Chapter 24

Electric Field

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3} \hat{r}$$

$$\downarrow$$

$$\nabla \times \vec{F} = 0 \quad \Rightarrow \quad \nabla \times \vec{E} = 0$$

$$\Downarrow$$

Electric field
is conservative

Chapter 24

Electric Potential

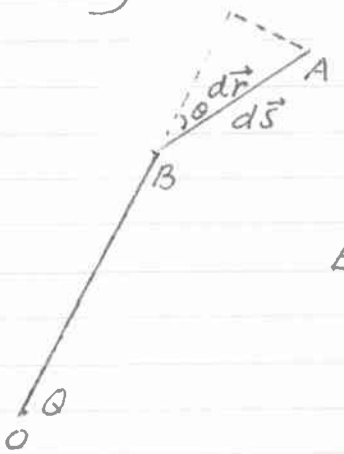
1. Electrostatic Field is Conservative

$\int \vec{E}\cdot d\vec{r}$ is independent of the path.

$\nabla \times \vec{E} = 0$

A. A charge Q at the origin.

(i) Take the straight line path.



$$\vec{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$$\vec{E}\cdot d\vec{s} = |\vec{E}|\underbrace{|ds|\cos\theta}_{dr}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{dr}{r^2}$$

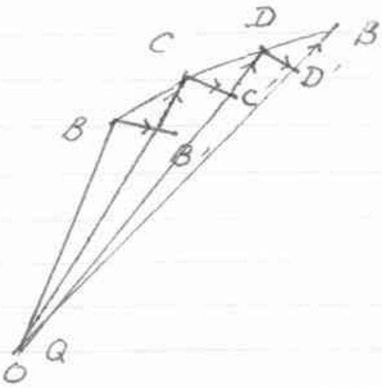
$$\int_B^A \vec{E}\cdot d\vec{s} = -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)_B^A$$

$$= -\frac{Q}{4\pi\epsilon_0 r_A} + \frac{Q}{4\pi\epsilon_0 r_B}$$

$$[\ = V_B - V_A\ ]$$

$$[\ V(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\ ]$$

(ii)



Stairway case

$r_B = r_{B'}$, $r_C = r_{C'}$, $r_D = r_{D'}$

	$\int \vec{E}\cdot d\vec{s}$
BB'	0
$B'C$	$\frac{Q}{4\pi\epsilon_0}\left(\frac{-1}{r_C} + \frac{1}{r_B}\right)$
CC'	0
$C'D$	$\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r_D} + \frac{1}{r_{C'}}\right)$
DD'	0
$D'A$	$\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r_A} + \frac{1}{r_{D'}}\right)$

$$\int_B^A \underset{\downarrow}{\vec{E}}\cdot d\vec{s} = -\frac{Q}{4\pi\epsilon_0 r_A} + \frac{Q}{4\pi\epsilon_0 r_B}$$

along the stair-way case

All pathes can be approximated, as close as one want, by the stairway case

For an electric field produced by a point charge Q located at the origin

$\Longrightarrow \quad \int_B^A \vec{E} \cdot d\vec{s}$ is independent of path

B. A charge Q_1 at $\vec{r}_1$ and a charge Q_2 at $\vec{r}_2$

$$\vec{E}(x, y, z) = \vec{E}(\vec{r})$$

$$= \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) \qquad \text{superposition principle.}$$

$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3}, \quad \vec{E}_2(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$\int_B^A \vec{E} \cdot d\vec{r} = \int_B^A \vec{E}_1 \cdot d\vec{r} + \int_B^A \vec{E}_2 \cdot d\vec{r}$$

$\int_B^A \vec{E}_1 \cdot d\vec{s}$ is independent of path

$\downarrow$

the proof use the same method as in A.

$\downarrow$

with $\vec{r} - \vec{r}_1 \longrightarrow \vec{r}$

One can, use the same method, to show that

$\int_B^A \vec{E}_2 \cdot d\vec{s}$ is independent of path.

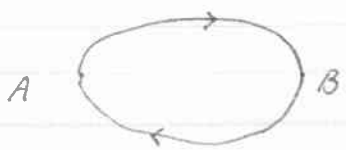
$\Downarrow$ generalization

For $\vec{E}$ produced by static any charge distribution

$\int_B^A \vec{E} \cdot d\vec{s}$ is independent of path.

$\Rightarrow$ Electrostatic Field is conservative

2. For Electrostatic Field $\quad \oint \vec{E} \cdot d\vec{s} = 0$



$$\oint \vec{E} \cdot d\vec{s} = \int_{A \atop r_1}^{B} \vec{E} \cdot d\vec{s} + \int_{B \atop r_2}^{A} \vec{E} \cdot d\vec{s}$$

$$= -\int_{B\,(r_1)}^{A} \vec{E}\cdot d\vec{s} \quad + \int_{B\,(r_2)}^{A} \vec{E}\cdot d\vec{s}$$

Since $\int_B^A \vec{E}\cdot d\vec{s}$ is independent of path,

$\oint \vec{E}\cdot d\vec{s} = 0$ for any closed loop.

3. Electric Potential Field

$$\int_B^A \vec{E}\cdot d\vec{s} = \underset{\substack{\downarrow \\ \text{electric potential field} \\ \text{at point } B.}}{V_B} - V_A$$

For a charge Q located at the origin

$$\int_B^A \vec{E}\cdot d\vec{s} = -\frac{Q}{4\pi\epsilon_0 r_A} + \frac{Q}{4\pi\epsilon_0 r_B}$$

$$= V_B - V_A$$

If B is at ∞
A is at r

$$\underset{\substack{\downarrow \\ \text{taken to be zero}}}{V(\infty)} - V_A = -\frac{Q}{4\pi\epsilon_0 r_A}$$

$$\Rightarrow \quad V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ \text{Electric potential field at } \vec{r} = (x,y,z)}}{V(x,y,z)} = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}$$

For a charge Q_1 at $\vec{r}_1$
Q_2 at $\vec{r}_2$

$$\Rightarrow \quad V(x,y,z) = \frac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|}$$

For Q_i at $\vec{r}_i$

$$\Rightarrow \quad V(x,y,z) = \sum_i \frac{Q_i}{4\pi\epsilon_0|\vec{r}-\vec{r}_i|}$$

For a charge distribution $\rho(x', y', z')$

$\rho(x', y', z')\, dx'\, dy'\, dz'$

$\downarrow$

charge in the volume enclosed by

$x' \to x' + dx'$

$y' \to y' + dy'$

$z' \to z' + dz'$

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x', y', z')\, dx'\, dy'\, dz'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}$$

$V(x, y, z)$ electric potential field is a scalar field $\longrightarrow$ superposition is easier to handle.

4. Equipotential Surface

$V(x, y, z) = V_0$ ← constant

$\downarrow$

equipotential surface (electric)

$\vec{E} \cdot d\vec{s} = 0$ if $d\vec{s}$ is along an equipotential surface

$\vec{E} \perp d\vec{s}$

$\Rightarrow$ $\vec{E}$ electric field $\perp$ to the equipotential surface.

5. Relation between $V(x, y, z)$ and $\vec{E}(x, y, z)$

$$\int_B^A \vec{E} \cdot d\vec{s} = V_B - V_A$$

A $(x + \ x,\ y + \ y,\ z + \ z)$

B $(x,\ y,\ z)$

$$E_x(x, y, z)\ x + E_y(x, y, z)\ y + E_z(x, y, z)\ z$$

$$= V(x, y, z) - V(x + \ x,\ y + \ y,\ z + \ z)$$

Take the special case $y = 0$, $z = 0$

$$E_x(x, y, z)\ x = V(x, y, z) - V(x + \ x,\ y,\ z)$$

$$E_x(x,y,z) = -\left[\frac{V(x+\Delta x, y, z) - V(x,y,z)}{\Delta x}\right]$$

$$\rightarrow -\frac{\partial V(x,y,z)}{\partial x}$$

Same method

$$E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

For a given $V(x,y,z)$

$$\vec{E}(x,y,z) = -\frac{\partial V(x,y,z)}{\partial x}\hat{i} - \frac{\partial V(x,y,z)}{\partial y}\hat{j} - \frac{\partial V(x,y,z)}{\partial z}\hat{k}$$

$$= -\nabla V(x,y,z)$$

To solve an electrostatic problem (i.e., given charge distribution to find $E(x,y,z)$)
we can
(i) Find $V(x,y,z)$ due to the charge distribution
$\downarrow$
this is easier since only scalar quantities are involved.

(ii) Find $\vec{E}(x,y,z)$ from $V(x,y,z)$ by partial differentiation.

6. Uniform Line Charge

λ = line charge density

$$V(x) = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\, dx'}{|x| + x'}$$

For $x' < 0$

$$= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\, dx'}{x' - x}$$

$y = x' - x \qquad dy = dx'$

$x' = 0 \qquad y = -x$

$x' = L \qquad y = L - x$

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-x}^{L-x}\frac{dy}{y}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln(L-x) - \ln(-x)\right]$$

$$-\frac{\partial V}{\partial x} = \frac{-\lambda}{4\pi\epsilon_0}\left[\frac{-1}{L-x} - \frac{-1}{-x}\right]$$

$$\parallel$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{L-x} + \frac{1}{x}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{L}{x(L-x)} < 0$$

toward the left as it should.

7. Uniform Disk

$$dq = \sigma(2\pi R')\,dR'$$

for the ring with radius between R' and $R' + dR'$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R')\,dR'}{\sqrt{z^2+R'^2}}$$

$$V(z) = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{R'\,dR'}{\sqrt{z^2+R'^2}}$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+R^2} - z\right]$$

$$E_z(z) = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{2z}{\sqrt{z^2+R^2}} - 1\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] > 0 \text{ (as it should)}$$

the same as Equation (23-24)

8. Electric Dipole (see the notes)

Electric dipole

$+Q$ located at $(0, 0, \frac{a}{2})$

$-Q$ located at $(0, 0, -\frac{a}{2})$

$$V(\vec{r}) \underset{\hookrightarrow x,y,z}{} = \frac{Q}{4\pi\epsilon_0 \sqrt{x^2+y^2+(z-\frac{a}{2})^2}} - \frac{Q}{4\pi\epsilon_0 \sqrt{x^2+y^2+(z+\frac{a}{2})^2}}$$

$$E_x = -\frac{\partial V}{\partial x}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[(-\tfrac{1}{2})\frac{2x}{[x^2+y^2+(z-\frac{a}{2})^2]^{3/2}} - (-\tfrac{1}{2})\frac{2x}{[x^2+y^2+(z+\frac{a}{2})^2]^{3/2}}\right]$$

$$= \frac{Qx}{4\pi\epsilon_0}\left\{[x^2+y^2+z^2-az+(\tfrac{a}{2})^2]^{-3/2} - [x^2+y^2+z^2+az+(\tfrac{a}{2})^2]^{-3/2}\right\}$$

$$= \frac{Qx}{4\pi\epsilon_0}\left[(r^2-az+(\tfrac{a}{2})^2)^{-3/2} - (r^2+az+(\tfrac{a}{2})^2)^{-3/2}\right]$$

$$= \frac{Qx}{4\pi\epsilon_0}\frac{1}{r^3}\left[\left(1-\frac{az}{r^2}+\frac{(\frac{a}{2})^2}{r^2}\right)^{-3/2} - \left(1+\frac{az}{r^2}+\frac{(\frac{a}{2})^2}{r^2}\right)^{-3/2}\right]$$

$$= \frac{Qx}{4\pi\epsilon_0 r^3}\left[1+\frac{3}{2}\frac{az}{r^2}+\cdots - (1-\frac{3}{2}\frac{az}{r^2}+\cdots)\right]$$

$$= \frac{3Q}{4\pi\epsilon_0}\frac{az}{r^5}x$$

Same method, we can find E_y, E_z

The result is, of course, the same as we obtained in Chapter 23.

9. Electric Potential, Electric Potential Energy

$$V_B - V_A = \int_B^A \vec{E}\cdot d\vec{s}$$

Here we shall be careful with the sign

Example, put a charge Q at the origin

Want to find the electric potential at $\vec{r}$

Use above formula, we shall take A to be at $\vec{r}$
B to be at ∞

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$$

$$V(\infty) - V(\vec{r}) = \int_\infty^r \frac{Q}{4\pi\epsilon_0}\frac{1}{r'^2}\,dr'$$

There are two − sign involved

$$\vec{E}\cdot d\vec{s} = -\frac{Q}{4\pi\epsilon_0}\frac{1}{r'^2}\,|dr'|$$

dr' is >0 when r' is increasing

$$V(\infty) - V(\vec{r}) = \int_\infty^r \frac{Q}{4\pi\epsilon_0}\frac{1}{r'^2}\,dr'$$

$$= \frac{-Q}{4\pi\epsilon_0}\frac{1}{r'}\Big|_\infty^{\vec{r}}$$

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

Potential Energy

For a particle with charge q moving from ∞ to r, the change in potential energy is

$$U(\vec{r}) - \underbrace{U(\infty)}_{\text{taken to be zero}} = q\,V(\vec{r}) - q\,\underbrace{V(\infty)}_{=0}$$

$$U(\vec{r}) = q\,V(\vec{r})$$

For hydrogen atom, with proton fixed at the origin, the electron potential energy at $\vec{r}$ is

$$U(\vec{r}) = -\frac{e^2}{4\pi\epsilon_0 r}$$

$$E = \underbrace{\frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r}}$$

the starting point for Bohr model.

For charge Q_i at $\vec{r}_i$, the electric potential at $\vec{r}$ is given by

$$V(\vec{r}) = \sum \frac{Q_i}{4\pi\epsilon_0|\vec{r}-\vec{r}_i|}$$

The potential energy (electric) for a charge q at $\vec{r}$ is given by

$$U(\vec{r}) = q V(\vec{r})$$
$$= q \sum \frac{Q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|}$$

For continuous distribution, for a charge q at $\vec{r}$ is given by

$$U(\vec{r}) = q \iiint \frac{\rho(x', y', z')\, dx'dy'dz'}{4\pi\epsilon_0 \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

10. Conductor and Electric Potential

Conductor as a whole is equal potential.

Inside the conductor $\vec{E} = 0 \Rightarrow$ equal potential.

Along the surface $\vec{E} \perp d\vec{s} \Rightarrow$ equal potential

An uncharged conductor suspended in an external field. $\vec{E}_1$

The charges in the conductor must rearrange themselve so that it will produce an electric field $\vec{E}_2$

$$\Rightarrow \quad \vec{E} = \underbrace{\vec{E}_1}_{\text{everywhere}} + \vec{E}_2$$

such that $\vec{E}_1 + \vec{E}_2 = 0$ inside the conductor

Fig. 25-21

- For an external electric field along x direction positive charge will be concentrated toward the $+x$ direction of the surface of the conductor negative charge will be concentrated toward the $-x$ direction of the surface of the conductor.
- $E_2 \to 0$ as $x \to \infty$ (follow from Coulomb's law)
  $\vec{E} \to \vec{E}_1$ as $x \to \infty$