$\sin(kx - \omega t)$

$= \sin kx \cos\omega t + \cos kx \sin\omega t$

Linear Combination.

趙凱華
李怡嚴

Mathematics

Wave Equation

$t=0$

$\underline{u(x,t)}$

Wave Equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = v^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \left.\begin{aligned} u(x,0) &= u(x) \\ \left.\frac{\partial u}{\partial x}\right|_{t=0} &= v(x) \end{aligned}\right] \text{ initial condition}$$

Ansatz $\quad u(x,t) = \phi(t)\psi(x)$

$$\psi(x)\frac{d^2\phi}{dt^2} = \phi(t)\frac{d^2\psi}{dx^2} \cdot \underset{\substack{\uparrow \\ \text{constant}}}{v^2}$$

$$\underset{\substack{\downarrow \\ \text{function of } t}}{\frac{1}{\phi}\frac{d^2\phi}{dt^2}} = \underset{\substack{\downarrow \\ \text{function of } x}}{\frac{1}{\psi}\frac{d^2\psi}{dx^2}} \cdot v^2 = -\omega^2$$

$$\Rightarrow \quad \frac{d^2\phi}{dt^2} = -\omega^2\phi \quad \Rightarrow \quad \phi = C\sin\omega t + D\cos\omega t$$

$$\underset{\substack{\downarrow \\ \text{known}}}{\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{v^2}\psi} \quad \Rightarrow \quad \psi = E\sin\frac{\omega}{v}x + F\cos\frac{\omega}{v}x$$

The method is known as seperation of variable

$$u(x,t) = [C\sin\omega t + D\cos\omega t] \cdot [\text{"}E\text{"}\sin\frac{\omega}{v}x + \text{"}F\text{"}\cos\frac{\omega}{v}x]$$

The coefficients (constants) are to be dertermined by the initial condition and boundary conditions

$\psi(0, t) = 0$

$\psi(L, t) = 0$

$u(0, t) = 0 \Rightarrow$ "F" $= 0$

$$u = (A \sin \omega t + B \cos \omega t) \sin \frac{\omega}{v} x$$

C · "E"   DE

$$0 = u(L, t) = (A \sin \omega t + B \cos \omega t) \sin \frac{\omega}{v} L$$

$$\Rightarrow (\omega / v) L = n\pi \qquad n = 0, 1, 2, 3, \dots$$

$$\omega_n = v \frac{n\pi}{L}$$

$$\psi_n(x, t) = u_n(x, t) = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin(\omega_n / v) x$$

$u_n(x, t)$ $n = 0, 1, \dots$

$n = 1 \rightarrow$ first harmonic

$n = 2 \rightarrow$ second harmonic

satisfies the wave equation and the boundary condition

If $\{u_n(x, t)\}$ are solutions

then the general solution can be written as

constant

$$u(x, t) = \sum_n A_n' \, u_n(x, t)$$

$$= \sum_n A_n' (A_n \sin \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi}{L} x$$

$$= \sum (K_n' \cos \omega_n t + K_n'' \sin \omega_n t) \sin \frac{n\pi}{L} x$$

K_n', K_n'' are constants to be determined by initial conditions

$u(x,0) = W(x)$ initial conditions

$\left(\frac{\partial u(x,t)}{\partial t}\right)_{t=0} = V(x)$

$u(x,0) = W(x) = \sum_n K_n' \sin \frac{n\pi}{L} x.$ (A)

$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0$ if $m \neq n$

$= \frac{L}{2}$ if $m = n.$

Proof: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$A = \frac{n\pi}{L}, \quad B = \frac{m\pi}{L}$

$\frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi}{L}x\right] dx$

$y \equiv \frac{\pi x}{L}$

$dy = \frac{\pi}{L} dx$

$x = 0 \quad y = 0$

$x = L \quad y = \pi$

$= \frac{1}{2}\frac{L}{\pi}\left[\int_0^\pi \cos (n-m)y \; dy + \int_0^\pi \cos(n+m)dy\right.$

$\underline{n \neq m}$ $n-m$, $n+m$ are all non-zero integrals

$\Rightarrow$ the integrals are all zero

if $n = m$ $n-m = 0$, the first integral $= \pi$

$\Rightarrow$ thus the proof

$\underline{\int_0^L} u(x,0) \sin \frac{m\pi}{L} \underline{dx} = \sum K_n' \underline{\int_0^L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx$

$\quad \overset{\shortparallel}{W(x)}$

$= K_m'$

$\downarrow$

have been found

Use the same technique to find K_m''

$V(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum K_n'' \omega_n \sin \frac{n\pi}{L} x$

$\int_0^L V(x) \sin \frac{m\pi}{L} x = \int_0^L \left[\sum K_n'' \omega_n \sin \frac{n\pi x}{L}\right] \sin \frac{m\pi}{L} x \, dx$

$= K_m'' \omega_n \frac{L}{2}$

$$\Rightarrow \quad K_m'' = \underbrace{\frac{2}{\omega_m L}}_{\text{known}} \int_0^L \overline{W}(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[\sin \frac{n\pi}{L} x \int_0^L \frac{2}{L} W(x) \sin \frac{n\pi}{L} \, dx \right] \cos \omega_n t \right.$$

$$\left. + \left[\sin \frac{n\pi}{L} x \int_0^L \frac{2}{\omega_n L} \overline{W}'(x) \sin \frac{n\pi}{L} \, dx \right] \sin \omega_n t \right\}$$

$$\omega_n = v \frac{n\pi}{L}$$

$$\sin \underbrace{\frac{n\pi}{L}}_{A} x \cdot \cos \underbrace{v \frac{n\pi}{L}}_{B} t$$

$$\cos A \cos B = \frac{1}{2} \left[\cos(A+B) + \cos(A-B) \right]$$

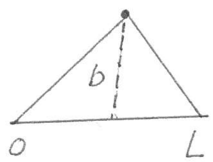
$$\sin A \cos B = \frac{1}{2} \left[\sin(A+B) + \sin(A-B) \right]$$

$$\rightarrow \frac{1}{2} \left[\underbrace{\sin \frac{n\pi}{L}(x+vt)}_{\text{function of } x+vt} + \underbrace{\cos \frac{n\pi}{L}(x-vt)}_{\text{function of } x-vt.} \right]$$

See P. 745 – P. 750 of 李怡嚴之"大學物理學

$$u(x) = \frac{2b}{L} x \qquad 0 < x < \frac{L}{2}$$

$$= \frac{2b}{L} (L - x) \qquad \frac{L}{2} < x < L$$



$$V(x) = \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = 0$$

Mathematics

Wave Equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \underset{\substack{\downarrow \\ \text{constant}}}{v^2} \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \underline{\text{Linear}}$$

Seperation of variable

Ansatz : $u(x,t) = \underset{\substack{\uparrow \\ \phi(t)}}{F(t)}\ \underset{\substack{\uparrow \\ \psi(x)}}{G(x)}$

$$\frac{1}{\phi}\frac{d^2\phi}{dt^2} = \frac{1}{\psi}\frac{d^2\psi}{dx^2} v^2 = -\omega^2$$

$$\frac{d^2\phi}{dt^2} = -\omega^2\phi \quad \Rightarrow \quad \phi = C\sin\omega t + D\cos\omega t$$

$$\frac{d^2\psi}{dt^2} = -\frac{\omega^2}{v^2}\psi \quad \Rightarrow \quad \psi = C'\sin\frac{\omega}{v}x + D'\cos\frac{\omega}{v}x$$

$$u(x,t) = [C\sin\omega t + D\cos\omega t][C'\sin\frac{\omega}{v}x + D'\cos\frac{\omega}{v}x]$$

any ω can satisfy the wave equation

Linear $\Rightarrow$ any linear combination of above equation are solution of the wave equation

$u(0,t) = 0$
$u(L,t) = 0$ fixed end point

$D' = 0,\quad \sin\omega \overset{L}{\uparrow}t = 0 \quad \boxed{\frac{\omega L}{v} = n\pi}$

$$\omega_n = \frac{v}{L}(n\pi)$$

only discrete values can satisfy the boundary condition

$$u_n(x,t) = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin(\omega_n/v)\ x$$

is a solution of the wave equation and the boundary condition

The general solution

$$u(x,t) = \sum_n \left(K_n' \cos \omega_n t + K_n'' \sin \omega t\right) \sin \frac{n\pi}{L} x$$

to be determined by the initial condition.

Initial Condition

$$u(x,0) = W(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=0} = \overline{W}(x)$$

the solution of the problem

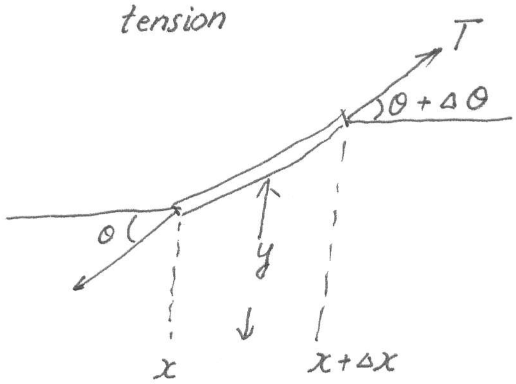
Comment

- Linear
- Boundary condition
- Initial condition
- Method of Seperation of Variable
- Fourier Series

Vibration String

μ, T ↑ tension

$y(x,t)$



Small

$$F_y = T\sin(\theta + \Delta\theta) - T\sin\theta$$

$$F_x = T\cos(\theta + \Delta\theta) - T\cos\theta$$

Transverse displacement y is small

$$F_y \simeq T\Delta\theta \quad ①$$

$$F_x \simeq 0$$

$$T\Delta\theta \simeq (\mu\Delta x)\, a_y \longrightarrow \text{Newton's second law}$$

$T\Delta\theta$ ↓ fixed t, $y(x)$

a_y ↓ fixed x, $y(t)$

The reason for partial derivatives

$$\tan\theta = \frac{\partial y}{\partial x}$$

$$\sec^2\theta\, \Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x \quad ②$$

$\sec^2\theta$ ↓ ① for small θ

$$a_y = \frac{\partial^2 y}{\partial t^2} \quad ③$$

$$\Rightarrow \quad \underbrace{T \frac{\partial^2 y}{\partial x^2} \Delta x}_{\text{combine ①, ②}} = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\boxed{T \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}}$$

$$\parallel$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}}$$

$$v = \sqrt{\frac{T}{\mu}}.$$

violen E string

$$f = 640\ Hz$$

$$\omega \text{"}T\text{"} = 2\pi$$

$$\omega \frac{1}{f} = 2\pi$$

$$\boxed{f = \frac{\omega}{2\pi}} \ne 640\ Hz$$

$$\omega = 2\pi \cdot 640\ Hz$$

$$\boxed{\omega_1 = v \frac{①\pi}{L}}$$

$$2\pi \cdot 640 = \sqrt{\frac{T}{\mu}} \frac{\pi}{L} \qquad \Rightarrow \sqrt{\frac{T}{\mu}}$$

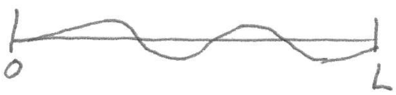
$$\mu = \frac{m}{L}$$

$$L = 33\ cm$$

$$m = 0.125\ g$$

$$\Rightarrow \quad T \cong 68\ N$$

Strected String



$\Rightarrow$ small oscillation in transverse direction

$\Rightarrow$ $\boxed{y(x,t)}$ described the transverse motion $\rightarrow$ wave function

Derivation of the Wave equation.

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} \qquad \boxed{v = \sqrt{\frac{T}{\mu}}}$$

This is the wave equation
Note it is linear.

Solution of the problem, mathematical problem

~~Calcula~~ Relation with Energy, Energy Density $\downarrow$ to go first.

Fundamental frequency of a string with both end fixed end at L

$$\boxed{\omega_n = v\frac{n\pi}{L}}$$

$\boxed{n=1}$ fundamental.

$$\boxed{\omega = \frac{v}{L}\pi}$$

application

$$= \frac{1}{L}\sqrt{\frac{T}{\mu}}\,\pi$$

Violen

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{L}\sqrt{\frac{T}{\mu}}\,\pi$$

$$= \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

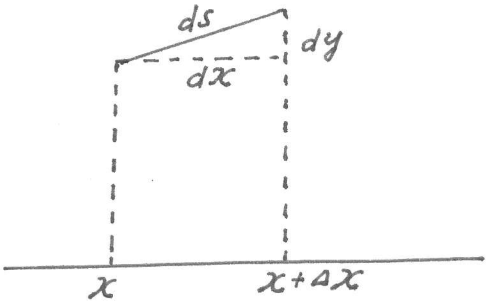
$f \downarrow$ 640 Hz

$L = 33$ cm

$\mu = \frac{m}{L}$ $\qquad m = 0.125g$

$\Rightarrow T \simeq 68\,N$

The Energy in a Mechanical Wave



The mass of the small segment is $\mu\, dx$

The transverse velocity is $\frac{\partial y}{\partial t}$

$\Rightarrow$ for this segment

the kinetic energy is $\frac{1}{2}\mu\, dx \left(\frac{\partial y}{\partial t}\right)^2$

$\Rightarrow$ kinetic energy per unit length.

$\equiv$ kinetic energy density

$$\frac{d(K.E)}{dx} = \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2$$

The potential energy can be calculated by finding the amount by which the string, when deformed, is longer than when it is straight

The extension . constant tension $\Rightarrow$ work done in the deformation

$\Rightarrow$ for this segment

potential energy $= T(ds - dx)$ $\boxed{y \leftrightarrow u}$

$$ds = ((dx)^2 + (dy)^2)^{1/2}$$

$$= dx \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}$$

$\frac{\partial y}{\partial x} \ll 1$

$$\downarrow$$

$$ds - dx \cong \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

potential energy $\simeq \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx$

potential energy density $\equiv \frac{dE_p}{dx} \simeq \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$

$\Rightarrow$ energy density $\cong \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$

Summary of Vibration String

1. Derivation of the Wave Equation

2. Find the wave function with the boundary condition + initial condition

$\downarrow$

linear Combination

$(\sin kx , \cos kx) \cdot (\omega t , \cos\omega t)$

or

$$\sin kx \cos \omega t = \frac{1}{2} [\sin(kx + \omega t) + \sin(kx - \omega t)]$$

$$k = \frac{\omega}{v}$$

3. Given the wave function

$\downarrow$

calculate the physical interesting quantities

Need to calculate the energy of a segement

$$\boxed{\frac{K.E.}{\Delta x} = \frac{1}{2}\mu\left[\frac{\partial y}{\partial t}\right]^2}$$

clearly a function of x, t.

$$\frac{P.E.}{\Delta x} = \frac{1}{2} T\left(\frac{\partial y}{\partial x}\right)^2$$

$\tan\theta$

clearly a function of x, t

$\Rightarrow$ describe the energy distribution of the string vibration as a function of x, t.

Key of the wave phenomenon

- Distribuance

  $y(x,t)$ Elastic wave (Mechanical wave)

  water wave
  Sound wave
  EM wave (Light)
  "Matter" wave

- Derivation of the wave equation

  Wave equation type.

Waves

Oscillations in space and time

$u(x,t)$

Transverse — stretced string

Longitude — sound wave

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

Note linear

Pulses of arbitrary shape

$$\underset{\underset{u}{\updownarrow}}{\zeta}(x,t) = f(\underbrace{x \pm vt}_{\zeta})$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial \zeta}(\pm v)$$

$$\frac{\partial^2 u}{\partial t^2} = (\pm v)^2 \frac{\partial^2 f}{\partial \zeta^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial \zeta^2}$$

$\Rightarrow$ the wave equation is satisfied.

$A^c \sin(kx - \omega t)$

$+ \sin(kx + \omega t)$

$= \underline{2\sin(kx)\cos(\omega t)}$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$=$

The same height

$x_2 - vt_2 = x_1 - vt_1 = \zeta$

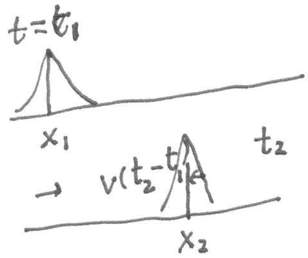
$\underline{f(\zeta)}$ the same

$\downarrow$

the same height.

$\underline{\underline{x_2}} = \underline{\underline{x_1}} + v(t_2 - t_1)$

moving with velocity



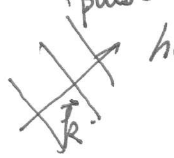
$f(x - vt)$ going to the right with velocity v

$f(x + vt)$ going to the left with velocity v

$f(\zeta)$ function can be ~~Progress~~ found at $t=0$

$f(x-vt)$

travelling $\underline{\text{wave}}$



harmonic pulses

$$\zeta(x,t) = \zeta_0 \cos(k(x \pm vt) + \phi)$$

$$\boxed{k = \frac{\omega}{v}}$$

$\boxed{f(x)\,g(t)}$

$A \sin k$

it is

$\underline{\zeta(x,t)} = A\sin(kx - \omega t)$

$= A \sin kx \cos \omega t - A \cos kx \sin \omega t$

Standing wave

$A\sin(kx - \omega t)$

$+ A\sin(kx + \omega t)$

$=$

$$P = \vec{F} \cdot \vec{v}_{tra} = -T \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial t}$$

$$= \mu v \omega^2 A^2 \cos^2(kx - \omega t)$$

notece the difference

$v \neq v_{trans}$

$$v = \sqrt{\frac{T}{\mu}}$$

$$\tan\theta \sim \theta \sim \frac{\Delta y}{\Delta x}$$

$T\sin\theta - T\sin\theta$

$T \Delta\theta = \vec{F}$

$\boxed{\dfrac{\omega}{v} = k}$ → useful definition

wave ~~numbe~~

This is an example of the Mechanical Wave

<u>one dimensional</u>

<u>linear</u>

General Discussion

Wave $\boxed{\text{Oscillation}}$ in space, time

what to describe

origin Mechanical Wave

- String
- Elastic rod
- Sound

Ocean Wave

⋮

EM Wave Matter Wave

transverse $y(x,t)$

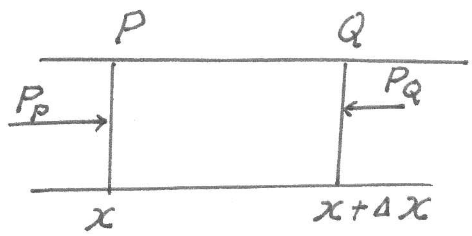
longitudinal $\xi(x,t)$ → sound wave

Wave equation

All partial differential equation involved ~~distrib~~

disturbance in space - time

Sound Wave

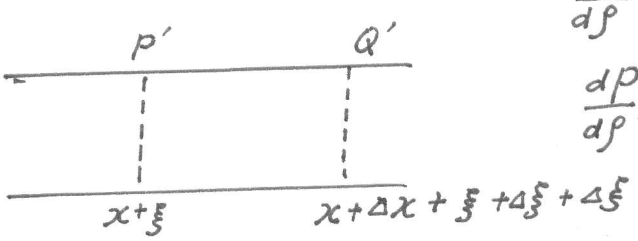


$$\Delta F = \underset{\substack{\downarrow \\ \text{area}}}{A}\,[-p(x+\Delta x) + p(x)]$$

$$= -A\,\frac{\partial p}{\partial x}\,\Delta x$$

$$\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x} A \Delta x$$

$$\Rightarrow \rho \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x} \simeq -\frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = \frac{dp}{d\rho}\rho_0\frac{\partial^2 \xi}{\partial x^2}$$



$$\rho(\Delta x + \Delta \xi)A = \rho_0 \Delta x A$$ continuity equation

$$\rho = \rho_0 \frac{1}{1+\frac{\Delta \xi}{\Delta x}}$$

$$\simeq \rho_0\left[1 - \frac{\partial \xi}{\partial x}\right]$$

$$\frac{\partial \rho}{\partial x} = -\rho_0 \frac{\partial^2 \xi}{\partial x^2}$$

$$\downarrow$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial p}{\partial \rho}\frac{\partial^2 \xi}{\partial x^2}$$

Adiabatic expansion of gas

$$pV^\gamma = C \qquad p = \frac{C}{m^\gamma}\rho^\gamma = k\rho^\gamma$$

$$\frac{dp}{d\rho} = \gamma k \rho^{\gamma-1} = \frac{\gamma k p}{\rho}\frac{\rho^\gamma}{p} = \frac{\gamma k p}{\rho}\frac{1}{k} = \frac{\gamma p}{\rho}$$

$$\Rightarrow \frac{\partial^2 \xi}{\partial t^2} = \frac{\gamma p}{\rho}\frac{\partial^2 \xi}{\partial x^2}$$

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$p = 1$ atm

$\gamma = \frac{c_p}{c_v} = 1.41$

$\rho = \frac{28}{22.4} Kg/m^3$

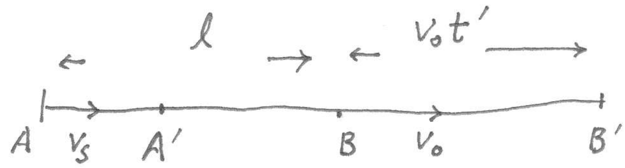
$\Rightarrow \quad v \sim 331$ m/sec

longitudinal wave

Doppler effect

V_o = velocity of the observer relative to the medium
observer

V_s = velocity of the source relative to the medium.
source



$\boxed{t=0}$ $\begin{cases}\text{source at } A \\ \text{observer at } B\end{cases}$ $AB = l$

Emit a wave
↓
reach the observer at t

observer $V_o t$

$Vt = \boxed{l + V_o t}$

$$\boxed{t = \frac{l}{V - V_o}}$$ the first signal arrive at the observer

$t = \tau$ source is at A'
↓
received by B' reach the receiver at t'

$(l - V_s \tau) + V_o t'$

actual distance travelled

$\boxed{V(t' - \tau)} = (l - V_s \tau) + V_o t'$

$Vt' - V\tau$
$= l - V_s \tau + V_o t'$

$$\boxed{t' = \frac{l + (V - V_s)\tau}{V - V_o}} \Leftarrow (V - V_o)t' = l + V\tau - V_s\tau$$

$0 \to \tau$ $\nu\tau$ number of pulse

$\tau' = t' - t$ $\parallel$ $\nu'\tau'$ number of pulse receved.

$$\nu' = \frac{\tau}{\tau'} \nu =$$

{ emitted frequency

$$\tau' = t' - t = \frac{l + (V - V_s)\tau}{V - V_o} - \frac{l}{V - V_o}$$

$$= \frac{(V - V_s)\tau}{(V - V_o)}$$

$$\boxed{\nu' = \frac{\tau}{\tau'} \nu}$$

observed by the observer.

emitted by the source

$\boxed{\text{pitch}}$ of moving ve

$$= \frac{\cancel{\tau}}{\frac{(V - V_s)\cancel{\tau}}{V - V_o}} \nu = \frac{V - V_o}{V - V_s} \nu$$

V_s

f

This is equation 14-58

The other

Use of the Doppler effect

sonic bome

$$\boxed{V = V_s}$$

~~shock~~ waves

Cerenkov radiation.

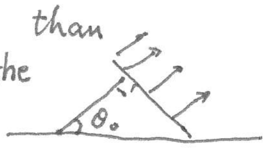
Light in vacuum c

Light in medium $v = \frac{c}{n}$ $\boxed{n > 1}$

if charged particle moving through a medium faster than

differential counter → vary the n

threshold counter



$$\cos\theta_o = \frac{v}{v_s}$$

v → velocity in the medium

v_s ↳ source

v could be the velocity of the electron ~~meds~~