Chapter 23

(1)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}$$

↓

Gauss' Law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

↓

differential form.

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \hat{r}$$

↓

Coulomb's Law

Chapter ~~4~~ 23

Gauss' Law

1. $$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0} \qquad \text{Gauss' Law}$$

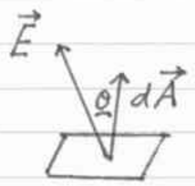
$$\Updownarrow$$

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \qquad \text{Coulomb's Law}$$

2. Surface Element, Electric Flux Element, Electric Flux

Choose a volume
↓
surface enclose this volume.

Divide the surface into small surfaces
↓
small enough to be considered as a plane element



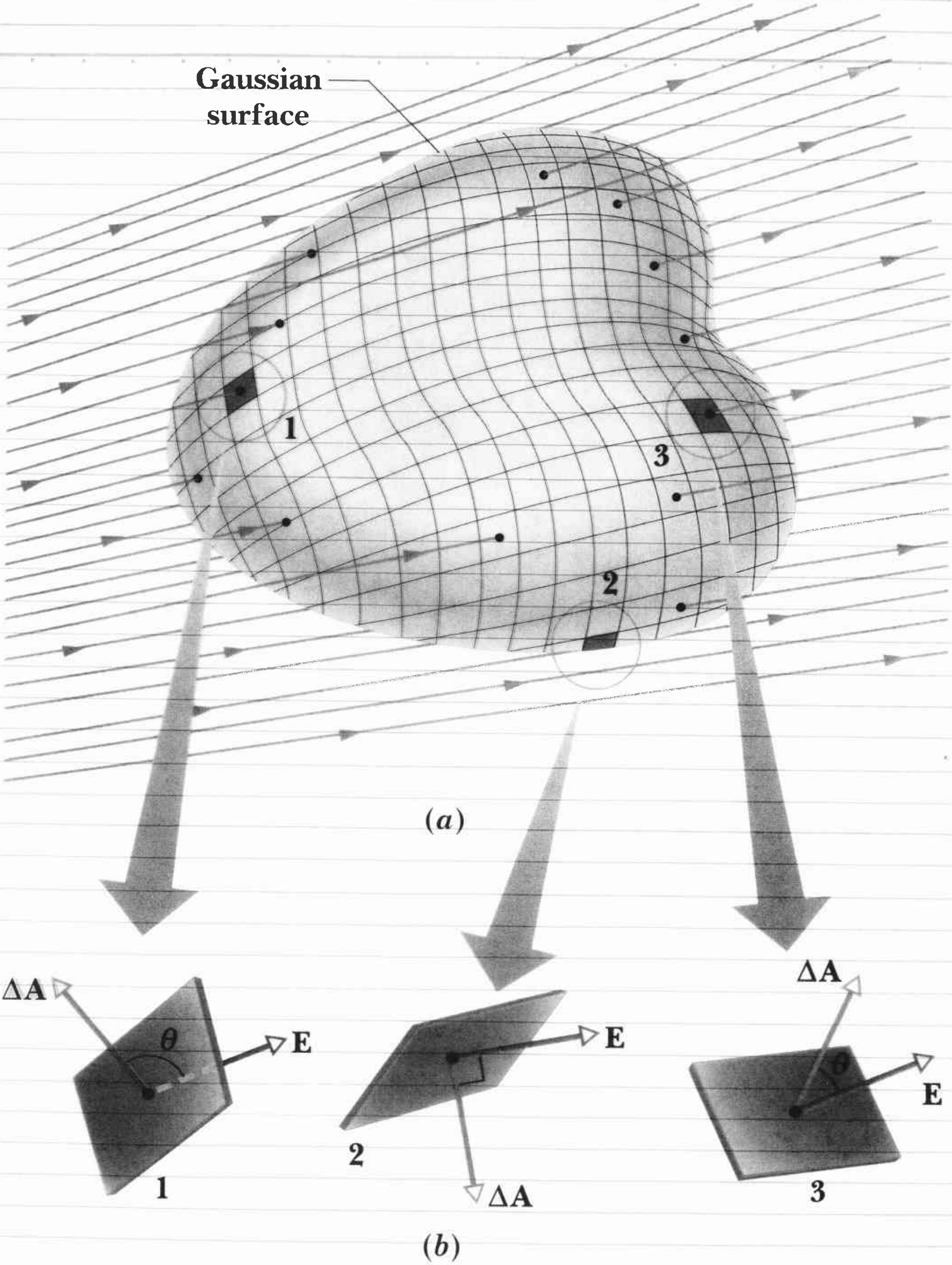
$d\vec{A}$ magnitude = area of the small surface dA
direction: ⊥ to the surface
sign: outward from the volume.
↓
surface element

$\vec{E}$ is the electric field at the surface element.
↓
the surface element is small enough
⇒ $\vec{E}$ is uniquely defined.

$d\Phi_E$ = electric field flux element
↓ we shall suppress the sub-script

$= \vec{E} \cdot d\vec{A}$ (scalar product between two vectors)

$= |\vec{E}||d\vec{A}| \cos\theta$

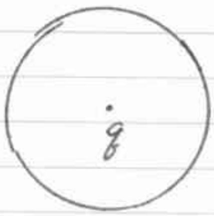

Gaussian surface
1
2
3
(a)
ΔA
θ
E
1
E
2
ΔA
ΔA
θ
E
3
(b)

$$\Phi = \oint d\phi$$

↓ over the whole surface.

Now the left-hand side of the Gauss' law is specified.

3. A Simple Example



A charge q is located at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$\hat{r}$ = unit vector, going radially outward.

Take a sphere with radius r

$$\vec{E}\cdot d\vec{A} = |\vec{E}||d\vec{A}|\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\, dA$$

$\vec{E}$, $d\vec{A}$ is along the same direction $\Rightarrow \cos\theta = 1$

$$\oint \vec{E}\cdot d\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\oint dA$$

↓ $4\pi r^2$

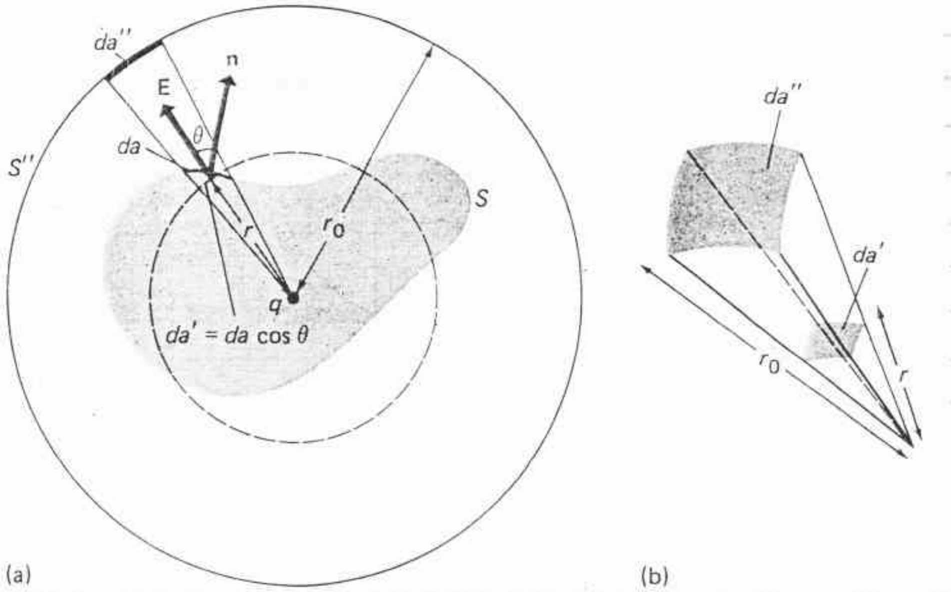
$$= \frac{q}{\epsilon_0}$$

- Provide an example of calculating the electric flux (independent of r)
- Show that in this simple case, we can prove

Coulomb's Law ⟶ Gauss' Law.

4. Coulomb's Law → Gauss' Law

- Calculation of the electric flux arising from a point charge inside an arbitrary surface S



Calculation of the electric flux arising from a point charge inside an arbitrary closed surface S.

$$da' = da\cos\theta$$

$$d\Phi = E\,da\cos\theta = E\,da'$$

$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

$$\frac{da'}{da''} = \frac{r^2}{r_0^2}$$

$$d\Phi = \frac{q}{4\pi\epsilon_0 r^2}\,da' = \frac{q}{4\pi\epsilon_0 r^2}\,da''\cdot\frac{r^2}{r_0^2}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{r_0^2}\,da''$$

$$\Phi = \oint_S \vec{E}\cdot\hat{n}\,da$$

$$= \oint \frac{q}{4\pi\epsilon_0}\frac{1}{r_0^2}\,da'' = \frac{q}{\epsilon_0}$$

This equation is valid for any q inside the volume.

If there are two charges q_1, q_2 inside the volume

$$\oint_S \vec{E}_1\cdot d\vec{A} = \frac{q_1}{\epsilon_0}$$ $\vec{E}_1$ is the electric field produced by q_1

$$\oint_S \vec{E}_2 \cdot d\vec{A} = \frac{q_2}{\epsilon_0}$$ $\vec{E}_2$ is the electric field produced by q_2

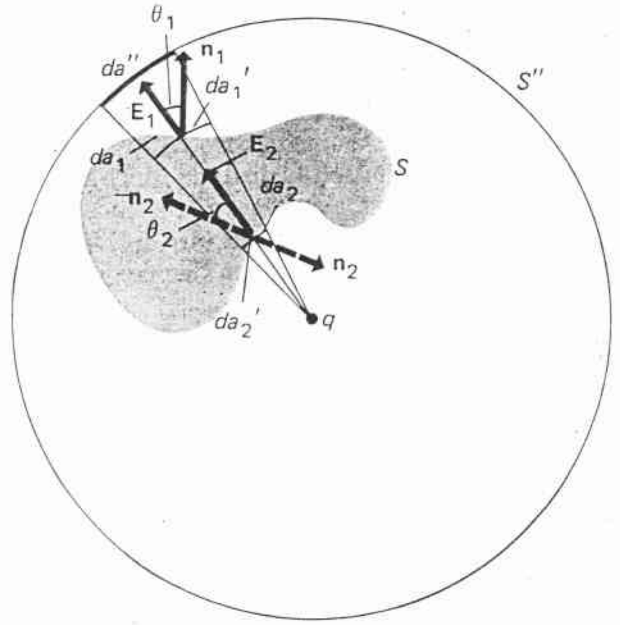
$$\oint_S \vec{E}_1 \cdot d\vec{A} + \oint_S \vec{E}_2 \cdot d\vec{A} = \frac{1}{\epsilon_1}(q_1 + q_2)$$

$\downarrow$ $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0}(q_1 + q_2) = \frac{1}{\epsilon_0} Q$$

$\downarrow$ total Q inside the volume

Calculation of the electric field flux arising from a point charge outside an arbitrary closed surface S



Calculation of the electric flux arising from a point charge outside an arbitrary closed surface S.

$$\vec{E}_a \cdot d\vec{A}_a = \frac{q}{4\pi\epsilon_0 r_0^2} da''$$

$$\vec{E}_b \cdot d\vec{A}_b = \frac{-q}{4\pi\epsilon_0 r_0^2} da''$$

$\downarrow$

$\hat{n}_2$ is pointing outward

$\downarrow$

produces a negative sign

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{A} = 0$$

A point charge outside the closed surface gives no contribution to the electric flux

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

↓

Gauss' Law

To apply Gauss' law, symmetry consideration is essential!

5. Electric Field Produced by a Uniformly Charged Sphere

- $r > R$ $\quad \vec{E}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r}$
- $r < R$ $\quad \vec{E}(\vec{r}) = \dfrac{Qr}{4\pi\epsilon_0 R^3}\hat{r}$

Note $r = R$ $\quad E(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 R^2}$

6. Electric Field Produced by a Charged Shell with Inner Radius R_1 and Outer Radius R_2

- $r > R_2$ $\quad \vec{E}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r}$
- $R_1 < r < R_2$ $\quad \vec{E}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r^2}\dfrac{r^3 - R_1^3}{R_1^3 - R_2^3}\hat{r}$
- $r < R_1$ $\quad \vec{E}(\vec{r}) = 0$

7. Electric Field Produced by a Uniform Line Charge.

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

$\hat{r}$ ↳ radial direction from the line

↓ this result has been obtained before

8. Electric Field Produced by a Uniform Surface Charge.

$$\vec{E}(r) = \frac{\sigma}{2\epsilon_0}\hat{r}$$

$\hat{r}$ ↳ ⊥ to the plane, pointing outward.

Note : the result is independent of d.

9. Given $\vec{E}$ at the Surface , to alculate the Total Charge Enclosed.

10. $\vec{E}$ Field in a Conductor

11. $\vec{E}$ Field Inside a Conducting Cavity.
($\vec{E}$ due to electrostatic charge ⇒ conservative force).

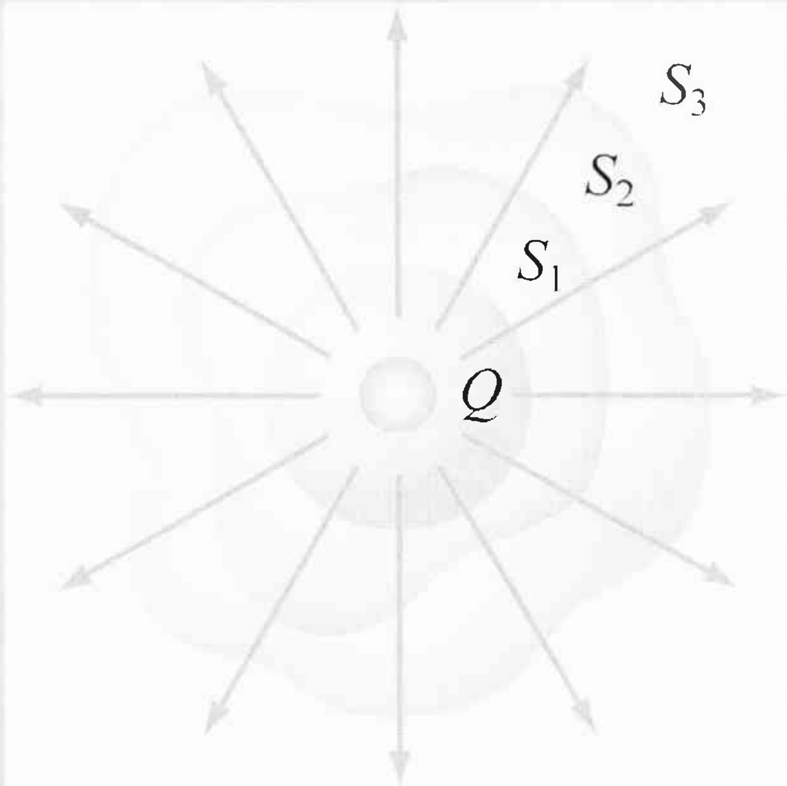
12. Van de Graaf Accelerator

See the lecture notes.

Gauss's Law

The first Maxwell Equation
A very useful computational technique
This is important!

Gauss's Law – The Idea



The total "flux" of field lines penetrating any of these surfaces is the same and depends only on the amount of charge inside

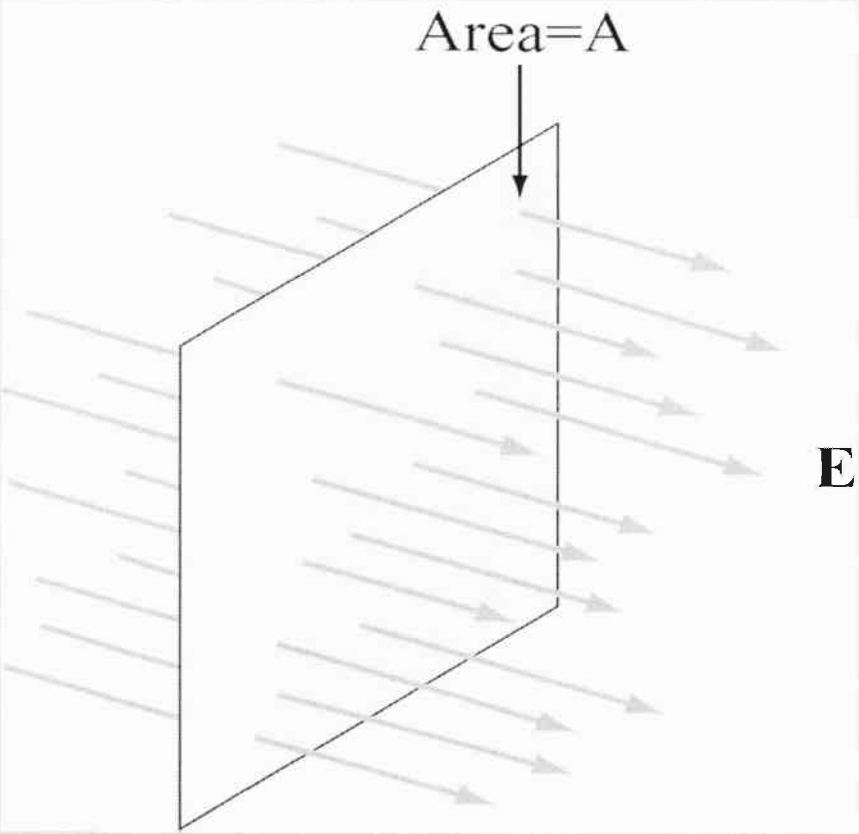
Gauss's Law – The Equation

$$\Phi_E = \underset{\substack{\text{closed}\\\text{surfaceS}}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Electric flux Φ_E (the surface integral of E over closed surface *S*) is proportional to charge inside the volume enclosed by *S*

Electric Flux Φ_E

Case I: E is constant vector field perpendicular to planar surface S of area A



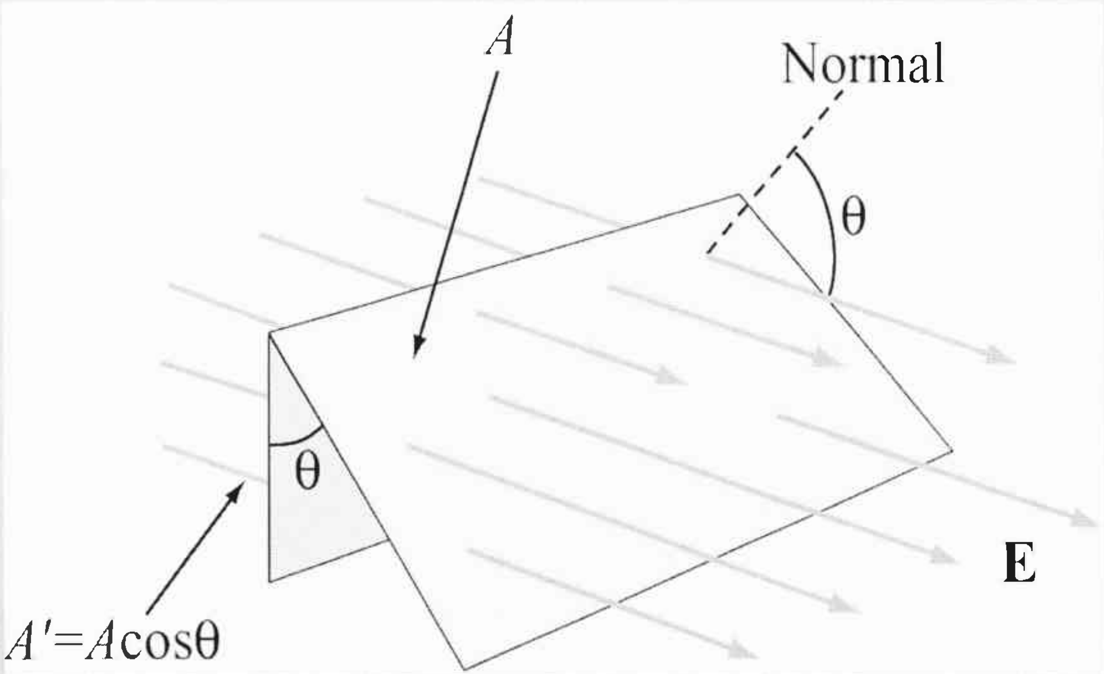
$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = +EA$$

Our Goal: Always reduce problem to this

Electric Flux Φ_E

Case II: E is constant vector field directed at angle θ to planar surface S of area A



$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

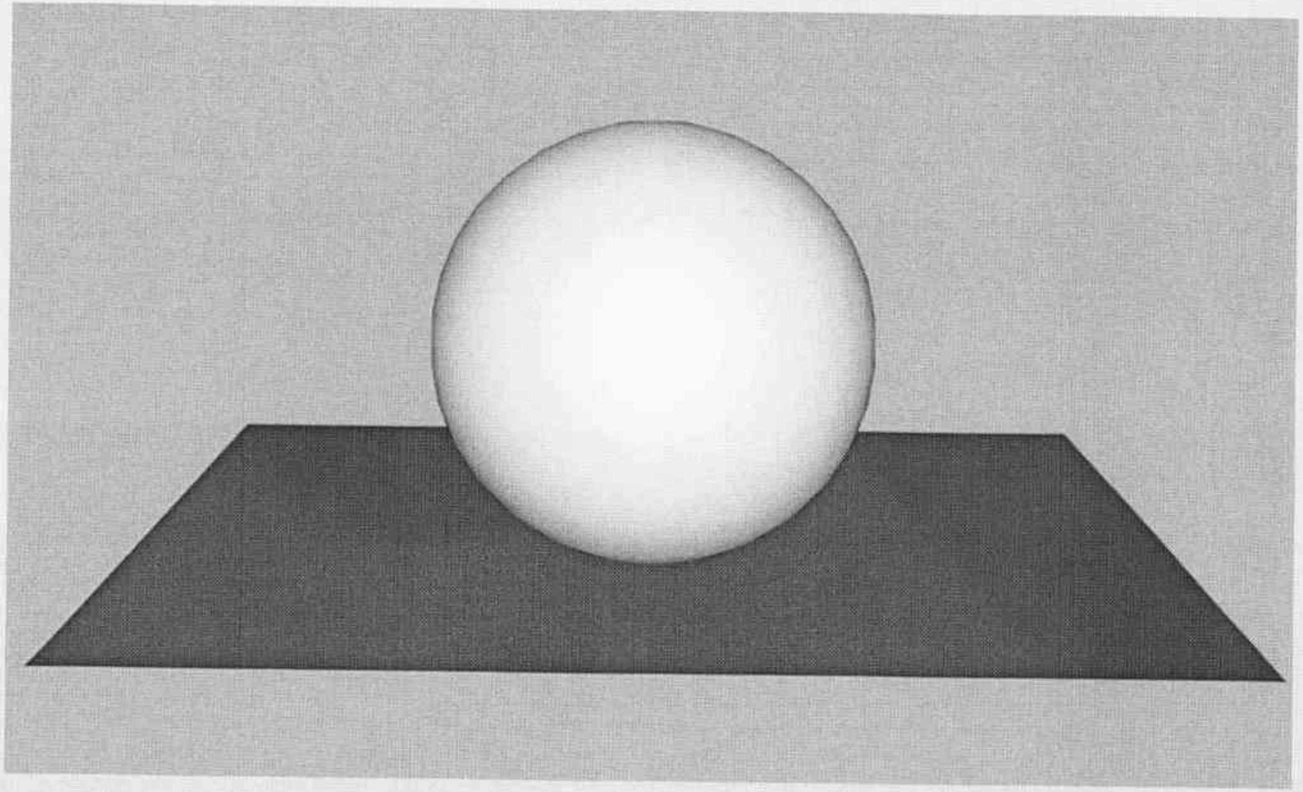
$$\Phi_E = EA\cos\theta$$

Gauss's Law

$$\Phi_E = \underset{\substack{\text{closed} \\ \text{surfaceS}}}{\oint\!\!\!\!\iint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Note: Integral must be over closed surface

Open and Closed Surfaces

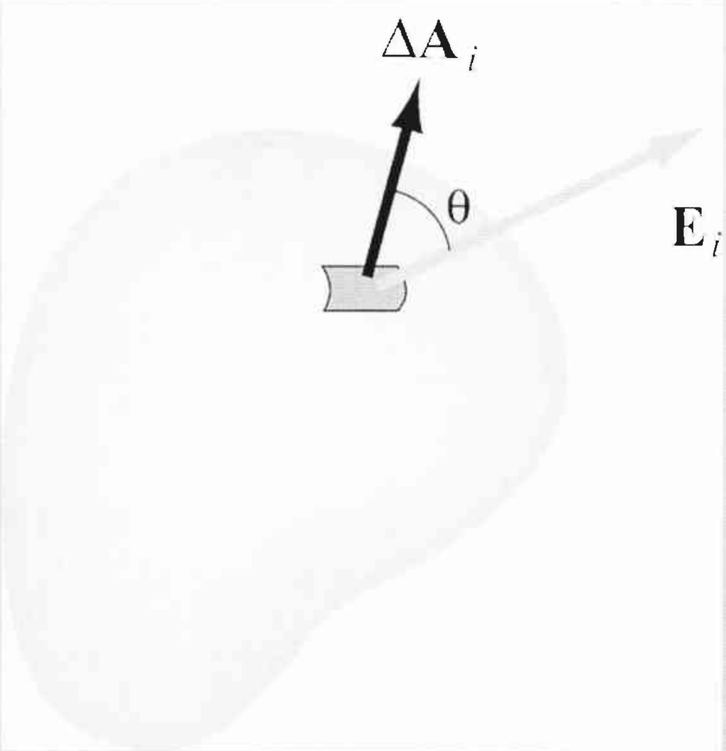


A rectangle is an open surface — it does NOT contain a volume

A sphere is a closed surface — it DOES contain a volume

Area Element dA: Closed Surface

For closed surface, d**A** is normal to surface and points outward
(from inside to outside)

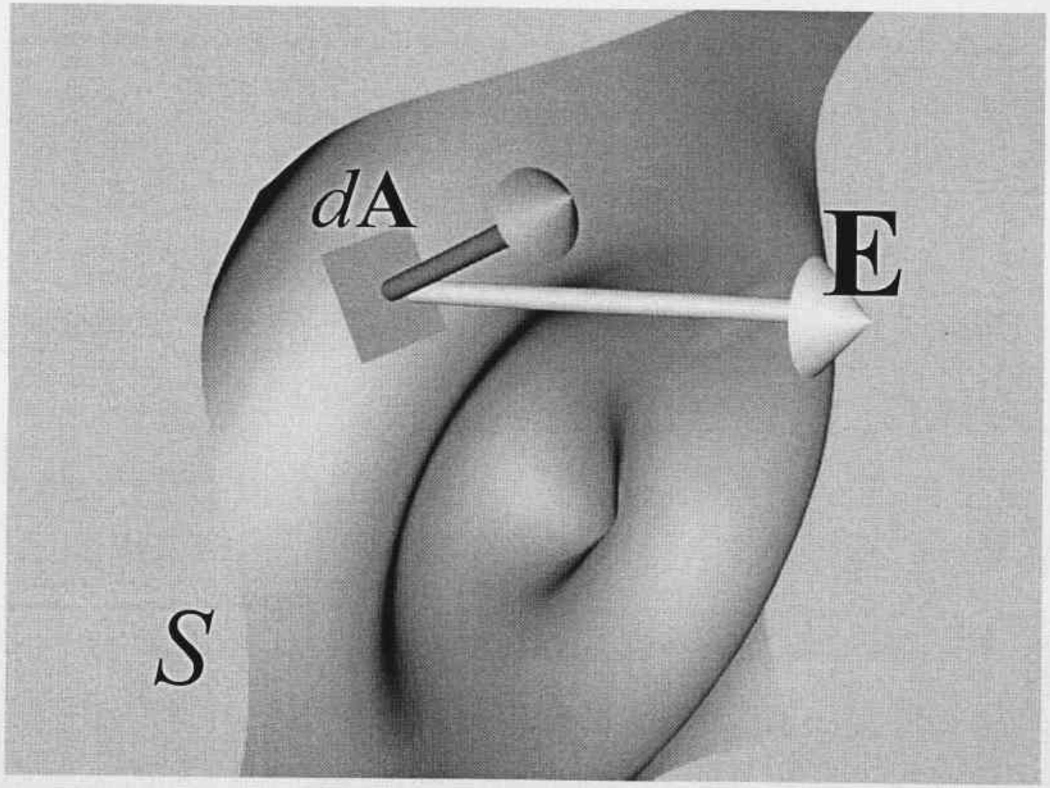


$\Phi_E > 0$ if E points out

$\Phi_E < 0$ if E points in

Electric Flux Φ_E

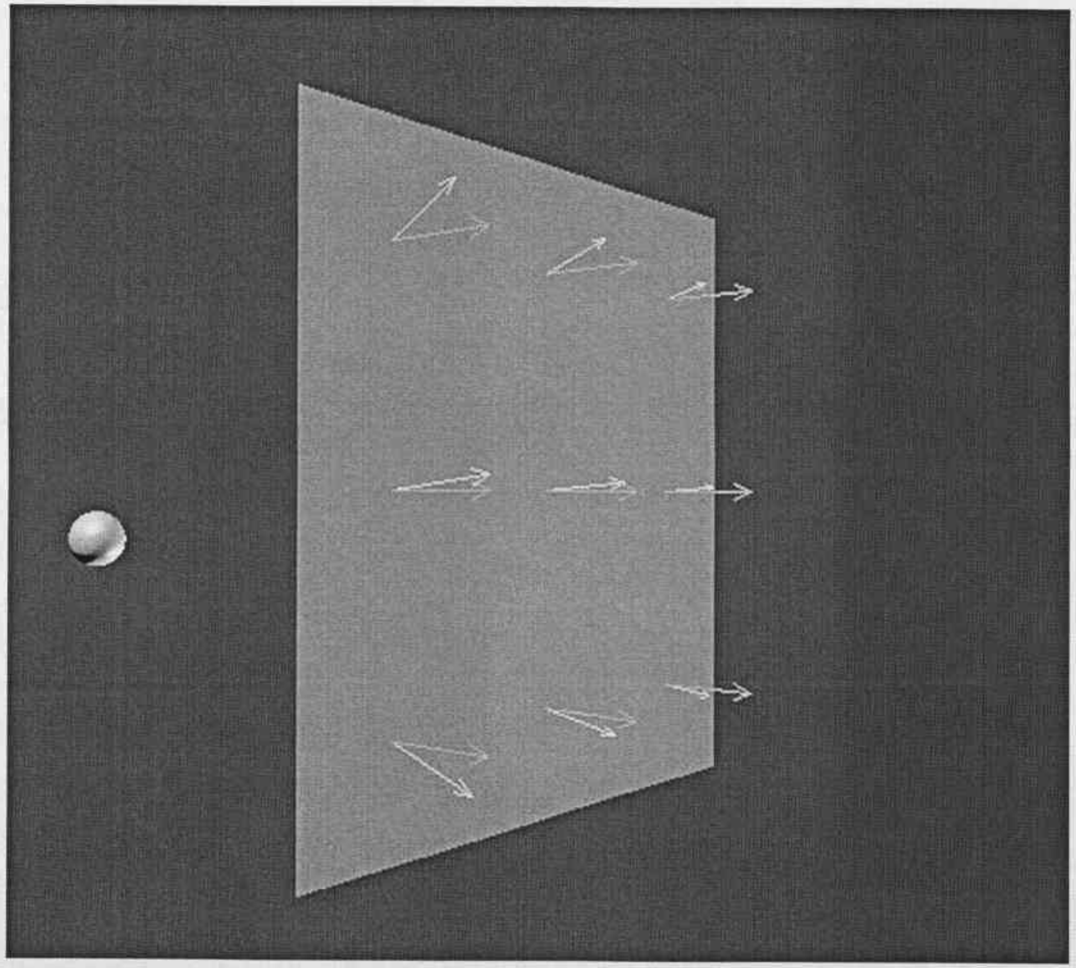
Case III: E not constant, surface curved



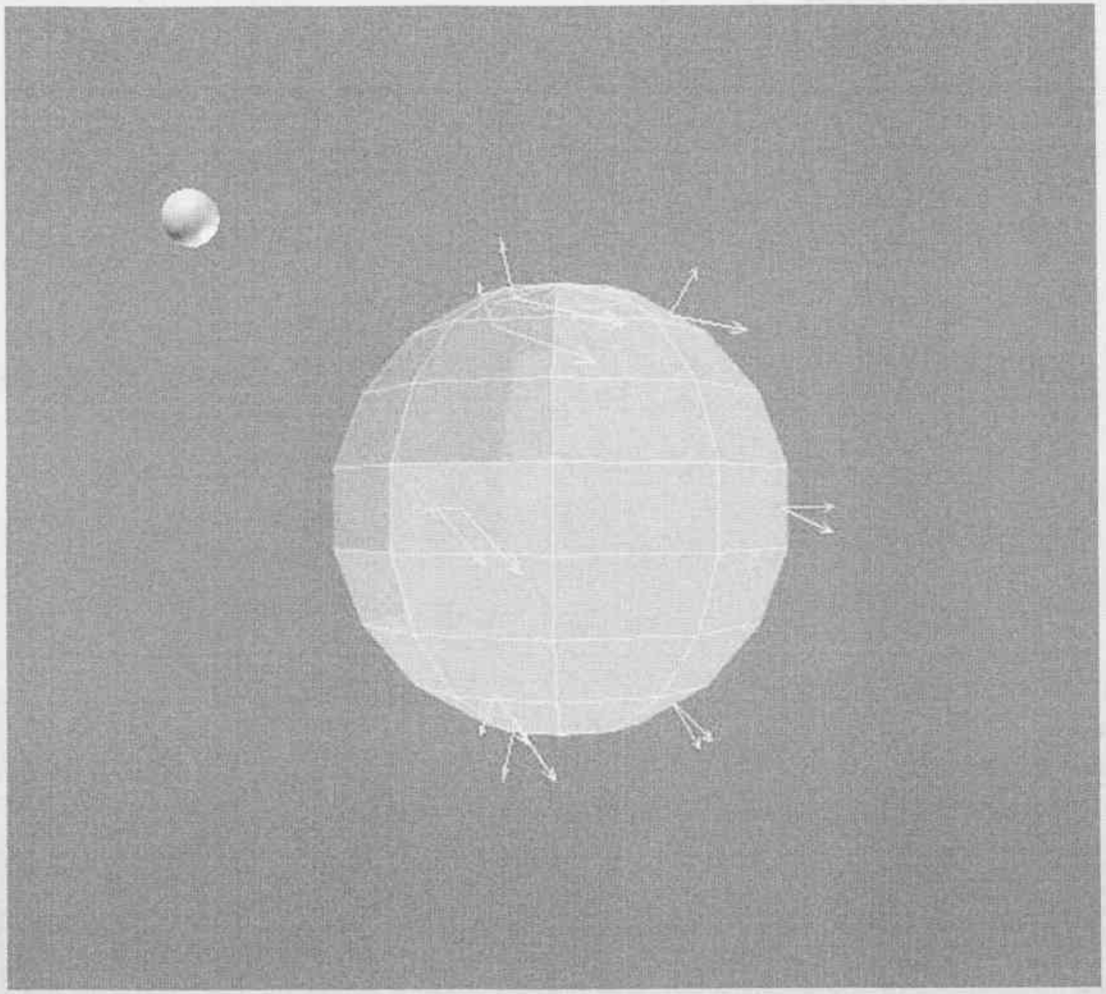
$$d\Phi_E = \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = \iint d\Phi_E$$

Example: Point Charge Open Surface

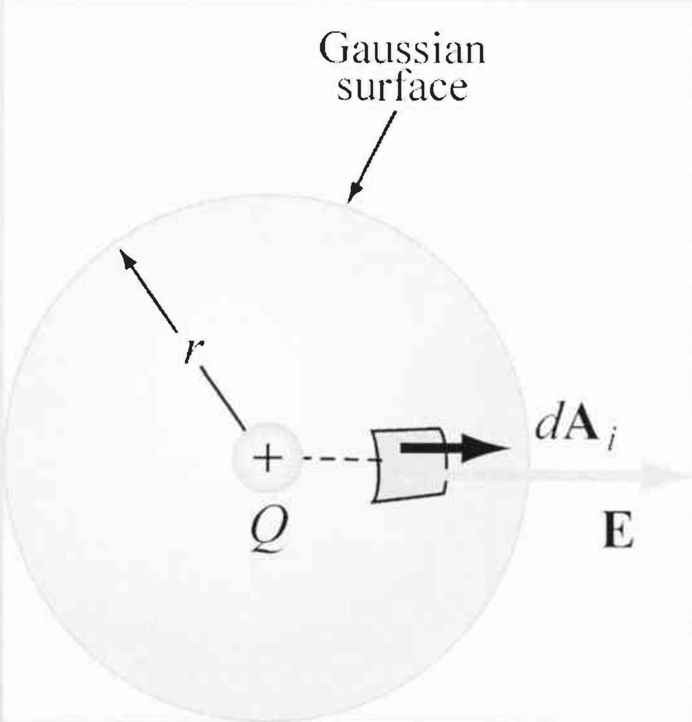


Example: Point Charge Closed Surface



Electric Flux: Sphere

Point charge Q at center of sphere, radius r



E field at surface:

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}$$

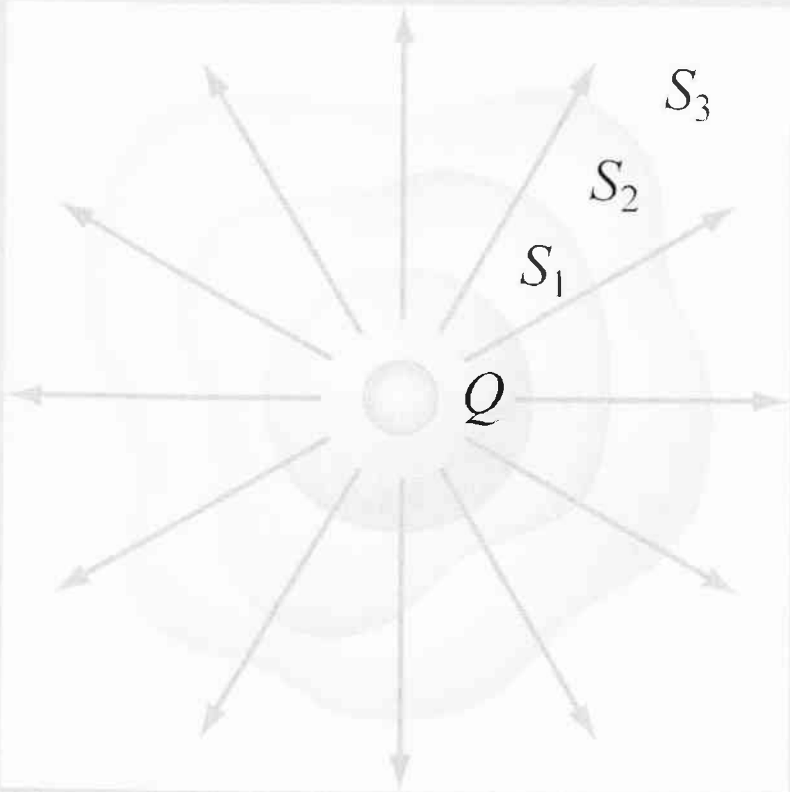
Electric flux through sphere:

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oiint_S \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \cdot dA\,\hat{\mathbf{r}}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2} \oiint_S dA = \frac{Q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$d\vec{\mathbf{A}} = dA\,\hat{\mathbf{r}}$$

Arbitrary Gaussian Surfaces



$$\Phi_E = \underset{\text{closed surface S}}{\oint\!\!\!\int} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0}$$

For all surfaces such as S_1, S_2 or S_3

Applying Gauss's Law

1. Identify regions in which to calculate E field.
2. Choose Gaussian surfaces S: Symmetry
3. Calculate $\Phi_E = \underset{S}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
4. Calculate q_{in}, charge enclosed by surface S
5. Apply Gauss's Law to calculate E:

$$\Phi_E = \underset{\text{closed surface} S}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Choosing Gaussian Surface

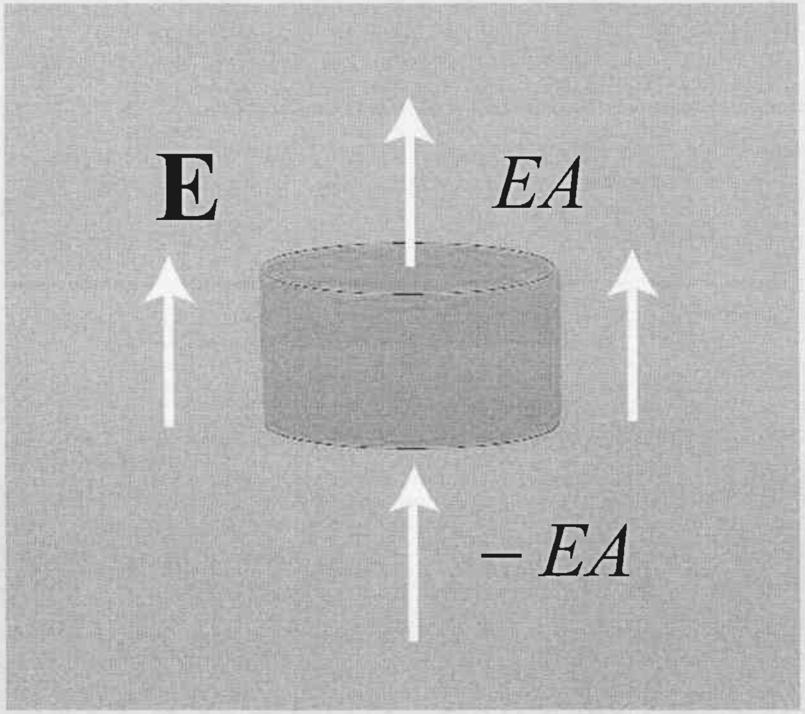
Choose surfaces where **E** is perpendicular & constant.

Then flux is EA or -EA.

OR

Choose surfaces where **E** is parallel.

Then flux is zero



Example: Uniform Field

Flux is EA on top

Flux is –EA on bottom

Flux is zero on sides

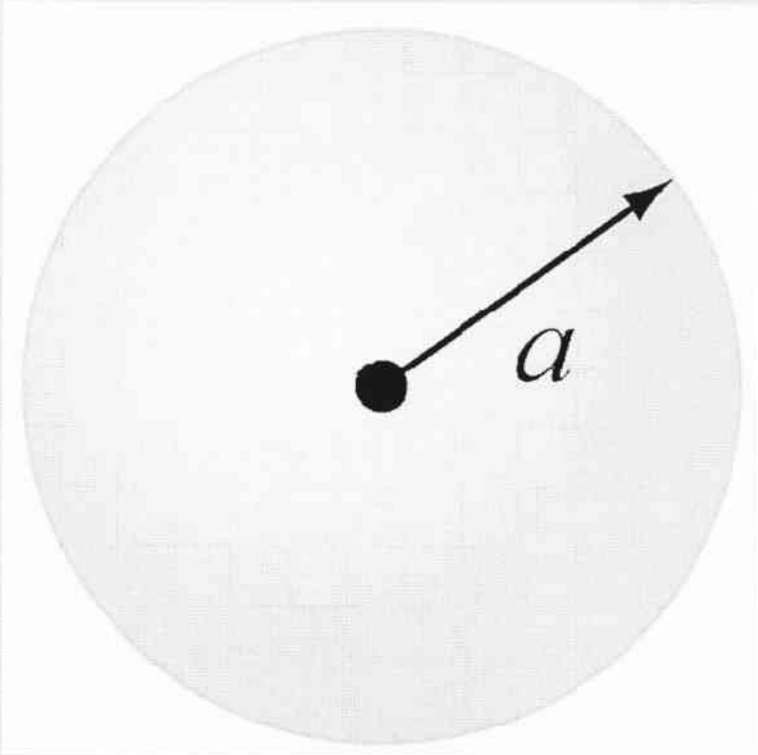
Symmetry & Gaussian Surfaces

Use Gauss's Law to calculate E field from highly symmetric sources

Symmetry	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

Gauss: Spherical Symmetry

+Q uniformly distributed throughout non-conducting solid sphere of radius a. Find **E** everywhere

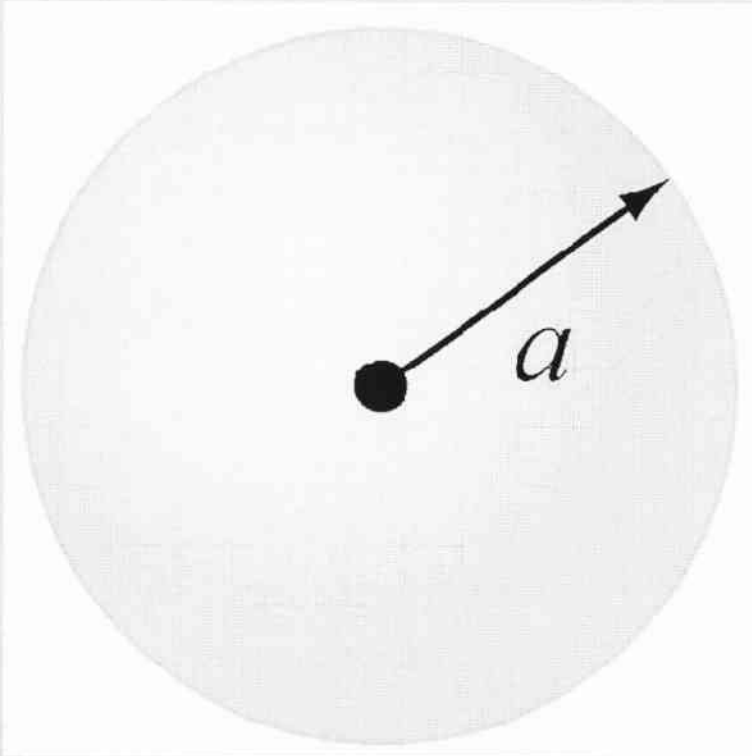


Gauss: Spherical Symmetry

Symmetry is Spherical

$$\vec{\mathbf{E}} = E\,\hat{\mathbf{r}}$$

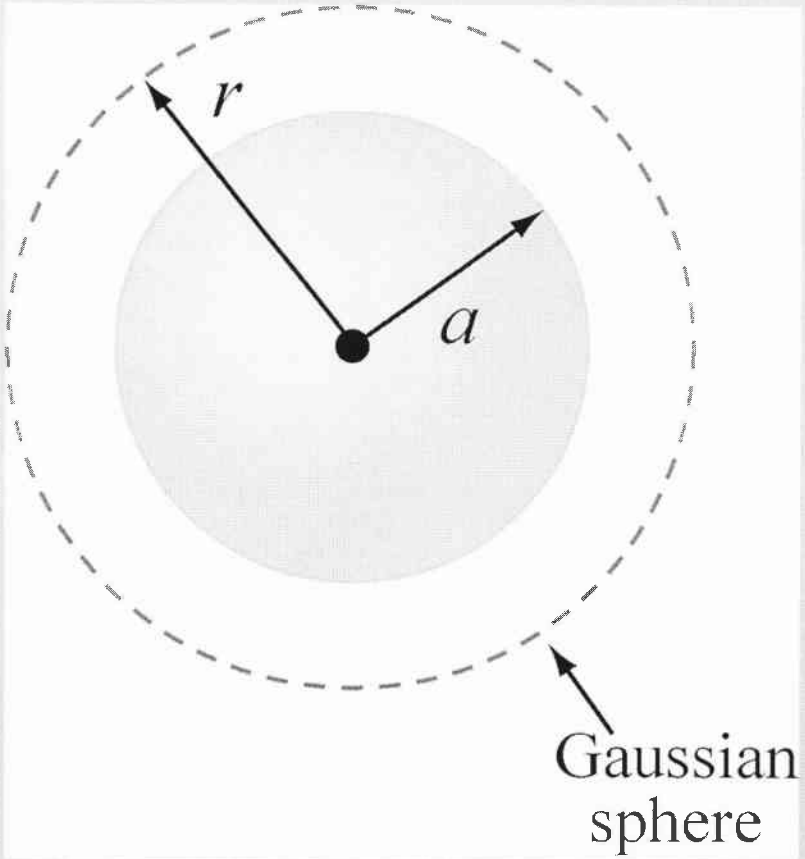
Use Gaussian Spheres



Gauss: Spherical Symmetry

Region 1: $r > a$

Draw Gaussian Sphere in Region 1 ($r > a$)



Note: r is arbitrary **but** is the radius for which you will calculate the E field!

Gauss: Spherical Symmetry

Region 1: $r > a$

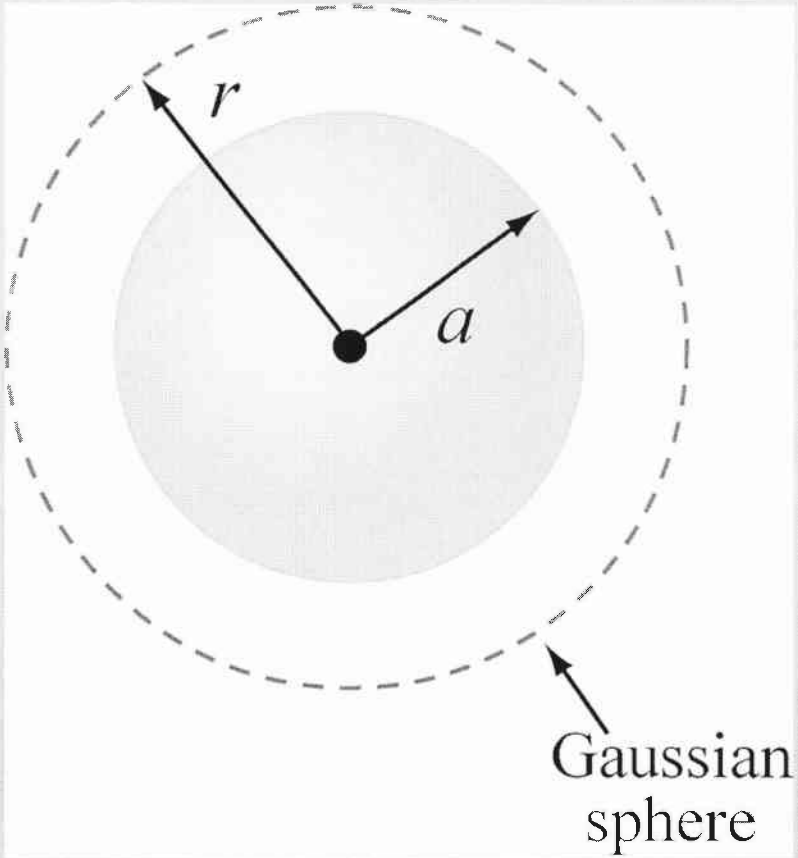
Total charge enclosed $q_{in} = +Q$

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA$$

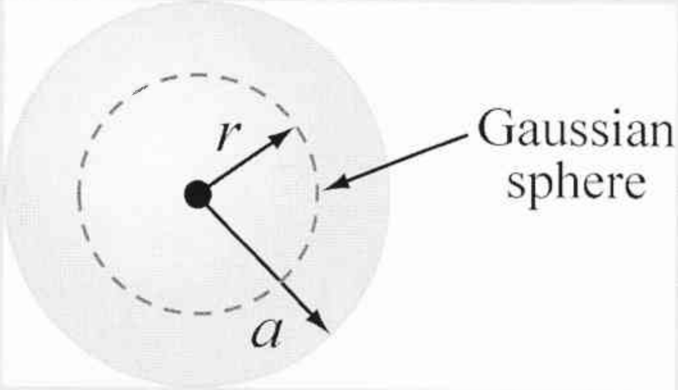
$$= E\left(4\pi r^2\right)$$

$$\Phi_E = 4\pi r^2 E = \frac{q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$



Gauss: Spherical Symmetry



<u>Region 2</u>: $r < a$

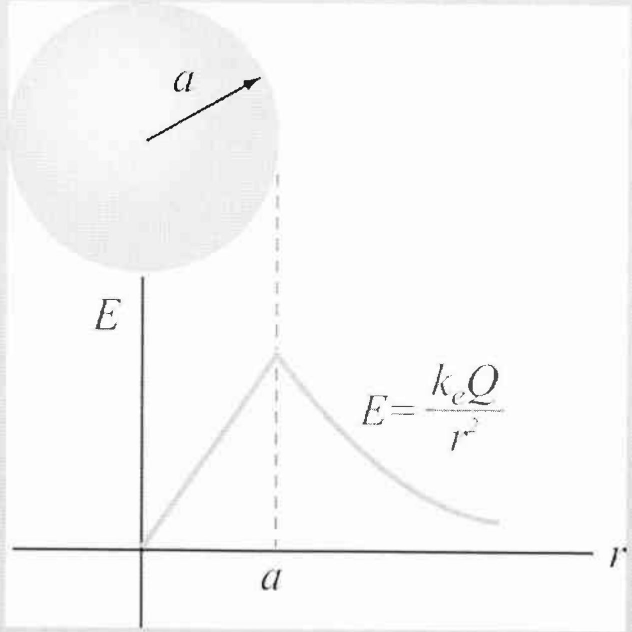
Total charge enclosed:

$$q_{in} = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3} \right) Q = \left(\frac{r^3}{a^3} \right) Q \quad \text{OR} \quad q_{in} = \rho V$$

Gauss's law:

$$\Phi_E = E\left(4\pi r^2\right) = \frac{q_{in}}{\varepsilon_0} = \left(\frac{r^3}{a^3} \right) \frac{Q}{\varepsilon_0}$$

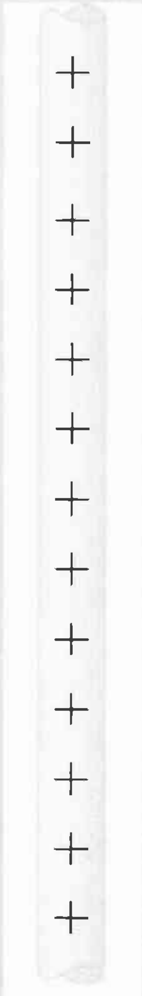
$$E = \frac{Q}{4\pi\varepsilon_0} \frac{r}{a^3} \Rightarrow \boxed{\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0} \frac{r}{a^3} \hat{\mathbf{r}}}$$



Gauss: Cylindrical Symmetry

Infinitely long rod with uniform charge density λ

Find **E** outside the rod.



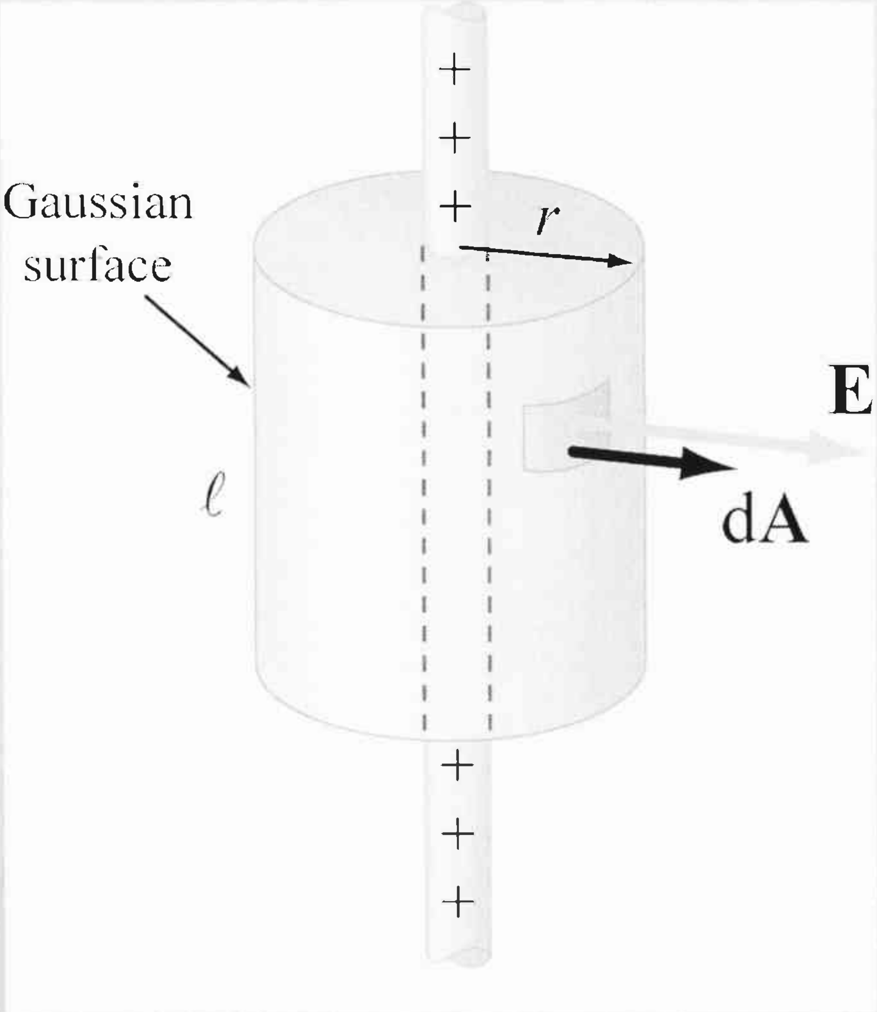
Gauss: Cylindrical Symmetry

Symmetry is Cylindrical

$$\vec{\mathbf{E}} = E\,\hat{\mathbf{r}}$$

Use Gaussian Cylinder

Note: *r* is arbitrary **but** is the radius for which you will calculate the E field!
ℓ is arbitrary and should divide out



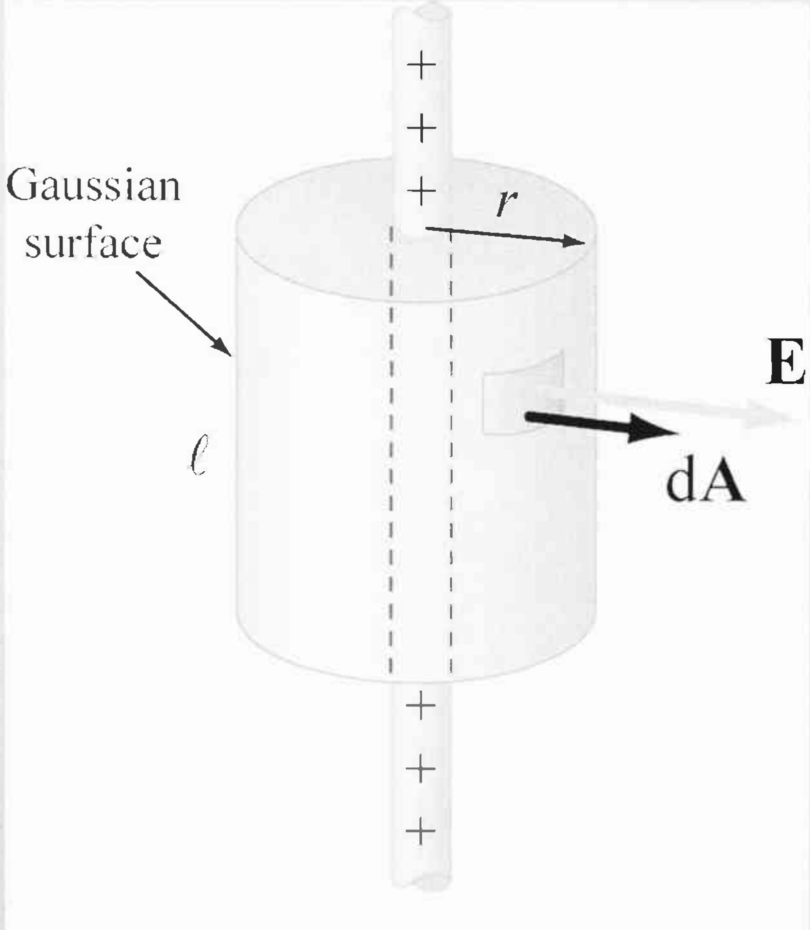
Gauss: Cylindrical Symmetry

Total charge enclosed: $q_{in} = \lambda\ell$

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA$$

$$= E\left(2\pi r\ell\right) = \frac{q_{in}}{\varepsilon_0} = \frac{\lambda\ell}{\varepsilon_0}$$

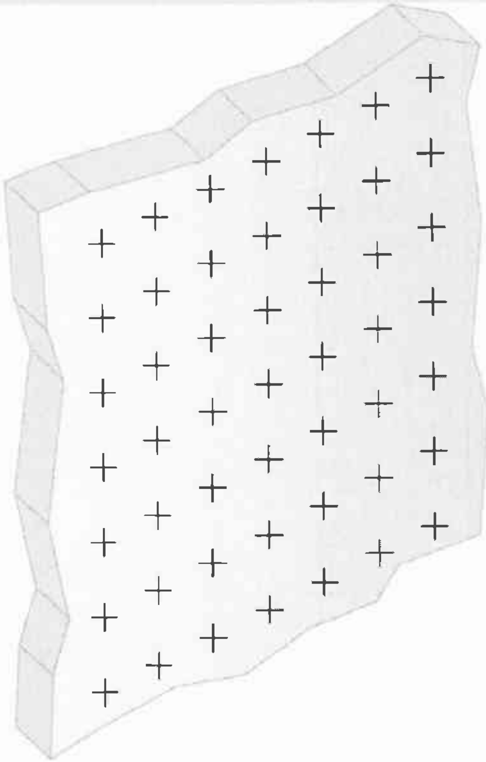
$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \boxed{\Rightarrow \vec{\mathbf{E}} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}}$$



Gauss: Planar Symmetry

Infinite slab with uniform charge density σ

Find **E** outside the plane



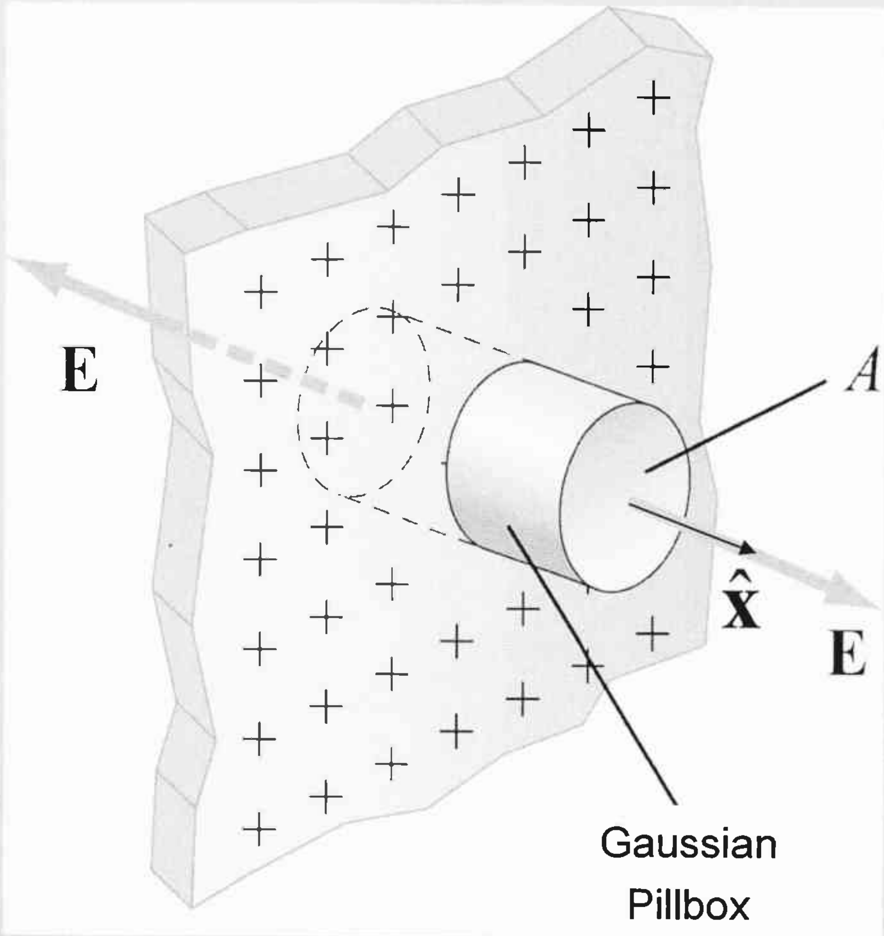
Gauss: Planar Symmetry

Symmetry is Planar

$$\vec{\mathbf{E}} = \pm E\,\hat{\mathbf{x}}$$

Use Gaussian Pillbox

Note: *A* is arbitrary (its size and shape) and should divide out



Gauss: Planar Symmetry

Total charge enclosed: $q_{in} = \sigma A$

NOTE: No flux through side of cylinder, only endcaps

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA_{Endcaps}$$

$$= E(2A) = \frac{q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{\mathbf{E}} = \frac{\sigma}{2\varepsilon_0} \begin{cases} \hat{\mathbf{x}} & \text{to right} \\ -\hat{\mathbf{x}} & \text{to left} \end{cases}$$