts that at each value of x the displacements become arbitrarily large as t increases. There are no physical systems which actually display this behavior and, therefore, (10.3.4) is not likely to be a wave of any physical interest. In Fig. 10.5, the above functions are illustrated for various equally spaced time intervals. Notice that **(a)**, **(b)**, and **(d)** maintain their *shape* as t increases, whereas **(c)** does not. Equation **(d)** represents a *sinusoidal* wave, which turns out to be the simplest and most important type of wave motion.

Equation (10.3.2) is a solution of a well-known differential equation called the *wave equation*. This equation relates the second partial derivative of y with respect to the variable x to the second partial derivative with respect to t. Partial derivatives[3] are needed because y depends on two variables, x and t. In calculating the partial derivatives of functions of x and t, it is important to note that in taking the partial derivative with respect to x, the quantity t is held constant, while in taking the partial derivative with respect to t, the quantity x is held constant. These derivatives may be obtained from (10.3.2) with the help of the "chain rule" of calculus. Letting $u(x, t) = x - vt$, we obtain

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} F(u) = \frac{dF(u)}{du}\frac{\partial u}{\partial x} = \frac{dF(u)}{du}\frac{\partial}{\partial x}(x - vt) = \frac{dF(u)}{du}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{dF(u)}{du}\right) = \left[\frac{d}{du}\left(\frac{dF(u)}{du}\right)\right]\frac{du}{\partial x}$$

$$= \frac{d^2F(u)}{du^2}\frac{\partial}{\partial x}(x - vt) = \frac{d^2F(u)}{du^2} \qquad (10.3.7)$$

[3] The basic mathematical facts about partial derivatives are set forth in Appendix D at the end of the book.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} F(u) = \frac{dF(u)}{du}\frac{\partial u}{\partial t} = \frac{dF(u)}{du}\frac{\partial}{\partial t}(x - vt)$$

$$= -v\frac{dF(u)}{du}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(-v\frac{dF(u)}{du}\right) = -v\left[\frac{\partial}{\partial u}\left(\frac{dF(u)}{du}\right)\right]\frac{\partial u}{\partial t}$$

$$= -v\frac{d^2F(u)}{du^2}\frac{\partial}{\partial t}(x - vt) = v^2\frac{d^2F(u)}{du^2} \qquad (10.3.8)$$

Dividing (10.3.8) by v^2 and comparing it to (10.3.7), we see that $y(x, t)$ satisfies the *partial differential equation*

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0} \qquad (10.3.9)$$

This equation is a *one-dimensional wave equation.* Any function $y(x, t)$ that has the properties of a wave as outlined above will satisfy this equation.

There are, of course, many systems in which waves do not propagate; their motion is not described by an equation of this form. In systems in which waves do propagate, we shall see that the wave equation arises directly from the physical principles that govern the system's motion. Thus, Newton's laws of motion should reveal that a string must satisfy a wave equation of the form given above rather than some other type of equation. Let us investigate this further by discussing the application of Newton's laws to a moving string.

Consider a small segment of a string which has been disturbed as in Fig. 10.1; this segment is shown in Fig. 10.6. The left end has been displaced an amount $y(x, t)$, while the right end has been displaced $y(x + \Delta x, t)$ at time t. The angles which the ends of this segment make with the x-axis are $\theta(x)$ and $\theta(x + \Delta x)$, respectively. These angles are assumed to be quite small, so that the piece of string has been displaced from equilibrium by only a small amount. It is also assumed that the tension in the entire string is uniform and has the value T_0. Now, according to Newton's second law, the resultant of all forces acting on this segment is equal to its mass multiplied by its acceleration. If the segment is taken to be very small (infinitesimal), then the concept of a single acceleration is meaningful since the segment approaches a point mass. Accordingly, we have

$$T_0 \sin\theta(x + \Delta x) - T_0 \sin\theta(x) = ma_y = (\mu\,\Delta x)a_y$$
$$= (\mu\,\Delta x)\frac{\partial^2 y}{\partial t^2} \qquad (10.3.10)$$

where μ is the linear mass density, that is, the mass per unit length. The acceleration a_y is written as a *partial* derivative since the quantity y is a function not only of time but *also* of position x along the string.

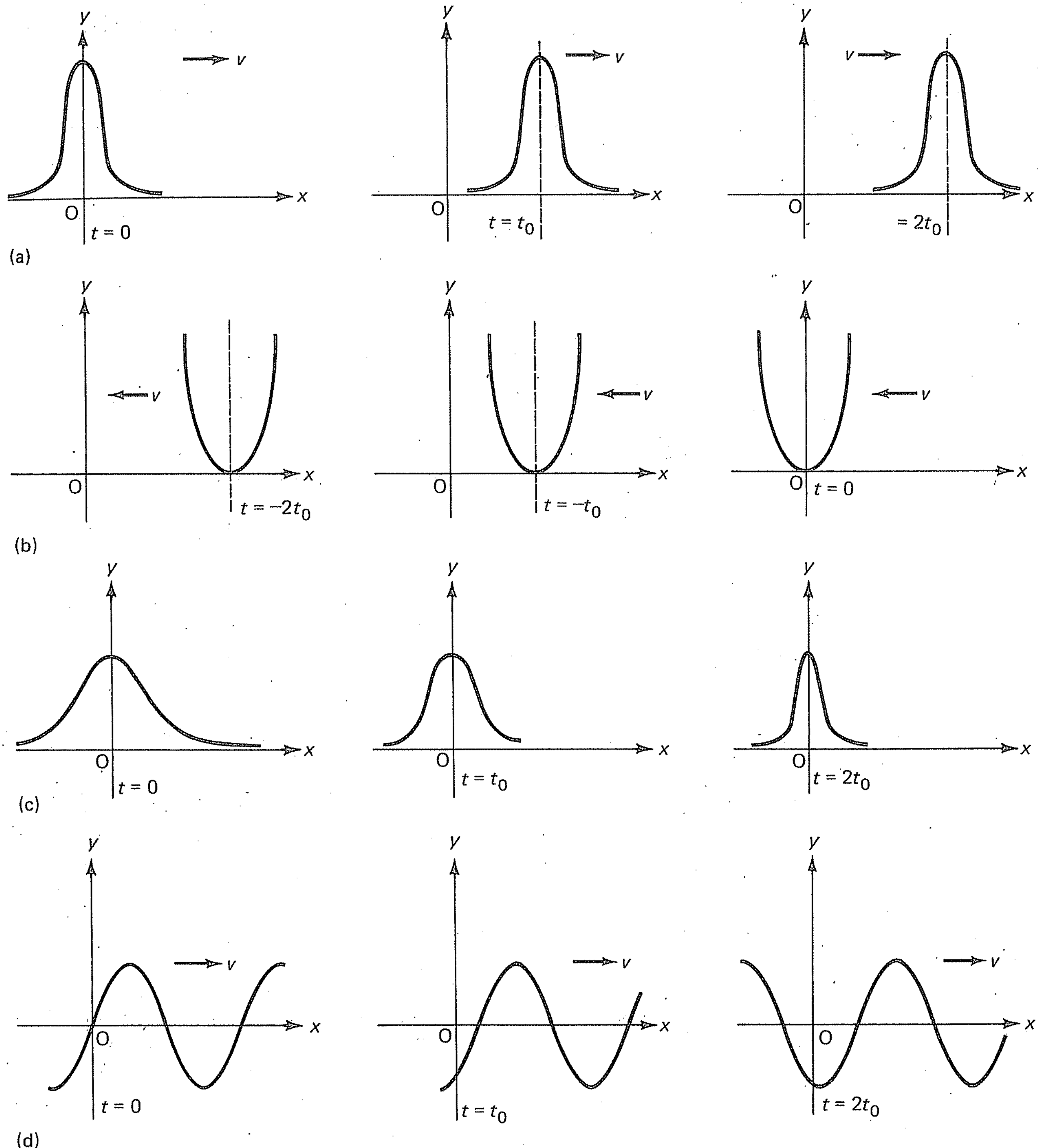


FIGURE 10.5

Since we have assumed that the angles are very small and that they are expressed in radians, it is legitimate to use the approximations

$$\sin\theta(x) \cong \theta(x) \cong \tan\theta(x) = \left(\frac{\partial y}{\partial x}\right)_x$$

recalling that the slope $\partial y/\partial x$ is the tangent of the angle θ. Also, at $x + \Delta x$,

$$\sin\theta(x+\Delta x) \cong \theta(x+\Delta x) \cong \tan\theta(x+\Delta x)$$

$$= \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \qquad (10.3.11)$$

Substituting these approximations in (10.3.10) and dividing both sides of the equation by Δx,

$$T_0 \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \mu \frac{\partial^2 y}{\partial t^2} \qquad (10.3.12)$$

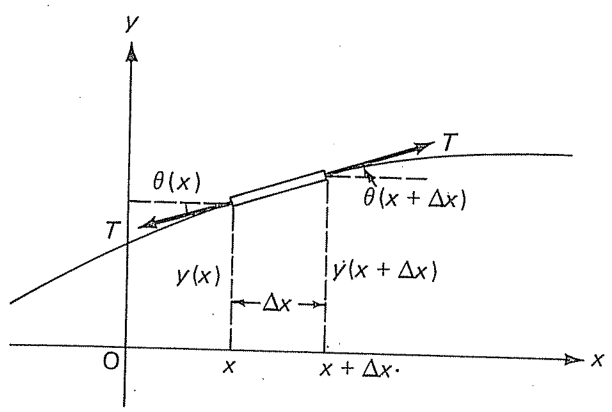


FIGURE 10.6. Forces and displacements associated with an infinitesimal element of a string along which a wave propagates.

Finally, if we allow Δx to become arbitrarily small, we obtain

$$T_0 \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \tag{10.3.13}$$

or

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{(T_0/\mu)} \frac{\partial^2 y}{\partial t^2} = 0 \tag{10.3.14}$$

since in the limit as $x \to 0$ the quantity on the left side of equation (10.3.13) becomes,[4] by definition, $T_0(\partial/\partial x)(\partial y/\partial x)$, or $T_0(\partial^2 y/\partial x^2)$.

This equation may now be compared to (10.3.8), the general equation for a one-dimensional wave. This comparison reveals that the string does, indeed, satisfy the wave equation and can therefore propagate waves. It also tells us that the waves which propagate must have a *wave velocity* such that $1/v^2 = \mu/T_0$, or

$$\boxed{v = \sqrt{\frac{T_0}{\mu}}} \tag{10.3.15}$$

Thus, we see that the emergence of wave motion is a direct consequence of Newton's laws applied to the string as well as some approximations which are quite valid in many instances. We see also that the velocity of the waves is related directly to the physical characteristics of the propagating medium, which in this case is the string. From (10.3.14), it is apparent that the wave velocity is directly proportional to the square root of the tension and inversely proportional to the mass per unit length.

The wave equation, (10.3.14), has an enormous number of possible solutions. The simplest familiar examples of these are *sinusoidal* solutions, having the form

$$\boxed{y(x, t) = y_0 \sin[k(x - vt) + \delta]} \tag{10.3.16}$$

The fundamental importance of sinusoidal solutions will become more apparent later when we discuss more complex solutions. It may be shown directly by differentiating (10.3.16) and substituting the resulting expressions into (10.3.14) that the sine wave (10.3.16) will be a solution of the wave equation, but it is not the only solution, since we have already demonstrated that *any* function of the variable $x - vt$ will be a solution, and (10.3.15) is such a function.

Let us now describe some of the properties of the sinusoidal waves described by (10.3.16). At a fixed time t_0, (10.3.16) specifies the displacement of every point of the string from its equilibrium position; we shall continue to use the string as an example although many of the remarks we make are applicable to other systems as well. In Fig. 10.7, we illustrate schematically the dependence of y on x at some time t_0. This figure shows a number of locations where y assumes a maximum value of y_0 or a minimum value of $-y_0$. These values occur whenever sine takes on a value of $+1$ or -1, respectively. Therefore, the quantity y_0, which is called the *amplitude* of the wave, represents the *maximum possible displacement from equilibrium.*

The entire argument of the sine function in (10.3.16) is $[k(x - vt) + \delta]$. This is called the *phase* of the wave. The constant δ appearing in the phase is determined from initial conditions. For example, it may be regarded as the phase angle at $x = 0$ and at the initial time $t = 0$.

Referring to Fig. 10.7 again, we see that the distance λ represents the distance between successive maximum positive displacements. This distance is called the *wavelength.* Now suppose a maximum of displacement occurs at $x = x_1$. If we substitute $y = y_0$, $x = x_1$, into (10.3.16), this tells us that

$$y_0 = y_0 \sin[k(x_1 - vt_0) + \delta]$$

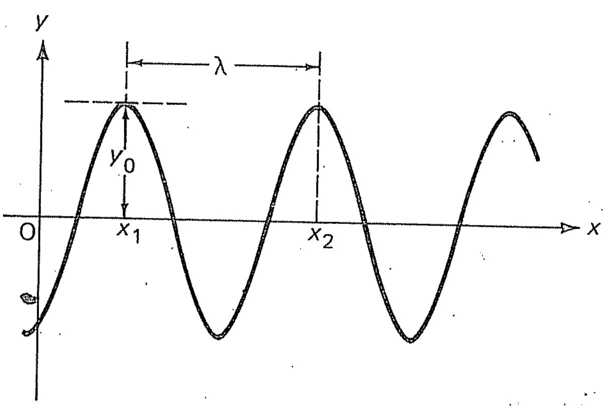


FIGURE 10.7. Sinusoidal wave, illustrating the wavelength and amplitude.

[4] See Appendix D for the definition of the partial derivative in terms of fundamental limiting processes.

or

$$k(x_1 - vt_0) + \delta = \frac{\pi}{2} \qquad (10.3.17)$$

If λ represents the wavelength, then the next maximum will occur at $x = x_1 + \lambda$, and in the interval between these two successive maxima, the phase angle $[k(x - vt) + \delta]$ will have advanced by precisely 2π radians. Equation (10.3.17) then becomes

$$k(x_1 + \lambda - vt_0) + \delta = \frac{\pi}{2} + 2\pi \qquad (10.3.18)$$

Subtracting (10.3.17) from (10.3.18), we now obtain

$$\boxed{k = \frac{2\pi}{\lambda}} \qquad (10.3.19)$$

The constant k, called the *propagation constant*, can, therefore, be expressed in terms of the wavelength by means of the above relation.

We have examined several features of a "snapshot" of the wave at time t_0 and have, therefore, studied the dependence of y on x for a *fixed value of t*. Let us now study the dependence of y on t for a *fixed value of x*, for example, $x = x_0$. Here, we are looking at only a tiny point on the string and we are watching how that point moves as time progresses. From (10.3.16) it is evident, setting $x = x_0$, that the motion is *simple harmonic motion*, with frequency f and period T given by the relation

$$\omega = kv = 2\pi f = \frac{2\pi}{T} \qquad (10.3.20)$$

But since $k = 2\pi/\lambda$ and $\omega = 2\pi f$, we may express the velocity v by

$$\boxed{v = \frac{\omega}{k} = \lambda f} \qquad (10.3.21)$$

which relates the frequency, the wavelength, and the wave velocity.

Using the various relations above, we can write the original wave equation (10.3.16) in a number of different equivalent forms. These include

$$\boxed{\begin{aligned} y(x, t) &= y_0 \sin[k(x - vt) + \delta] \\ &= y_0 \sin\left[\frac{2\pi}{\lambda}(x - vt) + \delta\right] \\ &= y_0 \sin[(kx - \omega t) + \delta)] \\ &= y_0 \sin\left(2\pi \frac{x}{\lambda} - ft + \delta\right) \\ &= y_0 \sin\left(2\pi \frac{x}{\lambda} - \frac{t}{T} + \delta\right) \end{aligned}} \qquad (10.3.22)$$

All of these expressions contain the same information. The student should be able to derive one form from any other. To reinforce some of these concepts, let us discuss a few examples.

EXAMPLE 10.3.1

A long string of mass density 0.001 slug/ft is stretched so that the tension is 0.5 pound. The left end is moved up and down with simple harmonic motion having a period of 0.5 sec and an amplitude of 1.5 ft. Assume the tension is constant throughout the motion. (a) Find the speed of the wave generated in the string. (b) What is the frequency of the wave? (c) Determine the wavelength. (d) Obtain a mathematical expression for the displacement $y(x, t)$ at any point.

The expression for the velocity of propagation has already been derived and is given by (10.3.15). Substituting the numbers given above, we obtain

$$v = \sqrt{\frac{T_0}{\mu}} = \sqrt{\frac{0.5}{0.001}} = 22.4 \text{ ft/sec} \qquad (10.3.23)$$

If we look at a small segment of the string and observe its motion, we will find that it is simple harmonic motion with a frequency given by the frequency of the source of the wave. Therefore, the frequency of the wave is

$$f = \frac{1}{T} = 2.0 \text{ Hz} \qquad (10.3.24)$$

Now that we know both the frequency and the speed of the wave, we can use (10.3.21) to determine the wavelength. In the present example, we obtain

$$\lambda = \frac{v}{f} = \frac{22.4}{2.0} = 11.2 \text{ ft} \qquad (10.3.25)$$

Finally, we may try to determine the mathematical expression for the wave. This can be done by using (10.3.23), (10.3.24), and (10.3.25) in conjunction with the expressions given by (10.3.22). We find that

$$\begin{aligned} y(x, t) &= y_0 \sin[(kx - \omega t) + \delta] \\ &= 1.5 \sin\left(\frac{2\pi}{11.2} x - 2\pi(2.0)t + \delta\right) \end{aligned}$$

or

$$y(x, t) = 1.5 \sin(0.561x - 12.57t + \delta) \text{ ft} \qquad (10.3.26)$$

Now, this cannot be a complete solution since the phase angle δ has not been specified, nor can it be determined from the conditions stated in the problem. We need to know something else. The reader should show that if it is specified that $y = 0$ and $\partial y/\partial t < 0$ at $x = 5$ ft and $t = 0.4$ sec, then this information is enough to give $\delta = 2.221$ rad. It is clear from this that a given wave motion is completely specified by amplitude, frequency or wavelength, velocity, and initial phase. The velocity may be obtained by knowing the physical properties of the medium (tension, mass per unit length, etc.); but the others can be ascertained only by determining, somehow, the position, velocity, and acceleration of any point of the system at some stage of the motion.

EXAMPLE 10.3.2

The equation for a transverse wave on a string is given by

$$y = 2\cos[\pi(x - 100t)]\ \text{m} \tag{10.3.27}$$

where x and y are in meters and t is in seconds. (a) What is the amplitude of this wave? (b) What is the phase angle δ? (c) Determine the frequency of vibration of the string. (d) Find the wave velocity. (e) At $t = 1$ sec, find the displacement, velocity, and acceleration of a small segment of the string located at $x = 2$ m.

The amplitude of a simple wave such as that expressed by (10.3.17) is the maximum possible displacement y_0. Now in (10.3.17) the wave is specified by means of a sine function, whereas in the present example we have used a cosine function. The simple trigonometric identity $\cos\theta = \sin(\theta + \frac{1}{2}\pi)$ may be used to convert one form to another. Thus, (10.3.27) may also be written as

$$y = 2\sin[\pi(x - 100t) + \pi/2] \tag{10.3.28}$$

Let us now compare this to (10.3.22). We find that the amplitude is $y_0 = 2$ m, the phase angle δ is $\pi/2$, the frequency of vibration is $f = 50$ Hz, and the wave velocity is $v = 100$ m/sec. Using (10.3.28), we can determine directly the displacement of any part of the string at any time t. We may also determine the velocity and acceleration of any portion of the string by evaluating appropriate derivatives of Eq. (10.3.28). Thus, the up-and-down velocity of a segment located at a given distance x from the origin will be given by the partial derivative $\partial y/\partial t$ evaluated at x, from which

$$v_y = \frac{\partial y}{\partial t} = -200\pi\cos[\pi(x - 100t) + \pi/2] \tag{10.3.29}$$

The acceleration is given by $\partial v_y/\partial t$ evaluated at x, or

$$a_y = \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -20{,}000\pi^2 \sin\left[\pi(x - 100t) + \frac{\pi}{2}\right] \tag{10.3.30}$$

The values at $t = 1$ sec and $x = 2$ m are, therefore,

$$y = 2\sin\left[\pi(2 - 100) + \frac{\pi}{2}\right] = 2\sin\left[-98\pi + \frac{\pi}{2}\right]$$

$$= 2\sin\frac{\pi}{2} = 2\ \text{m}$$

$$\frac{\partial y}{\partial t} = -200\pi\cos\left[-98\pi + \left(\frac{\pi}{2}\right)\right] = -200\pi\cos\left(\frac{\pi}{2}\right) = 0$$

and

$$\frac{\partial^2 y}{\partial t^2} = -20{,}000\pi^2 \sin\left[-98\pi + \left(\frac{\pi}{2}\right)\right]$$

$$= -20{,}000\pi^2\ \text{m/sec}^2 = -1.97 \times 10^5\ \text{m/sec}^2$$

This is a very large acceleration. It is evident from the above calculation that the maximum magnitude of the acceleration is always given by

$$(a_y)_{\max} = \omega^2 y_0 \tag{10.3.31}$$

Therefore, waves of large amplitude and high frequency can possess a very large acceleration, as in the present example.

So far, we have been discussing the physics of one-dimensional transverse waves, using a string as our primary example. Transverse waves can be produced and propagated in other ways. For example, if a pebble is dropped into a pond, a familiar circular pattern of waves emanates from the source. A stretched membrane also provides a medium in which two-dimensional waves can propagate. For two-dimensional waves, the displacements z from equilibrium depend on the time t as well as the segment of the medium undergoing displacements. Now each part of the undisturbed medium must be located by means of two spatial coordinates x and y. If z represents the displacement of water molecules located at (x, y) when a wave is present, then z can be shown to satisfy the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} = 0 \tag{10.3.32}$$

The above equation is a natural generalization of (10.3.9) and is a *two-dimensional wave equation.*

In the case of one-dimensional wave motion, a disturbance propagates *undiminished* (in the absence of dissipation), and the amplitude of the simple harmonic motion described by (10.3.16) is unchanged as the wave progresses. This is a consequence of the fact that the initial energy given to the system must be transferred through every point x. The situation is quite different for a two-dimensional circular wave. The initial disturbance created by a stone falling into a pond imparts a certain energy to the system. As this energy propagates outward from its source, more and more particles must share that energy. As a consequence, each particle will have an amplitude of vibration that *decreases* with increasing distance from the source.

Waves also propagate in three-dimensional media and are governed by the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{10.3.33}$$

where u represents the displacement from equilibrium. An example of a solution of this equation is provided by sound propagation in three dimensions. Sound waves propagate in all directions from a point source of sound in the atmosphere. Since the velocity is the same in all directions, the wavefronts emitted by such a source are spherical. Once again, since the energy emitted by the source is spread over a larger and larger area as the wavefronts propagate outward, the wave amplitude decreases as the distance r from the source increases. In this instance, the amplitude varies inversely with the distance from the source. For this reason, sound seems fainter and fainter to our ears as we move further away from the source. Light waves emitted from a point source behave in the same way.

If we are very far from a point source that emits spherical wavefronts, the curvature of the wavefronts becomes very small, and for practical purposes the wavefronts can be regarded as *planes* rather than spheres. For example, the sun emits light in the form of spherical electromagnetic waves. But when these wavefronts reach the earth, their radius is nearly 150,000,000 km. When we consider that the earth's radius is less than 6400 km, it is apparent that the earth intercepts only a very tiny portion of the spherical wavefronts emitted by the sun; this restricted portion of such an enormous sphere is for all intents and purposes flat, just as the small part of the earth's surface we see in our immediate vicinity is essentially flat. Plane waves are the simplest and most important examples of waves that propagate in three-dimensional systems. Indeed, even though they propagate in three dimensions, their amplitude can be described by the *one-dimensional* wave equation (10.3.9) or by the solutions given in Eq. (10.3.22). We shall have occasion to study the characteristics of plane waves in great detail, not only in this chapter, but also in our later study of electromagnetic waves and optics.

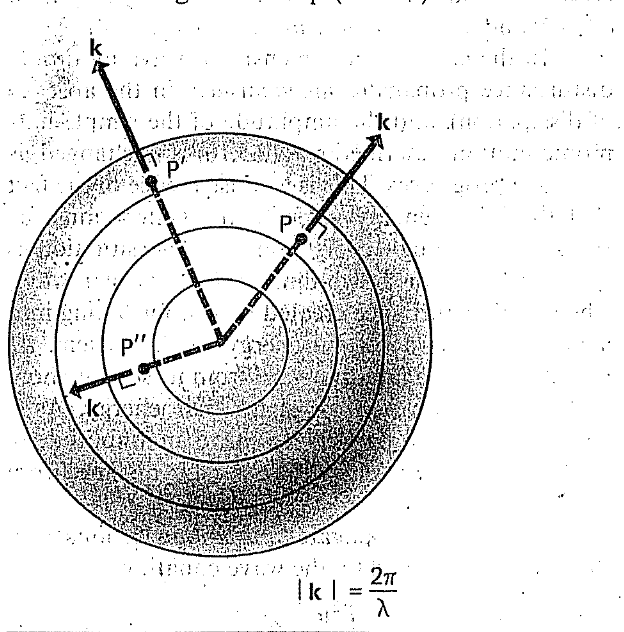


FIGURE 10.8. The concept of the propagation vector.

In the case of a wave that travels in a three-dimensional medium, the characteristic features of its propagation may be conveniently described by a *propagation vector* **k**. The propagation vector is a vector which is perpendicular to the wavefront at every point and whose magnitude is given by $k = 2\pi/\lambda$. The propagation vector at every part of the wavefront points in the direction of propagation at that point, as illustrated by Fig. 10.8. The propagation vector is a natural generalization of the propagation constant introduced previously for one-dimensional wave motion. It enables us to visualize the way in which the wavefronts are moving at every point. It is a concept that we shall have occasion to refer to often in our future work concerning sound waves, electromagnetic waves, and the propagation of light.

10.4 Longitudinal Waves: Sound Waves

The emphasis in the previous section has been placed on transverse wave motion even though much of the mathematics is also applicable to the case of longitudinal waves. In this section, we consider the propagation of longitudinal waves and, in particular, sound waves, which are the most important example of longitudinal waves. There is a profound difference between the waves previously discussed and the sound waves we will study. If you have seen waves traveling on water or moving along a string, you know that the displacements of particles from equilibrium are macroscopic ones that are usually visible to the human eye. On the other hand, the displacements occurring in sound waves are displacements of individual molecules or atoms from equilibrium positions and can be extremely minute, in some cases as small as 10^{-8} cm.

Newton's second law was used to show that a string under tension can support wave motion. The derivation of its one-dimensional wave equation also described the velocity of wave propagation. We would like to ask now whether Newton's second law can also be used to predict wave motion in a solid bar which has been disturbed by being struck with a hammer. If wave motion does result, what is the wave velocity and how does it depend on the properties of solids? To answer these questions, it is necessary first to know a little about elastic deformation of solids.

Let us consider a solid bar of length l_0 and cross-sectional area A, as shown in Fig. 10.9. If this rod experiences forces of magnitude F on both of its faces, the rod is said to be under longitudinal stress, and the amount of stress is defined by

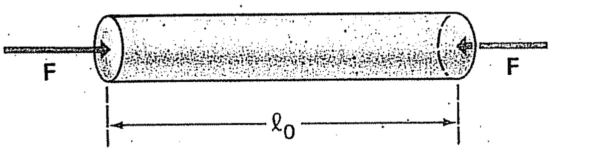


FIGURE 10.9. Solid elastic bar in which longitudinal waves are excited.

$$\text{stress} = \frac{F}{A} \tag{10.4.1}$$

This stress can be due to compression, as in Fig. 10.9, or it can be due to tension. An idealized "rigid object" will not exhibit any deformation at all when under stress; but real substances, even those that are quite "rigid" in the conversational sense of the word, will undergo a slight deformation when stress is applied. Therefore, when longitudinal stress is applied to the bar shown in Fig. 10.9, its length will change by a small amount Δl. The change in length divided by the length l_0 is known as the *longitudinal strain:*

$$\text{strain} = \frac{\Delta l}{l_0} \tag{10.4.2}$$

Now, if the stress is not too large, the relationship between stress and strain for most substances is a linear one. The stress is directly proportional to the strain and the proportionality constant is given by a number Y, which is called Young's modulus. The magnitude of Young's modulus will depend upon the mechanical properties of the substance and is, in fact, a quantitative measure of its rigidity. The equation relating stress and strain may thus be written

$$\frac{F}{A} = Y\frac{\Delta l}{l_0} \tag{10.4.3}$$

This equation, in which the stress varies linearly with the strain, is simply another form of Hooke's law.

Equation (10.4.3) may be used to study the propagation of a disturbance in a bar of finite length. Let us consider a *very small* section of such a bar, having mass Δm and length Δx when in equilibrium, and located at position x. This section is shown in Fig. 10.10. When the bar is struck by a hammer, this section is no longer in equilibrium. Its left end, at some time t, has undergone a displacement $u(x, t)$ from its equilibrium position. Its right end has experienced a similar displacement $u + \Delta u$. Its length, therefore, due to applied stress, has changed by an amount Δu from Δx to $\Delta x + \Delta u(x)$. Accordingly, at the left end, where the force is $F(x)$ and the elastic elongation $\Delta u(x)$, (10.4.3) becomes

$$\frac{F(x)}{A} = Y\frac{\Delta u(x)}{\Delta x} \tag{10.4.4}$$

At the right end, the force is $F(x + \Delta x)$ and the elastic elongation is $\Delta u(x + \Delta x)$. Under these circumstances, (10.4.3) becomes

$$\frac{F(x + \Delta x)}{A} = Y\frac{\Delta u(x + \Delta x)}{\Delta x} \tag{10.4.5}$$

The total force acting on the element is, therefore,

$$F(x + \Delta x) - F(x) = AY\left[\frac{\Delta u(x + \Delta x)}{\Delta x} - \frac{\Delta u(x)}{\Delta x}\right] \tag{10.4.6}$$

If Δx approaches zero, $\Delta u(x)/\Delta x$ represents the partial derivative of u with respect to x, evaluated at x. Using Newton's second law, we find, therefore, that

$$AY\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = (\Delta m)\frac{\partial^2 u}{\partial t^2} = \rho A\,\Delta x\,\frac{\partial^2 u}{\partial t^2} \tag{10.4.7}$$

where ρ is the density, or mass per unit volume, and $\partial^2 u/\partial t^2$ is the acceleration of the element. If we now

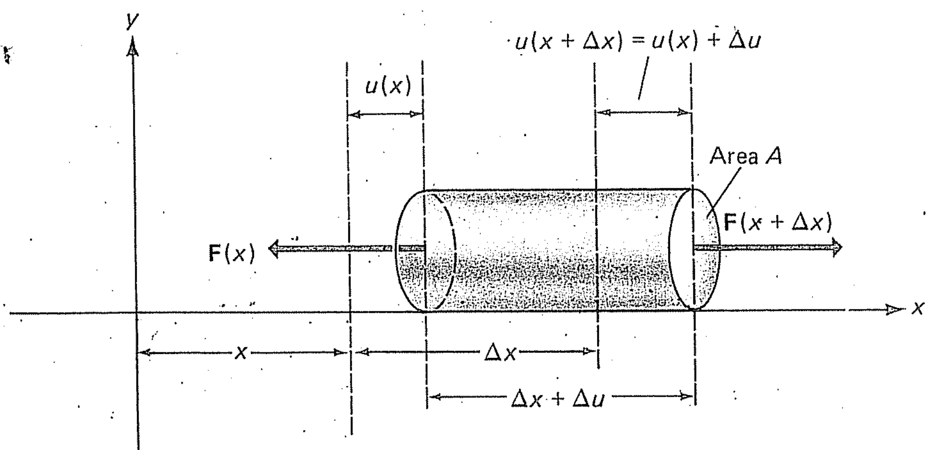


FIGURE 10.10. Forces and displacements associated with an infinitesimal element of an elastic bar along which longitudinal waves propagate.

divide both sides of (10.4.7) by $Y \Delta x$ and take the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{Y/\rho}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (10.4.8)$$

Equation (10.4.8) is a one-dimensional wave equation describing the displacement u of a small segment located at x. The wave velocity is given by (10.3.9) as

$$\boxed{v = \sqrt{Y/\rho}} \qquad (10.4.9)$$

We find, therefore, that any initial displacement of molecules at the left end of the bar, the end struck by the hammer, can propagate as a *longitudinal wave* described by a solution of (10.4.8). Such a wave is nothing more nor less than a *sound wave*. A typical sinusoidal solution of Eq. (10.4.8) can be written as

$$u(x, t) = A \sin(kx - \omega t) \qquad (10.4.10)$$

with

$$v = \omega/k = \sqrt{Y/\rho}$$

It is important to realize that the displacement $u(x, t)$ of any part of the bar from its equilibrium position takes place *along the x-direction*. The particles of the bar execute simple harmonic motion about their equilibrium positions, vibrating back and forth along the x-axis, while the wave itself is, according to Eq. (10.4.10), propagating also along the x-direction. The situation is very similar to that illustrated in Fig. 10.2 by the passage of a compressional pulse along a spring. This type of wave motion, in which the particle displacements take place parallel to the direction in which the wave itself travels, is called *longitudinal* wave motion and is to be contrasted with the transverse type of wave motion studied previously, in which particle motions took place perpendicular to the direction of travel of the wave.

EXAMPLE 10.4.1

A metallic bar of length 1 m and cross-sectional area 2 cm^2 has a density of 3 g/cm^3. It is subjected to compression by forces of 10^3 newtons at its ends, and as a result of this compression, the bar contracts by 0.1 mm. (a) Find the longitudinal strain on the bar. (b) What is the longitudinal stress? (c) Determine Young's modulus. (d) What is the velocity of propagation of longitudinal waves traveling in the bar?

The strain is given by the expression

$$\text{strain} = \frac{\Delta l}{l_0} = \frac{0.1 \text{ mm}}{1000 \text{ mm}} = 10^{-4}$$

while the stress is given by

$$\text{stress} = \frac{F}{A} = \frac{10^8}{2} = 0.5 \times 10^8 \text{ dyn/cm}^2$$

Using these two results in conjunction with the definition of Young's modulus, we obtain

$$Y = \frac{\text{stress}}{\text{strain}} = 0.5 \times 10^{12} \text{ dyn/cm}^2$$

To find the speed of wave propagation, we use the formula given by Eq. (10.4.9) to find

$$v = \left(\frac{0.5 \times 10^{12}}{3}\right)^{1/2} \doteq 0.41 \times 10^6 \text{ cm/sec} = 4100 \text{ m/sec}$$

This wave velocity is typical of the speed of longitudinal waves in metallic objects and gives an idea of what we might expect for the velocity of sound in solid substances.

When a longitudinal disturbance propagates in a solid as a wave, the form of the disturbance depends on its source. The blow of a hammer causes a sharp pulse to propagate from one end to the other. On the other hand, a periodic disturbance of a definite frequency causes a periodic or sinusoidal wave to propagate. Just as we found that the wave equation for a one-dimensional string could exhibit sinusoidal solutions of a definite frequency and wavelength, we also find that one-dimensional longitudinal waves possess these same characteristics.

Longitudinal waves propagating through a solid (or liquid or gas) are also called *sound waves*, although not all waves of this type are audible to the human ear. Those waves which have frequencies below 20 Hz are called infrasonic waves and are inaudible. These waves are usually generated by very big sources since their wavelength is generally very long. For example, a longitudinal wave of 1/20 Hz traveling through the earth, where the wave speed is about 5000 m/sec, would have a wavelength $\lambda = v/f = 100{,}000$ meters. Such waves can be produced inside the earth's mantle by seismic activity such as earthquakes. In fact, longitudinal waves with a wavelength as large as the diameter of the earth have been detected.

When longitudinal waves contain frequencies in the range of 20 to 20,000 Hz, they are in the audible range and can be heard as sound by the human ear.[5] Such waves may originate in vibrating strings, air columns, membranes, loudspeaker diaphragms, and other sources. Their wavelength depends on the sound velocity in the medium in which the waves are traveling as well as their frequency. In air, where the sound velocity is about 330 m/sec, 20-Hz waves have

[5] Other animal species can hear much higher frequencies. For example, bats can hear up to 120,000 Hz and porpoises up to 240,000 Hz. However, at the low-frequency end, bats and porpoises are limited to 10,000 Hz and would, therefore, not be able to appreciate our music. Although many animals hear frequencies that are ultrasonic, few can match the human sensitivity in the low-frequency end of the spectrum.

a wavelength of about 16 m, while 20,000-Hz sound waves have a wavelength of around 1.6 cm.

High-frequency longitudinal waves (above 20,000 Hz) cannot be detected by the human ear. These waves may be produced by electrically induced high-frequency vibrations of crystals; they can also by produced and detected by bats and porpoises, and are extremely important for navigation and food gathering for both of these species. They are referred to as *ultrasonic waves.*

We have discussed to some extent the propagation of sound in a solid. Sound also travels through gases and liquids, although the speed is usually smaller than in solids. Liquids and gases are fluids rather than rigid substances, and, therefore, the description of sound propagation in them is different in one important respect. Instead of using Young's modulus to characterize their elastic properties, we must use instead the *bulk modulus B* of the substance, which is defined by

$$\Delta P = B\frac{\Delta V}{V_0} \quad \text{or} \quad B = \frac{\Delta P}{(\Delta V/V_0)} \qquad (10.4.11)$$

where $\Delta V/V_0$ is the fractional volume change a volume V_0 of the substance undergoes when subjected to an externally applied change in pressure (force per unit area) ΔP. The bulk modulus B or, more properly, its inverse, is a direct measure of the *compressibility* of the substance. The velocity of sound in a liquid or a gas is, therefore, expressed by

$$\boxed{v = \sqrt{\frac{B}{\rho}}} \qquad (10.4.12)$$

In water, this velocity is approximately 1500 m/sec, while in air at normal atmospheric pressure and at a temperature of 0°C, the sound velocity is 331.4 m/sec, or 1089 ft/sec.

For a gas, both the bulk modulus B and the density depend upon the temperature and pressure, and, therefore, the velocity of sound varies significantly if these parameters change to any large extent. In normal terrestrial surroundings, the atmospheric temperature and pressure variations that arise from changing climatic conditions are not sufficiently great to alter the sound velocity very much from the figure given above. However, we shall learn more about the variation of the sound velocity with temperature and pressure in Chapter 13.

10.5 Energy Transport by Waves

Energy may be transported from one place to another in two ways. In the first of these, the energy is carried between two places because of the motion of a body that possesses energy. For example, the energy of a thrown baseball is carried by the ball from the pitcher's hand to the catcher's glove. The second means of transporting energy is by wave motion, and it is this mechanism which is of primary interest in this section.

To illustrate the basic concepts, let us consider the very idealized problem of transferring energy between two peaks of a mountain range by using the transverse waves that can propagate along a rope strung between the two peaks. On one of the peaks, an energy source is available in the form of an oscillator which can vibrate up and down at a definite frequency. The end of the rope is attached to the oscillator, and, therefore, energy is transferred to the rope. Figure 10.11 illustrates a number of snapshots of the rope at various time intervals. In the absence

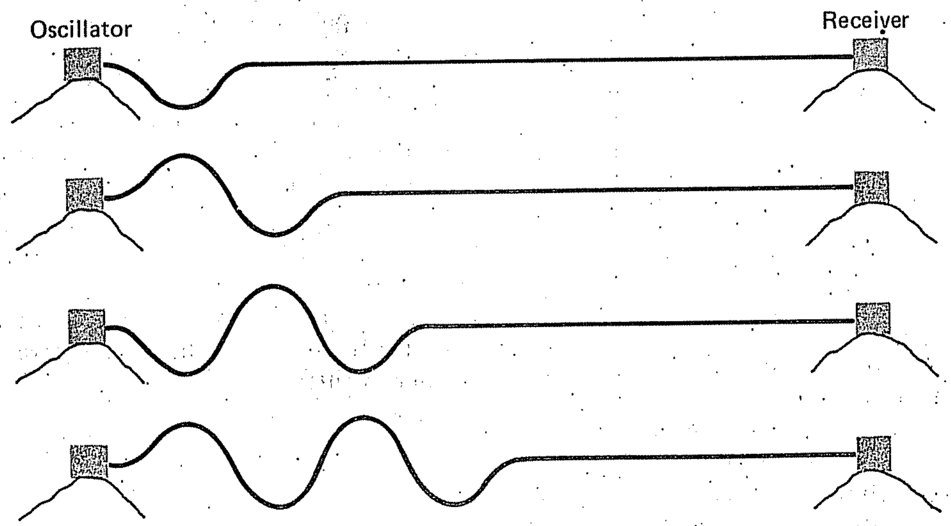


FIGURE 10.11. Successive stages in the transport of energy from an oscillator to a distant receiver via transverse waves in a rope.

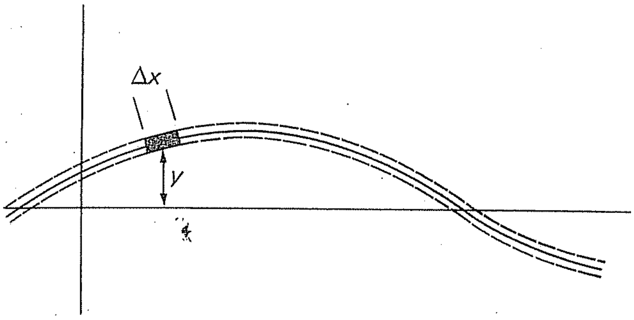


FIGURE 10.12. Typical mass element of the rope illustrated in Fig. 10.11.

of any mechanism for loss of energy, we have a rope which ultimately delivers energy to the second peak at the same rate that it receives it. If the rope oscillates in such a manner as to ultimately establish a wave of the form

$$y = y_0 \sin(kx - \omega t) \tag{10.5.1}$$

then at what rate is energy being transferred between the two peaks?

To answer this question, let us consider a small element of the rope of length Δx whose mass is $\Delta m = \mu\,\Delta x$, where μ is the linear mass density, or mass per unit length. Such an element, as illustrated in Fig. 10.12, moves up and down with simple harmonic motion of amplitude y_0 and frequency ω. Its contribution to the kinetic energy of the rope will be

$$\Delta U_k = \tfrac{1}{2}(\Delta m)v_y^2 = \tfrac{1}{2}(\mu\,\Delta x)\left(\frac{\partial y}{\partial t}\right)^2 \tag{10.5.2}$$

since $v_y = \partial y/\partial t$. But $y(t)$ is given by (10.5.1), from which

$$\frac{\partial y}{\partial t} = \omega y_0 \cos(kx - \omega t) \tag{10.5.3}$$

Substituting this into (10.5.2), we obtain

$$\Delta U_k = \tfrac{1}{2}\mu\omega^2 y_0^2\,\Delta x \cos^2(kx - \omega t) \tag{10.5.4}$$

But we may use the well-known trigonometric identity

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \tag{10.5.5}$$

to write (10.5.4) in the form

$$\Delta U_k = \tfrac{1}{4}\mu\omega^2 y_0^2\,\Delta x + \tfrac{1}{4}\mu\omega^2 y_0^2\,\Delta x \cos 2(kx - \omega t) \tag{10.5.6}$$

If we average this over a length of time corresponding to many periods of the harmonic motion, we will obtain the time-average kinetic energy associated with this portion of the rope, $\Delta\bar{U}_k$. This is easily accomplished, since the first term in the above equation is constant with respect to time, and the time average of the second term is zero because the average value of $\cos 2\omega t$ is zero over many cycles. We then obtain

$$\boxed{\Delta\bar{U}_k = \tfrac{1}{4}\mu\omega^2 y_0^2\,\Delta x} \tag{10.5.7}$$

It is possible, though somewhat less straightforward, to calculate directly the average potential energy associated with this element of the system. To simplify matters, we shall recall from our previous work in section 9.3 that the average potential energy of a simple harmonic oscillator is equal to its average kinetic energy. Therefore,

$$\boxed{\Delta\bar{U}_p = \Delta\bar{U}_k} \tag{10.5.8}$$

and the average total energy $\Delta\bar{U}$ can, therefore, be written

$$\Delta\bar{U} = \Delta\bar{U}_k + \Delta\bar{U}_p = 2\Delta\bar{U}_k = \tfrac{1}{2}\mu\omega^2 y_0^2\,\Delta x \tag{10.5.9}$$

In the limit where $\Delta x \to 0$, this can be expressed as

$$d\bar{U} = \tfrac{1}{2}\mu\omega^2 y_0^2\,dx \tag{10.5.10}$$

We may interpret this result in the following way. Equation (10.5.9) tells us that every segment of length dx has total energy $d\bar{U}$. Since the system executes a wave motion of constant amplitude and frequency, all the energy received by every such segment from its neighbor on the left in any given time interval must be balanced by an equal amount that is passed on to the neighboring segment on the right. At the extreme left end of the rope, energy is fed in by an externally powered oscillator. The rate at which this energy is absorbed from the external source by the rope is simply the power output of the oscillator. At the same time, if there is no internal energy dissipation in the rope, energy will be transferred *at the same rate* into whatever energy-absorbing system is fastened to the right-hand end. In any time interval dt, therefore, the first segment of the rope of length dx on the left-hand end—which, you will recall, receives *all* its energy from the externally powered source—will receive an amount of energy $d\bar{U}$ as given by (10.5.10), and the last segment of length dx on the right will give up an equal amount to whatever it is connected to. Clearly, then, the amount of energy transported by the system during this time interval must be

$$d\bar{U} = \tfrac{1}{2}\mu\omega^2 y_0^2\,dx = \tfrac{1}{2}\mu\omega^2 y_0^2\left(\frac{dx}{dt}\right)dt \tag{10.5.11}$$

Since dx/dt corresponds to the velocity with which energy is transported by the system, hence the wave velocity v, this can be written

$$d\bar{U} = \tfrac{1}{2}\mu v\omega^2 y_0^2\,dt \tag{10.5.12}$$

or finally,

$$\boxed{\bar{P} = \frac{d\bar{U}}{dt} = \tfrac{1}{2}\mu v\omega^2 y_0^2} \tag{10.5.13}$$

In this equation, $\bar{P}$ represents the *average power* transmitted by the wave motion that has been set up in the system.

Equation (10.5.13) exhibits some very important features of energy transfer in wave propagation. In particular, we note that the power depends on the *square of the amplitude and on the square of the frequency*. It also has a linear variation with the wave velocity. Although the above features were derived for a transverse wave traveling in one dimension, they are also valid for one-dimensional longitudinal waves.

In the case of one-dimensional waves, in which there is no dissipation of energy, any energy which flows past any given point in 1 sec must also flow past any other point in that same time interval. The situation is very much like water flowing through a pipe; the rate of flow is constant everywhere.

For waves in two or in three dimensions, the energy goes out in many directions, and, as a consequence, the energy flowing through a given unit area perpendicular to the direction of propagation depends crucially on the exact location of the area. For this reason it is useful to define a new term called the *intensity*. The intensity I at a given location is defined as the average energy which crosses a unit area perpendicular to the propagation direction in unit time, that is,

$$I = \frac{\Delta \bar{U}}{(\Delta A)(\Delta t)} = \frac{\bar{P}}{\Delta A} \qquad (10.5.14)$$

as illustrated by Fig. 10.13. The value of I depends upon the energy input by the source of the wave as well as the geometry which relates the given area to the source. Typical units used to measure I are watts/cm^2 or watts/m^2. In the case of a sound wave propagating in a gaseous medium, Eq. (10.5.13) is still correct, even though the wave is longitudinal rather than transverse. The average power crossing an area A per unit time, therefore, is

$$\frac{\bar{P}}{A} = \frac{\mu v \omega^2 {y_0}^2}{2A} \qquad (10.5.15)$$

In this expression, the quantity μ/A represents mass divided by (length × area), or mass per unit volume. It may, therefore, be replaced by the volume density ρ of the medium, which expresses its mass per unit volume. Making this substitution, (10.5.15) becomes

$$I = \tfrac{1}{2}\rho v \omega^2 {y_0}^2 \qquad (10.5.16)$$

EXAMPLE 10.5.1

A point source emits energy with a power of 10 watts in the form of spherical sound waves. Find the intensity 5 m from the source, and also find the energy

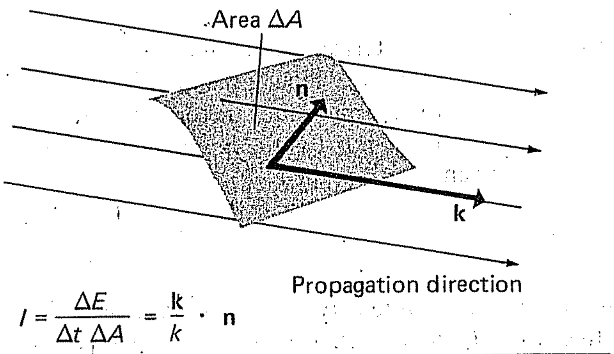


FIGURE 10.13. Transport of wave energy across an arbitrarily oriented area element.

which crosses an area 3 cm^2 (perpendicular to the direction of propagation of the wave) every 5 sec if the area is 5 m away from the source.

Let $\bar{P}$ be the average power emitted by the source. Thus, $\bar{P}$ joules are emitted each second, where $\bar{P}$ has the numerical value 10 in the present case. This amount of energy must pass through any sphere whose center is at the source (assuming no absorption); and therefore, from (10.5.13),

$$I = \frac{\Delta U}{(\Delta t)(\Delta A)} = \frac{\bar{P}}{\Delta A} = \frac{\bar{P}}{4\pi r^2} \qquad (10.5.17)$$

where I is the intensity a distance r from the source. It is evident from this equation that the intensity of spherical sound waves radiating from a point source falls off *inversely as the square of the distance* from the source. Substituting numbers, we obtain

$$I = \frac{10}{4\pi(5)^2} = 3.2 \times 10^{-2} \text{ watt/m}^2$$
$$= 3.2 \times 10^{-6} \text{ watt/cm}^2$$

This expression may now be utilized to find the energy passing through any area on a sphere of radius 5 m. In 5 sec, the energy passing through 3 cm^2 is

$$U = (3.2 \times 10^{-6})(3)(5) = 4.8 \times 10^{-5} \text{ J.}$$

It should be remarked that $\bar{P} = 10$ watts is the *acoustic* power output, which is considerably smaller than the electrical power input needed to create this acoustic power. The conversion of electrical power in devices such as loudspeakers to useful acoustic power is usually quite inefficient, much of the electrical energy being wasted in the form of heat.

The faintest sound the human ear can detect has an intensity (at the ear) of 10^{-16} watts/cm^2. This represents an extremely small amount of energy received by the ear each second, and it reminds us that the ear is an enormously sensitive organ. The ear can respond effectively to intense sound waves but generally cannot tolerate intensities larger than about 10^{-4} watts/cm^2 without experiencing pain. Due to the large range of intensities we hear, it has become customary to characterize sound intensities by means of a logarithmic scale which gives the intensity of sound relative to the minimum discernible intensity $I_0 = 10^{-16}$ watts/cm². If I is the intensity of a sound wave at a given location, then

$$\beta = 10 \log \frac{I}{I_0} \tag{10.5.18}$$

gives a measure of intensity in so-called *decibels*, or db. Thus, if $I = I_0$, $\beta = 0$; whereas if $I = 10^{-4}$ watts/cm², $\beta = 120$ db. A value of 10 db implies an intensity 10 times as large as I_0, while 20 db corresponds to 100 times I_0. Ordinary conversation occurs at about 65 db, while whispering corresponds to 20 db. It should be kept in mind that intensity is a quantitative measure of energy received and is not synonymous with the sensation of loudness, although there is, to be sure, a correlation. It could happen that sound waves of the same intensity appear to have different loudness since they may be of different frequency. The sensation of loudness is a qualitative physiological response which depends on many factors. Two people can disagree on the "loudness' of a TV set, and often do, but they will not disagree on its intensity if they have the proper instruments with which to measure it.

EXAMPLE 10.5.2
At a football stadium, the plays are announced from a single loudspeaker, since the university cannot afford a more elaborate system. The most distant spectator is 600 ft from the speaker. If an intensity level of 70 db is desired at this point, what is the minimum acoustic power required?

The intensity I may be determined from Eq. (10.5.18), according to which $70 = 10 \log I/I_0$, which implies (noting that I_0 represents the minimum audible intensity of 1.00×10^{-16} watt/cm²),

$$I = 10^7 I_0 = 10^7 \times 10^{-16} = 10^{-9} \text{ watt/cm}^2$$

Substituting this numerical value into Eq. (10.5.17), we obtain, for $r = 600$ ft $= 1.83 \times 10^4$ cm,

$$P = 4\pi r^2 I = (4\pi)(1.83 \times 10^4)^2(10^{-9}) = 4.21 \text{ watts}$$

We can compare the relative sound intensities I and I' at different distances r and r' from a point source that radiates at a constant power level $\bar{P}$ by noting from (10.5.17) that

$$\bar{P} = 4\pi r^2 I = 4\pi r'^2 I' \tag{10.5.19}$$

from which

$$\frac{I'}{I} = \frac{r^2}{r'^2} \tag{10.5.20}$$

This equation is simply another way of expressing the inverse square law for a point source. In this example, we know that for $r = 600$ ft, $I = 1.0 \times 10^{-9}$ watt/cm². Using (10.5.20), it is easy to show that for $r = 30$ ft, $I' = 4.0 \times 10^{-7}$ watt/cm²; and for $r = 3$ ft, $I' = 4.0 \times 10^{-5}$ watt/cm². Both of these figures are still below the pain threshold of 10^{-4} watt/cm², although in the latter case the intensity level would certainly be uncomfortable.

EXAMPLE 10.5.3
At 10,000 cycles per second, the threshold of audibility occurs at about 11 db. Find the amplitude of molecular vibrations in air (at standard temperature and pressure) at this frequency. The density of air is $\rho = 1.29 \times 10^{-3}$ g/cm³ and the wave speed is $v = 3.31 \times 10^4$ cm/sec. Note that 1 watt = 1 joule/sec = 10^7 ergs/sec = 10^7 dyne-cm/sec = 10^7 g cm²/sec³.

From (10.5.18), we can determine the intensity at 11 db. We may write, therefore,

$$11 = 10 \log \frac{I}{I_0}$$

which implies $I/I_0 = 12.6$, or

$$I = 12.6 \times 10^{-16} \text{ watts/cm}^2$$
$$= 12.6 \times 10^{-16} \text{ g cm}^2/\text{sec}^3 \text{ cm}^2$$

since $I_0 = 1.00 \times 10^{-9}$ watt/cm². Then, from (10.5.16),

$$y_0{}^2 = \frac{2I}{\rho v \omega^2} = \frac{(2)(12.6 \times 10^{-9})}{(1.29 \times 10^{-3})(3.31 \times 10^4)(2\pi \times 10^4)^2}$$
$$= 14.9 \times 10^{-20} \text{ cm}^2$$

$$y_0 = 3.86 \times 10^{-10} \text{ cm}$$

This is an incredibly small displacement amplitude. In fact, the displacement of molecules or atoms from their equilibrium positions is considerably smaller than the moleoules or atoms themselves, for they are generally larger than 10^{-8} cm. It is almost incredible, but nonetheless true, that the human ear can detect motions of such minute proportions.

10.6 Superposition of Waves

One of the most profound principles underlying wave motion is the so-called *superposition principle*. It states that two or more waves may travel through the same region of space in a completely independent way and that the displacement of particles in the medium is obtained by direct addition of the displacements which each of the separate waves would produce in the absence of all others. This principle is confirmed by experiment, provided the amplitudes of the various waves are not too large. For very large amplitudes, the restoring forces on the particles do not obey Hooke's law, and this simple result is no longer true.

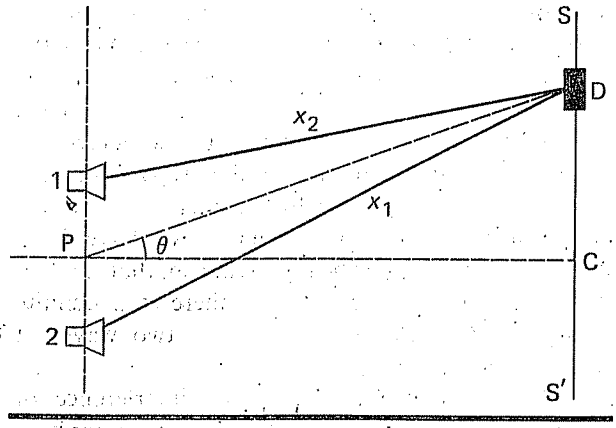


FIGURE 10.14. Spatial interference of sound waves emitted by two loudspeakers.

As a consequence of the superposition principle, waves exhibit an effect called interference. When two or more waves interfere, they do not in any way affect one another. It is unfortunate, therefore, that the word interference has become universally adopted, for in the present context it merely means that two or more waves add or superpose to provide a resultant which may produce effects that were not inherent in any of the individual waves. Thus, interference is really synonymous with superposition. There are two main classes of interference phenomena, since waves can interfere either spatially or temporally. The distinction between these classes will be made below.

Consider an example illustrated by Fig. 10.14 in which two loudspeakers emit sound waves of the same frequency and amplitude, and let these waves be denoted by

$$y_1 = y_0 \sin(kx_1 - \omega t)$$
$$y_2 = y_0 \sin(kx_2 - \omega t) \qquad (10.6.1)$$

In these expressions, x_1 and x_2 are the distances, respectively, from speaker 1 and speaker 2 to a detector D, and y_1 and y_2 are the molecular displacements produced there at time t by each speaker acting *by itself*. The movable detector D, which is far from the speakers in comparison to their separation, is free to move along the line SS′. What is the resultant wave received by the detector and how does the intensity of sound vary as the detector is moved?

The geometry of the situation is illustrated in Fig. 10.14. According to the superposition principle, the molecular displacements at the detector are given by

$$y = y_1 + y_2 = y_0[\sin(kx_1 - \omega t) + \sin(kx_2 - \omega t)] \qquad (10.6.2)$$

The above sum may be evaluated by utilizing the trigonometric identity

$$\sin a + \sin b = 2 \sin \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b) \qquad (10.6.3)$$

If this is used, we obtain

$$y = 2y_0 \sin[\tfrac{1}{2}k(x_1 + x_2) - \omega t] \cos \tfrac{1}{2}k(x_1 - x_2) \qquad (10.6.4)$$

for the result. Since we have assumed that the speaker separation is small compared to the distance from the detector, the distance $\frac{1}{2}(x_1 + x_2)$ is approximately the distance from the detector to the point P midway between the speakers.

Let us now rewrite Eq. (10.6.4) as

$$y = Y_0 \sin(kX - \omega t) \qquad (10.6.5)$$

where

$$X = \tfrac{1}{2}(x_1 + x_2) \quad \text{and} \quad Y_0 = 2y_0 \cos \tfrac{1}{2}k(x_1 - x_2) \qquad (10.6.6)$$

Now, Eq. (10.6.5) looks like a single wave, except that its amplitude Y_0 *depends on the position of the detector*. If x_1 and x_2 are equal, which corresponds to having the detector midway between the speakers, the amplitude Y_0 assumes its maximum value of $2y_0$. As we have seen previously, the intensity of a wave is proportional to the square of the amplitude, and therefore an amplitude of $2y_0$ gives an intensity four times as large as the intensity which would have been present if the sound from only one speaker were received. Now, as the distance $x_2 - x_1$ increases, the amplitude and therefore the intensity decreases until, when $\frac{1}{2}k(x_1 - x_2) = \pi/2$, it has the value *zero*. At this point, the difference in path length $x_1 - x_2$ is π/k, which (since $2\pi/\lambda$) amounts to *one half the wavelength* λ. There is, then, a difference in phase of π radians, or 180°, between the two waves. When the amplitude is a maximum, the waves reinforce one another and we have what is known as *constructive* interference. When the amplitude assumes its minimum value of zero, the two waves cancel one another and we have complete *destructive* interference. Whether the interference is constructive or destructive depends upon the path difference $x_1 - x_2$ and thus upon the spatial position of the detector. Therefore, this phenomenon is known as *spatial interference* of the waves.

Constructive interference occurs whenever

$$\tfrac{1}{2}k(x_1 - x_2) = 0, \pi, 2\pi, 3\pi, \text{etc.,}$$

thus, whenever

$$x_1 - x_2 = n\lambda, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \qquad (10.6.7)$$

The condition for destructive interference is

$$\tfrac{1}{2}k(x_1 - x_2) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

or, in other words,

$$x_1 - x_2 = (n + \tfrac{1}{2})\lambda \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{10.6.8}$$

These conditions simply state that constructive interference occurs whenever the path difference $x_1 - x_2$ amounts to an integral number of wavelengths, or whenever the phase difference is an integral multiple of 2π radians. Destructive interference takes place whenever the path difference amounts to an odd number of half-wavelengths, or when the phase difference is an odd multiple of π radians.

There are a number of subtle features concerning the above derivation of interference which we have glossed over but which should at least be mentioned. We have assumed that the amplitude of each wave, y_0, does not depend upon position. This is actually not true; the amplitude is much larger near a speaker than far away. We may expect, however, that Eqs. (10.6.1) are valid expressions for giving the displacements along the line SS′, provided this line is so far from the speakers that, as D is varied, the distance x_1 changes by an amount much smaller than x_1 itself. The amplitude y_0 then will not vary appreciably as the detector is moved. The assumption that the detector is far away is needed for still another reason. Since the waves are longitudinal the displacements characterized by y_1 and y_2 are not really parallel, as we have assumed. If, however, the distance to the detector is large in comparison to the speaker separation, the displacements are practically parallel, and our assumption is a good one. Finally, we have been assuming, without specifically saying so, that the two waves are *coherent*. This means there is a definite constant phase relation between the two waves at every instant of time.

A second example of spatial interference of waves is illustrated by Fig. 10.15, a photograph of "double slit interference" of water waves. At the bottom of the picture, waves are generated by two mechanically driven oscillators excited at the same frequency and phase. Circular waves spread out from each of the two sources and superpose to produce a pattern in which light and dark spots correspond, respectively, to destructive and constructive interference. This example is very similar to the preceding one, except that now the waves are transverse rather than longitudinal.

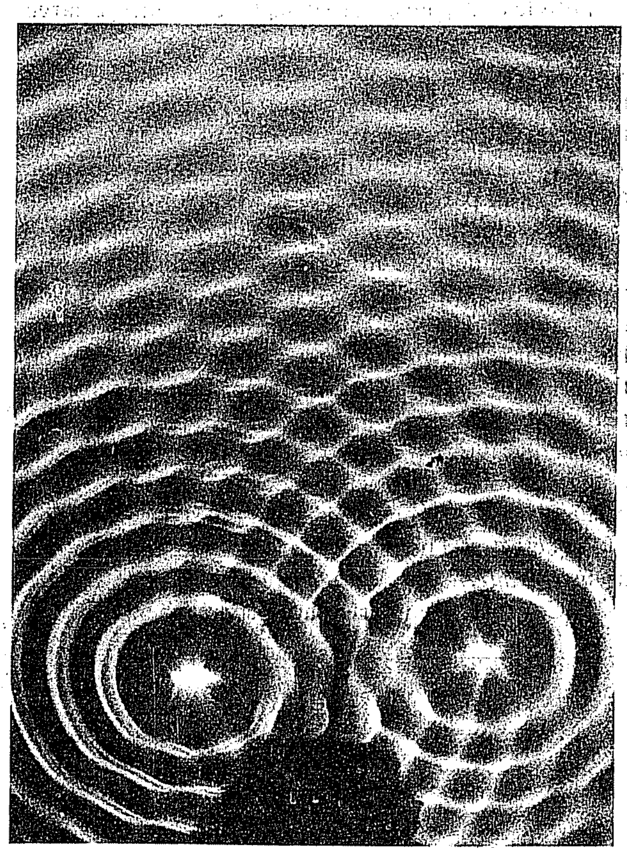


FIGURE 10.15. Spatial interference of water waves excited by two point sources. (Ripple tank photo courtesy of Professor T. A. Wiggins, Pennsylvania State University)

EXAMPLE 10.6.1

The two sources in a "ripple tank" are separated by a distance d, as illustrated in Fig. 10.16. A wave of angular frequency ω is incident. Discuss the pattern produced by the superposition of waves. Assume that each wave has an amplitude which is a function of distance from the source given by $y_0(r_1)$ and $y_0(r_2)$.

Let P be the point of observation. The superposition principle gives a resultant wave at P as

$$y = y_0(r_1)\sin(kr_1 - \omega t) + y_0(r_2)\sin(kr_2 - \omega t) \tag{10.6.9}$$

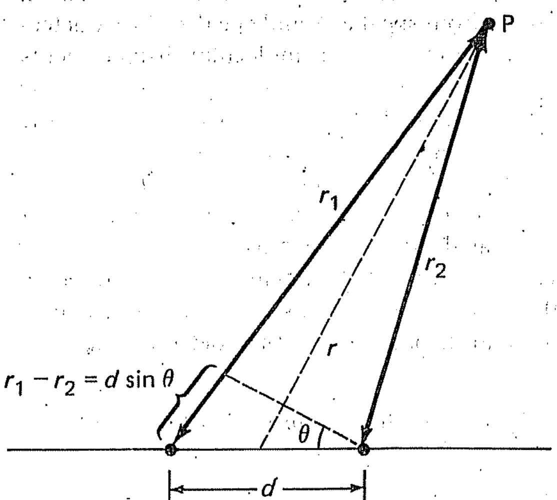


FIGURE 10.16. Geometry of spatial interference from two point sources.

The complete interference pattern is very difficult to calculate explicitly because $y_0(r_1)$ and $y_0(r_2)$ may be quite different at an arbitrary point. Let us, therefore, simplify the problem by assuming that r_1 and r_2 are very large in comparison to the distance d. Under this condition, r_1 and r_2 are almost equal. In fact, as an approximation, we may assume that $y_0(r_1) \cong y_0(r_2) \cong y_0(r)$, where $r = \frac{1}{2}(r_1 + r_2)$. Under this assumption, Eq. (10.6.9) may be transformed, once again using the identity (10.6.3), to yield the displacement

$$y = 2y_0(r) \cos k\left(\frac{r_1 - r_2}{2}\right) \sin(kr - \omega t) \qquad (10.6.10)$$

Here, we have a resultant wave with an effective amplitude

$$A = 2y_0(r) \cos \tfrac{1}{2}k(r_1 - r_2) \qquad (10.6.11)$$

The individual waves are transverse two-dimensional waves, and, therefore, there is no difficulty in numerically adding the displacements. This would be true even without the approximation made above.

The amplitude A will vary slowly with r. The reader should show that for large r it is reasonable to expect a dependence $y_0(r) \sim r^{-1/2}$. Figure 10.16 shows the geometry of the situation. When r_1 and r_2 are large, their difference may be expressed in terms of the separation of the sources and the angular variable θ. It is easy to see from the figure that

$$r_1 - r_2 = d \sin \theta \qquad (10.6.12)$$

The amplitude, therefore, can be expressed in terms of r and θ by means of the equation

$$A(r, \theta) = 2y_0(r) \cos \tfrac{1}{2}(kd \sin \theta). \qquad (10.6.13)$$

For a fixed value of r, this amplitude varies significantly with the angle θ. The amplitude is zero whenever

$$\boxed{\frac{k}{2} d \sin \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots} \qquad (10.6.14)$$

and has its maximum value when

$$\boxed{\frac{k}{2} d \sin \theta = 0, \pi, 2\pi, 3\pi, \ldots} \qquad (10.6.15)$$

When the amplitude is zero, we have complete destructive interference. The points at which it vanishes are called *nodes*. Constructive interference is implied by the condition (10.6.15). These locations are called *antinodes*. At all other points, the waves superpose to produce neither a minimum nor a maximum in the amplitude.

Let us rewrite the condition for a *minimum amplitude* by using the relation $k = 2\pi/\lambda$ discussed earlier. This gives

$$\boxed{d \sin \theta = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots} \qquad (10.6.16)$$

Now, the value of $\sin \theta$ lies between zero and unity. Therefore, if $d < \lambda/2$, it is impossible to obtain any minimum in the interference pattern. If d is very much larger than $\lambda/2$, the maxima and minima may be located too close to one another to reveal a discernible interference pattern. Thus, the condition for a nice, visible pattern is that d be larger than $\lambda/2$, perhaps five or ten times as large, but certainly not very much larger (for example, not 100 times as large).

In much the same way, the condition for *maximum amplitude* can be written as

$$\boxed{d \sin \theta = 0, \lambda, 2\lambda, 3\lambda, \ldots} \qquad (10.6.17)$$

We see that the variation of amplitude with θ is quite pronounced. For a fixed angle θ, the amplitude does vary with r but the variation is quite gradual. This variation is due to the fact that at large values of r, the available energy is distributed over a larger area and, therefore, the displacement amplitude at each point will be correspondingly reduced.

We have discussed two similar examples of superposition of waves to produce an interference pattern that exhibits a strong spatial variation of the displacement amplitude. The first case involved a longitudinal wave, while the second involved a transverse wave. Let us now consider a third example of spatial interference, this one resulting from waves traveling in opposite directions with the same speed and frequency. This situation is one which commonly occurs in systems in which a wave may be *reflected*. It is illustrated in Fig. 10.17. A string is given a pulse that travels to the right toward the support to which its right end is tied. If no energy is lost at the support, the pulse is reflected and *inverted* as shown in the figure. This inversion and reflection is a consequence of Newton's third law. As the pulse approaches the

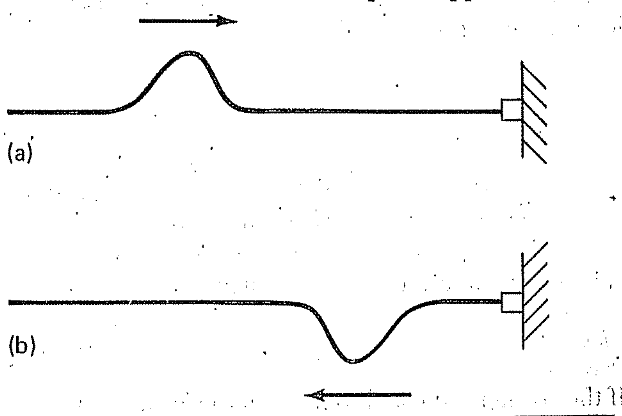


FIGURE 10.17. Reflection of a transverse wave in a string by a fixed boundary, illustrating the 180° phase change.

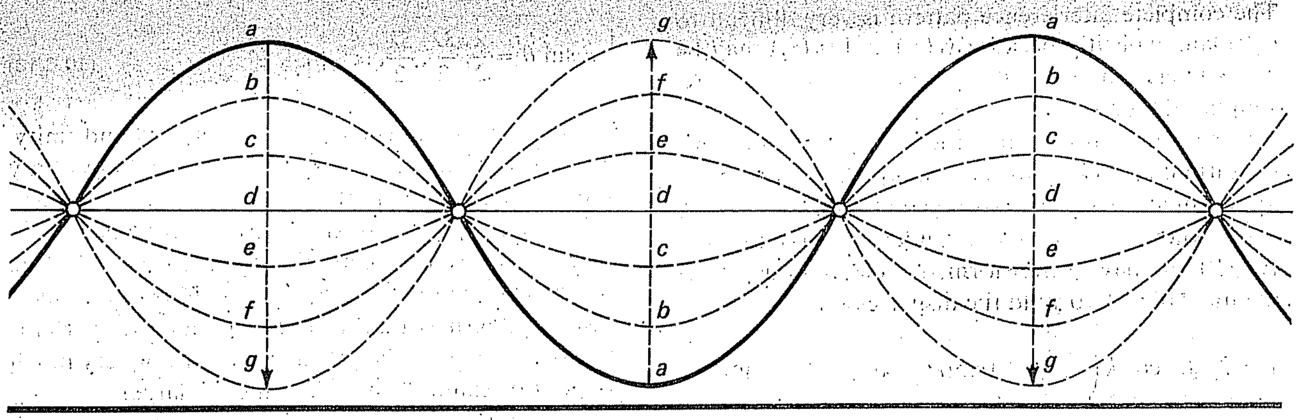


FIGURE 10.18. System of standing waves in a vibrating string.

support; the string exerts a force on the support while the support exerts an equal and opposite force on the string. The force the support exerts has a time variation and a magnitude exactly such as to cause the inversion. This same phenomenon of inversion also occurs for a continuous wave and in this case can be regarded as a phase change of 180° upon reflection.

Consider now the case of a string fixed at *both* of its ends. In this case, reflection occurs at both ends. Let us suppose, for simplicity, that the string is vibrating with a single frequency ω. The motion must then be a superposition of waves traveling to the right and waves traveling in the opposite direction, to the left. Mathematically, this superposition can be written as[6]

$$y = y_0 \sin(kx - \omega t) + y_0 \sin(kx + \omega t) \qquad (10.6.18)$$

Once again the trigonometric identity, (10.6.3), can be utilized to transform this expression into

$$y = 2y_0 \sin kx \cos \omega t \qquad (10.6.19)$$

The superposition of waves has produced a situation in which the string element at position x undergoes a simple harmonic vibration with amplitude $2y_0 \sin kx$. This amplitude varies with x but not with time. It attains the maximum value $2y_0$ whenever the following condition is fulfilled:

$$\boxed{kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots} \qquad (10.6.20)$$

These locations of maximum amplitude of vibration are known as *antinodes*. There are also positions for which the amplitude of vibration vanishes. These are called *nodes* and they occur whenever

$$\boxed{kx = 0, \pi, 2\pi, \ldots} \qquad (10.6.21)$$

If the string is attached to fixed supports at $x = 0$ and at $x = L$, then these locations must be nodes since the string cannot move at these points. The expression (10.6.19) certainly guarantees the existence of a node at the left end. The condition to have a node at the right end requires that for $x = L$, $\sin kL = 0$. This implies, in turn, that

$$\boxed{kL = \pi, 2\pi, \ldots}$$

or

$$\frac{2\pi L}{\lambda} = \pi, 2\pi, \ldots \qquad (10.6.22)$$

or

$$\boxed{\lambda = 2L, \frac{2L}{2}, \frac{2L}{3}, \frac{2L}{4}, \frac{2L}{5}, \ldots}$$

The wavelength and the frequency are related by the condition $f\lambda = v$, where v is the velocity of the component traveling waves. Therefore, the equation above also implies that a string fixed at both ends cannot vibrate with any arbitrary frequency, but only those frequencies that result when condition (10.6.22) is satisfied. The only possible frequencies are, therefore, given by

$$\boxed{f = \frac{v}{2L}, \frac{2v}{2L}, \frac{3v}{2L}, \frac{4v}{2L}, \ldots} \qquad (10.6.23)$$

These frequencies are all multiples of the lowest frequency, $v/2L$, which is known as the *fundamental*.

The disturbance produced by means of this superposition is not a traveling wave at all, but is referred to as a standing wave. The reason for this terminology is clear when one examines Fig. 10.18. Here, the whole string is viewed at a number of instants in time. Since the nodes are places which are at rest, energy cannot be transmitted across a node. Therefore, the energy contained in the region between two adjacent nodes "stands" there, although it alternates between kinetic and potential energy during the

[6] Recall that when we replace v by $-v$ in the expression $y = F(x - vt)$, we reverse the direction in which the wave travels.

period of vibration. The standing wave pattern may also be understood from a slightly different viewpoint. Two component waves have been superposed, and each transports energy. The wave moving to the right transports energy in that direction, while the wave moving to the left carries energy to the left. The total effect is no energy transfer at all, since both waves are transferring equal energies each second but in opposite directions.

The various frequencies which are possible, as given by Eq. (10.6.22), are known as the *natural frequencies* of the string, or the harmonics of the string. In addition to the fundamental frequency $f_1 = v/2L$, also called the first harmonic, we may have all integral multiples of it. These are known as the overtones, or the higher harmonics. Thus, $f_2 = 2f_1$ and $f_3 = 3f_1$ are the second harmonic (first overtone) and third harmonic (second overtone), respectively. The series continues in the same manner for all other harmonics.

In the preceding chapter on simple harmonic motion, we encountered various systems which vibrate at only one natural frequency. Examples of this were provided by a mass attached to a simple spring and also by a pendulum. We have now seen that other systems can have many natural frequencies of vibration. It is very fortunate that this is the case, for otherwise the music emanating from a stringed instrument such as a violin would be incredibly dull. The *quality* of sound emanating from a violin or other musical instrument depends cirtically on the relative amplitudes of the various overtones as well as upon the fundamental vibration frequency. A typical sound spectrum from a violin is shown in Fig. 10.19. It should be mentioned that our ears differentiate the sound of an oboe, for example, from the sound of a violin playing the same fundamental notes completely on the basis of the different spectrum of harmonics or overtones emitted by the two instruments.

Let us now turn our attention to a very different type of interference, namely, interference in time. In the case of spatial interference of waves, the amplitude of the resultant disturbance is strongly dependent on *position*. The resultant displacement in any wave motion also has a time variation, of course, which is characterized by the frequency, but this variation is generally too rapid for our ears to detect. The frequency of most sound waves is simply too high to allow identification of the individual maxima and minima of the displacements. The ear responds, therefore, only to the time-averaged intensity. It is, however, true that different frequencies cause distinctly different physiologic responses which we identify as the *pitch*, and therefore the individual waves are not entirely without their effect.

By superposition of waves, it is nevertheless possible to create sounds in which distinct *time variations* of intensity are readily detected by the ear. The phenomenon leads to the identification of *beats*. Let us suppose, for example, that we have two tuning forks which when struck emit sound waves of frequency f_1 and f_2. At the position of an observer's ear, the net displacement of molecules of the medium is given by

$$y = y_0 \sin(k_1 x - \omega_1 t) + y_0 \sin(k_2 x - \omega_2 t) \qquad (10.6.24)$$

provided the amplitude of each of the waves is the same. The resultant wave, Eq. (10.6.24), can be re-expressed as

$$\begin{aligned} y &= 2y_0 \sin \tfrac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t] \\ &\quad \times \cos \tfrac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t] \\ &= 2y_0 \sin(\bar{k}x - \bar{\omega}t) \cos \tfrac{1}{2}[(\Delta k)x - (\Delta\omega)t] \end{aligned} \qquad (10.6.25)$$

where

$$\begin{aligned} \bar{k} &= \tfrac{1}{2}(k_1 + k_2) \qquad \bar{\omega} = \tfrac{1}{2}(\omega_1 + \omega_2) \\ \Delta k &= k_1 - k_2 \qquad \Delta\omega = \omega_1 - \omega_2 \end{aligned} \qquad (10.6.26)$$

The quantities $\bar{\omega}$ and $\bar{k}$ represent the average of the

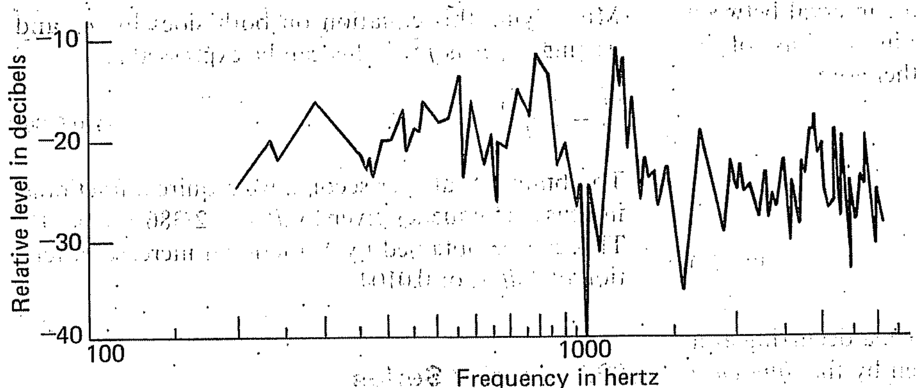


FIGURE 10.19. Frequency spectrum of sound emitted by a violin. Sound intensity level, in decibels, is plotted on the vertical axis, while frequency is plotted along the horizontal axis. (From M. V. Mathews and J. Kohut, *J. Acoust. Soc. Amer.* 53, 1620, 1973)

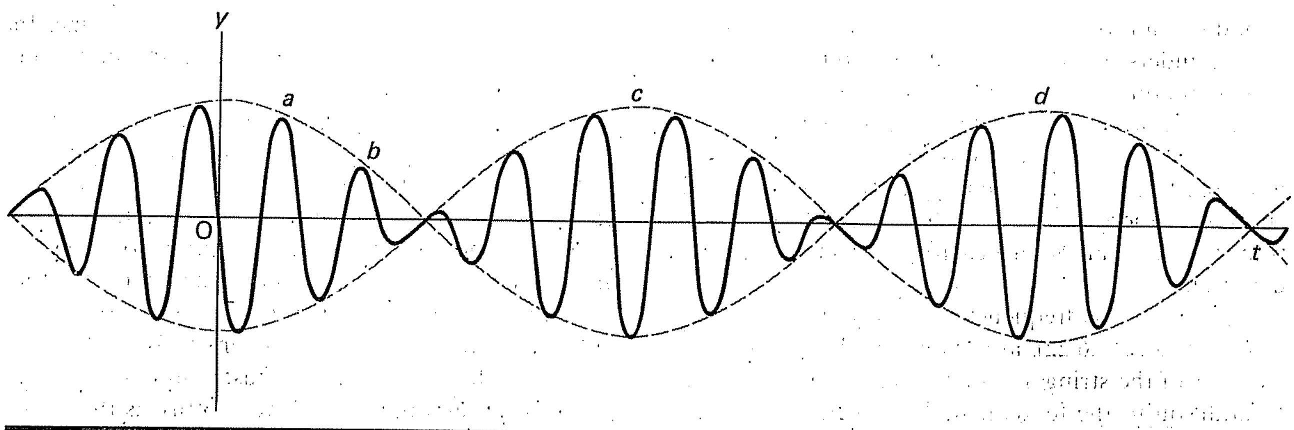


FIGURE 10.20. Temporal interference of two waves having slightly different frequencies, illustrating the phenomenon of beats.

individual frequencies and propagation constants, while $\Delta\omega$ and Δk represent their differences.

At a *fixed position* x, this expression displays a *variation in time* in both the sine factor as well as the cosine factor. Let us now assume that the two angular frequencies ω_1 and ω_2 are very nearly equal. Then, the first factor contains a very rapid time variation determined by the average of both frequencies. However, since $\Delta\omega$ is now small, the variation in time governed by the second factor may indeed by quite noticeable and may show up as *beats*, or successions of maximum amplitude spaced far enough in time to be distinguished by the ear. The situation is illustrated by means of Fig. 10.20, in which a graph of y as a function of t is given at the position $x = 0$. Any other value for x would, of course, illustrate the same general characteristics. This graph exhibits two distinct time scales. The time between any two relative maxima such as a and b is the period T, which is the reciprocal of the average frequency f. On the other hand, there is also a time T' between two absolute maxima such as c and d. These maxima occur whenever the cosine function becomes $+1$ or -1. This time interval between maxima corresponds to a change in the phase of the cosine of 180°, or π radians, and therefore

$$\tfrac{1}{2}(\Delta\omega)T' = \pi$$

or

$$\boxed{T' = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{f_1 - f_2} = \frac{1}{f_b}} \qquad (10.6.27)$$

There are, therefore, absolute maxima occurring at a "beat frequency" of $f_b = 1/T'$ given by the *difference between the frequencies f_1 and f_2*. This phenomenon is one which most of us have noticed while listening to musical instruments that are slightly "out of tune." The phenomenon is not restricted to sound waves but is also present for other forms of waves. In a subsequent section, its application to radar will be discussed.

EXAMPLE 10.6.2

Two identical piano wires of equal length have a frequency of 386 Hz when kept under the same tension. What fractional increase in tension will lead to 2 beats per second when the wires vibrate at the same time?

A change in tension will have the effect of modifying the wave speed v, which is related to the tension T_0 by means of $v = \sqrt{T_0/\mu}$. The product of frequency and wavelength is the wave speed v, and, therefore, $f\lambda = \sqrt{T_0/\mu}$, or

$$f^2\lambda^2 = T_0/\mu \qquad (10.6.28)$$

This implies that frequency is proportional to the square root of the tension since the wavelength is fixed by geometry. Differentiating the above equation with respect to T_0, regarding λ as fixed, we find

$$2f\lambda^2\frac{df}{dT_0} = \frac{1}{\mu} \qquad (10.6.29)$$

Multiplying this equation on both sides by T_0 and writing T_0/μ as $f^2\lambda^2$, this can be expressed as

$$\frac{df}{f} = \frac{1}{2}\frac{dT_0}{T_0} \qquad (10.6.30)$$

To obtain 2 beats per second, we require a fractional increase in frequency given by $df/f = 2/386 = 0.00518$. This can be obtained by a fractional increase in tension of $2\,df/f$, or 0.0104.

10.7 Fourier Series

In studying the vibrating string, we have tried to restrict the discussion to vibrations of a single frequency. The same restriction was also made for sound waves. The superposition principle tells us, however, that a very complicated wave may be built by means of the addition of many waves with differing amplitudes and frequencies. Now suppose we have a very complicated wave like that which is excited when a violin string is bowed or plucked. Is it possible to decompose this wave into a sum of simple sinusoidal waves each of which vibrates at its own definite frequency? Or, turning the question around, is it possible to synthesize a complex nonsinusoidal wave by superposing simple sinusoidal components? The answers to these questions were provided many years ago, in 1807, by the brilliant French mathematician J. B. J. Fourier in the course of his studies of heat conduction. Fourier showed that an "arbitrary" periodic function could be described as an infinite sum of sinusoidal functions.

Specifically, he discovered that if $f(x)$ is a function which is periodic over an interval $-L \le x \le L$, then $f(x)$ can be written as an infinite sum of sine and cosine functions of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{10.7.1}$$

The coefficients a_n and b_n are called the Fourier coefficients, and their values are given by the formulas

$$a_n = \frac{1}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \tag{10.7.2}$$

$$b_n = \frac{1}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \tag{10.7.3}$$

For these integrals to exist, the function $f(x)$ must be reasonably well behaved.

In Fig. 10.21, a few illustrations of the above method are given for various simple periodic functions. In principle, an infinite number of terms are needed in the series to reproduce exactly the function $f(x)$. In practice, however, it usually happens that a very good approximation to the actual function may be obtained by keeping a limited finite number of terms. If a small number of terms suffices as an approximation, we say that the series converges rapidly, while if many terms are needed, we say that the convergence is slow.

If we have a function which is not periodic but is limited in its domain of definition to an interval from $-L$ to L, we may extend the function to make it periodic, and then the above formulas are valid for describing the function in the region between $-L$ to L. As an example, consider the human profile traced from a photograph in Playboy magazine and shown in Fig. 10.22. Professor G. Ward of Penn State University has Fourier analyzed this profile numerically. The sequence of pictures illustrates various approximations to the profile by taking one, two, three, five, ten, 20, 30, and 40 terms in the Fourier series. It is clear from these figures that 40 terms constitute a very good approximation to the actual profile. Even so, 40 is a lot less than the infinite number needed for its exact reconstruction.

The technique of Fourier analysis is extremely powerful and marks a significant advance in the development of mathematical physics. Since the method may be somewhat beyond the mathematical level of many of the students for whom this text is intended, we shall try to convey an appreciation of the basic ideas involved without a full presentation of the mathematics. To fix a framework for these ideas, let us attempt to describe the general motion of a string of length L which is fixed at both ends. We have previously shown in (10.6.19) that if waves of a definite

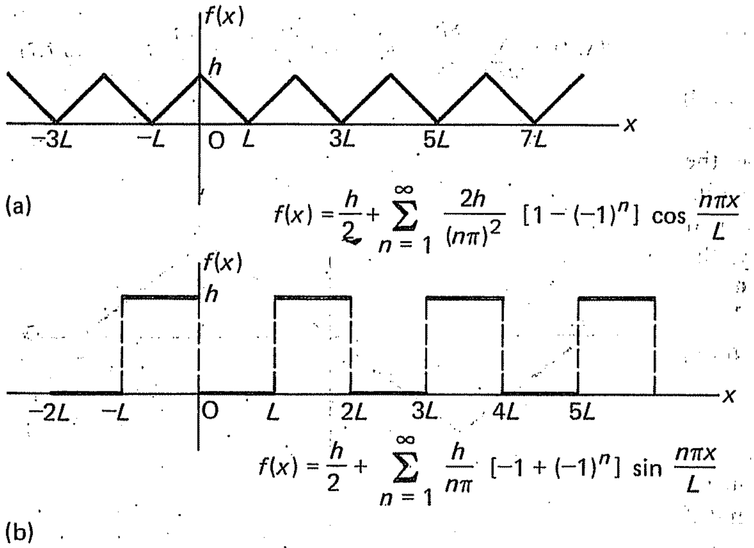


FIGURE 10.21. (a) Fourier series representing "sawtooth" wave of wavelength $2L$. (b) Fourier series representing "square" wave of wavelength $2L$.

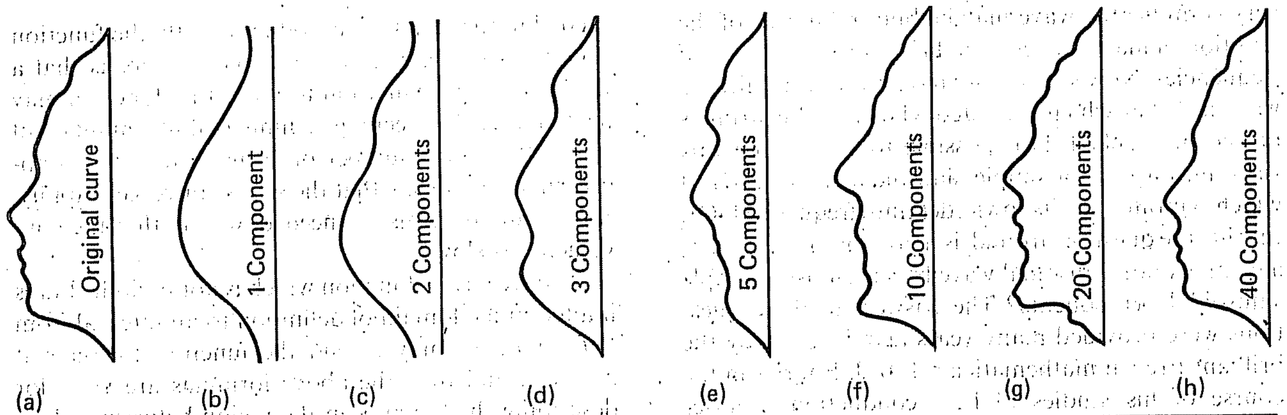


FIGURE 10.22. Synthesis of a human profile by a Fourier Series. (a) Original profile. (b)–(h) Fourier series representation, retaining 1, 2, 3, 5, 10, 20, and 40 components. (Courtesy of G. Ward, Pennsylvania State University)

frequency moving in opposite directions are superposed, a *standing wave* of a definite frequency is established. The possible frequencies were given by $f_n = nv/2L$. According to the superposition principle, the most general standing wave would, therefore, have the form of a sum of waves like those shown in (10.6.19), that is,

$$y(x, t) = \sum_{n=1}^{\infty} (b_n \sin k_n x) \cos \omega_n t$$

In this equation, according to (10.6.22) and (10.3.21), we must choose

$$k_n = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots \quad \text{or} \quad k_n = \frac{n\pi}{L}$$

and

$$\omega_n = vk_n = \frac{n\pi v}{L}$$

The above equation may, therefore, be written as

$$y(x, t) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v}{L} t\right) \tag{10.7.4}$$

Suppose now that at $t = 0$, the string has the form shown in Fig. 10.23, corresponding to a violin string that is plucked and suddenly released. Then, at $t = 0$, since $\cos(n\pi vt/L)$ has the value of unity, the function

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{10.7.5}$$

must somehow represent the triangular form of the plucked string shown in Fig. 10.23. This can be accomplished by making the proper choice of the Fourier coefficients in Eq. (10.7.1). Indeed, it is evident that if we choose a_0 and all the coefficients a_n of the cosine terms in that expression to be zero, (10.7.1) reduces to the equation above. The numerical values of the remaining coefficients b_n are calculated from (10.7.3), using for $f(x)$ the function

$$f(x) = \frac{2hx}{L} \quad (0 < x < L/2)$$

$$f(x) = \frac{2h}{L}(L - x) \quad (L/2 < x < L),$$

which represents the triangular shape of the plucked string at $t = 0$. We shall not go through the details of the calculation, but instead simply state that the coefficients b_n are found to have the form

$$b_n = \frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2} \tag{10.7.6}$$

In this expression, we should note that $\sin(n\pi/2) = 1, 0, -1, 0, \ldots$ for $n = 1, 2, 3, 4, \ldots$. According to (10.7.4), then, the form of the string at any time t can be represented as

$$y(x, t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{\pi n}{L} x \cos \frac{\pi n v}{L} t \tag{10.7.7}$$

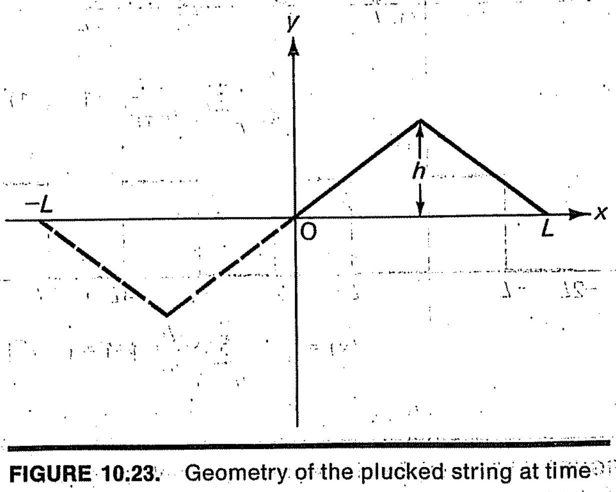


FIGURE 10.23. Geometry of the plucked string at time $t = 0$.

Now let us review what has been accomplished. We started with a snapshot of a plucked violin string at $t = 0$, at which time the string was not moving. We went through some fairly complicated mathematics to obtain the result shown above as Eq. (10.7.7). This equation tells us, however, exactly what a snapshot of the string would look like at any instant of time greater than zero. We were able to determine the entire time dependence of the motion of the string by knowing the time dependence of the component standing waves as well as the Fourier coefficients, which provide a measure of *how much of each standing wave is present in the entire sum.*

It should be noted from Eq. (10.7.6) that the Fourier coefficients contain the factor $1/n^2$ and, therefore, drop off rapidly with increasing n. This is an example of a series which has rapid convergence, and we might expect that the first ten terms already represent an excellent approximation to the actual function. We have carried out a numerical evaluation by keeping ten nonvanishing terms and have plotted a number of different snapshots of what the string would look like at different times. The results are given in Fig. 10.24. The idea that a complex nonsinusoidal wave can always be represented as a superposition of simple sinusoidal components is an important concept and one that we may on occasion ask you to recall.

10.8 The Doppler Effect and Some Related Phenomena

Whenever there is relative motion between the source of a wave and an observer, there are some very important and easily observable effects that occur. The siren of a speeding police car or ambulance appears to increase in pitch as the vehicle approaches and to decrease when it is moving away. This alteration in the frequency of detected sound is even more pronounced when the source is a train whistle or a jet plane. In the case of electromagnetic waves, the frequencies are also modified by relative motion. The frequency of light waves coming from distant galaxies is thus lowered as a result of the recession of the galaxies from our solar system.

The effect in which the emitted and detected frequencies differ due to relative motion is known as the *Doppler effect.* It was named after the Austrian physicist Christian Johann Doppler who discovered the effect in 1842 for light waves.

Consider the simple case of emission of a sound wave of definite frequency f from a fixed point source. The source may be considered to emit f disturbances or f wavefronts per second which then travel outward at the speed of sound. An observer who is stationary in the medium will detect f wavefronts per second. In other words, the number emitted and the number received are identical, as shown in Fig. 10.25a. In Fig. 10.25b, the observer is moving toward the source of sound, and in 1 second he now receives more wavefronts as a result of his motion. In t seconds, he moves a distance $d = v_0 t$, and as a consequence he detects $d/\lambda = v_0 t/\lambda$ additional wavefronts. Since the frequency f' detected by him is merely the number of wavefronts he receives per second,

$$f' = f + \frac{v_0}{\lambda} \tag{10.8.1}$$

where λ is the wavelength of the wave. But since $\lambda = v/f$, where v is the wave velocity, this equation can be written as

$$f' = f + f\frac{v_0}{v} = \left(1 + \frac{v_0}{v}\right)f \tag{10.8.2}$$

By the same reasoning, if the observer recedes from the source of sound as in Fig. 10.25c, he receives fewer disturbances per second and, therefore, the detected frequency would be

$$f' = \left(1 - \frac{v_0}{v}\right)f \tag{10.8.3}$$

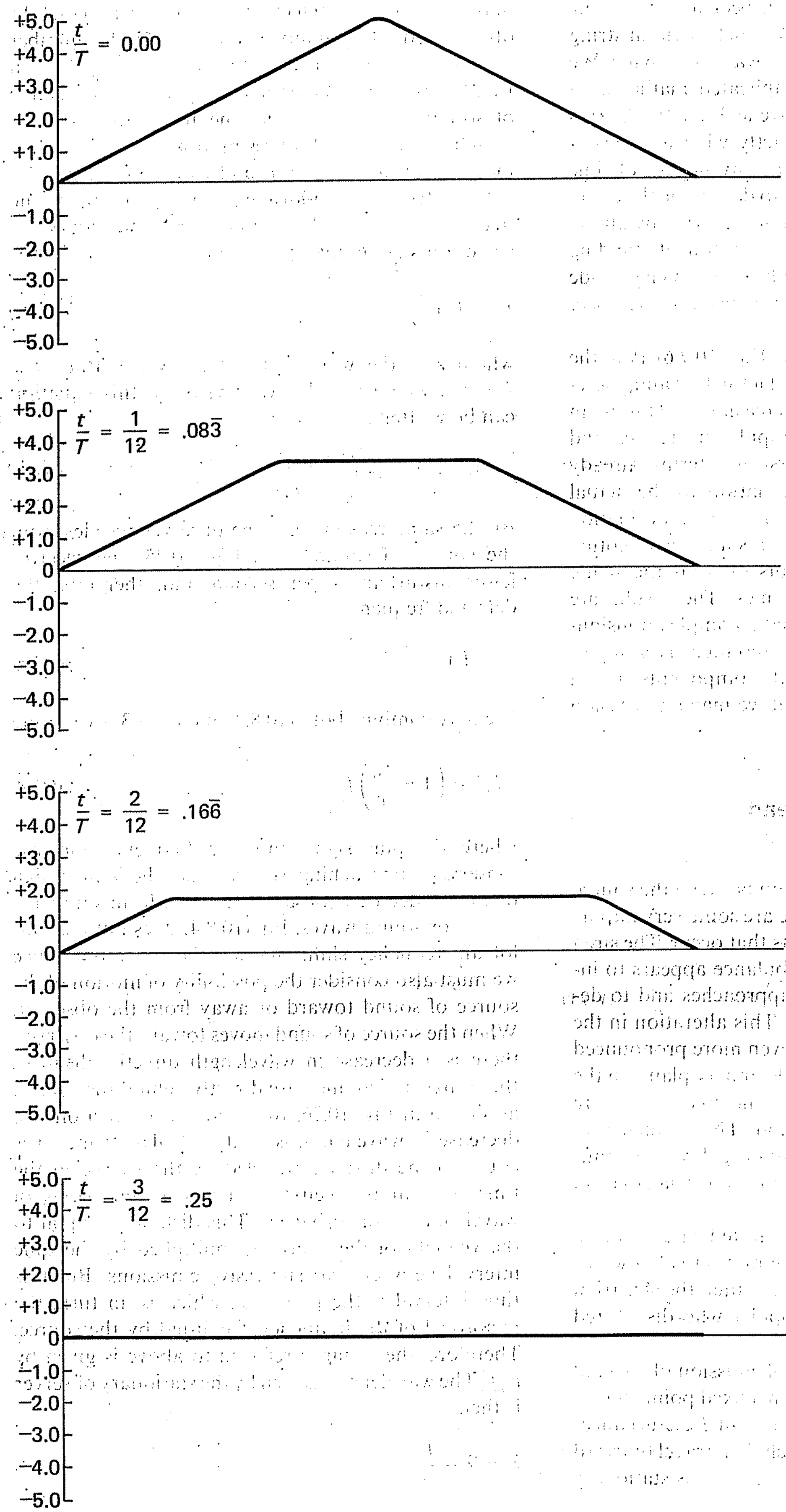
We may combine both (10.8.2) and (10.8.3) by writing

$$\boxed{f_\pm' = \left(1 \pm \frac{v_0}{v}\right)f} \tag{10.8.4}$$

where the plus sign denotes a frequency increase (observer approaching source) and the minus sign denotes a decrease (observer receding from source).

For sound waves, Eq. (10.8.4) does not account for all frequency shifts due to relative motion since we must also consider the possibility of motion of the source of sound toward or away from the observer. When the source of sound moves toward the observer, there is a decrease in wavelength directly ahead of the source and an increase directly behind the source, as shown in Fig. 10.26. In the forward direction, the decrease in wavelength is easily calculated since it is equal to the distance traveled by the source in the time interval between two successive emissions of wavefronts or disturbances. This distance is equal to the velocity of the source v_s multiplied by the time interval between two successive emissions. But this time interval is the period T, which is, in turn, the reciprocal of the frequency f emitted by the source. Therefore, the distance referred to above is given by v_s/f. The wavelength as seen by the stationary observer is then

$$\lambda' = \lambda - \frac{v_s}{f} \tag{10.8.5}$$


t/T = 0.00
t/T = 1/12 = .083̅
t/T = 2/12 = .166̅
t/T = 3/12 = .25
+5.0
+4.0
+3.0
+2.0
+1.0
0
−1.0
−2.0
−3.0
−4.0
−5.0

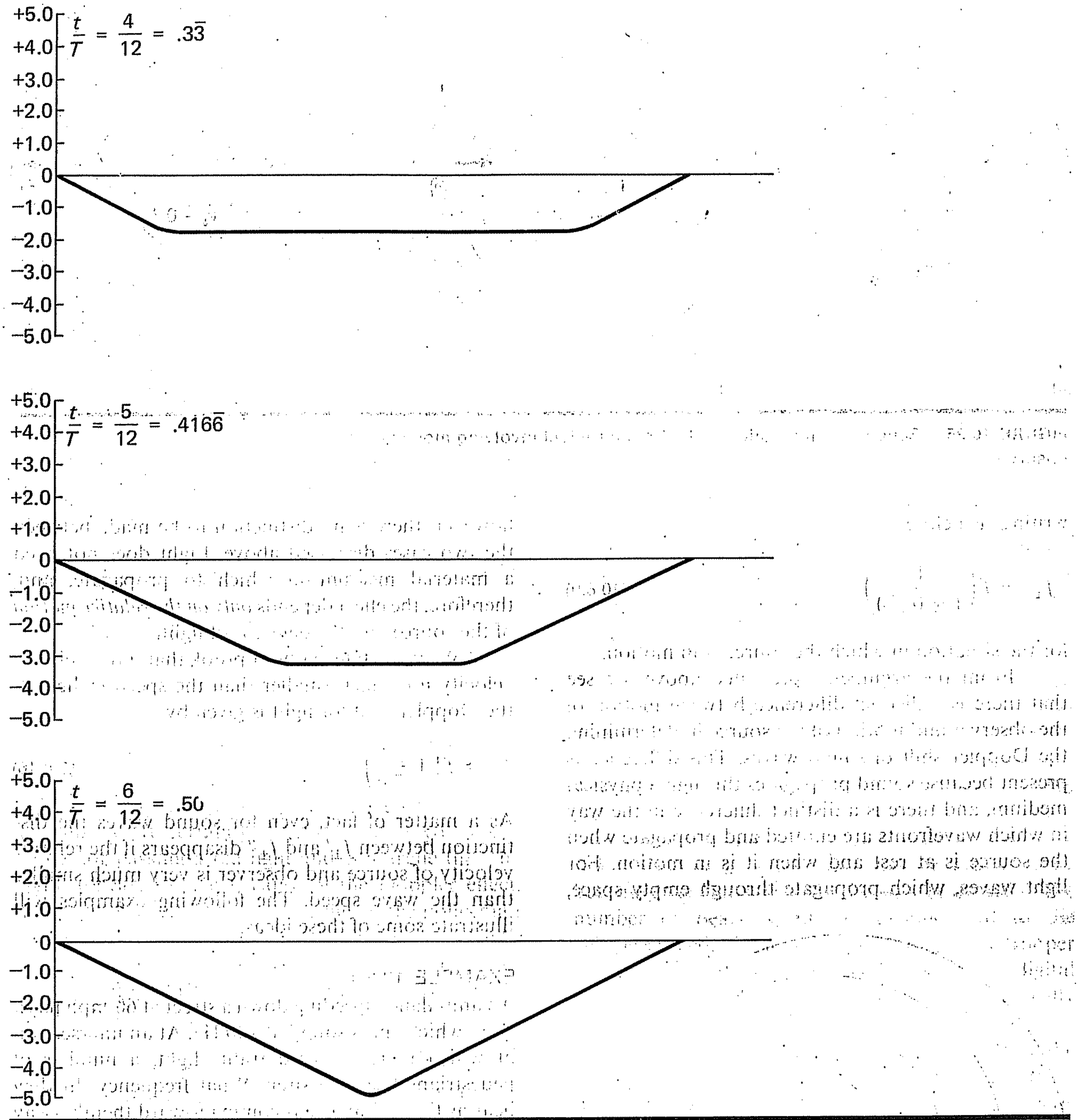


FIGURE 10.24. Fourier series solutions, according to (10.7.7), showing the plucked string in various stages of its vibrational cycle. Each succeeding illustration corresponds to a time interval of $\frac{1}{12}$ the period of vibration.

and hence the frequency of sound heard by the observer is

$$f'' = \frac{v}{\lambda'} = \frac{v}{\lambda - (v_s/f)} \tag{10.8.6}$$

But since λ is given by v/f, we may write

$$f'' = f\left(\frac{1}{1 - (v_s/v)}\right) \tag{10.8.7}$$

as the frequency detected by the observer at P. An observer located at Q will hear a smaller frequency, given by

$$f'' = f\left(\frac{1}{1 + (v_s/v)}\right) \tag{10.8.8}$$

since the wavelength he receives is greater than that emitted.

Let us summarize Eqs. (10.8.7) and (10.8.8) by

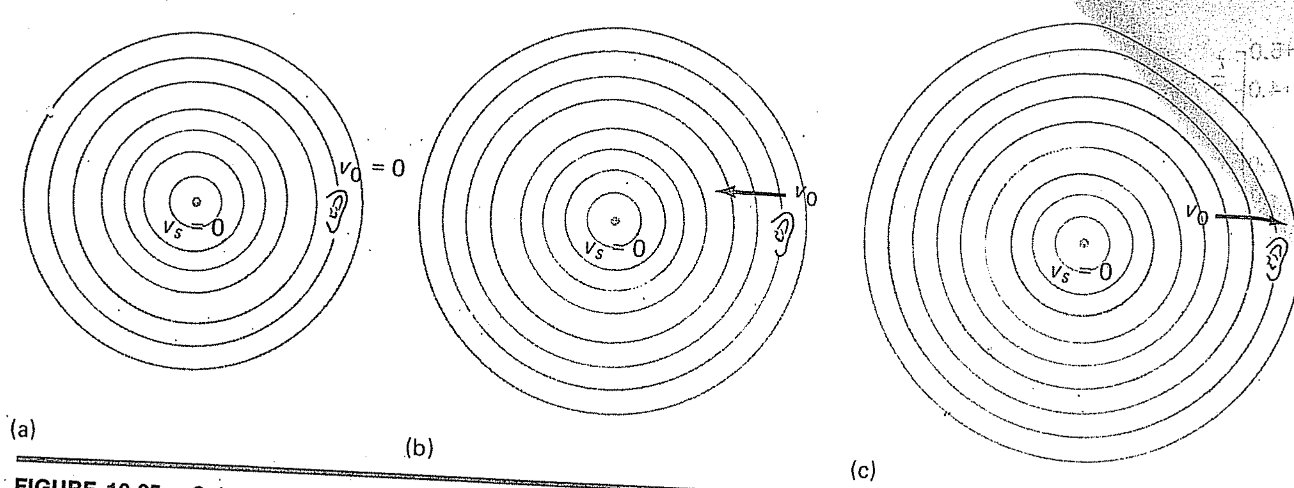


FIGURE 10.25. Schematic illustration of the Doppler effect involving motion of the observer.

writing, as before,

$$f_{\pm}'' = f\left(\frac{1}{1 \pm (v_s/v)}\right) \qquad (10.8.9)$$

for the situation in which the source is in motion.

From the arguments presented above, we see that there is a distinct difference between motion of the observer and motion of the source in determining the Doppler shift of sound waves. This difference is present because sound propagates through a physical medium, and there is a distinct difference in the way in which wavefronts are emitted and propagate when the source is at rest and when it is in motion. For light waves, which propagate through empty space, however, there is no distinction to be made between the two cases discussed above. Light does not need a material medium in which to propagate, and, therefore, the effect depends *only on the relative motion* of the source and the detector of light.

We may state, without proof, that if this relative velocity u is much smaller than the speed of light c, the Doppler shift for light is given by

$$f_{\pm}' \cong f\left(1 \pm \frac{u}{c}\right) \qquad (10.8.10)$$

As a matter of fact, even for sound waves the distinction between $f_{\pm}'$ and $f_{\pm}''$ disappears if the relative velocity of source and observer is very much smaller than the wave speed. The following examples will illustrate some of these ideas.

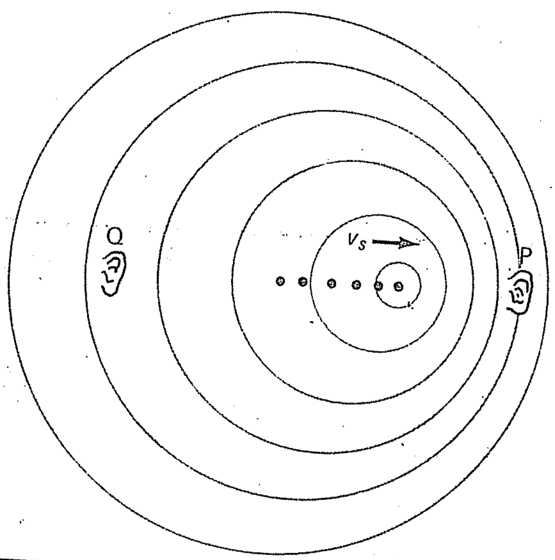


FIGURE 10.26. Schematic illustration of the Doppler effect involving motion of the source.

EXAMPLE 10.8.1

An ambulance speeding down a street at 60 mph has a siren which emits sound at 440 Hz. At an intersection at which there is a red traffic light, a number of pedestrians hear the siren. What frequency do they hear as the ambulance is coming toward them? Away from them? How fast must the ambulance go in order for the red light to appear green to the driver? The numerical information required is $v(\text{sound}) = 331$ m/sec, $v(\text{light}) = c = 3 \times 10^8$ m/sec, $\lambda(\text{green}) = 540 \times 10^{-9}$ m, and $\lambda(\text{red}) = 660 \times 10^{-9}$ m, 1 mph = 0.45 m/sec.

In this example, we have a case in which the source of sound is in motion. When the ambulance is moving toward the pedestrians, the frequency they detect is

$$f_+'' = 440 \frac{1}{1 - \dfrac{60 \times 0.45}{331}} = 479.1 \text{ Hz} \qquad (10.8.11)$$

while when it is receding from them, they hear

$$f_-'' = 440 \frac{1}{1 + \dfrac{60 \times 0.45}{331}} = 406.8 \text{ Hz} \quad (10.8.12)$$

These frequency shifts are quite substantial and are easily noticed.

When the ambulance is moving toward the red light, we have a Doppler shift of *light* due to the relative motion of source and observer. We expect a frequency increase so that the detected light frequency will be

$$f_+' = f\left(1 + \frac{u}{c}\right) \quad (10.8.13)$$

where u is the speed of the ambulance and c is the speed of light. The frequencies are related to the wavelengths by $f\lambda = c$, and, therefore, (10.8.13) may be rewritten as

$$\frac{c}{540 \times 10^{-9}\text{ m}} = \frac{c}{660 \times 10^{-9}\text{ m}}\left(1 + \frac{u}{c}\right) \quad (10.8.14)$$

This is easily solved for u/c:

$$\frac{u}{c} = \frac{66}{54} - 1 = 0.22 \quad (10.8.15)$$

If we now substitute for c, we find that the ambulance must be going at a speed of 0.66×10^8 m/sec, or 1.47×10^8 mph. This is a pretty good speed for an ambulance. It certainly suggests that if you get a ticket for passing a red light and you argue that the light appeared green because of the Doppler effect, you are not likely to be convincing.[7] If the policeman wants to show off his knowledge of physics, he might very well say, "Perhaps you are right, Sir, about seeing a green light, but if you did, then you surely deserve a speeding ticket."

The numerical answer to the latter part of the above example may seem a bit absurd. It is undoubtedly true that if you can detect any light frequency shift at all with your eyes, your speed is extremely high. There are, however, instruments much more sensitive than the human eye, and these are capable of measuring very small frequency shifts and, therefore, low speeds. The radar sets that are used to measure auto speeds are examples of such refined instruments. Their operation is only possible because of the Doppler effect. The following example illustrates the method.

[7] Although R. W. Wood, formerly Professor of Physics at Johns Hopkins University, is reputed to have used this argument on a judge and gotten away with it!

EXAMPLE 10.8.2

On Interstate Route 80, a radar trap operates by sending out electromagnetic waves of frequency 2400×10^6 cycles per second. This frequency is in the so-called microwave region. A station wagon moving at 85 mph gets pulled over for speeding. Explain how the radar system based on the Doppler effect works.

The state trooper knows that the station wagon was speeding because some of the microwaves he sent out were reflected by the approaching vehicle and came back with a higher frequency. The station wagon first receives microwaves which are shifted in frequency by Δf, where Δf, according to (10.8.13) is given by

$$\frac{\Delta f}{f} = \frac{f_+' - f}{f} = \frac{u}{c} = \frac{85 \times 0.45}{3 \times 10^8} = 1.28 \times 10^{-7}$$

The station wagon, in reflecting the microwaves back to the radar set, acts once again as a moving source of microwave energy, introducing a *second* Doppler shift of the same magnitude. The total frequency shift in the waves received by the patrol car is, therefore,

$$2\,\Delta f = 2(u/c)f = (2)(1.28 \times 10^{-7})(2.4 \times 10^9) = 614.4 \text{ Hz}$$

The speed of light is so high that if the trooper's radar hits your car when you are 100 m from him, it takes only 0.66×10^{-6} sec for the reflected radar waves to return to his car. During this time interval, your vehicle, moving 85 mph, has only moved a distance of 2.5×10^{-5} m, or 0.025 mm. Fantastic, but true!

The actual *shift* in frequency between the emitted radar and the detected radar may be measured by detecting the beats between the two frequencies. The number of beats is directly proportional to the vehicle's speed, and in a typical situation the trooper in the radar car reads your speed directly on a digital display panel. The entire process is practically instantaneous.

There are many other important uses of the Doppler effect, especially for light. We will return to the subject again in the chapters on electromagnetic waves and will discuss more examples at that time. The Doppler effect provides an example of a physical phenomenon that arises from the relative motion between a source of waves and a receiver of waves. Other interesting phenomena also emerge as a consequence of this relative motion. As one example, we shall consider the sonic boom observed when the speed of an aircraft or any other object is in excess of the speed of sound.

EXAMPLE 10.8.3

In Fig. 10.27, a projectile is in motion with a constant velocity v_s in excess of the speed of sound v in the

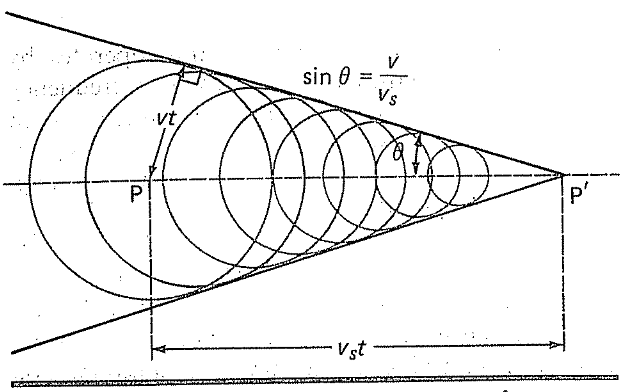


FIGURE 10.27. Wavefronts emitted by an object traveling at a speed in excess of the speed of sound, illustrating the formation of a shock front.

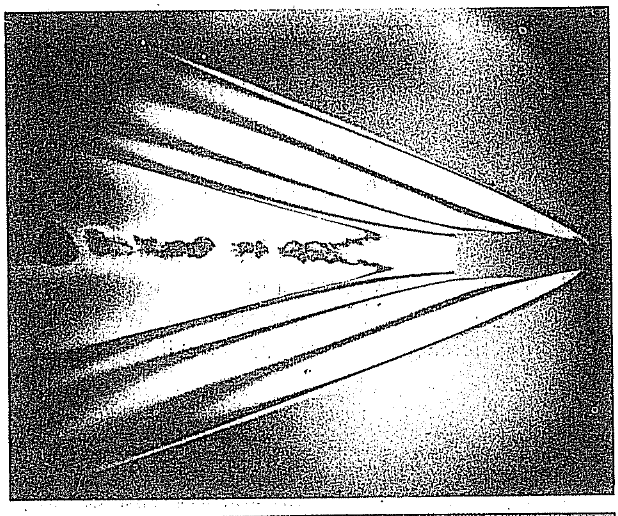


FIGURE 10.28. Photograph of shock wave associated with a bullet moving at Mach 2.5. Reconstructed image of a double-exposed holographic interferogram of a high-velocity 22-caliber bullet. The first exposure was made just before the firing, and the second exposure was made when the bullet was passing through the scene volume. A pulsed ruby laser was used. The interference fringes give the change in optical path which, in turn, gives the air density behind the shock wave. (Photo courtesy of L. O. Heflinger, R. F. Wuerker, and R. E. Brooks; TRW Systems)

medium. At $t = 0$, the projectile is at a point P, while at some later time t, it is at point P′. In this figure, the spherical wavefronts which have been emitted in the time of flight between P and P′ are shown. In the present case, in which $v_s > v$, the wavefront has a conical shape, with the angle θ at the apex given by $\sin\theta = v/v_s$. The reciprocal of this number, v_s/v, is called the *Mach number*. An actual photograph of this conical shape is illustrated in Fig. 10.28 for a bullet moving at Mach 2.5. When the wavefronts of a sound wave have this form as a result of supersonic speeds, we have what is known as a *shock wave*. This shock wave is responsible for the so-called sonic boom or explosion caused by supersonic aircraft.

Let us now refer to the situation shown in Fig. 10.29. At time t when an airplane (or other supersonic object) is at point P′, let us consider the sound which arrives at a point A on the conical front, which might represent some given point on the ground. Points C and D are former locations of the airplane separated by a distance $v_s\,\Delta t$, where Δt is the time of flight between C and D. Now let us calculate the difference in arrival times at A of a sound wave emitted at C and one at D. If t_C and t_D denote these times, we find

$$t_C = \frac{CA}{v}$$

and

$$t_D = \Delta t + \frac{DA}{v} \tag{10.8.16}$$

where CA and DA are the distances. Therefore,

$$t_D - t_C = \Delta t + \frac{1}{v}(DA - CA)$$

$$= \Delta t + \frac{1}{v}(DA - EA - C\overset{E}{D}) \tag{10.8.17}$$

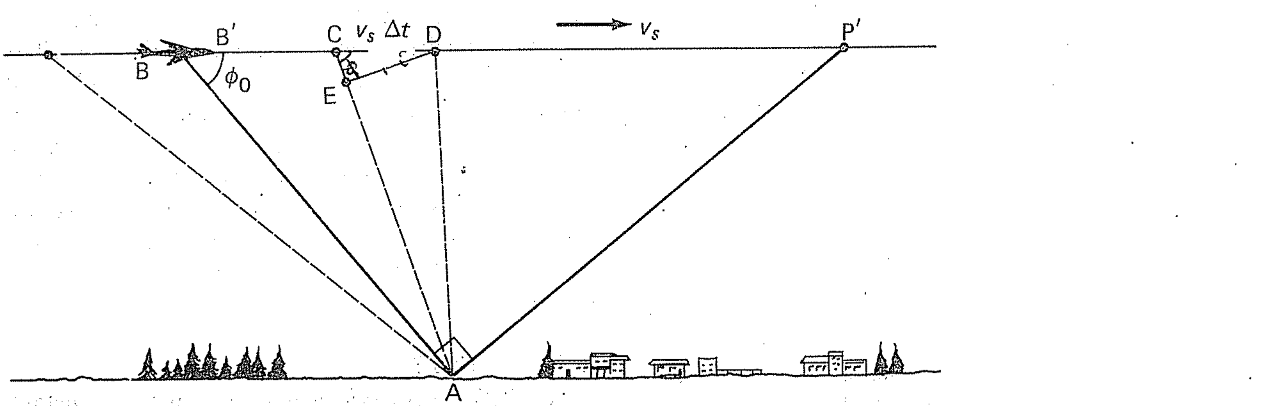


FIGURE 10.29. Geometry of shock wave excited by a supersonic aircraft.

Now, if the distance CD is much smaller than CA, it follows that $DA \cong EA$. We also note that $CE = CD \cos\phi = (v_s\,\Delta t)\cos\phi$, and, therefore, the difference in arrival times is

$$t_D - t_C = \Delta t\left(1 - \frac{v_s}{v}\cos\phi\right) \qquad (10.8.18)$$

Several conclusions can now be drawn from Eq. (10.8.18), an equation which would also be valid at subsonic speeds. If $v_s < v$ (subsonic), there is no possibility of simultaneous arrival at point A. Waves emitted at different times arrive at A at different times. However, if $v_s > v$, then if the angle ϕ is chosen to be ϕ_0, where $\cos\phi_0 = v/v_s$, the arrival times are *identical.* In fact, subject to the approximations made, the arrival time of *all* sound waves emitted between points B and B′ are the same. This implies that a very substantial constructive interference of sound occurs at point A, and as a result a *sonic boom* or loud explosion is heard at A. The only condition is that B and B′ should be points whose separation is substantially smaller than their distances to the detector (at A). It is seen that the angle ϕ_0 obtained above is the complement of the previously defined angle θ.

The very large intensity at A and the correspondingly large pressure variations which result pose a very serious problem. The sonic boom produced by such a shock wave can shatter glass and do considerable damage to the structure of buildings. It is also extremely unpleasant to hear the intense sound wave. Actually, supersonic aircraft produce a double boom. One is caused by the nose and the other by the tail of the airplane. This is illustrated in Fig. 10.30.

Sonic boom cannot be eliminated. Supersonic aircraft are designed so as to minimize the shock waves, but there seems to be no way in which they can be done away with altogether. The only solution to the problem seems to be to require aircraft to fly subsonically until they attain enough altitude so that the sonic boom will not pose any difficulty at ground level.

10.9 Ocean Waves

We began our discussion of wave motion by referring to the water waves that travel across the surface of the sea. Although these waves are familiar to all of us, their behavior is very complex. We shall, therefore, present only a brief qualitative survey of their more important properties.

Ocean waves originate in the action of the wind. Winds create small ripples on a calm sea. The profile of these ripples presents additional surfaces that absorb energy from the air currents and further increases their size. The ultimate amplitude of the waves that are formed depends upon the average wind velocity, the time during which the wind acts, and the distance over which it is effective.

Ocean waves can carry huge amounts of energy and can, therefore, cause severe damage to coastal installations when they come ashore. At Wick, Scotland, for example, in a great storm that occurred in 1872, the designer of the breakwater watched from a nearby cliff as the sea wrought its destruction on his handiwork. The end of the breakwater was capped by an 800-ton piece of concrete secured to the foundation by iron columns 3.5 inches in diameter. Both cap and foundation, weighing a total of 1350 tons were removed as a unit and swept into the sea. The designer, undaunted by this, rebuilt the installation, using a larger cap that weighed 2600 tons. The new breakwater, however, suffered a similar fate a few years

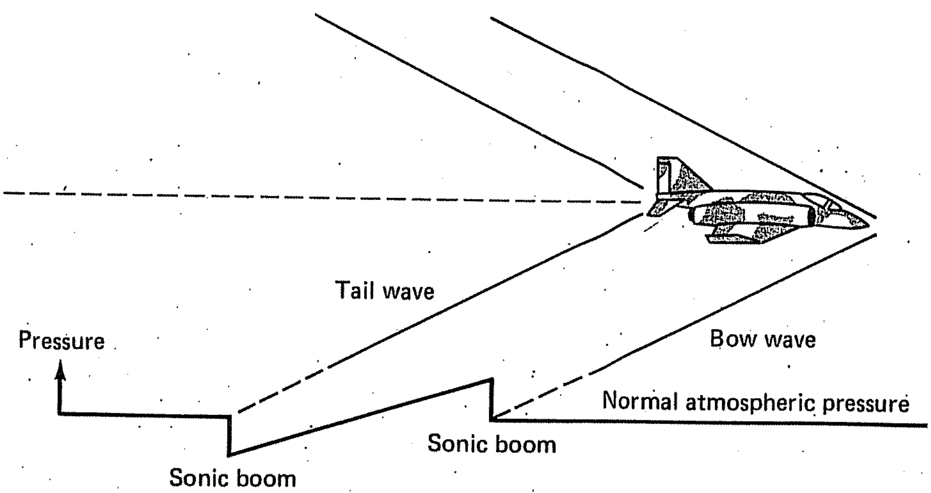


FIGURE 10.30. Double "sonic boom" excited by nose and tail of aircraft.

later. Whether he kept his job and made a third try is not documented.[8]

Although most ocean waves are generated by the wind, underwater earthquakes, landslides, and volcanic activity can also cause huge waves. Waves such as these are often referred to as *tidal waves*, or *tsunamis*, which is the Japanese equivalent. On August 27, 1883, an eruption of the volcano Krakatoa in the East Indies caused a tremendous tsunami in which waves 100 feet high swept away the town of Merak, killing 36,380 persons. Tidal waves can travel thousands of miles at speeds of 300 to 400 mph, yet are hardly visible in the open sea. Only when they approach the shore can their impact, which is usually deadly, be realized.

One of the main reasons for the complex behavior of water waves is that the wave velocity is not a single fixed number but is instead a function of the wavelength. A water wave of short wavelength travels slower than a wave of longer wavelength, the relation between velocity and wavelength being something like $v^2 = 5.12\lambda$, if λ is measured in feet and v in ft/sec. This behavior is distinctly different from the way in which waves propagate in stretched strings or in gases, where the wave velocity is completely independent of wavelength or frequency. Waves such as water waves, whose velocity varies with wavelength, are said to exhibit the phenomenon of *dispersion*.

Dispersion plays an important role in physics, particularly in the propagation of light and electromagnetic waves in dense transparent substances like glass. However, light waves traveling in a vacuum or in a rarified gaseous medium such as air do not exhibit dispersion; they travel with the same speed (2.997×10^8 m/sec) regardless of wavelength. In a dense, transparent substance such as glass or water, on the other hand, their speed is significantly reduced and depends on the wavelength. Because the phenomenon of dispersion is present when light propagates through glass or water, these substances are often referred to as *dispersive media* with respect to the propagation of light. Dispersion is responsible for the separation of white light into its constituent spectral colors in the rainbow or in a glass prism. We shall investigate the subject of dispersion in more detail in our later work on light and optics.

Another complexity in the study of water waves arises from the fact that they are neither purely transverse nor purely longitudinal, but rather a mixture of the two. For plane water waves in an infinitely large body of very deep water, the motion of individual volume elements within the medium is circular, the size of the circular paths decreasing with increasing depth. But in shallow water, this picture is no longer valid and the situation becomes even more complex. As the water depth decreases, the wave amplitude increases, the wavelength decreases, and the wave velocity is reduced. This decrease in wave velocity accounts for the fact that wavefronts are nearly always parallel to the coast when they reach shore. When a wave approaches shallow water at an angle to the coastline, the part nearest the shore, where the water is shallowest, begins to travel more slowly than before. The part farther out, in deeper water, therefore, catches up, so to speak, until the entire wavefront is practically parallel with the shoreline. Ultimately, the effect of shallow water in reducing the wavelength and at the same time increasing the amplitude causes the waves to crest and break up in what we recognize as surf. In this process, the originally circular orbits of water elements first become elliptical, and finally extremely irregular as the waves break up.

Our understanding of the complex dynamics of water waves is still incomplete, and a better understanding of waves in fluid media is necessary to resolve existing problems in ship design, harbor planning, and the construction of coastal structures. Research in the behavior of water waves is therefore an active field of endeavor which utilizes sophisticated experimental tanks, testing basins, and computing equipment.

[8] See, for example, W. Bascom, "Ocean Waves," *Scientific American* (August 1959). The reader will find many other interesting facts about ocean waves in this comprehensive article.

SUMMARY

Waves are self-propagating disturbances that travel through material media, transporting energy without transport of the medium itself. In the case of light and electromagnetic waves, transport of energy through empty space can be accomplished by interaction of electric and magnetic fields.

Periodic waves are characterized by amplitude, phase angle, wavelength, frequency, and propagation velocity: *Amplitude* refers to the maximum possible displacement of a point in the medium from its equilibrium position. The *phase angle* specifies the part of the periodic cycle that is at hand for any point in the medium at any given time. *Wavelength* λ refers to the distance between successive maxima or minima, or between any two neighboring points having equivalent phase angles.

Frequency f refers to the number of cycles of motion executed by a given particle per unit time *angular frequency* ω is simply 2π times the frequency. *Propagation velocity* v refers to the rate at which a point having a given phase (such as a maximum or minimum of displacement) travels through the medium. Propagation velocity, frequency, and wavelength are related by $v = f\lambda$.

The *propagation constant* k is a quantity defined by $k = 2\pi/\lambda$. The wave velocity can be expressed in terms of the angular frequency and the propagation constant as

$$v = f\lambda = \frac{\omega}{k}$$

Wavefronts represent the locus of points in the medium of equal phase, such as points corresponding to maximum displacement or zero displacement. Wavefronts propagate through the medium with wave velocity v. In general, the shape of the wavefront depends upon the nature of the source of wave motion and the distance of the wavefront from the source, but there are three particularly simple cases: *point source*, giving rise to *spherical* wavefronts; *line source*, giving rise to *cylindrical* wavefronts; and *plane source*, giving rise to *plane* wavefronts.

Plane wavefronts also approximate the behavior of cylindrical and spherical wavefronts at large distances from the source of the waves. The direction in which the wave disturbance propagates is everywhere perpendicular to the wavefront.

Mathematically, for one-dimensional waves or plane waves in three dimensions, the displacement $u(x, t)$ of the medium from equilibrium at any point x and any time t can be expressed as

$$u(x, t) = f(x - vt)$$

where v is the wave velocity and f represents any function of the argument $x - vt$. The most important special case of this situation is the sinusoidal plane waves, where

$$u(x, t) = A \sin[(kx - \omega t) + \delta]$$
$$= A \sin\left[2\pi\left(\frac{x}{\lambda} - ft\right) + \delta\right]$$

with $\omega/k = f\lambda = v$. In the case of *longitudinal waves*, the displacement of the medium is parallel to the direction of propagation, while for *transverse waves*, the displacement is normal to the propagation direction. Any of the three functions written above satisfies a partial differential *wave equation* of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where v is the wave velocity.

The propagation velocity v may always be expressed in terms of the physical properties of the medium in which the waves travel. For example, for transverse waves in a stretched string where the tension is T_0 and the linear mass density is μ,

$$v = \sqrt{T_0/\mu}$$

while for longitudinal acoustic waves in an elastic solid substance

$$v = \sqrt{Y/\rho}$$

where Y is Young's modulus defined by (10.4.3) and ρ is the density. For longitudinal sound waves in gases, the velocity is

$$v = \sqrt{B/\rho}$$

where B is the bulk modulus defined by (10.4.11) and ρ is the density.

The average energy $\bar{U}$ transmitted by a one-dimensional wave, such as might propagate along a rope, in time Δt is given by

$$\Delta\bar{U} = \tfrac{1}{2}\mu v\omega^2 {y_0}^2\,\Delta t$$

where y_0 is the amplitude and μ is the mass per unit length.

The intensity I of a wave is the average energy which crosses unit area normal to the propagation direction in unit time. For a rope, as discussed above,

$$I = \frac{\Delta\bar{U}}{(\Delta A)(\Delta t)} = \frac{\mu v\omega^2 {y_0}^2}{2\,\Delta A}$$

while for a wave in a three-dimensional medium of density ρ, the intensity is

$$I = \tfrac{1}{2}\rho v\omega^2 {y_0}^2$$

Waves can be superposed or added by adding the separate displacements caused by each at every point in the medium at every instant of time.

Spatial interference occurs when the phase relationships between two waves of equal frequency are such as to cause reinforcement or cancellation of wave amplitudes at points whose distances from the two sources that emit the waves are different. Reinforcement, referred to as *constructive interference*, occurs at points where the difference in path between the two sources amounts to an integral number of wavelengths; while cancellation, called *destructive interference*, occurs at points where the difference in path between the two sources is an odd-integral number of half wavelengths.

Temporal interference occurs when waves emitted from two different sources have slightly different frequencies. The intensity of the received wave then varies in amplitude periodically with a "beat" frequency equal to the difference in the two source frequencies.

The *Doppler effect* is an alteration in perceived frequency caused by motion of source or observer. For sound waves, in the case where the observer is moving, the perceived frequency f' is related to the source frequency f by

$$f' = \left(1 \pm \frac{v_0}{v}\right)f$$

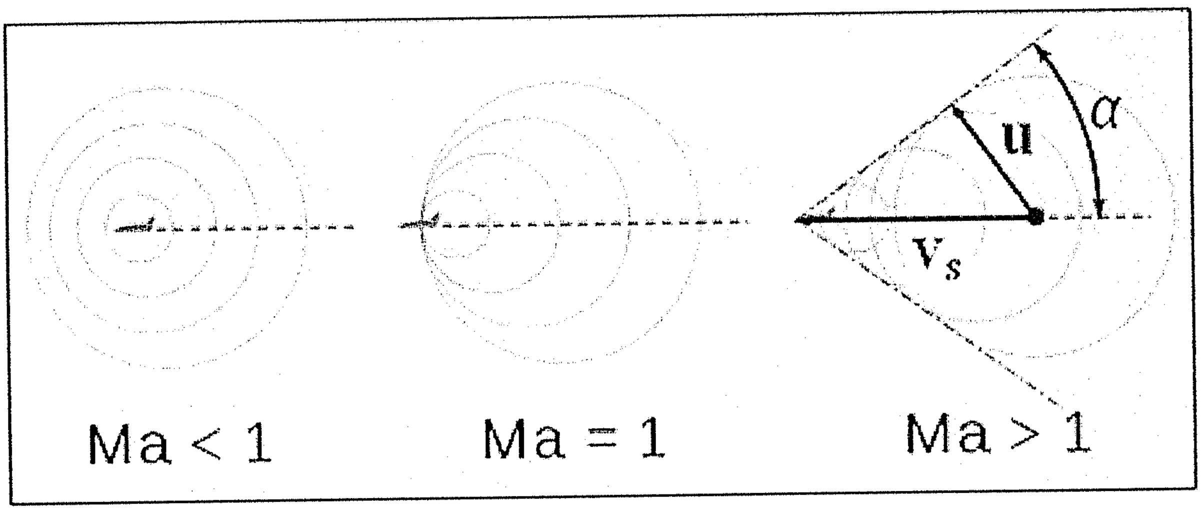
衝擊波（Shock Waves）與馬赫數（Mach Number□ » 高瞻自然科學教學資源平台

衝擊波（**Shock Waves**）與馬赫數（**Mach Number**）

天主教曉明女子高級中學物理科李忠義老師/國立臺灣師範大學物理系蔡志申教授責任編輯

衝擊波又稱爲衝擊面（shock front），跟一般的波動一樣，是一種具有傳播特性的擾動，可以在固體、液體或氣體中傳播並傳遞能量。

當物體移動的速度超過聲音的速度時，物體產生的聲波波前將會重疊在一起，如下圖所示，而重疊的部分便是衝擊波。就立體空間而言，衝擊波會形成一個圓錐角，我們稱爲馬赫錐（Mach cone）。馬赫錐的中心軸線爲是波源（如飛機）運動的軌跡線，而此線與錐面所夾的角（如圖中 α 角）稱爲馬赫角。



馬赫數代號爲M或Ma，其定義爲『物體移動速率與聲音速率的比值』：

$$\text{馬赫數} \ M = \frac{v_s}{u}$$

其中 v_s 表示聲源（飛機）移動速率□ u 表示聲音在介質（空氣）中傳遞的速率。

透過馬赫角，我們可以進一步將馬赫數表示爲：

The Doppler Effect and Sonic Booms

Daniel A. Russell, Kettering University

The sudden change in pitch of a car horn as a car passes by (source motion) or in the pitch of a boom box on the sidewalk as you drive by in your car (observer motion) was first explained in 1842 by Christian Doppler. His **Doppler Effect** is the shift in frequency and wavelength of waves which results from a source moving with respect to the medium, a receiver moving with respect to the medium, or even a moving medium.

The perceived frequency (f') is related to the actual frequency (f_0) and the relative speeds of the source (v_s), observer (v_o), and the speed (v) of waves in the medium by

$$f' = f_0\left(\frac{v \pm v_o}{v \pm v_s}\right)$$

The choice of using the plus (+) or minus (-) sign is made according to the convention that if the source and observer are moving *towards* each other the perceived frequency (f') is *higher* than the actual frequency (f_0). Likewise, if the source and observer are moving *away from* each other the perceived frequency (f') is *lower* than the actual frequency (f_0).

Although first discovered for sound waves, the Doppler effect holds true for all types of waves including light (and other electromagnetic waves). The Doppler effect for light waves is usually described in terms of colors rather than frequency. A red shift occurs when the source and observer are moving away from eachother, and a blue shift occurs when the source and observer are moving towards eachother. The red shift of light from remote galaxies is proof that the universe is expanding.

The animations below will illustrate this phenomena for a moving source and stationary observer.

Stationary Sound Source

The movie at left shows a stationary sound

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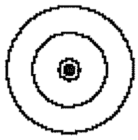
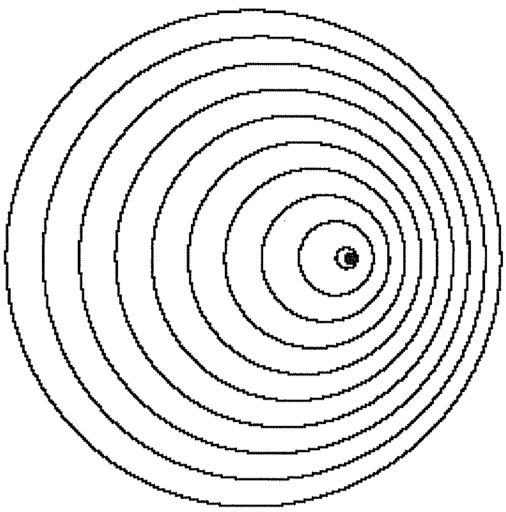


propagate symmetrically away from the source at a constant speed *v*, which is the speed of sound in the medium. The distance between wavefronts is the wavelength. All observers will hear the same frequency, which will be equal to the actual frequency of the source.

For a movie showing how circular waves can be created (in terms of particle motion and wave motion) go here.

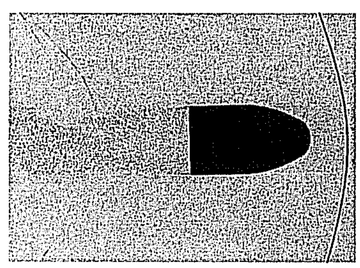
Source moving with $v_{source} < v_{sound}$ (Mach 0.7)

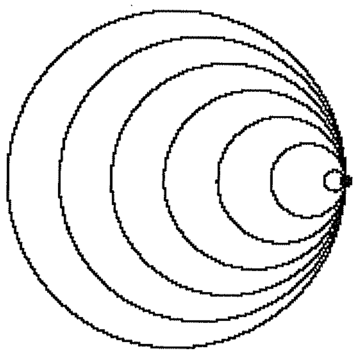


In the movie at left the same sound source is radiating sound waves at a constant frequency in the same medium. However, now the sound source is moving to the right with a speed v_s = 0.7 v (Mach 0.7). The wavefronts are produced with the same frequency as before. However, since the source is moving, the center of each new wavefront is now slightly displaced to the right. As a result, the wavefronts begin to bunch up on the right side (in front of) and spread further apart on the left side (behind) of the source. An observer in front of the source will hear a higher frequency $f' > f_0$, and an observer behind the source will hear a lower frequency $f' < f_0$.

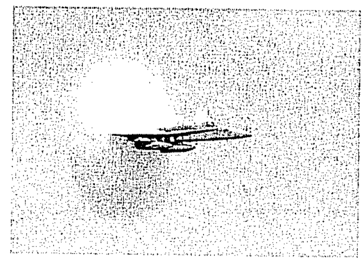
Source moving with $v_{source} = v_{sound}$ (Mach 1 - breaking the sound barrier)

Now the source is moving at the speed of sound in the medium





air at sea level is about 340 m/s or about 750 mph. The wavefronts in front of the source are now all bunched up at the same point. As a result, an observer in front of the source will detect nothing until the source arrives. The pressure front will be quite intense (a shock wave), due to all the wavefronts adding together, and will not be percieved as a pitch but as a "thump" of sound as the pressure wall passes by. The figure at right shows a bullet travelling at Mach 1.01. You can see the shock wave front just ahead of the bullet.

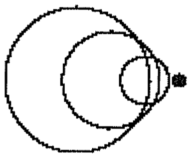


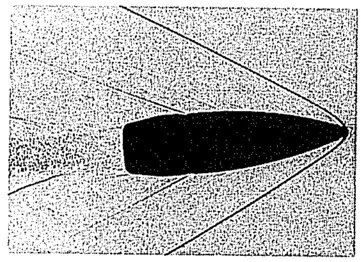
Jet pilots flying at Mach 1 report that there is a noticeable "wall" or "barrier" which must be penetrated before achieving supersonic speeds. This "wall" is due to the intense pressure front, and flying within this pressure front produces a very turbulent and bouncy ride. Chuck Yeager was the first person to break the sound barrier when he flew faster than the speed of sound in the X-1 rocket-powered aircraft on October 14, 1947. Check out the movie The Right Stuff for more about this significant milestone, and the beginnings of the US space project. The figure at right shows a n F-18 at the exact instant it goes supersonic. Click on the figure to see more information and a MPEG movie of this event.

Source moving with $v_{source} > v_{sound}$ (Mach 1.4 - supersonic)

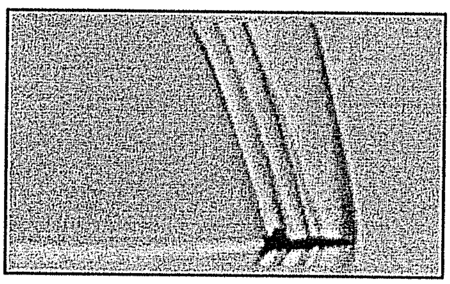
The sound source has now broken through the
d db i di t li t14

source is moving faster than the sound waves it creates, it actually leads the advancing wavefront. The sound source will pass by a stationary observer before the observer actually hears the sound it creates.



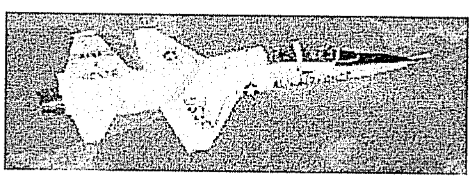


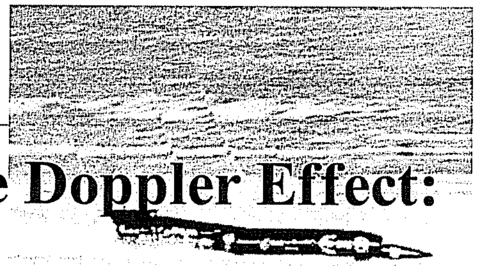
As you watch the animation, notice the clear formation of the *Mach cone*, the angle of which depends on the ratio of source speed to sound speed. It is this intense pressure front on the Mach cone that causes the *shock wave* known as a **sonic boom** as a supersonic aircraft passes overhead. The shock wave advances at the speed of sound v, and since it is built up from all of the combined wave fronts, the sound heard by an observer will be quite intense. A supersonic aircraft usually produces two sonic booms, one from the aircraft's nose and the other from its tail, resulting in a double thump. The figure at right shows a bullet travelling at Mach 2.45. The mach cone and shock wavefronts are very noticeable.



The picture at the left shows the shock wave front generated by a T-38 Talon, a twin-engine, high-altitude, supersonic jet trainer (below).

This picture shows a sonic boom created by the THRUST SSC team car as it broke the land speed record (and also broke the sound barrier on land). Click on the image to download a larger version of the image.





Other important applications of the Doppler Effect:

- Doppler Radar uses the doppler effect for electromagetic waves to predict the weather.
- The Doppler shift for light is used to help astronomers discover new planets and binary stars.
- Echocardiography - a medical test using ultrasound and Doppler techniques to visualize the structure of the heart.
- Radio Direction Finding Systems
- There is also an instrumental rock group called The Doppler Effect

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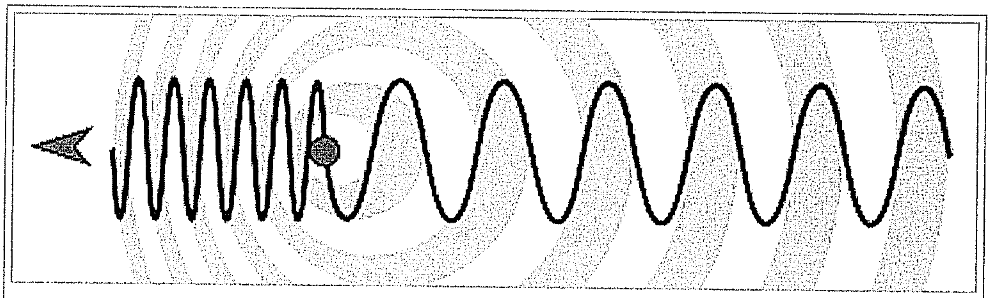
Back to the Vibration and Waves Demos Page

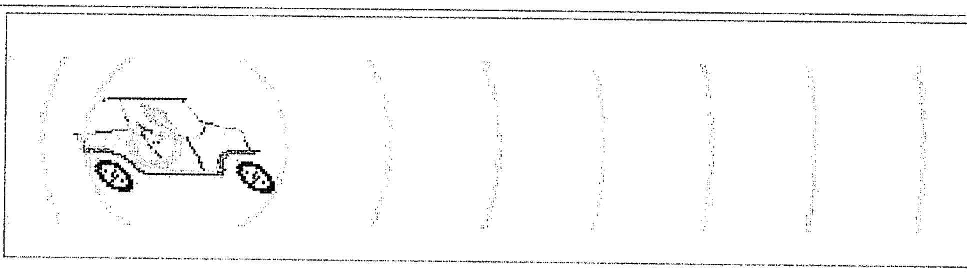
Doppler effect

From Wikipedia, the free encyclopedia

The **Doppler effect** (or **Doppler shift**), named after Austrian physicist Christian Doppler who proposed it in 1842 in Prague, is the change in frequency of a wave for an observer moving relative to the source of the wave. It is commonly heard when a vehicle sounding a siren or horn approaches, passes, and recedes from an observer. The received frequency is higher (compared to the emitted frequency) during the approach, it is identical at the instant of passing by, and it is lower during the recession.

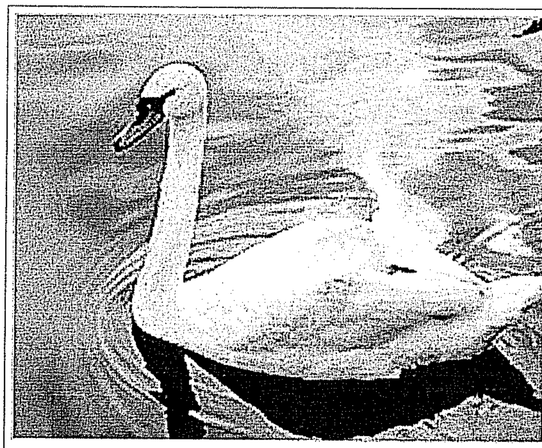


Change of wavelength caused by motion of the source.



An animation illustrating how the Doppler effect causes a car engine or siren to sound higher in pitch when it is approaching than when it is receding. The pink circles are sound waves. When the car is moving to the left, each successive wave is emitted from a position further to the left than the previous wave. So for an observer in front *(left)* of the car, each wave takes slightly less time to reach him than the previous wave. The waves "bunch together", so the time between arrival of successive wavefronts is reduced, giving them a higher frequency. For an observer in back *(right)* of the car, each wave takes a slightly longer time to reach him than the previous wave. The waves "stretch apart", so the time between the arrival of successive wave-fronts is increased slightly, giving them a lower frequency.

The relative changes in frequency can be explained as follows. When the source of the waves is moving toward the observer, each successive wave crest is emitted from a position closer to the observer than the previous wave. Therefore each wave takes slightly less time to reach the observer than the previous wave. Therefore the time between the arrival of successive wave crests at the observer is reduced, causing an increase in the frequency. While they are travelling, the distance between successive wave fronts is reduced; so the waves "bunch wave, so the arrival time between successive waves is increased, reducing the frequency. The distance between successive wave fronts is increased, so the waves "spread out".



Doppler effect of water flow around a swan.

For waves that propagate in a medium, such as sound waves, the velocity of the observer and of the source are relative to the medium in which the waves are transmitted. The total Doppler effect may therefore result from motion of the source, motion of the observer, or motion of the medium. Each of these effects are analyzed separately. For waves which do not require a medium, such as light or gravity in general relativity, only the relative difference in velocity between the observer and the source needs to be considered.

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  - 5.5 Medical imaging and blood flow measurement
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Development

Doppler first proposed the effect in 1842 in his treatise "*Über das farbige Licht der Doppelsterne und einiger anderer Gestirne des Himmels*" (On the coloured light of the binary stars and some other stars of the heavens).[1] The hypothesis was tested for sound waves by Buys Ballot in 1845. He confirmed that the sound's pitch was higher than the emitted frequency when the sound source approached him, and lower than the emitted frequency when the sound source receded from him. Hippolyte Fizeau discovered independently the same phenomenon on electromagnetic waves in 1848 (in France, the effect is sometimes called "l'effet Doppler-Fizeau" but that name was not adopted by the rest of the world as Fizeau's discovery was three years after Doppler's). In Britain, John Scott Russell made an experimental study of the Doppler effect (1848).[2]

An English translation of Doppler's 1842 treatise can be found in the book '*The Search for Christian Doppler* by Alec Eden.[1]

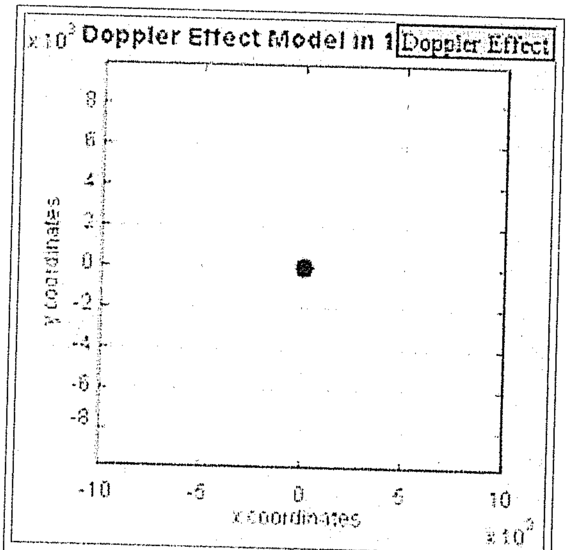
General

In classical physics, where the speeds of source and the receiver relative to the medium are lower than the velocity of waves in the medium, the relationship between observed frequency f and emitted frequency f_0 is given by:[3]

$$f = \left(\frac{c + v_r}{c - v_s}\right) f_0$$

where

c is the velocity of waves in the medium
v_r is the velocity of the receiver relative to the medium; positive if the



Stationary sound source produces sound waves at a constant frequency f, and the wave-fronts propagate symmetrically away from the source at a constant speed c (assuming speed of to the medium; positive if the source is moving away from the receiver.

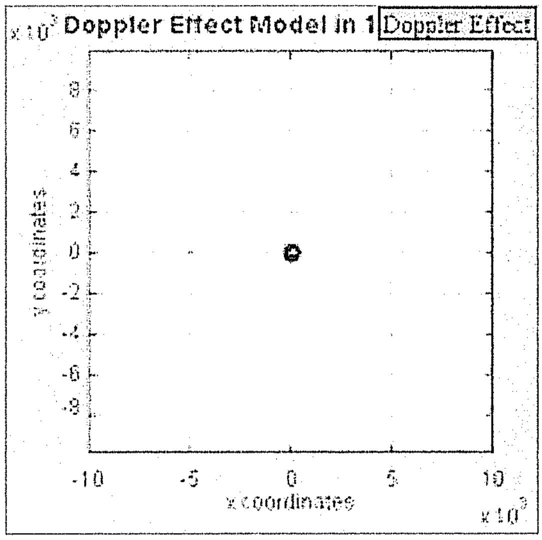
The frequency is decreased if either is moving away from the other.

The above formula works for sound waves if and only if the speeds of the source and receiver relative to the medium are slower than the speed of sound. See also Sonic boom.

The above formula assumes that the source is either directly approaching or receding from the observer. If the source approaches the observer at an angle (but still with a constant velocity), the observed frequency that is first heard is higher than the object's emitted frequency. Thereafter, there is a monotonic decrease in the observed frequency as it gets closer to the observer, through equality when it is closest to the observer, and a continued monotonic decrease as it recedes from the observer. When the observer is very close to the path of the object, the transition from high to low frequency is very abrupt. When the observer is far from the path of the object, the transition from high to low frequency is gradual.

In the limit where the speed of the wave is much greater than the relative speed of the source and observer (this is often the case with electromagnetic waves, e.g. light), the relationship between observed frequency f and emitted frequency f_0 is given by:[3]

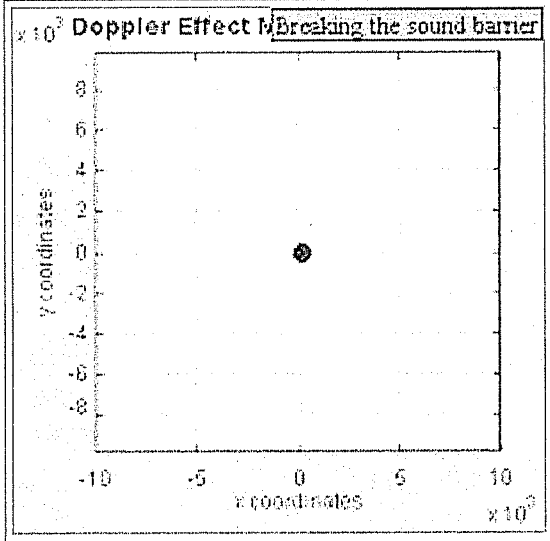
speed of sound in the medium. The distance between wave-fronts is the wavelength. All observers will hear the same frequency, which will be equal to the actual frequency of the source where $f = f_0$



The same sound source is radiating sound waves at a constant frequency in the same medium. However, now the sound source is moving to the right with a speed $v_s = 0.7\ c$ (Mach 0.7). The wave-fronts are produced with the same frequency as before. However, since the source is moving, the centre of each new wavefront is now slightly displaced to the right. As a result, the wave-fronts begin to bunch up on the right side (in front of) and spread further apart on the left side (behind) of the source. An observer in front of the source will hear a higher frequency

$$f = \left(\frac{c + v_r}{\;} \right) f_0 = 3.33\, f_0$$

source will hear a lower frequency $f = \left(\frac{c - v_r}{c + v_s}\right) f_0 = 0.59 f_0$



Now the source is moving at the speed of sound in the medium ($v_s = c$, or Mach 1). assuming the speed of sound in air at sea level is about 330 m/s . The wave fronts in front of the source are now all bunched up at the same point. As a result, an observer in front of the source will detect nothing until the source arrives where $f = \left(\frac{c + v_r}{c - v_s}\right) f_0 = \infty Hz.$ An observer behind of the source $f = \left(\frac{c - v_r}{c + v_s}\right) f_0 = 0.5 f_0.$

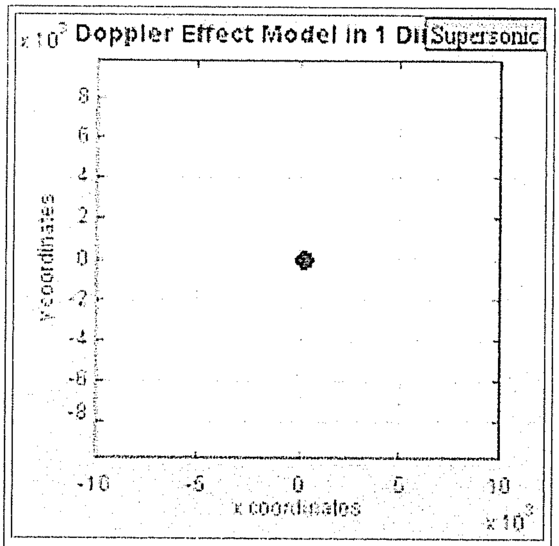
Observed frequency	**Change in frequency**
$f = \left(1 - \frac{v_{s,r}}{c}\right) f_0$	$\Delta f = -\frac{v_{s,r}}{c} f_0 = -\frac{v_{s,r}}{\lambda_0}$

where

$v_{s,r} = v_s - v_r$ is the velocity of the source relative to the receiver: it is positive when the source and the receiver are moving further apart

travelling in a vacuum)
λ_0 is the wavelength of the transmitted wave in the reference frame of the source.

These two equations are only accurate to a first order approximation. However, they work reasonably well when the speed between the source and receiver is slow relative to the speed of the waves involved and the distance between the source and receiver is large relative to the wavelength of the waves. If either of these two approximations are violated, the formulae are no longer accurate.



The sound source has now broken through the sound speed barrier, and is traveling at 1.4 times the speed of sound, c (Mach 1.4). Since the source is moving faster than the sound waves it creates, it actually leads the advancing wavefront. The sound source will pass by a stationary observer before the observer actually hears the sound it creates. As a result, an observer in front of the source will detect

$$f = \left(\frac{c + v_r}{c - v_s}\right) f_0 = \infty Hz$$

and an observer behind the source

$$f = \left(\frac{c - v_r}{c + v_s}\right) f_0 = 0.42 f_0$$

.

Analysis

The frequency of the sounds that the source *emits* does not actually change. To understand what happens, consider the following analogy. Someone throws one ball every second in a man's direction. Assume that balls travel with constant velocity. If the thrower is stationary, the man will receive one ball every second. However, if the thrower is moving towards the man, he will receive balls more frequently because the balls will be less spaced out. The inverse is true if the thrower is moving away from the man. So it is actually the *wavelength* which is affected; as a consequence, the received frequency is also affected. It may also be said that the velocity of the wave remains constant whereas wavelength changes; hence frequency also changes.

If the source moving away from the observer is emitting waves through a medium with an actual frequency f_0, then an observer stationary relative to the medium detects waves with a frequency f given by

$$f = \left(\frac{c}{\quad}\right) f_0$$

where v_s is positive if the source is moving away from the observer, and negative if the source is moving towards the observer.

A similar analysis for a moving *observer* and a stationary source yields the observed frequency (the receiver's velocity being represented as v_r):

$$f = \left(\frac{c + v_r}{c}\right) f_0$$

where the similar convention applies: v_r is positive if the observer is moving towards the source, and negative if the observer is moving away from the source.

These can be generalized into a single equation with both the source and receiver moving.

$$f = \left(\frac{c + v_r}{c + v_s}\right) f_0$$

or, alternatively:

$$f\left(1 + \frac{v_{s,r}}{c + v_r}\right) = f_0$$

where $v_{s,r} = v_s - v_r$.

However the limitations mentioned above still apply. When the more complicated exact equation is derived without using any approximations (just assuming that source, receiver, and wave or signal are moving linearly relatively to each other) several interesting and perhaps surprising results are found. For example, as Lord Rayleigh noted in his classic book on sound, by properly moving it would be possible to hear a symphony being played backwards. This is the so-called "time reversal effect" of the Doppler effect. Other interesting conclusions are that the Doppler effect is time-dependent in general (thus we need to know not only the source and receivers' velocities, but also their positions at a given time), and in some circumstances it is possible to receive two signals or waves from a source, or no signal at all. In addition there are more possibilities than just the receiver approaching the signal and the receiver receding from the signal.

All these additional complications are derived for the classical, i.e., non-relativistic, Doppler effect, but hold for the relativistic Doppler effect as well [*citation needed*]

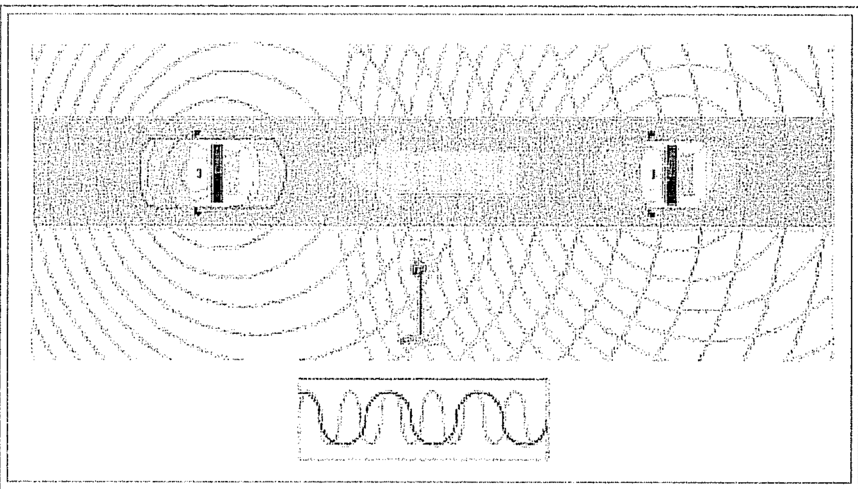
A common misconception

Craig Bohren pointed out in 1991 that some physics textbooks erroneously state that the observed frequency *increases* as the object approaches an observer and then decreases only as the object passes the observer.[4] In most cases, the observed frequency of an approaching object declines monotonically from a value above the emitted frequency, through a value equal to the emitted frequency when the object is closest to the observer, and to values increasingly below the emitted frequency as the object recedes from the observer. Bohren proposed that this common misconception might occur because the *intensity* of the sound increases as an object approaches an observer and decreases once it passes and recedes from the observer and that this change in intensity is misperceived as a change in frequency. Higher sound pressure levels make for a small decrease in perceived pitch in low frequency sounds, and for a small increase in perceived pitch for high frequency sounds.[5]

Application

Sirens

The siren on a passing emergency vehicle will start out higher than its stationary pitch, slide down as it passes, and continue lower than its stationary pitch as it recedes from the observer. Astronomer John Dobson explained the effect thus:



A stationary microphone records moving police sirens at different pitches depending on their relative direction.

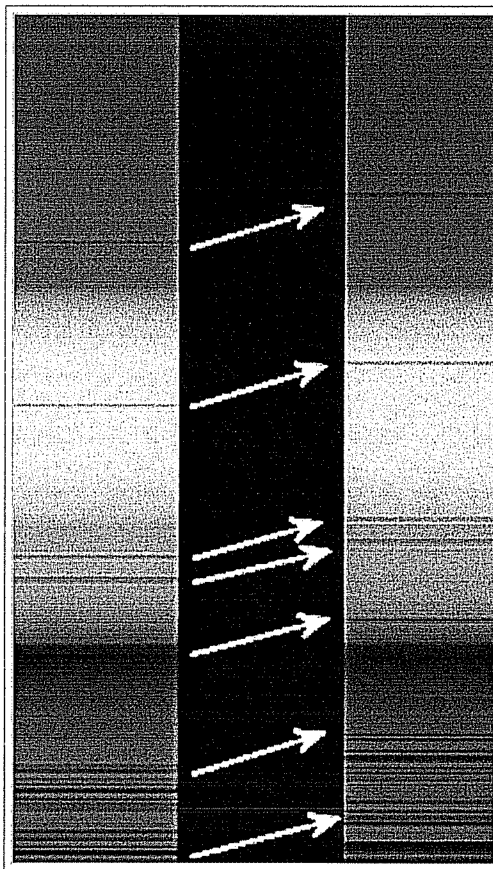
> "The reason the siren slides is because it doesn't hit you."

In other words, if the siren approached the observer directly, the pitch would remain constant (as $v_{s,\,r}$ is only the radial component) until the vehicle hit him, and then immediately jump to a new lower pitch. Because the vehicle passes by the observer, the radial velocity does not remain constant, but instead varies as a function of the angle between his line of sight and the siren's velocity:

where v_s is the velocity of the object (source of waves) with respect to the medium, and θ is the angle between the object's forward velocity and the line of sight from the object to the observer.

Astronomy

The Doppler effect for electromagnetic waves such as light is of great use in astronomy and results in either a so-called red shift or blue shift. It has been used to measure the speed at which stars and galaxies are approaching or receding from us, that is, the radial velocity. This is used to detect if an apparently single star is, in reality, a close binary and even to measure the rotational speed of stars and galaxies.



Redshift of spectral lines in the optical spectrum of a supercluster of distant galaxies (right), as compared to that of the Sun (left).

The use of the Doppler effect for light in astronomy depends on our knowledge that the spectra of stars are not continuous. They exhibit absorption lines at well defined frequencies that are correlated with the energies required to excite electrons in various elements from one level to another. The Doppler effect is recognizable in the fact that the absorption lines are not always at the frequencies that are obtained from the spectrum of a stationary light source. Since blue light has a higher frequency than red light, the spectral lines of an approaching astronomical light source exhibit a blue shift and those of a receding astronomical light source exhibit a redshift.

Among the nearby stars, the largest radial velocities with respect to the Sun are +308 km/s (BD-15°4041, also known as LHS 52, 81.7 light-years away) and -260 km/s (Woolley 9722, also known as Wolf 1106 and LHS 64, 78.2 light-years away). Positive radial velocity means the star is receding from the Sun, negative that it is approaching.

Temperature measurement

plasma) which is emitting a spectral line. Due to the thermal motion of the emitters, the light emitted by each particle can be slightly red- or blue-shifted, and the net effect is a broadening of the line. This line shape is called a Doppler profile and the width of the line is proportional to the square root of the temperature of the emitting species, allowing a spectral line (with the width dominated by the Doppler broadening) to be used to infer the temperature.

Radar

Main article: Doppler radar

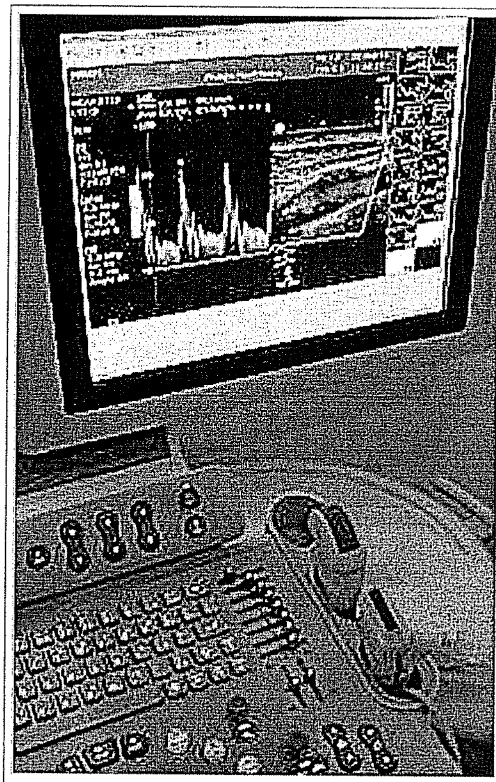
The Doppler effect is used in some types of radar, to measure the velocity of detected objects. A radar beam is fired at a moving target — e.g. a motor car, as police use radar to detect speeding motorists — as it approaches or recedes from the radar source. Each successive radar wave has to travel farther to reach the car, before being reflected and re-detected near the source. As each wave has to move farther, the gap between each wave increases, increasing the wavelength. In some situations, the radar beam is fired at the moving car as it approaches, in which case each successive wave travels a lesser distance, decreasing the wavelength. In either situation, calculations from the Doppler effect accurately determine the car's velocity. Moreover, the proximity fuze, developed during World War II, relies upon Doppler radar to explode at the correct time, height, distance, etc.[*citation needed*]

Medical imaging and blood flow measurement

An echocardiogram can, within certain limits, produce accurate assessment of the direction of blood flow and the velocity of blood and cardiac tissue at any arbitrary point using the Doppler effect. One of the limitations is that the ultrasound beam should be as parallel to the blood flow as possible. Velocity measurements allow assessment of cardiac valve areas and function, any abnormal communications between the left and right side of the heart, any leaking of blood through the valves (valvular regurgitation), and calculation of the cardiac output. Contrast-enhanced ultrasound using gas-filled microbubble contrast media can be used to improve velocity or other flow-related medical measurements.

Although "Doppler" has become synonymous with "velocity measurement" in medical imaging, in many cases it is not the frequency shift (Doppler shift) of the

Velocity measurements of blood flow are also used in other fields of medical ultrasonography, such as obstetric ultrasonography and neurology. Velocity measurement of blood flow in arteries and veins based on Doppler effect is an effective tool for diagnosis of vascular problems like stenosis.[6]



colour flow ultrasonography (Doppler) of a carotid artery - scanner and screen

Flow measurement

Instruments such as the laser Doppler velocimeter (LDV), and acoustic Doppler velocimeter (ADV) have been developed to measure velocities in a fluid flows. The LDV emits a light beam and the ADV emits an ultrasonic acoustic burst, and measure the Doppler shift in wavelengths of reflections from particles moving with the flow. The actual flow is computed as a function of the water velocity and phase. This technique allows non-intrusive flow measurements, at high precision and high frequency.

Velocity profile measurement

Developed originally for velocity measurements in medical applications (blood flow), Ultrasonic Doppler Velocimetry (UDV) can measure in real time complete velocity profile in almost any liquids containing particles in suspension such as dust, gas bubbles, emulsions. Flows can be pulsating, oscillating, laminar or turbulent, stationary or transient. This technique is fully non-invasive.

Satellite communication

Fast moving satellites can have a Doppler shift of dozens of kilohertz relative to a ground station. The speed, thus magnitude of Doppler effect, changes due to earth curvature. Dynamic Doppler compensation, where the frequency of a signal is changed multiple times during transmission, is used so the satellite receives a constant frequency signal. [7]

Underwater acoustics

In military applications the Doppler shift of a target is used to ascertain the speed of a submarine using both passive and active sonar systems. As a submarine passes by a passive sonobuoy, the stable frequencies undergo a Doppler shift, and the speed and range from the sonobuoy can be calculated. If the sonar system is mounted on a moving ship or another submarine, then the relative velocity can be calculated.

Audio

The Leslie speaker, associated with and predominantly used with the Hammond B-3 organ, takes advantage of the Doppler Effect by using an electric motor to rotate an acoustic horn around a loudspeaker, sending its sound in a circle. This results at the listener's ear in rapidly fluctuating frequencies of a keyboard note.

Vibration measurement

A laser Doppler vibrometer (LDV) is a non-contact method for measuring vibration. The laser beam from the LDV is directed at the surface of interest, and the vibration amplitude and frequency are extracted from the Doppler shift of the laser beam frequency due to the motion of the surface.

See also

- Relativistic Doppler effect
- Dopplergraph
- Fizeau experiment
- Fading
- Inverse Doppler effect
- Photoacoustic Doppler effect
- Differential Doppler effect
- Rayleigh fading

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1. ^ ***a b*** Alec Eden *The search for Christian Doppler*,Springer-Verlag, Wien 1992. Contains a facsimile edition with an English translation.
2. ^ Scott Russell, John (1848). "On certain effects produced on sound by the rapid motion of the observer" (http://www.ma.hw.ac.uk/~chris/doppler.html) . *Report of the Eighteen*

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3. ^ *a* *b* Rosen, Joe; Gothard, Lisa Quinn (2009). *Encyclopedia of Physical Science* (http://books.google.com/books?id=avyQ64LIJa0C) . Infobase Publishing. p. 155. ISBN 0-816-07011-3. http://books.google.com/books?id=avyQ64LIJa0C., Extract of page 155 (http://books.google.com/books?id=avyQ64LIJa0C&pg=PA155) Note that the sign of the relative speed in the source (u) is opposite from the sign in the article ($v_{s,r}$)
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5. ^ Olson, Harry F. (1967). *Music, Physics and Engineering*. Dover Publications. pp. 248–251. ISBN 0486217698.
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7. ^ Qingchong, Liu (1999), "Doppler measurement and compensation in mobile satellite communications systems" (http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=822695) , *Military Communications Conference Proceedings / MILCOM* **1**: 316-320, http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=822695

Further reading

- "Doppler and the Doppler effect", E. N. da C. Andrade, *Endeavour* Vol. XVIII No. 69, January 1959 (published by ICI London). Historical account of Doppler's original paper and subsequent developments.
- Adrian, Eleni (24 June 1995). "Doppler Effect" (http://archive.ncsa.uiuc.edu/Cyberia/Bima/doppler.html) . NCSA. http://archive.ncsa.uiuc.edu/Cyberia/Bima/doppler.html. Retrieved 2008-07-13.

External links

- Doppler Effect (http://scienceworld.wolfram.com/physics/DopplerEffect.html) , ScienceWorld
- Java simulation of Doppler effect (http://www.falstad.com/ripple/ex-doppler.html)
- Doppler Shift for Sound and Light (http://www.mathpages.com/rr/s2-04/2-04.htm) at MathPages
- The Doppler Effect and Sonic Booms (D.A. Russell, Kettering University) (http://www.kettering.edu/~drussell/Demos/doppler/doppler.html)

- Wave Propagation (http://math.ucr.edu/~jdp/Relativity/WaveDancer.html) *from John de Pillis.* An animation showing that the speed of a moving wave source does not affect the speed of the wave.
- EM Wave Animation (http://math.ucr.edu/~jdp/Relativity/EM_Propagation.html) *from John de Pillis.* How an electromagnetic wave propagates through a vacuum
- Doppler Shift Demo (http://astro.unl.edu/classaction/animations/light/dopplershift.html) - Interactive flash simulation for demonstrating Doppler shift.
- Interactive applets (http://www.colorado.edu/physics/2000/applets_New.html) at Physics 2000

Categories: Doppler effects | Radio frequency propagation | Wave mechanics | Radar signal processing

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