Wave function and Wave Equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$$

<u>Linear</u> ξ_1, ξ_2 are solution

$a\xi_1 + b\xi_2$ is also a solution

$$\frac{\partial^2 \xi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi_1}{\partial t^2}$$

$$\frac{\partial^2 \xi_2}{\partial x^2} = \frac{1}{v}\frac{\partial^2 \xi_2}{\partial t^2}$$

$$\frac{\partial^2}{\partial x^2}(a\xi_1 + b\xi_2) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(a\xi_1 + b\xi_2)$$

$$\Downarrow \checkmark \qquad\qquad\qquad \checkmark$$

$$a\left[\frac{\partial^2 \xi_1}{\partial x^2}\right] + \underline{\underline{b\frac{\partial^2 \xi_2}{\partial x^2}}} = a\frac{1}{v^2}\frac{\partial^2}{\partial t}\xi_1 + \underline{\underline{b\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\xi_2}}$$

This wave equation is <u>linear</u>

See Problem 15 - 59

$$\frac{\partial \psi_1}{\partial t} = 2\psi_1\frac{\partial \psi_1}{\partial t} + 3\psi_1$$

$$\frac{\partial \psi_2}{\partial t} = 2\psi_2\frac{\partial \psi_2}{\partial t} + 3\psi_2$$

$$\frac{\partial}{\partial t}(a\psi_1 + b\psi_2) \overset{?}{=} 2(a\psi_1 + b\psi_2)\frac{\partial}{\partial t}(a\psi_1 + b\psi_2) + 3(a\psi_1 + b\psi_2)$$

$$\downarrow$$

no !

We shall only discuss the wave equation

$$\frac{\partial^2 \xi\,(y)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi\,(y)}{\partial t^2}$$

$y(x - vt)$ is a travelling wave moving toward right

$y(x + vt)$ " " " " " " left

Chapter 14

↓

harmonic waves

↓

idealization

Superposition and interference

↓

understand more complex waves
in terms of simple
harmonic waves

Different ways waves can be combined

Superposition Principle

Wave equations

Coherence

↓

An interference pattern occurs only when the
waves have a definite, stable
relation between their
frequencies and phase

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = A\sin(kx - \omega t) = A\sin k\left(x - \frac{\omega}{k}t\right)$$

$$= A\sin k(x - vt)$$

$$v = \frac{\omega}{k}.$$

Harmonic wave

$$y(x,t) = A\sin(kx-\omega t) = A\sin k(x-\underbrace{\frac{\omega}{k}}_{=\,v}t)$$

$$A\cos(kx-\omega t)$$

Sinusodial wave

$$y = \underbrace{A}_{\text{amplitude}}\cos(\underbrace{\omega t - kx}_{\theta} + \underbrace{\phi}_{\text{phase}})$$

$$y_1 = A_1\cos(\theta+\phi_1), \quad y_2 = A_2\cos(\theta+\phi_2)$$

addition of similar Sinusodial Waves

Addition of Similar Sinusodial Waves

$$y = \underbrace{A}_{\text{amplitude}} \cos(\omega t - kx + \underbrace{\phi}_{\text{phase}})$$

$$y(x,t) = y_1 + y_2 = A_1 \cos(\theta + \phi_1) + A_2 \cos(\theta + \phi_2)$$

$$\underline{\theta = \omega t - kt}$$

$$\cos(\theta \pm \phi) = \cos\theta \cos\phi \mp \sin\theta \sin\phi$$

$$y = (A_1 \cos\phi_1 + A_2 \cos\phi_2)\cos\theta - (A_1 \sin\phi_1 + A_2 \sin\phi_2)\sin\theta$$

$$y = C \cos(\underbrace{\omega t - kx}_{\theta} + \delta)$$

$$= C \cos\delta \cos\theta - C \sin\delta \sin\theta$$

$$C \cos\delta = A_1 \cos\phi_1 + A_2 \cos\phi_2$$

$$C \sin\delta = A_1 \sin\phi_1 + A_2 \sin\phi_2$$

$$\Rightarrow \quad C^2 = (A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2$$

$$\tan\delta = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}$$

$\Rightarrow$ Any two sinusoidal function superimpose into yet another sinusoidal function, different from the original components, only in amplitude and phase

$$C^2 = A_1^2(\sin^2\phi_1 + \cos^2\phi_1) + A_2^2(\sin^2\phi_2 + \cos^2\phi_2) + 2A_1A_2(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$$

$$= A_1^2 + A_2^2 + \underbrace{2A_1A_2 \cos(\phi_1 - \phi_2)}_{\text{interference}}$$

note the importance of phase difference

$$\phi_1 = \phi_2 \Rightarrow C^2 = A_1^2 + A_2^2 + 2A_1A_2 \rightarrow \text{maximum}$$

$$\phi_1 - \phi_2 = \pi \qquad C^2 = A_1^2 + A_2^2 - 2A_1A_2 \rightarrow \text{minimum.}$$

Generalize to any number of components

$$y = \sum_i y_i = \sum_i A_i \cos(\omega t - kx + \phi_i)$$

$$y = C \cos(\omega t - kx + \delta)$$

$$C^2 = \left(\sum_i A_i \cos\phi_i\right)^2 + \left(\sum_{i=1}^{n} A_i \sin\phi_i\right)^2$$

$$\tan\delta = \frac{\sum_i A_i \sin\phi_i}{\sum_i A_i \cos\phi_i}$$

Any number of sinusoidal waves in superposition constitute still another sinusoidal wave

The Energy of Two Waves

The intensity of any sinusoidal wave is proportional to the square of the amplitude

Two such wave (different only in amplitude and phase

$$C^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_1 - \phi_2)$$

Average intensity over any number of complete cycles

$$I = I_1 + I_2 + \underbrace{2\sqrt{I_1 I_2} \cos(\phi_1 - \phi_2)}_{\text{interference}}$$

$\phi_1 - \phi_2 = \frac{\pi}{2}$ waves appear independent of one another

$$I = I_1 + I_2$$

If $A_1 = A_2$

$$I = 2I_1 [1 + \cos(\phi_1 - \phi_2)]$$

$\phi_1 - \phi_2 = 0$ $4I_1$

$\phi_1 - \phi_2 = \pi$ 0

appearance or disapparence of the wave energy

↓

Young's interference

↓

coherence

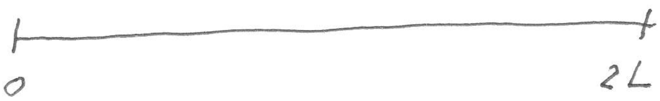
Interference of two oppositely travelling waves

$$y = y_1 + y_2 = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

$$= 2A\cos kx \cos \omega t$$

fixed points of zero distance

$$kx_n = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \cdots$$



$$f_n(x) = A_n \cos(k_n x + \phi_n)$$

$$f_n(x) = f_n(x + 2L) \quad \text{periodic.}$$

$$k_n(2L) = 2n\pi$$

$$\Rightarrow \quad k_n = \frac{n\pi}{L}$$

Fourier Series.

$$F(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x + \phi_n)$$

Fourier's theorem

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\frac{n\pi x}{L} = n\theta$$

$$\theta = \frac{\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_x^{x+2L} F(x)\,dx$$

$$a_n = \frac{1}{L}\int_x^{x+2L} F(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_x^{x+} F(x)\sin\frac{n\pi x}{L}\,dx$$

$$\int_\theta^{\theta+2\pi} \sin n\theta \sin m\theta \, d\theta = \pi\delta_{n,m}$$

$$\int_\theta^{\theta+2\pi} \sin n\theta \cos m\theta \, d\theta = 0$$

$$\int_\theta^{\theta+2\pi} \cos n\theta \cos m\theta \, d\theta = \pi\delta_{m,n}$$

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Define $\quad k_1 = \frac{\pi}{L}$

$$\cos n k_1 x = \frac{1}{2} \left(e^{ink_1 x} + e^{-ink_1 x} \right)$$

$$\sin n k_1 x = \frac{i}{2} \left(e^{-ink_1 x} - e^{ink_1 x} \right)$$

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{ink_1 x}$$

$$c_0 = a_0$$

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2}$$

$$c_n = \frac{1}{2L} \int_x^{x+2L} F(x) e^{-ink_1 x} dx$$

$$\Rightarrow \quad F(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{ink_1 x} \int_{-L}^{L} F(x') e^{-ink_1 x'} dx'$$

$$k_1 \rightarrow \frac{\pi}{L}$$

$$F(x) = \lim_{L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} k_1 \int_{-L}^{L} F(x') e^{-ink_1 (x' - x)} dx'$$

$$\Delta k = k_{n+1} - k_n = \frac{\pi}{L} = k_1$$

$$k_n = n k_1$$

$$\lim_{L \to \infty} n k_1 = \lim_{L \to \infty} k_n = k$$

$$\sum_{n=-\infty}^{\infty} \Delta k \, G(k_n)$$

$$F(x) = \lim_{L \to \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta k \int_{-L}^{L} F(x') e^{-ik_n (x' - x)} dx'$$

$$= \underbrace{\frac{1}{2\pi}}_{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} dk \Big[\underbrace{\int_{-\infty}^{\infty} F(x') e^{-ikx'} dx'}_{= G(k)} \Big] e^{ikx}$$

$$\Rightarrow \quad F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ikx} dk$$

Fourier integral

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-ikx} dx$$

$$\int_{-\infty}^{\infty} e^{ikx}\, e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha}$$

$$\parallel$$

$$\int_{-\infty}^{\infty} \cos kx\, e^{-\alpha x^2} dx + i\underbrace{\int_{-\infty}^{\infty} \sin kx\, e^{-\alpha x^2} dx}_{0}$$

even - odd
argument

$$I_1 = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$I_2^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy$$

$$= \int_{-\infty}^{+\infty} e^{-\alpha(x^2+y^2)} dx\, dy$$

$$= \int_0^{\infty} r\, dr \int_0^{2\pi} d\theta\ e^{-\alpha r^2} r\, dr$$

$$= 2\pi \int_0^{\infty} e^{-\alpha r^2} r\, dr$$

$$z = \alpha r^2$$

$$dz = 2r\,dr \cdot \alpha$$

$$= 2\pi \int_0^{\infty} \frac{1}{2} e^{-z} \frac{dz}{\alpha}$$

$$= \frac{\pi}{\alpha} \int_0^{\infty} e^{-z} dz$$

$$= \frac{\pi}{\alpha} \left[-e^{-z}\right]_0^{\infty} = \frac{\pi}{\alpha}$$

$$I_1^2 = \frac{\pi}{\alpha}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$I_2 = \int_{-\infty}^{\infty} \cos kx \; e^{-\alpha x^2} dx$$

$$\cos kx = 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \cdots$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi}\frac{d}{d\alpha}\alpha^{-\frac{1}{2}}$$

$$\parallel$$

$$-\int_{-\infty}^{\infty} e^{-\alpha x^2} x^2 dx = \sqrt{\pi}\left(-\frac{1}{2}\right)\alpha^{-3/2}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-\alpha x^2} x^2 dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

Take derivative with respect to α again

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} x^4 dx = \frac{3}{2^2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$I_2 = \sqrt{\frac{\pi}{\alpha}}\left[1 - \frac{k^2}{4\alpha} + \frac{3k^4}{\alpha^2 4!\, 2^2} - \frac{3\cdot 5 k^6}{\alpha^3 6!\, 2^3} + \cdots\right]$$

$$= \sqrt{\frac{\pi}{\alpha}}\left[1 - \frac{k^2}{4\alpha} + \left(\frac{k^2}{4\alpha}\right)^2\Big/2! - \left(\frac{k^2}{4\alpha}\right)^3\Big/3! + \cdots\right]$$

$$\downarrow$$

$$e^{-k^2/4\alpha}$$

$$F(x) = e^{-\alpha x^2} \qquad\qquad \sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha}$$

$$e^{-\alpha\tilde{x}^2} = e^{-1} \qquad\qquad e^{-\tilde{k}_1^2/4\alpha} = e^{-1}$$

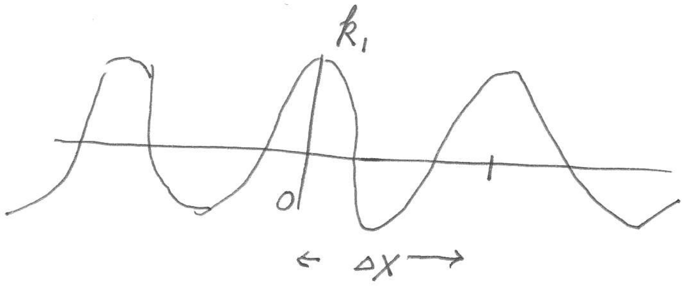
$$\tilde{x}^2 = \frac{1}{\alpha} \qquad\qquad \tilde{k}_1^2/4\alpha = 1$$

$$\boxed{\tilde{k}_1 = 2\sqrt{\alpha}}$$

$$\Delta x = \frac{2}{\sqrt{\alpha}} \qquad\qquad \Delta k = 4\sqrt{\alpha}$$

$$\Delta k\,\Delta x \sim O(1)$$

uncertainty principle



$x > \Delta x$ want the wave to "vanish" or small

The sinuodinal wave with k_1 outside

must be destructively interfered by a sinuoidal wave with wave number k_2 such that the phase difference

$$k_1 \Delta x - k_2 \Delta x \sim \pi$$

$$\boxed{(\Delta k)(\Delta x) \sim \pi}$$

uncertainty priniple

Δx smaller

Δk must grow

$\boxed{p \sim \hbar k}$ in matter wave

$\boxed{\Delta p \Delta x \gtrsim h}$

Heisenberg's uncertainty principle.

The solution of the problems for Chapter 14 - 16 is on line

Final Jan 10 7:00 pm

Chapter 11 – 16

Thursday

$$F(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk$$

$$F(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x,0)\, e^{-ikx}\, dx.$$

Energy associated with the wave

$$\int_x^{x+2L} [F(x)]^2 dx$$

$$= \int_x^{x+2L} \Big[\Big[a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\Big)\Big]$$

$$\Big[a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\Big]\Big] dx$$

$$\sim a_0^2 (2L) + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) L$$

$\swarrow$

Fourier Energy theorem.

Advantages of use Fourier Series Expansion.

Problem 46 of Chapter 15

Applica~~ti~~ble to any periodic function.
(even with finite discontinunity)

Similar to Taylor expansion

Addition of two different k values

$$v = \frac{\omega}{k}$$

So far we have consider the case $\quad$ k vary

$v = \text{constant}$

ω, v may either one or both

$$y_1 = A e^{i[(k+dk)x - (\omega + d\omega)t]}$$

$$y_2 = A e^{i[(k-dk)x - (\omega - d\omega)t}$$

$$y_1 = y_1 + y_2 = A e^{i(kx - \omega t)} \underbrace{\left[e^{i(dkx - d\omega t} + e^{-i(dkx - d\omega t} \right]}_{2\cos(dkx - d\omega t}$$

Real of y

$$y = 2A \cos(dkx - d\omega t) \cos(kx - \omega t)$$

$\cos(dkx - d\omega t)$ → waves propagating with group velocity $\frac{d\omega}{dk}$

$\cos(kx - \omega t)$ → moving with phase velocity v_p

No dispersive medium

$$v = \frac{\omega}{k} \Rightarrow \omega = ck$$

$$\frac{d\omega}{dk} = v$$

The group velocity is the same as the phase velocity

Dispersive medium

v is a function of $k = \frac{2\pi}{\lambda}$

λ: Example different color light propagating with different velocity

$$v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk}$$

$$= v + k \underbrace{\frac{dv}{dk}}_{<0}$$

$$v_g < v_p$$

The energy carried by the wave would propagate at the group velocity

11'

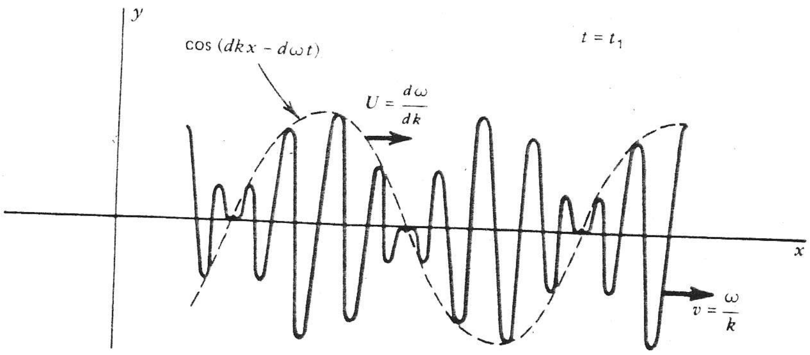


Figure 3.18. The sum of two waves of slightly differing angular frequency and wave number.

Gaussian wave packet

$$A(k) = \sigma\, e^{-(k-k_0)^2\sigma^2/2}$$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(k-k)^2\sigma^2/2}\, e^{i(kx-\omega t)}\, dk$$

$$\omega(k) \simeq \underbrace{\omega_0 + \left(\frac{d\omega}{dk}\right)_{k_0}(k-k_0)}_{\text{keep}} + \frac{1}{2}\frac{d^2\omega}{dk^2}(k-k_0)^2 + \cdots$$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}}\, e^{i(k_0x-\omega t)} \int_{-\infty}^{\infty} e^{-(k-k_0^2)\sigma^2 - (k_0-k)(x-Ut)}\, dk$$

Let $a^2 = \sigma^2/2$ $\qquad U = \left(\frac{d\omega}{dk}\right)_{k_0}$

$b = -i(x-Ut)$

$$F(x,t) = \frac{\sigma}{\sqrt{2\pi}}\, e^{i(k_0x-\omega_0 t)}\, \frac{e^{b^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= e^{i(k_0x-\omega_0 t)}\, e^{-(x-Ut)^2/2\sigma^2}$$

center of the Gaussian is moving with group velocity $\frac{d\omega}{dk}$