Chapter 25

Electric Potential Energy

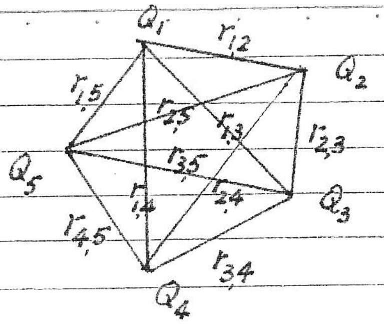
The Potential Energy $\mathcal{E}$ of a Set of Point Charges

Electrostatic field is conservative

$\Rightarrow$ Electric potential

$\downarrow$

Electric potential energy



N point charge

The potential energy are defined to be zero when the charges are infinitely removed from each other.

The potential energy of the system is amount of work needed to assemble this collection of particles.

Bring the charge Q_1 to r_1 needs no work

$$\mathcal{E}_1 = \frac{Q_1}{4\pi\epsilon_0}(0+0+\cdots\ 0)$$

Bring the charge Q_2 to r_2 need work $= \dfrac{Q_2}{4\pi\epsilon_0}\dfrac{1}{r_{12}}$

The force on $Q_2 = Q_2\vec{E}_{12}$

Need force $-Q_2\vec{E}_{12}$ to opposite the electric field

$$W_{12} = -\int_{\infty}^{r_{12}} Q_2 \frac{1}{4\pi\epsilon_0}\frac{Q_1}{r^2}\,\hat{r}\cdot d\vec{r}$$

$\hookrightarrow$ done by the external force

$$= \frac{Q_2}{4\pi\epsilon_0}\frac{Q_1}{r_{12}} = Q_2 V_{12}$$

$$\mathcal{E}_2 = \frac{Q_2}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{12}} + 0 + 0 + \cdots\ 0\right)$$

$\Rightarrow$ Total potential energy $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots\ \mathcal{E}_N$$

$$= \frac{Q_1}{4\pi\epsilon_0}(0+0+\cdots\cdots+0)$$

$$+ \frac{Q_2}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{12}} + 0 + \cdots\ 0\right)$$

$$+ \frac{Q_3}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{1,3}} + \frac{Q_2}{r_{2,3}} + \qquad \right)$$

$$+ \cdots + \frac{Q_N}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{1,N}} + \frac{Q_2}{r_{2,N}} + \cdots \frac{Q_{N-1}}{r_{N-1,N}} + 0\right)$$

However, we can assemble the charges in reverse order and do the same calculation to find $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_N + \mathcal{E}_{N-1} + \cdots + \mathcal{E}_3 + \mathcal{E}_2 + \mathcal{E}_1)$$

$$= \frac{Q_N}{4\pi\epsilon_0}(0 + 0 + \cdots + 0)$$

$$+ \frac{Q_{N-1}}{4\pi\epsilon_0}\left(\frac{Q_N}{r_{N-1,N}} + 0 + 0 + \cdots 0\right)$$

$$+ \cdots$$

$$+ \frac{Q_3}{4\pi\epsilon_0}\left(\frac{Q_N}{r_{3,N}} + \frac{Q_{N-1}}{r_{3,N-1}} + \cdots \frac{Q_4}{r_{3,4}} + 0 + 0 + 0\right)$$

$$+ \frac{Q_2}{4\pi\epsilon_0}\left(\frac{Q_N}{r_{2,N}} + \frac{Q_{N-1}}{r_{2,N-1}} + \cdots \frac{Q_4}{r_{2,4}} + \frac{Q_3}{r_{2,3}} + 0 + 0\right)$$

$$+ \frac{Q_1}{4\pi\epsilon_0}\left(\frac{Q_N}{r_{1,N}} + \frac{Q_{N-1}}{r_{1,N-1}} + \cdots + \frac{Q_3}{r_{1,3}} + \frac{Q_2}{r_{1,2}} + 0\right)$$

Add these two results

$$2\mathcal{E} = \frac{Q_1}{4\pi\epsilon_0}\left(0 + \frac{Q_2}{r_{1,2}} + \frac{Q_3}{r_{1,3}} + \cdots \frac{Q_N}{r_{1,N}}\right)$$

$$+ \frac{Q_2}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{1,2}} + 0 + \frac{Q_3}{r_{2,3}} + \cdots \frac{Q_N}{r_{2,N}}\right)$$

$$+ \cdots$$

$$+ \frac{Q_N}{4\pi\epsilon_0}\left(\frac{Q_1}{r_{1,N}} + \frac{Q_2}{r_{2,N}} + \frac{Q_3}{r_{3,N}} + \cdots \frac{Q_{N-1}}{r_{N-1,N}} + 0\right)$$

$$= \sum_{i=1}^{N} Q_i V_i$$

$$\Rightarrow \quad \mathcal{E} = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

Comments.

(i) Note the factor of $\frac{1}{2}$

(ii) The potential energy does not include the energy required to assemble the individual charge themselve

(iii) Can be positive, negative, or zero

2-24 The Potential Energy of a Continuous Charge Distribution

For a continuous electric charge distribution

$$Q_i \to \rho \, dV \quad \text{for volume charge}$$

$$\mathcal{E} = \frac{1}{2}\int \rho(\vec{r})\, V(\vec{r})\, \underbrace{dV}_{d^3r}$$

$$Q_i \to \sigma \, dS \quad \text{for surface charge}$$

$$\mathcal{E} = \frac{1}{2}\int \sigma \cdot V \, dS$$

The potential energy expressed in terms of $\vec{E}$

$$\nabla\cdot(V\vec{E}) = \nabla V\cdot\vec{E} + V\,\nabla\cdot\vec{E}$$

$$\int \nabla\cdot(V\vec{E})\, dV = \int \nabla V\cdot\vec{E}\, dV + \int V\,\nabla\cdot\vec{E}\, dV$$

$$\downarrow$$

$$\int V\vec{E}\cdot d\vec{S}$$

$$\downarrow$$

take the surface to be a spherial shell with radius $r\to\infty$

$$\int \underbrace{\frac{1}{r}}_{V} \cdot \underbrace{\frac{1}{r^2}}_{\vec{E}}\, r^2 \underbrace{d\Omega}_{\sin\theta\, d\theta\, d\phi}$$

$$\to 0$$

$$\Rightarrow \int V \underbrace{\nabla\cdot\vec{E}}_{\frac{\rho}{\epsilon_0}}\, dV = -\int(\nabla V\cdot\vec{E})\, dV = \int |\vec{E}|^2 dV$$

$$\Rightarrow \mathcal{E} = \frac{1}{2}\int V\rho\, dV = \int \frac{1}{2}\underbrace{\epsilon_0 |E|^2}_{\text{energy stored in the field}} dV$$

$\frac{1}{2}\epsilon_0|\vec{E}|^2$ can be considered as electric potential energy density

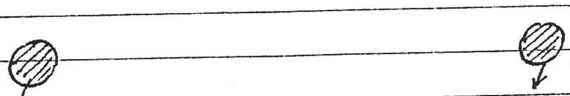
2-25 Discussions on Potential Energy

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$ can be positive, negative, or zero

$$\mathcal{E} = \int \frac{\epsilon_0}{2} |\vec{E}|^2 \, dv$$ is positive definite

the difference is due to the fact that we have now included the energies required to assemble the individual macroscopic charge

This point can be illustrated by the following example



ρ_1 → $\vec{E}_1 = -\nabla V_1$ $\qquad$ ρ_2 → $\vec{E}_2 = -\nabla V_2$

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

principle of superposition

$$\begin{aligned}
\mathcal{E} &= \int \frac{\epsilon_0}{2} |\vec{E}|^2 \, dv \\
&= \int \frac{\epsilon_0}{2} |\vec{E}_1 + \vec{E}_2|^2 \, dv \\
&= \int \frac{\epsilon_0}{2} |\vec{E}_1|^2 \, dv + \int \frac{\epsilon_0}{2} |\vec{E}_2|^2 \, dv + \underbrace{\int \epsilon_0 \vec{E}_1 \cdot \vec{E}_2 \, dv}_{\mathcal{E}^{int}}
\end{aligned}$$

clearly, these two terms represent the energy required to assemble the individual macroscopic charge.

Look at the last term in more detail

$$\begin{aligned}
\int \epsilon_0 \vec{E}_1 \cdot \vec{E}_2 \, dv &= -\epsilon_0 \int \nabla V_1 \cdot \vec{E}_2 \, dv \\
&= -\epsilon_0 \int \nabla \cdot (V_1 \vec{E}_2) \, dv + \epsilon_0 \int V_1 \nabla \cdot \vec{E}_2 \, dv
\end{aligned}$$

$$\int \nabla\cdot(V_1\vec{E}_2)\,dv = \oint V_1\vec{E}_2\cdot d\vec{S}$$

$$\downarrow$$

$$\int \frac{1}{r}\frac{1}{r^2}\, r^2\, d\Omega \quad \text{as } r\to\infty$$

$$\downarrow$$

$$0$$

$$\Rightarrow \quad \mathcal{E}^{int} = \epsilon_0 \int V_1 \frac{\rho_2}{\epsilon_0}\, dv$$

$$= \int V_1\,\rho_2\, dv$$

V_1 is the potential produced by charge distribution ρ_1

$$V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_1(\vec{r}')\,d^3r'}{|\vec{r}-\vec{r}'|}$$

$$\mathcal{E}^{int} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_1(\vec{r}')\,d^3r'\;\rho_2(\vec{r})\,d^3r'}{|\vec{r}-\vec{r}'|}$$

For the case $\rho_1(\vec{r}) = Q_1\,\delta^3(\vec{r}-\vec{r}_1)$

$$\rho_2(\vec{r}) = Q_2\,\delta^3(\vec{r}-\vec{r}_2)$$

$\Rightarrow$ point particle

$$\mathcal{E}^{int} = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{r_{12}}$$

$$= \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\epsilon_0 r_{12}}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 r_{12}}\right)\right]$$

$$= \frac{1}{2}\sum_{i=1}^{2} Q_i\,V_i$$

$$\downarrow$$

this terms correspond to
the interacting potential
energy

For a point particle, we can choose the origin to be

the location of Q_1

$$\int \frac{\epsilon_0}{2}|\vec{E}_1|^2\,dv$$

$$= \frac{\epsilon_0 Q_1^2}{2(4\pi\epsilon_0)^2}\int_0^\infty \left(\frac{1}{r^2}\right)^2 r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{Q_1^2}{2\cdot 4\pi\epsilon_0}\int_0^\infty \frac{1}{r^4}\, r^2\, dr \;\rightarrow\; \infty$$

The self-energy of point charge diverges.

Since, in practice, the point charge are given to us ready-made all we do is to move them around; the self-energy contribution cancel in the energy conservation equation.

it is equivalent to shifting the zero point of the potential energy.

The difference arises from

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{N} q_i V_i \qquad\qquad \mathcal{E} = \frac{1}{2}\int \rho V \, dv$$

V_i: potential due to all the other charges but not q_i

V: full potential produced by all charge distribution

2-26 Uniform Spherical Charge Distribution

(1) $\mathcal{E} = \int \frac{\epsilon_0}{2} |\vec{E}|^2 dv$

$r < R \qquad |\vec{E}| = \dfrac{Qr}{4\pi\epsilon_0 R^3}$

$$\mathcal{E}_I = \frac{\epsilon_0}{2}\int_0^R \left(\frac{Qr}{4\pi\epsilon_0 R^3}\right)^2 4\pi r^2 dr$$

$$= \frac{\epsilon_0}{2}\frac{Q^2}{(4\pi)^2\epsilon_0^2 R^6}\cdot 4\pi \int_0^R r^2 \cdot r^2 dr$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 R^6}\cdot\frac{1}{5}R^5$$

$$= \frac{1}{10}\frac{Q^2}{4\pi\epsilon_0 R}$$

$r > R \qquad |\vec{E}| = \dfrac{Q}{4\pi\epsilon_0 r^2}$

$$\mathcal{E}_{II} = \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{\epsilon_0}{2}\frac{Q^2}{(4\pi)^2\epsilon_0^2} 4\pi \int_R^\infty \frac{1}{r^2} dr$$

$$= \frac{\epsilon_0}{2}\frac{Q^2}{(4\pi)^2\epsilon_0^2} 4\pi\cdot\frac{1}{R} = \frac{Q^2}{4\pi\epsilon_0 R}\left(\frac{1}{2}\right)$$

$$\mathcal{E} = \mathcal{E}_{I} + \mathcal{E}_{II} = \frac{Q^2}{4\pi\epsilon_0 R}\left(\frac{1}{10} + \frac{1}{2}\right)$$

$$= \frac{3}{5}\,\frac{Q^2}{4\pi\epsilon_0 R}$$

(2) $\mathcal{E} = \frac{1}{2}\int V\rho\, dv$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \quad \text{for } r < R$$

For $r < R$

$$V = \frac{Q}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$$

V has the form $A + Br^2$

A, B is determined by requiring

(i) $E_r = -\frac{\partial V}{\partial r} = \frac{Qr}{4\pi\epsilon_0 R^3}$

(ii) At $r = R$, $V = \frac{Q}{4\pi\epsilon_0 R}$, and (iii) $V(r) \to 0$ as $r \to \infty$

$$\mathcal{E} = \frac{1}{2}\rho\frac{Q}{4\pi\epsilon_0 R}\left[\int_0^R \frac{3}{2}\,4\pi r^2 dr - \int_0^R \frac{r^2}{2R^2}\,4\pi r^2 dr\right]$$

$$= \frac{1}{2}\rho\,\frac{Q}{\epsilon_0 R}\left[\frac{3}{2}\cdot\frac{1}{3}R^3 - \frac{1}{2R^2}\cdot\frac{1}{5}R^5\right]$$

$$= \frac{1}{5}\,\frac{Q}{\frac{4}{3}\pi R^3}\,\frac{Q}{\epsilon_0 R}\,R^3$$

$$= \frac{3}{5}\,\frac{Q^2}{4\pi\epsilon_0 R}$$

(3) To make a charged sphere with charge density ρ to grow from r to $r + dr$

$\Rightarrow$ The energy needed is

$$\underbrace{V(r)}_{\text{due to charge sphere with radius } r}\ \rho(r)\cdot 4\pi r^2 dr$$

$$= \frac{\rho\cdot\overbrace{\frac{4}{3}\pi r^3}^{\text{charge inside sphere with radius } r}}{4\pi\epsilon_0 r}\cdot 4\pi r^2 dr$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$d\mathcal{E} = \frac{Q}{4\pi\epsilon_0 r}\,\frac{r^3}{R^3}\,\rho\, 4\pi r^2\, dr$$

To assemble the uniform spherical charge distribution of radius R, the total energy needed is

$$\mathcal{E} = \int_0^R \frac{1}{4\pi\epsilon_0 r}\, Q\,\frac{r^3}{R^3}\,\rho\, 4\pi r^2\, dr$$

$$= \frac{Q}{\epsilon_0 R^3}\,\rho \int_0^R r^4\, dr$$

$$= \frac{Q}{\epsilon_0 R^3}\,\frac{Q}{\frac{4}{3}\pi R^3}\cdot\frac{1}{5}R^5$$

$$= \frac{3}{5}\,\frac{Q^2}{4\pi\epsilon_0 R}$$

The three different methods give the same answer.

$\mathcal{E}$ is the energy required to assemble a uniform charge sphere with charge Q and radius R

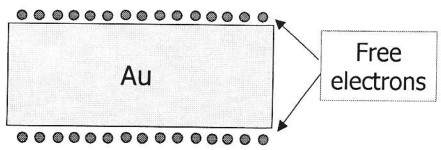
As $R \to 0$, $\mathcal{E} \to \infty \Rightarrow$ self-energy of point charge produces divergent problem.

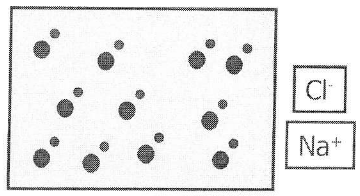
Conductors

Conductors and Insulators

Conductor: a material with free electrons

- Excellent conductors: metals such as Au, Ag, Cu, Al,...
- OK conductors: ionic solutions such as NaCl in H_2O



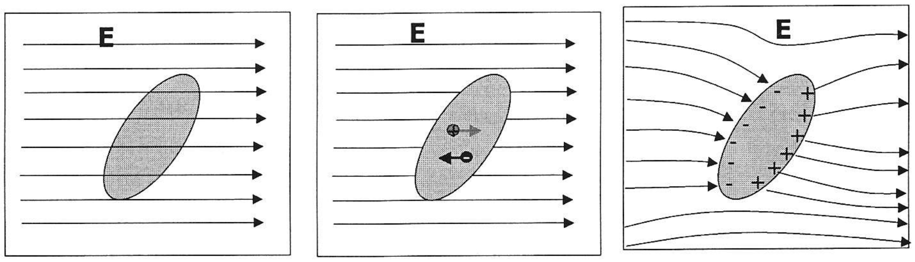


Insulator: a material without free electrons

- Organic materials: rubber, plastic,...
- Inorganic materials: quartz, glass,...

Electric Fields in Conductors (1)

- A conductor is assumed to have an infinite supply of electric charges
  - Pretty good assumption…
- Inside a conductor, **E**=0
  - Why? If **E** is not 0 → charges will move from where the potential is higher to where the potential is lower; migration will stop only when E=0.
  - How long does it take? 10^{-17}-10^{-16} s (typical resistivity of metals)

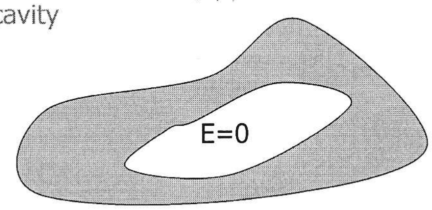


Electric Fields in Conductors (2)

- Electric potential inside a conductor is constant
  - Given 2 points inside the conductor P_1 and P_2 the $\Delta\phi$ would be:
    $\Delta\phi = \int_{P1}^{P2} \vec{E} \cdot d\vec{s} = 0$ since E=0 inside the conductor.
- Net charge can only reside on the surface
  - If net charge inside the conductor → Electric Field .ne.0 (Gauss's law)
- External field lines are perpendicular to surface
  - E// component would cause charge flow on the surface until $\Delta\phi$=0
- Conductor's surface is an equipotential
  - Because it's perpendicular to field lines

Corollary 1

In a hollow region inside conductor, ϕ=const and E=0 if there aren't any charges in the cavity



Why?

- Surface of conductor is equipotential
- If no charge inside the cavity → Laplace holds → ϕ_{cavity} cannot have max or minima

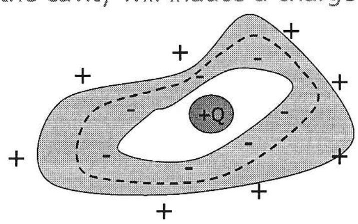
→ ϕ must be constant → E=0

Consequence:

- Shielding of external electric fields: Faraday's cage

Corollary 2

A charge +Q in the cavity will induce a charge +Q on the outside of the conductor



Why?

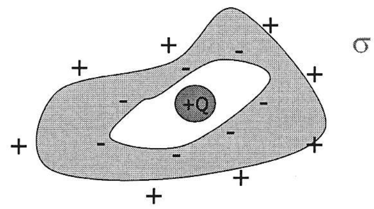
- Apply Gauss's law to surface - - - inside the conductor

$$\begin{cases} \oint \vec{E} \bullet d\vec{A} = 0 \text{ because E=0 inside a conductor} \\ \oint \vec{E} \bullet d\vec{A} = 4\pi(Q + Q_{inside}) \text{ Gauss's law} \end{cases}$$

$$\Rightarrow Q_{inside} = -Q \quad \Rightarrow Q_{outside} = -Q_{inside} = Q \text{ (conductor is overall neutral)}$$

Corollary 3

The induced charge density on the surface of a conductor caused by a charge Q inside it is $\sigma_{induced}=E_{surface}/4\pi$



Why?

- For surface charge layer, Gauss tells us that $\Delta E=4\pi\sigma$
- Since $E_{inside}=0 \rightarrow E_{surface}=4\pi\sigma_{induced}$

Uniqueness theorem

Given the charge density $\rho(x,y,z)$ in a region and the value of the electrostatic potential $\phi(x,y,z)$ on the boundaries, there is only one function $\phi(x,y,z)$ which describes the potential in that region.

Prove:

- Assume there are 2 solutions: ϕ_1 and ϕ_2; they will satisfy Poisson:

$$\nabla^2\phi_1(\vec{r}) = 4\pi\rho(\vec{r})$$
$$\nabla^2\phi_2(\vec{r}) = 4\pi\rho(\vec{r})$$

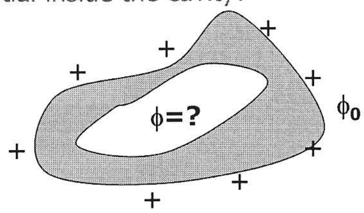
- Both ϕ_1 and ϕ_2 satisfy boundary conditions: on the boundary, $\phi_1=\phi_2=\phi$
- Superposition: any combination of ϕ_1 and ϕ_2 will be solution, including $\phi_3=\phi_2-\phi_1$: $\nabla^2\phi_3(\vec{r}) = \nabla^2\phi_2(\vec{r}) - \nabla^2\phi_1(\vec{r}) = 4\pi\rho(\vec{r}) - 4\pi\rho(\vec{r}) = 0$
- ϕ_3 satisfies Laplace: no local maxima or minima inside the boundaries
- On the boundaries $\phi_3=0 \rightarrow \phi_3=0$ everywhere inside region

$\rightarrow \phi_1=\phi_2$ everywhere inside region

Why do I care?
A solution is THE solution!

Uniqueness theorem: application 1

- A hollow conductor is charged until its external surface reaches a potential (relative to infinity) $\phi=\phi_0$.
  What is the potential inside the cavity?

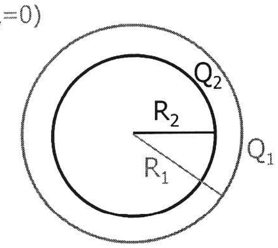


Solution

$\phi=\phi_0$ everywhere inside the conductor's surface, including the cavity.
Why? $\phi=\phi_0$ satisfies boundary conditions and Laplace equation
→ The uniqueness theorem tells me that is THE solution.

Uniqueness theorem: application 2

- Two concentric thin conductive spherical shells or radii R1 and R2 carry charges Q_1 and Q_2 respectively.
  - What is the potential of the outer sphere? ($\phi_{\text{infinity}}=0$)
  - What is the potential on the inner sphere?
  - What at r=0?



Solution

- Outer sphere: $\phi_1=(Q_1+Q_2)/R_1$
- Inner sphere $\phi_2-\phi_1=-\int_{R2}^{R1}\vec{E}\bullet d\vec{s}=-\int_{R2}^{R1}\frac{Q_2}{r^2}dr=\frac{Q_2}{R_1}-\frac{Q_2}{R_2}$

$\Rightarrow \phi_2=\frac{Q_2}{R_2}+\frac{Q_1}{R_1}$ Because of uniqueness: $\phi(r)=\phi_2 \forall r<R_2$

which electric charge can flow freely

For electrostatics charges have reached their equilibrium positions and are fixed in space

$\Rightarrow$ inside the conductor, $\vec{E}=0$, otherwise charge will move

$\downarrow$ no longer, electrostatic situation

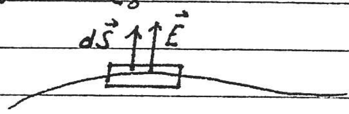
In electrostatics, $\vec{E}=0$ inside the conductor

$\downarrow$

all points inside the conductor are at the same potential

(iii) $\vec{E}$ is normal to the surface, otherwise charges will flow along the surface. Furthermore, $\vec{E}$ is normal to equipotential surface.

(iv) $|\vec{E}| = \frac{\sigma}{\epsilon_0}$



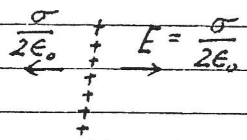
$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon_0}$

Only "up" surface will contribution

On the "down" surface $\vec{E}=0$ (inside the conductor)

On the other surface, $\vec{E} \perp d\vec{s} \Rightarrow \vec{E}\cdot d\vec{s} = 0$

$$|\vec{E}|\,dS = \frac{\sigma\, dS}{\epsilon_0} \Rightarrow |\vec{E}| = \frac{\sigma}{\epsilon_0}$$

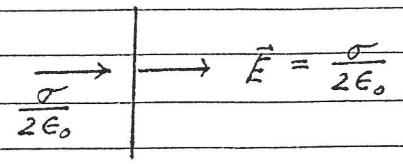


no other charge around

For conductor, there must be "other" charges to make $\vec{E}=0$ inside the conductor

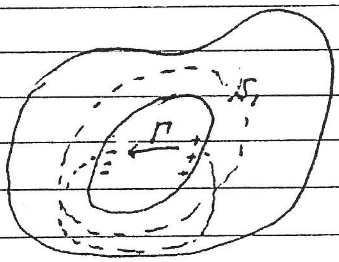
The charges on the surface give $|\vec{E}| = \frac{\sigma}{2\epsilon_0}$ both inside and outsid the surface.

Rest of the charges on the conductor "conspire" to produce an additional field



to make $\vec{E} = \frac{\sigma}{\epsilon_0}$ outside the conductor, $\vec{E} = 0$ inside the conductor.

2-20 Hollow Conductor Enclosing on Empty Cavity



$\oint \vec{E} \cdot d\vec{S} = 0$

↓

Gaussian surface is taken inside the conductor where $\vec{E} = 0$ everywhere

↓

$Q_{total,\ inside} = 0$

However, there still could be a positive surface charge on one part and a negative surface charge on somewhere else on the inner surface of the conductor

↓

this cannot be ruled out by Gauss's law

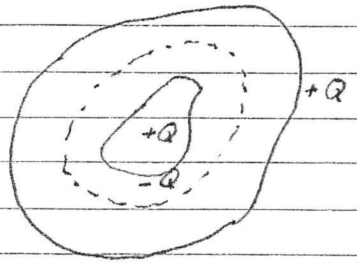
↓

ut it is ruled by the $\int \vec{E} \cdot d\vec{r} = 0$ required by electrostatics, for otherwise $\oint_{\Gamma} \vec{E} \cdot d\vec{r} \neq 0$

$\Rightarrow \vec{E} = 0$ inside the cavity

$\Rightarrow$ If an empty cavity is enclosed completely by a conductor no static distribution can produce any field inside.

2-21 Hollow Conductor Enclosed a Charge



$\oint_{S_1} \vec{E} \cdot d\vec{S} = 0$

$\downarrow$

$\vec{E} = 0$ inside the conductor

$\downarrow$

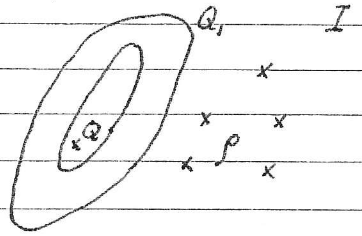
Gaussian surface encloses a zero net charge

$\downarrow$

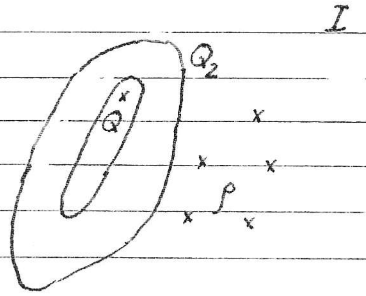
surface charge on the inside surface of the conductor is $-Q$

If the conductor carries a zero net charge, then the total charge on the outside surface is Q independent of location $+Q$ inside the cavity

2-22 Uniqueness Theorem



V_1, $\vec{E}_1$ outside the conductor



V_2, $\vec{E}_2$ outside the conductor

ρ is the same for both cases

We want to show that if $Q_1 = Q_2$, then $\vec{E}_1 = \vec{E}_2$ outside the conductor independent the location of Q inside the cavity

Define $\vec{E}_0 = \vec{E}_1 - \vec{E}_2$, $V_0 = V_1 - V_2$

Take the region I to be the region enclosed by the outer surface of the conductor and a sphere with radius $r \to \infty$

In region I $\nabla \cdot \vec{E}_0 = \nabla \cdot (\vec{E}_1 - \vec{E}_2) = 0$ since for the two cases ρ is the same.

At $r \to \infty$ $\vec{E}_1 \to \vec{E}_2 \to \frac{1}{r^2}$

$V_1 \to V_2 \to \frac{1}{r}$

$$\Rightarrow \int_{\substack{r\to\infty \\ \text{surface}}} \vec{E}_o V_o \cdot d\vec{S} = 0$$

$$\int \nabla\cdot(V_o\vec{E}_o)\,dv \overset{\text{divergence theorem}}{=} \underbrace{\int \vec{E}_o V_o \cdot d\vec{S}}_{\text{outside surface of the conductor}} + \underbrace{\int \vec{E}_o V_o \cdot d\vec{S}}_{0}$$

~~V_o = constant on this surface~~, ~~since V_1, V_2 are constant on~~ this surface

$$= V_o \int \vec{E}_o \cdot d\vec{S}$$

$$\underbrace{\int \nabla\cdot(V_o\vec{E}_o)\,dv}_{V_o\int \vec{E}_o\cdot d\vec{S}} = \underbrace{\int \nabla V_o \cdot \vec{E}_o\,dv}_{-\int |\vec{E}_o|\,dv} + \int V_o \underbrace{\nabla\cdot\vec{E}_o}_{0}\,dv$$

$$\Rightarrow -\int |\vec{E}_o|^2\,dv = V_o\left[\int \vec{E}_1\cdot d\vec{S} - \int \vec{E}_2\cdot d\vec{S}\right]$$

$$\vec{E}_1\cdot d\vec{S} = -|\vec{E}_1|\,d\vec{S} = -\frac{\sigma_1}{\epsilon_o}\,dS$$

$$\int_{\substack{\text{outside} \\ \text{surface of the conductor}}} \vec{E}_1\cdot d\vec{S} = -\frac{1}{\epsilon_o}\int \sigma_1\,dS = -Q_1/\epsilon_o$$

Similar argument $\Rightarrow \int_{\text{outside surface of the conductor}} \vec{E}_2\cdot d\vec{S} = -Q_2/\epsilon_o$

If $Q_1 = Q_2$, then $\int |\vec{E}_o|^2\,dv = 0 \Rightarrow \vec{E}_o = 0$ in region I

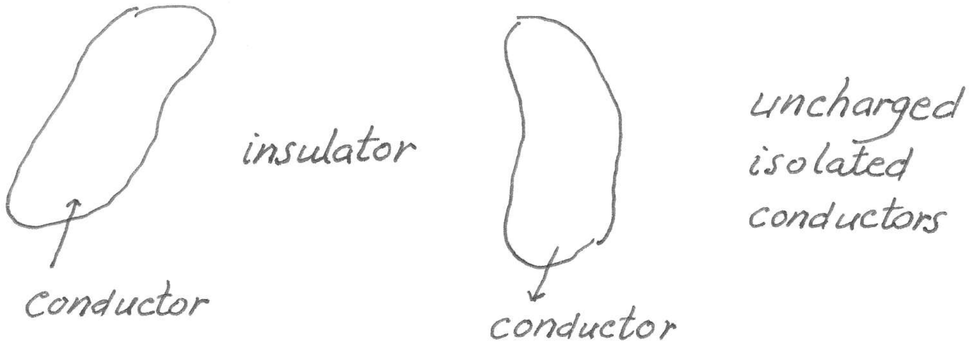
$\Rightarrow \vec{E}_1 = \vec{E}_2$ in region I

As long as $Q_1 = Q_2$ and ρ is the same, the $\vec{E}$ in region I is uniquely determined $\Rightarrow \vec{E}$ in region I is determined by ρ and surface charge on the surface of the conductor $\Rightarrow$ the electric field $\vec{E}$ in region I is independent of the location of Q inside the cavity.

Similarly, the field inside the cavity is determined by the charge distribution inside the cavity and is independent of the field outside the conductor. Note, there is no line of force in the conductor where $\vec{E} = 0$.

Capacitor

Capacitance between Two Conductors



Transfer charge Q from one to other establishes a potential difference between them.

$V \propto Q$

$\downarrow$

superposition principle

$\vec{E} \propto Q$

$V \propto \vec{E}$

$$\Rightarrow \quad V = \frac{Q}{C}$$

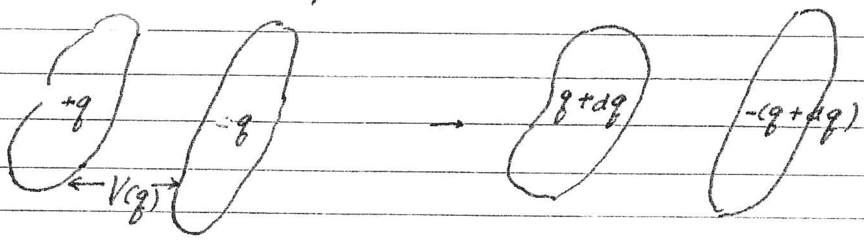
$\uparrow$ capacitance

C is a function of geometrical arrangement, independent of Q and V

$$\Rightarrow \quad C = \frac{Q}{V}$$

definition of capacitor

Energy stored in the capacitor



If we want to transfer dq from the conductor on the right to the conductor to the left, the work needed is $V(q)\,dq$

To set up a capacitor with Q, $-Q$ on the left, right conductors, respectively, requires

$$\mathcal{E} = \int_0^Q V(q)\,dq = \int_0^Q \frac{q}{C}\,dq$$

$\downarrow$

energy stored in the capacitor

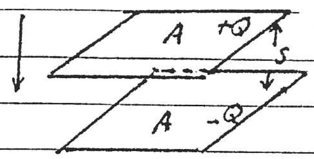
$$= \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2} C V^2$$

$$= \frac{1}{2} Q V$$

Note: The factor of $\frac{1}{2}$

Parallel plate capacitor



Neglect the edge effect

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{k} = \frac{Q}{\epsilon_0 A}$$

$$V_s = -\int_0^s \vec{E}\cdot dz\,\hat{k}$$

$$= Es = \frac{Qs}{\epsilon_0 A}$$

$$\Rightarrow \quad C = \frac{\epsilon_0 A}{s}$$

$$\mathcal{E} = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2 s}{2\epsilon_0 A}$$

Alternatively, $\mathcal{E} = \int \frac{\epsilon_0}{2}|\vec{E}|^2 dV$

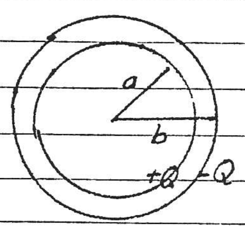
$$= \int \frac{\epsilon_0}{2}\frac{Q^2}{\epsilon_0^2 A^2} A\, dz$$

$$= \frac{Q^2 s}{2\epsilon_0 A}$$

$\downarrow$

the same result as before

Two spherical shell capacitor



$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{r}$$

$$= -\frac{Q}{4\pi\epsilon_0}\frac{1}{b} + \frac{Q}{4\pi\epsilon_0}\frac{1}{a}$$

$$\Rightarrow \quad V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow \quad C = \frac{Q}{V}$$

$$= \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\mathcal{E} = \frac{1}{2}CV^2 = \frac{1}{2}\frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}\frac{Q^2}{(4\pi\epsilon_0)^2}\left(\frac{1}{a} - \frac{1}{b}\right)^2$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Alternatively

$$\mathcal{E} = \int_a^b \frac{\epsilon_0}{2} |\vec{E}|^2 4\pi r^2 dr$$

$$= \int_a^b \frac{\epsilon_0}{2} \frac{Q^2}{(4\pi\epsilon_0^2)^2 r^4} 4\pi r^2 dr$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2} dr$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right)$$

$\downarrow$

same answer as before

The capacitance of an isolated conductor can be considered as the limit that the conductor $-Q$ is at ∞, where V now is the difference between the potential on the conductor and the potential at ∞.

The capacitance of a conducting sphere with radius R is thus $4\pi\epsilon_0 R = C$

If the charge on this spherical conductor is Q, then

$$\mathcal{E} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 R}$$

Alternatively

$$\mathcal{E} = \int_R^\infty \frac{\epsilon_0}{2} \frac{Q^2}{(4\pi\epsilon_0)^2 r^4} 4\pi r^2 dr$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0 R}$$

Chapter 25

Capacitance

1. The Uses of Capacitors

   Storage of energy
   Tunning the radio and TV
   On-off switches

2. Definition of Capacitor

3. Charging a Capacitor

4. Calculation of the Capacitance

   (i) Calculate the electric field
   (ii) Calculate the potential difference

$$\downarrow$$

$$Q = CV.$$

5. Parallel-Plate Capacitor

   (i) Argue it is uniform between the plate

   (ii) Choose a Guassian surface

$$\oint \vec{E} \cdot d\vec{s} = E \cdot A = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{A\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

$$\int_0^d \frac{\sigma}{\epsilon_0}\, dx = \frac{\sigma d}{\epsilon_0} = \text{potential difference } V$$

$$\frac{\sigma d}{\epsilon_0} = \frac{Qd}{A\epsilon_0} = V$$

$$\Rightarrow \quad C = \frac{A\epsilon_0}{d} \quad (\text{parallel-plate capacities})$$

6. Cylindrical Capacitor

$$\vec{E} \cdot d\vec{s} = E \underset{\substack{\downarrow \\ E\,2\pi r L}}{A} = \frac{Q}{\epsilon_0}$$

$$V = \int_a^b E\,dr$$

$$= \int_a^b \frac{Q}{\epsilon_0 2\pi r L}\,dr$$

$$= \frac{Q}{\epsilon_0 2\pi L}\int_a^b \frac{dr}{r}$$

$$= \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}.$$

7. Capacitors in Parallel and in Series.

Parallel — potential difference across the capacitors is the same

$C_{eq} = \sum C_i$

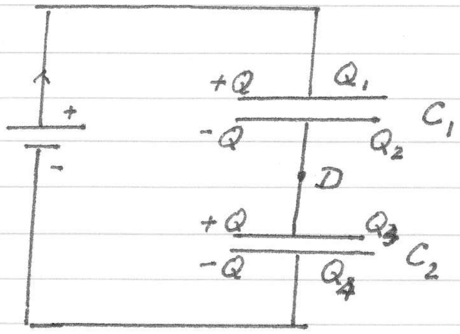
↓

when the same potential difference V is applied across it, a capacitor C_{eq} will store the same total charge q as is stored in the combination being replaced.

Series — $V = \sum V_i$, all have the same charge.

↓

explanation



Superposition principle

$Q_1 = -Q_4$

$Q_2 = -Q_3$

In order to have the E field at D to be zero
The field produced by Q_1, Q_4 must cancel the field produced by Q_2, Q_3 there
$\Rightarrow Q_1 = Q_3$

Thus we get the situation indicated at the left.

With this understood, it can be easily shown

$$\frac{1}{C_{eq}} = \sum \frac{1}{C_i}$$

8. Storage of Energy in an Electric Field

$$\begin{array}{c} q \\ -q \end{array} \longrightarrow \begin{array}{c} q+dq \\ q-dq \end{array}$$

$$V = \frac{q}{C}$$

$$dW = \frac{1}{C} q \, dq$$

$$W = \frac{1}{2}\frac{Q^2}{C}$$

$$\parallel$$

$$\int_0^Q \frac{1}{C} q \, dq$$

$$= \frac{1}{2} C V^2$$

For the parallel plate case

$$C = \frac{A\epsilon_0}{d}$$

U = energy stored in electric field

$$= \frac{1}{2}\frac{A\epsilon_0}{d} V = \underbrace{u}_{\text{energy density}} A d$$

$$u = \frac{1}{2}\epsilon_0 \underbrace{\left(\frac{V}{d}\right)^2}_{E^2}$$

Although, we have shown this only for the parallel case, the result is valid in general.

9. Example: Spherical capacity

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{using Gauss' law}$$

$$V = \int_a^b \frac{q}{4\pi\epsilon_0 r^2} dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow \quad C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

$$\text{Energy stored} = U = \frac{1}{2}\frac{q^2}{C}$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0 \frac{ab}{b-a}}$$

$$\text{Energy stored} = \int_a^b \frac{1}{2}\epsilon_0 \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{r^4} 4\pi r^2 dr$$

$$= \frac{1}{2}\epsilon_0 \frac{q^2}{16\pi^2\epsilon_0^2} \cdot 4\pi \int_a^b \frac{1}{r^2} dr$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,dr$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{b}+\frac{1}{a}\right]$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\frac{b-a}{ab}$$

↓

the same answer as before

[Medical defibrillator]

↓

rapid irregular connections of muscle fibers (as of the heart]

10. Capacitor with a Dielectric

$C_0 = \frac{Q}{V}$ in vacuum

$C = \kappa C_0$

↳ dielectric constant

Usually $\kappa > 1$

dielectric

Existence of V_{max}

↓

breakdown potential.

dielectric strength → maximum value of the electric field that it can tolerate without breakdown.

polar molecular

↓

have permanent electric dipole moment

without exteral field

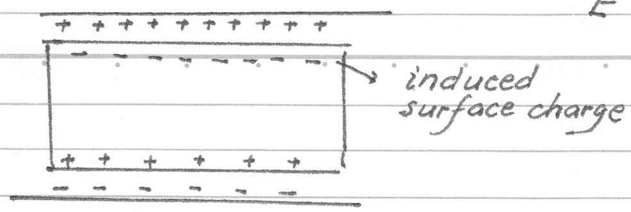
randomly oriented

with external field.

aligned

⇒ induced surface charge

$$\vec{E} = \vec{E}_0 + \vec{E}_1$$



$$\vec{E} = \frac{\vec{E}_0}{\kappa} = \vec{E}_0\left(1 + \frac{\vec{E}_1}{E_0}\right)$$

$$\vec{E}_1 \propto -\vec{E}_0$$

$$\Delta V = Ed = \frac{|\vec{E}_0|}{\kappa} d$$

$$\frac{\Delta V}{d} = \frac{|\vec{E}_0|}{\kappa} = \frac{\Delta V_0/d}{\kappa}$$

$$\Delta V = \frac{\Delta V_0}{\kappa}$$

$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{(\Delta V_0/\kappa)} = \kappa C_0$$

nonpolar

Without external field

no dipole moment

with external field

aligned dipole moment

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{(\kappa C_0)} = \frac{U_0}{\kappa}$$

$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}, \quad \vec{E}_1 = -\frac{\sigma_i}{\epsilon_0}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_1 = \frac{\sigma_0}{\epsilon_0} - \frac{\sigma_i}{\epsilon_0}$$

$$\parallel$$

$$\frac{\vec{E}_0}{\kappa}$$

$$\parallel$$

$$\frac{\sigma_0}{\epsilon_0}\frac{1}{\kappa} = \frac{\sigma_0}{\epsilon_0} - \frac{\sigma_i}{\epsilon_0}$$

$$\sigma_i = \left(\frac{\kappa - 1}{\kappa}\right)\sigma_0$$

$$E = \frac{Q_0 - Q_i}{A\epsilon}$$

$$\underset{\displaystyle\downarrow \atop \displaystyle D}{\frac{Q_0}{A}} = \epsilon_0 E + \underset{\displaystyle\downarrow \atop \displaystyle P}{\frac{Q_i}{A}}$$

$$D = \epsilon_0 E + P$$

D ↓ displacement field

$$P = \frac{Q_i}{A} = \frac{Q_i d}{A d}$$

P ↓ dipole moment per volume
↓
polarization.

$$\kappa = \frac{Q_0 / A\epsilon_0}{(Q_0 - Q_i)/A\epsilon_0}$$

$$\kappa(Q_0 - Q_i) = Q_0$$

$$Q_0 - Q_i = \frac{Q_0}{\kappa}$$

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q_0}{\kappa}$$

$$\Rightarrow \oint \kappa \vec{E}\cdot d\vec{s} = \frac{Q_0}{\epsilon_0}$$

parallel-plate

$$E = \frac{E_0}{\kappa} = \frac{Q_0}{A\kappa\epsilon_0} =$$

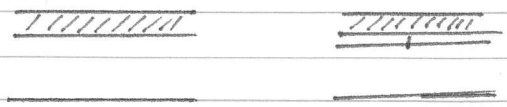
$$\vec{E} = \frac{Q_0}{\kappa\epsilon_0} = \underbrace{\frac{Q_0}{A}}_{=D}\frac{1}{\kappa\epsilon_0} \Rightarrow D\frac{1}{\kappa\epsilon_0}$$

$$\Rightarrow \oint \vec{D}\cdot d\vec{S} = Q_0$$

II. Example

A partial filled capacitor

$\frac{d}{4}$



$$C_1 = \frac{A\kappa\epsilon_0}{d/4} \qquad C_2 = \frac{A\epsilon_0}{3d/4}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\Rightarrow C = \frac{4A\epsilon_0}{d}\left(\frac{3\kappa+1}{\kappa}\right)$$

$$Q = C_1 \Delta V = \frac{4A\kappa\epsilon_0 \Delta V}{d}$$

$$\sigma_0 = \frac{Q}{A} \qquad \sigma_0 = \frac{4\kappa\epsilon_0 \Delta V}{d}$$

$$\sigma_i = \left(\frac{\kappa-1}{\kappa}\right)\sigma_0 = \frac{4\epsilon_0 \Delta V}{d}(\kappa-1)$$

Note: the linear assumption is crucial.

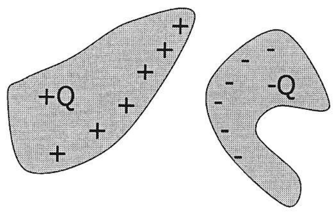
8.022 (E&M) – Lecture 6

Topics:

- More on capacitors
- Mini-review of electrostatics
  - (almost) all you need to know for Quiz 1

Last time…

- Capacitor:
  - System of charged conductors
- Capacitance: $C = \dfrac{Q}{V}$
  - It depends only on geometry



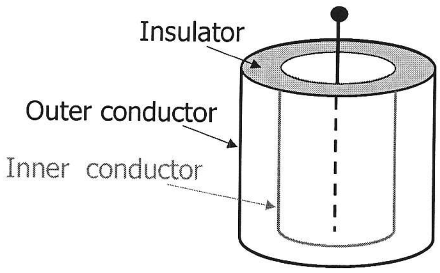
- Energy stored in capacitor:
  - In agreement with energy associated with electric field

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

- Let's now apply what we have learned…

Wimshurst machine and Leyden Jars (E1)

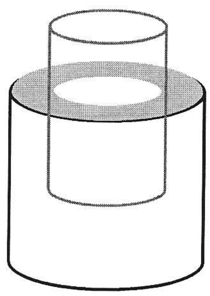
- A Wimshurst machine is used to charge 2 Leyden Jars
- Leyden Jars are simple cylindrical capacitors



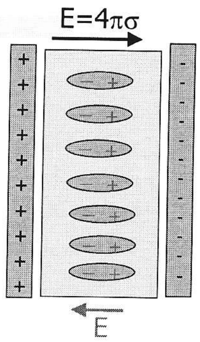
- What happens when we connect the outer and the outer surface?
- Why?

Dissectible Leyden Jar (E2)

- A Wimshurst machine is used to charge a Leyden Jar
- Where is the charge stored?
  - On the conductors?
  - On the dielectric?
- Take apart capacitor and short conductors
  - Nothing happens!
- Now reassemble it
  - Bang!
- Why?
  - Because it's "easier" for the charges to stay on dielectric when we take conductors apart or energy stored would have to change: $U=Q^2/2C$, and moving plates away C would decrease → U increase



Capacitors and dielectrics



- Parallel plates capacitor:

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{A}{4\pi d}$$

- Add a dielectric between the plates:
  - Dielectric's molecules are not spherically symmetric
  - Electric charges are not free to move

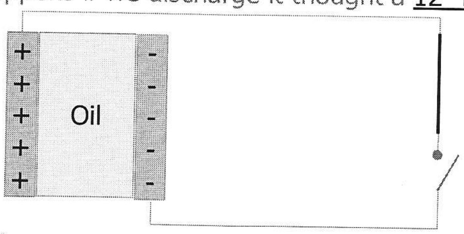
→ **E** will pull + and – charges apart and orient them // **E**

- $E_{dielectric}$ is opposite to $E_{capacitor}$
  - Given Q → V decreases
  - Given V → Q increases

  } → C increased!

Energy is stored in capacitors (E6)

- A 100 μF oil filled capacitor is charged to 4KV
- What happens if we discharge it thought a <u>12" long iron wire</u>?



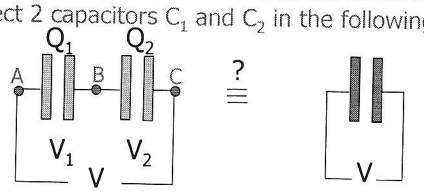
- How much energy is stored in the capacitor?
  - $U = \frac{1}{2} CV^2 = 800$ J Big!
- Resistance of iron wire: very small, but >> than the rest of the circuit

  → All the energy is dumped on the wire in a small time

  → Huge currents! → Huge temperatures! → The wire will explode!

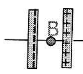
Capacitors in series

- Let's connect 2 capacitors C_1 and C_2 in the following way:



- What is the total capacitance C of the new system?

$$V_1 + V_2 = V$$
$$Q_1 = Q_2 = Q$$
$$\frac{1}{C} = \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{1}{C_1} + \frac{1}{C_2}$$

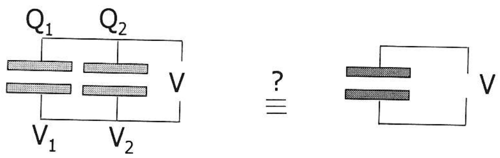


This is 1 conductor that starts electrically neutral → $Q_1=Q_2$

$$\boxed{C = \left(\sum_i \frac{1}{C_i}\right)^{-1}}$$

Capacitors in parallel

- Let's connect 2 capacitors C_1 and C_2 in the following way:



- What is the total capacitance C of the new system?

$$V_1 = V_2 = V$$
$$Q_1 + Q_2 = Q$$
$$C = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V_1} + \frac{Q_2}{V_2} = C_1 + C_2$$

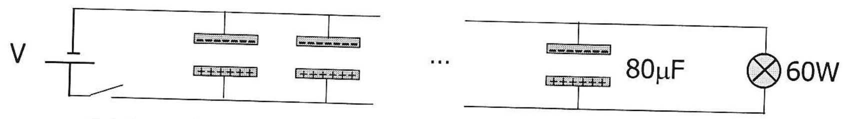
→ $\boxed{C = \sum_{i=1}^{i=N} C_i}$

Application

- Why are capacitors useful?
  - ...among other things...
  - They can store large amount of energy and release it in very short time
- Energy stored: $U = \frac{1}{2} CV^2$
  - The larger the capacitance, the larger the energy stored at a given V
- How to increase the capacitance?
  - Modify geometry
    - For parallel plates capacitors $C=A/(4\pi d)$: increase A or decrease d
  - Add a dielectric in between the plates
  - Add capacitors in parallel

Bank of capacitors (E7)

- Bank of 12 x 80 μF capacitors is parallel



  - Total capacitance: 960 μF
- Discharged on a 60 W light bulb when capacitors are charged at:
  - V = 100 V, 200 V, V = 300 V
- What happens?
  - Energy stored in capacitor is $U = \frac{1}{2} CV^2$
    $\rightarrow V = V_0 : 2xV_0 : 3xV_0 \rightarrow U = U_0 : 4xU_0 : 9xU_0$
  - R is the same → time of discharge will not change with V
  - The power will increase by a factor 9! ($P=RI^2$ and $I=V/R$)
  - Will the bulb survive?
    - Remember: light bulb designed for 120 V...

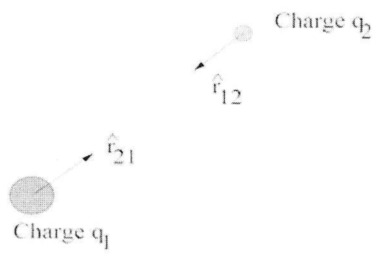
Review of Electrostatics for Quiz 1

Disclaimer:

- Can we review all of the electrostatics in less than 1 hour?
  - No, but we will try anyway…
- Only main concepts will be reviewed
  - Review main formulae and tricks to solve the various problems
  - No time for examples
    - Go back to recitations notes or Psets and solve problems again

The very basic:

Coulomb's law



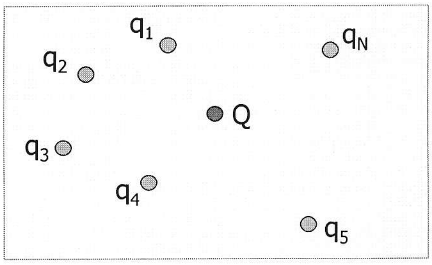
$$\vec{F}_2 = \frac{q_1 q_2}{|r_{21}|^2}\hat{r}_{21}$$

where F_2 is the force that the charge q_2 feels due to q_1

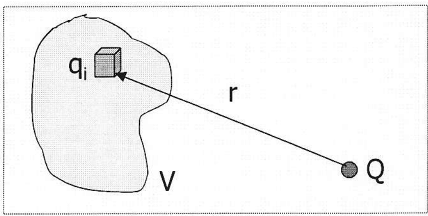
NB: this is in principle the only thing you have to remember: all the rest follows from this an the superposition principle

The very basic:

Superposition principle



$$\vec{F}_Q = \sum_{i=1}^{i=N} \frac{q_i Q}{|r_i|^2} \hat{r}_i$$

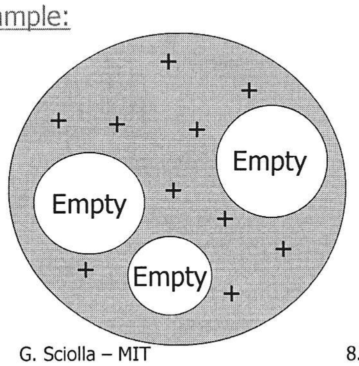


$$\vec{F}_Q = \int_V \frac{dq\ Q}{|r|^2}\ \hat{r} = \int_V \frac{\rho\ dV\ Q}{|r|^2}\ \hat{r}$$

The Importance of Superposition

Extremely important because it allows us to transform complicated problems into sum of small, simple problems that we know how to solve.

Example:



Calculate force F exerted by this distribution of charges on the test charge q

Electric Field and Electric Potential

- Solving problems in terms of $F_{coulomb}$ is not always convenient
  - F depends on probe charge q
- We get rid of this dependence introducing the Electric Field

$$\vec{E} = \frac{\vec{F}_q}{q} = \frac{Q}{|r|^2}\hat{r}$$

- Advantages and disadvantages of **E**
  - **E** describes the properties of space due to the presence of charge Q ☺
  - It's a vector → hard integrals when applying superposition… ☹
- Introduce Electric Potential ϕ
  - ϕ(P) is the work done to move a unit charge from infinity to P(x,y,z)

$$\phi(x,y,z) = -\int_{\infty}^{P} \vec{E} \bullet d\vec{s}$$

NB: true only when ϕ(inf)=0

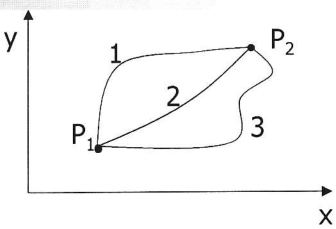
  - Advantages: superposition still holds but simpler calculation (scalar) ☺

Energy associated with **E**

- Moving charges in E requires work:

$$W_{1->2} = -\int_{1}^{2} \vec{F}_C \bullet d\vec{s}$$

$$\text{where} \quad F_{Coulomb} = \frac{Qq\hat{r}}{r^2}$$



- NB: integral independent of path: force conservative!

- Assembling a system of charges costs energy. This is the energy stored in the electric field:

$$U = \frac{1}{2}\int_{\substack{\text{Volume}\\ \text{with}\\ \text{charges}}} \rho\phi dV = \int_{\substack{\text{Entire}\\ \text{space}}} \frac{E^2}{8\pi} dV$$

Electrostatics problems

- In electrostatics there are 3 different ways of describing a problem:

$\rho(x,y,z)$

$E(x,y,z)$ $\qquad\qquad$ $\phi(x,y,z)$

- Solving most problem consists in going from one formulation to another. All you need to know is: how?

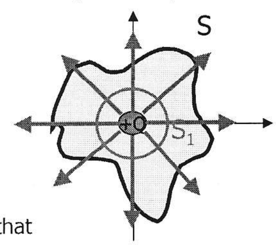
From $\rho \rightarrow$ E

- General case:
  - For a point charge: $\vec{E} = \dfrac{q}{|r|^2}\hat{r}$
  - Superposition principle: $\vec{E} = \int_V d\vec{E} = \int_V \dfrac{dq}{|r|^2}\hat{r}$

Solving this integral may not be easy...

- Special cases:
  - Look for symmetry and thank Mr. Gauss who solved the integrals for you
    - Gauss's Law: $\Phi_{\vec{E}} = 4\pi Q_{enc}$

$$\oint_S \vec{E}\bullet d\vec{A} = 4\pi \int_V \rho\, dV$$



- N.B.:
  - Gauss's law is always true but not always useful: Symmetry is needed!
  - Main step: choose the "right" gaussian surface so that E is constant on the surface of integration

From ρ → φ

- General case:
  - For a point charge: $\phi = \frac{q}{r}$
  - Superposition principle: $\phi = \int_V \frac{dq}{r}$

NB: implicit hypothesis: φ(infinity)=0

The problem is simpler than for **E** (only scalars involved) but not trivial...

- Special cases:
  - If symmetry allows, use Gauss's law to extract **E** and then integrate **E** to get φ:

$$\phi_2 - \phi_1 = -\int_1^2 \vec{E} \bullet d\vec{s}$$

  - N.B.: The force is conservative → the result is the same for any path, but choosing a simple one makes your life much easier....

From φ to E and ρ

Easy! No integration needed!

- From φ to **E** $\boxed{\vec{E} = -\nabla \phi}$
  - One derivative is all it takes but... make sure you choose the best coordinate system
  - You will not loose points but you will waste time...
- From φ to ρ
  - Poisson tells you how to get from potential to charge distributions directly:

$$\boxed{\nabla^2 \phi = -4\pi\rho}$$

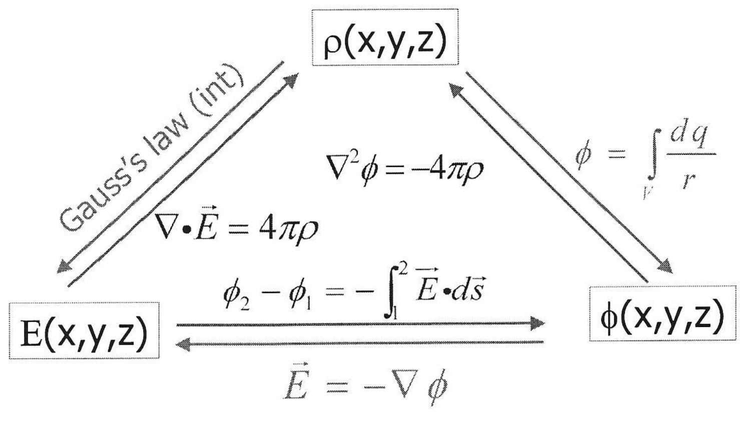
  - Uncomfortable with Laplacian? Get there in 2 steps:
    - First calculate E: $\vec{E} = -\nabla \phi$
    - The use differential form of Gauss's law: $\boxed{\nabla \bullet \vec{E} = 4\pi\rho}$

Thoughts about ϕ and **E**

- The potential ϕ is always continuous
- E is not always continuous: it can "jump"
  - When we have surface charge distributions
    - Remember problem #1 in Pset 2

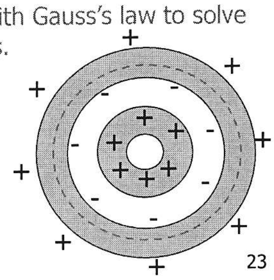
→ When solving problems always check for consistency!

Summary



Conductors

- Properties:
  - Surface of conductors are equipotential
  - E (field lines) always perpendicular to the surface
  - $E_{inside}=0$
  - $E_{surface}=4\pi\sigma$
- What's the most useful info?
  - $E_{inside}=0$ because it comes handy in conjunction with Gauss's law to solve problems of charge distributions inside conductors.
  - Example: concentric cylindrical shells
    - Charge +Q deposited in inner shell
    - No charge deposited on external shell
    - What is E between the 2 shells?
      - - Q induced on inner surface of inner cylinder
      - +Q induced on outer surface of outer cylinder

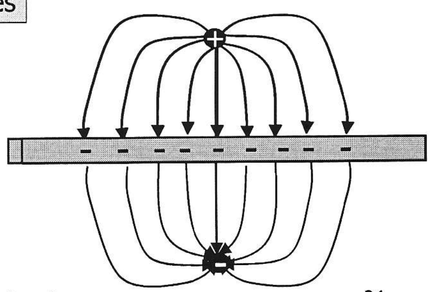


E due to Charges and Conductors

- How to find E created by charges near conductors?
  - Uniqueness theorem:
    - A solution that satisfies boundary conditions is THE solution
  - Be creative and think of distribution of point charges that will create the same filed lines:

Method of images

  - Example:



Capacitors

- Capacitance
  - Two oppositely charged conductors kept at a potential difference V will have capacitance C

$$C = \frac{Q}{V}$$

  - NB: capacitance depends only on the geometry!
- Energy stored in capacitor

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

- What should you remember?
  - Parallel plate capacitor: very well
  - Be able to derive the other standard geometries

Conclusion

- Material for Quiz #1:
  - Up to this lecture (Purcell chapters 1/2/3)
- Next lecture:
  - Charges in motion: currents
  - NB: currents are not included in Quiz 1!

Applications

See the textbook P.703 ~ P705.