Chapter 26

Current and Resistance

1. Definition

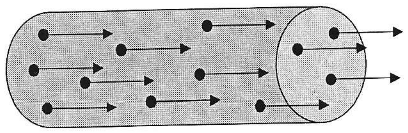
   Electric Current I Q

   Current Density $\vec{J}$ ρ

2. Conservation of Charge and Continuity Equation.

Electric current I

- Consider a region in which there is a flow of charges:
  - E.g. cylindrical conductor



- We define a current:
  the charge/unit time flowing through a certain surface

$$I = \frac{dQ}{dt}$$

- Units:
  - cgs: esu/s
  - SI: C/s=ampere (A)
  - Conversion: 1 A = 2.998 x 10^9 esu/s

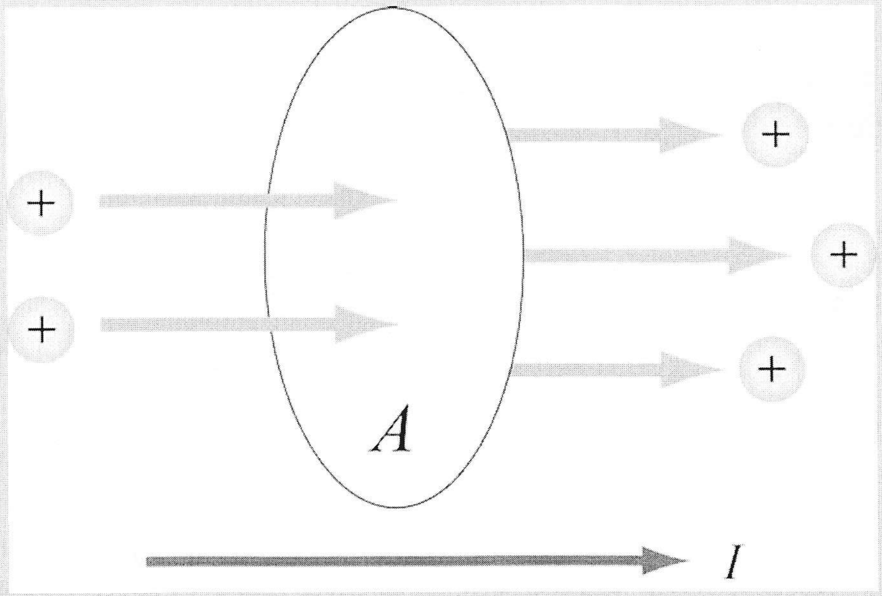
Current: Flow Of Charge

Average current I_{av}: Charge ΔQ flowing across area A in time Δt

$$I_{av} = \frac{\Delta Q}{\Delta t}$$

Instantaneous current: differential limit of I_{av}

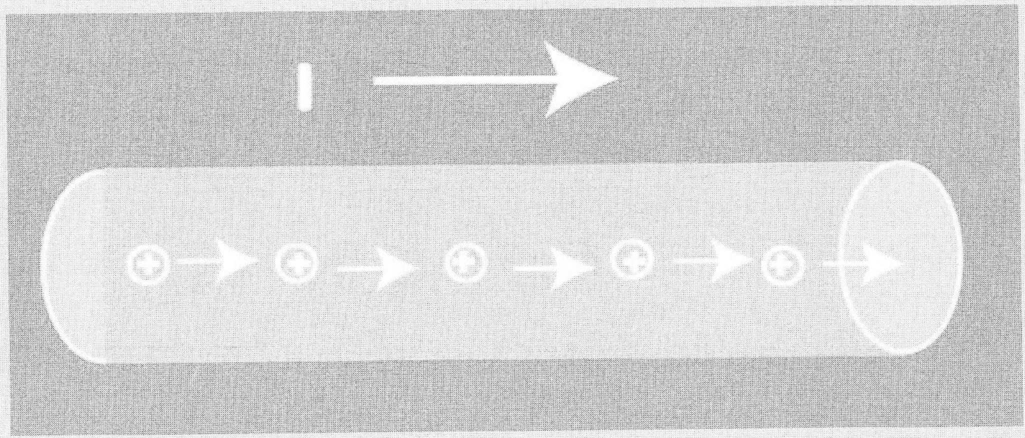
$$I = \frac{dQ}{dt}$$



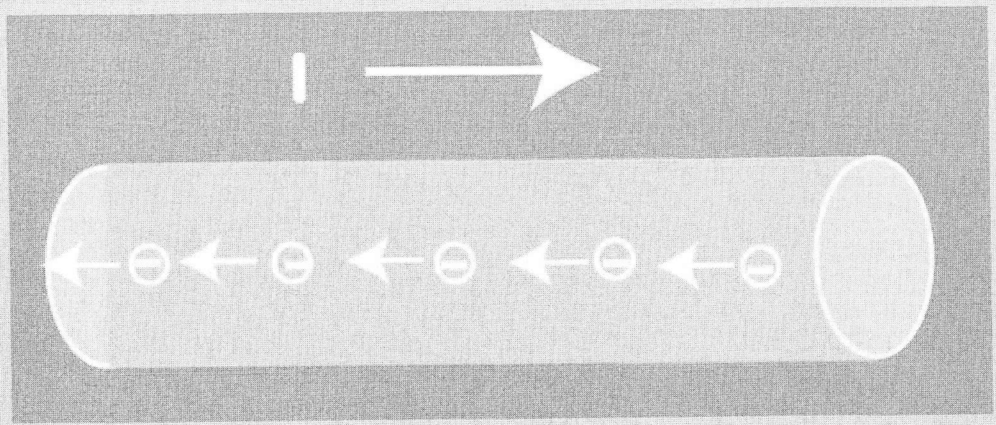
Units of Current: Coulombs/second = Ampere

Direction of The Current

Direction of current is direction of flow of pos. charge



or, opposite direction of flow of negative charge

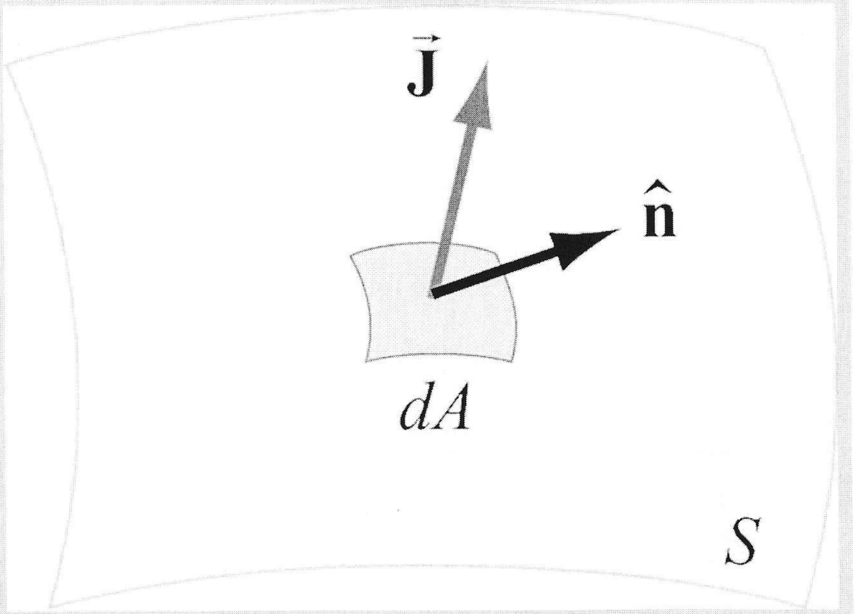


Current Density J

J: current/unit area

$$\vec{\mathbf{J}} \equiv \frac{I}{A}\hat{\mathbf{I}}$$

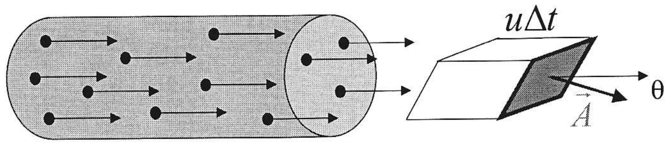
$\hat{\mathbf{I}}$ points in direction of current



$$I = \int_S \vec{\mathbf{J}} \cdot \hat{\mathbf{n}}\, dA = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$$

Current density J

- Number density: n = #charges / unit volume
- Velocity of each charge: **u**



- Current flowing through area A: I = ΔQ / Δt
  - where ΔQ= q x number of charges in the prism

$$\rightarrow \quad I = \frac{\Delta Q}{\Delta t} = \frac{q\Delta N}{\Delta t} = \frac{qnV_{prism}}{\Delta t} = \frac{qnA\cos\theta u\Delta t}{\Delta t} = qn\vec{u}\bullet\vec{A} = \vec{J}\bullet\vec{A}$$

- Where we defined the current density J as: $\boxed{\vec{J} \equiv qn\vec{u} \equiv \rho\vec{u}}$

G. Sciolla – MIT 8.022 – Lecture 7 NB: ρ=volume charge density

More realistic case…

- We made a number of unrealistic assumptions:
  - only 1 kind of charge carriers: we could have several, e.g.: + and – ions
  - **u** assumed to be the same for all particles: unrealistic!
  - regular surface with J constant on it
- Multiple charge carriers: $\vec{J} \equiv \sum_k q_k n_k \vec{u}_k \equiv \sum_k \rho_k \vec{u}_k$
  - E.g.: solution with different kind of ions
  - NB: + ion with velocity u_k is equivalent to – ion with velocity $-u_k$
- Velocity:
  - Not all charges have the same velocity → average velocity $\langle \vec{u}_k \rangle = \frac{1}{N_k}\sum_i (\vec{u}_k)_i$

$$\vec{J} \equiv \sum_k q_k n_k \langle \vec{u}_k \rangle \equiv \sum_k \rho_k \langle \vec{u}_k \rangle$$

- Arbitrary surface S, arbitrary **J**: $I = \int_S \vec{J}\bullet d\vec{A}$

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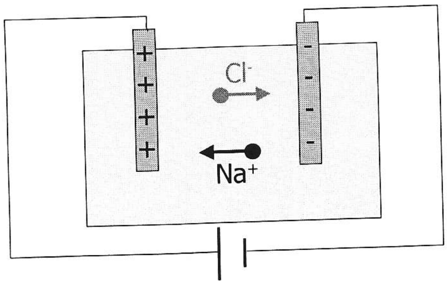
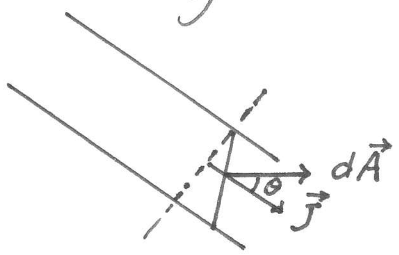
Non standard currents

- We usually think of currents as electrons moving inside a conductor
  - This is only one of the many examples!
- Other kinds of currents
  - Ions in solution such as Salt (NaCl) in water (Demo F5)



$\vec{J}$ direction → motion of the positive charge

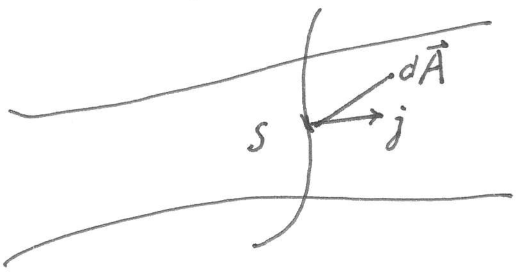
magnitude → charge passing per unit time per unit area ⊥ to the direction of $\vec{J}$

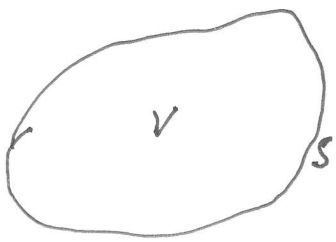


$$\vec{J} \cdot d\vec{A} = |\vec{J}||dA| \cos\theta$$

$$\int_{surface} \vec{J} \cdot d\vec{A} = I$$

↳ current cross the surface





$$\oint \vec{J} \cdot d\vec{A}$$

= Total charge crossing the surface enclosed the unit time

= Total charge leaving the volume per unit time $= -\frac{dQ_{ins}}{dt}$

charge conservation

$$\oint \vec{J} \cdot dA = -\frac{dQ_{in}}{dt}$$

$$\int \nabla \cdot \vec{J} \, dV = -\int \frac{\partial}{\partial t} \rho \, dV$$

$$\Downarrow$$

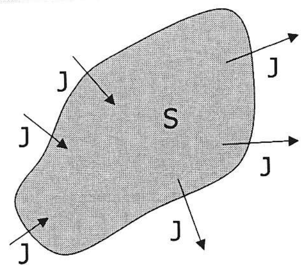
$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

↓ continuity equation

- Continuity equation will play important role when we discuss the Maxwell equation

conservation of Charge and The continuity equation

- A current I flows through the closed surface S:
  - Some charge enters
  - Some charge exits



- What happens to the charge after it enters?
  - Piles up inside
  - Leaves the surface

$$\oint_S \vec{J} \bullet d\vec{A} = -\frac{\partial Q_{inside}}{\partial t}$$

NB: - because dA points outside the surface

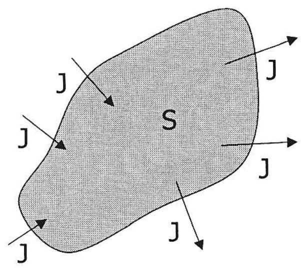
- Apply Gauss's theorem and obtain continuity equation:

$$\begin{cases} \oint_S \vec{J} \bullet d\vec{A} = \oint_V \nabla \bullet \vec{J} dV \\ -\frac{\partial}{\partial t} Q_{inside} = -\frac{\partial}{\partial t} \int_V \rho dV \end{cases} \Rightarrow \int_V \left(\nabla \bullet \vec{J} + \frac{\partial}{\partial t} \rho \right) dV = 0 \Rightarrow \boxed{\nabla \bullet \vec{J} + \frac{\partial}{\partial t} \rho = 0}$$

Thoughts on continuity equation

- Continuity equation:

$$\nabla \bullet \vec{J} + \frac{\partial}{\partial t} \rho = 0$$



- What does it teach us?
  - Conservation of electric charges in presence of currents
  - For steady currents:
    - no accumulation of charges inside the surface: dρ/dt=0

$$\rightarrow \quad \boxed{\nabla \bullet \vec{J} = 0}$$

Microscopic Ohm's law

- Electric fields cause charges to move
- Experimentally, it was observed by Ohm that

$$\vec{J} = \sigma \vec{E}$$

  - Microscopic version of Ohm's law:
    - It reflects the proportionality between E and J in each point
  - Proportionality constant: conductivity σ

More stuff here pleas

To find the drift velocity of the electrons, we first note that an electron in the conductor experiences an electric force $\vec{\mathbf{F}}_e = -e\vec{\mathbf{E}}$ which gives an acceleration

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}_e}{m_e} = -\frac{e\vec{\mathbf{E}}}{m_e} \tag{6.1.6}$$

Let the velocity of a given electron immediate after a collision be $\vec{\mathbf{v}}_i$. The velocity of the electron immediately before the next collision is then given by

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = \vec{\mathbf{v}}_i - \frac{e\vec{\mathbf{E}}}{m_e}t \tag{6.1.7}$$

where t is the time traveled. The average of $\vec{\mathbf{v}}_f$ over all time intervals is

$$\langle \vec{\mathbf{v}}_f \rangle = \langle \vec{\mathbf{v}}_i \rangle - \frac{e\vec{\mathbf{E}}}{m_e}\langle t \rangle \tag{6.1.8}$$

which is equal to the drift velocity $\vec{\mathbf{v}}_d$. Since in the absence of electric field, the velocity of the electron is completely random, it follows that $\langle \vec{\mathbf{v}}_i \rangle = 0$. If $\tau = \langle t \rangle$ is the average characteristic time between successive collisions (the *mean free time*), we have

$$\vec{\mathbf{v}}_d = \langle \vec{\mathbf{v}}_f \rangle = -\frac{e\vec{\mathbf{E}}}{m_e}\tau \tag{6.1.9}$$

The current density in Eq. (6.1.5) becomes

$$\vec{\mathbf{J}} = -ne\vec{\mathbf{v}}_d = -ne\left(-\frac{e\vec{\mathbf{E}}}{m_e}\tau \right) = \frac{ne^2\tau}{m_e}\vec{\mathbf{E}} \tag{6.1.10}$$

Note that $\vec{\mathbf{J}}$ and $\vec{\mathbf{E}}$ will be in the same direction for either negative or positive charge carriers.

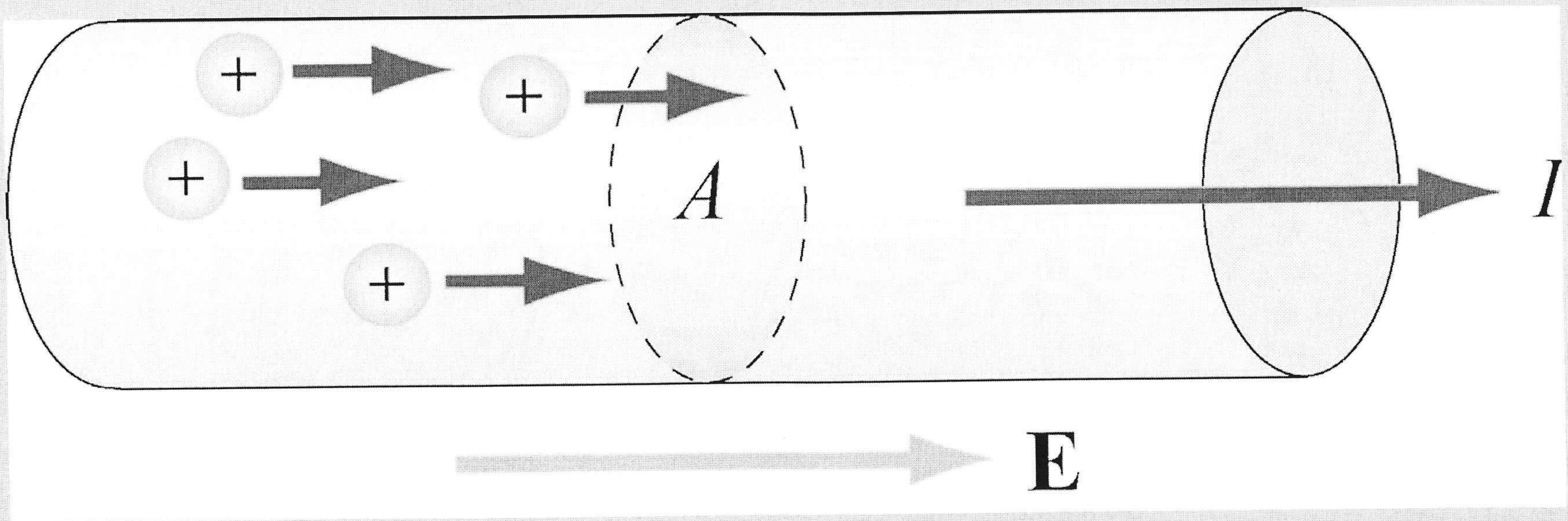
6.2 Ohm's Law

In many materials, the current density is linearly dependent on the external electric field $\vec{\mathbf{E}}$. Their relation is usually expressed as

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}} \tag{6.2.1}$$

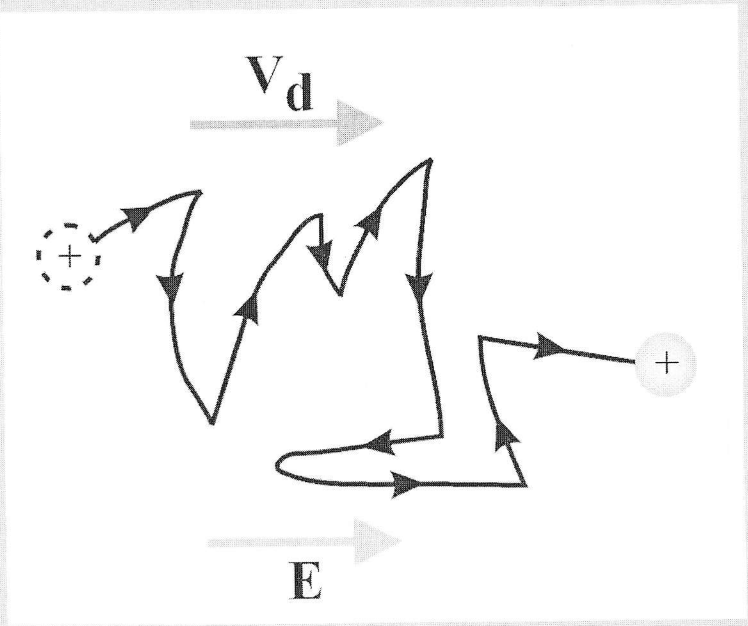
Why Does Current Flow?

If an electric field is set up in a conductor, charge will move (making a current in direction of E)



Note that when current is flowing, the conductor is not an equipotential surface (and $E_{inside} \neq 0$)!

Microscopic Picture



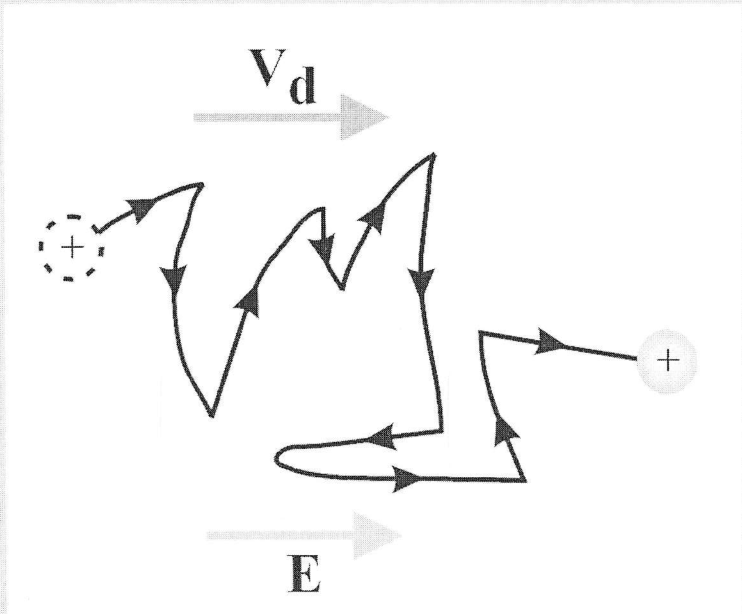
Drift speed is velocity forced by applied electric field in the presence of collisions.

It is typically 4×10^{-5} m/sec, or 0.04 mm/second!

To go one meter at this speed takes about 10 hours!

How Can This Be?

Conductivity and Resistivity



Ability of current to flow depends on density of charges & rate of scattering

Two quantities summarize this:

σ: conductivity

ρ: resistivity

Microscopic Ohm's Law

$$\vec{\mathbf{E}} = \rho \vec{\mathbf{J}} \quad \text{or} \quad \vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$$

$$\rho \equiv \frac{1}{\sigma}$$

ρ and σ depend only on the microscopic properties of the material, not on its shape

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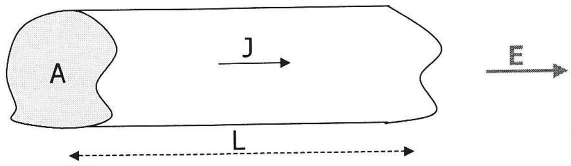
Macroscopic Ohm's law

- Current is flowing in a uniform material of length L in uniform electric field **E** // L



- Potential difference between two ends: V=EL
- Ohm's law $J=\sigma E$ holds in every point:

$$J = \sigma E \Rightarrow \frac{I}{A} = \sigma \frac{V}{L} \quad \Rightarrow \quad \boxed{V = IR} \quad \text{where} \quad \boxed{R \equiv \frac{L}{\sigma A}}$$

Resistance R

- Proportionality constant between V and R in Ohm's law

$$R \equiv \frac{L}{\sigma A} \equiv \frac{\rho L}{A}$$

- Units: [V]=[R][I]
  - SI: Ohm (Ω) = V/A
  - cgs: s/cm
- Dependence on the geometry:
  - Inversely proportional to A and proportional to L
- Dependence on the property of the material:
  - Inversely proportional to conductivity

Resistivity

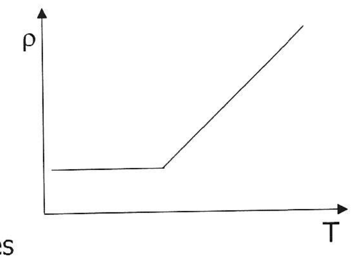
- Resistivity $\rho = 1/\sigma$
  - Describes how fast electrons can travel in the material
  - Units: in SI: Ω m; in cgs: s

Material	Resistivity (Ω-m)	Resistivity (sec)
Silver	1.6×10^{-8}	1.8×10^{-17}
Copper	1.7×10^{-8}	1.9×10^{-17}
Gold	2.4×10^{-8}	2.6×10^{-17}
Iron	1.0×10^{-7}	1.1×10^{-16}
Sea water	0.2	2.2×10^{-11}
Polyethylene	2.0×10^{11}	220
Glass	$\sim 10^{12}$	$\sim 10^{3}$
Fused quartz	7.5×10^{17}	8.3×10^{8}

  - Depends on chemistry of material, temperature,...
    - Demos F1 and F4

Resistivity vs. Temperature

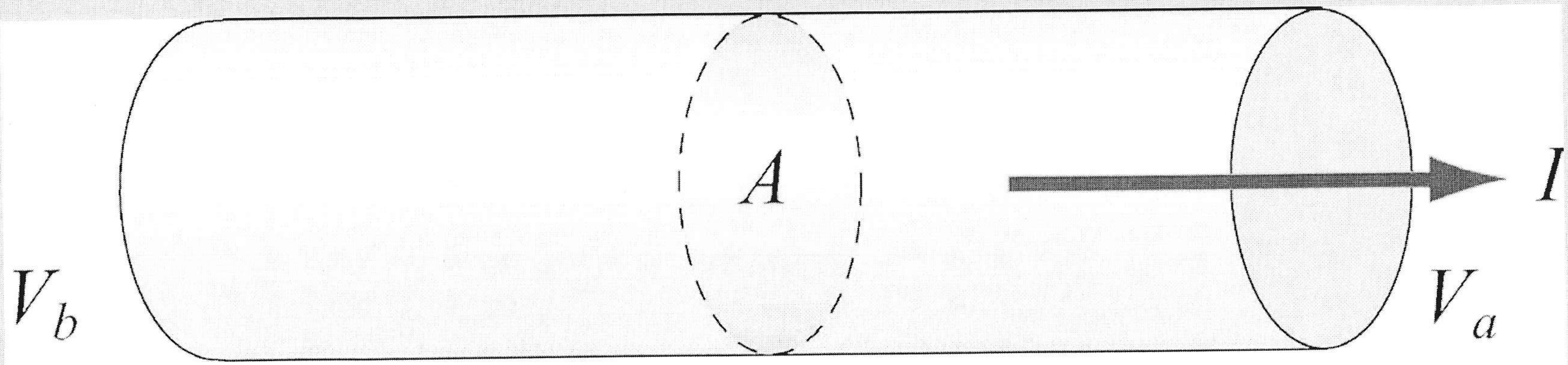
- Does resistivity depend on T?
  - Demos F1 and F4



- Why?
  - Room temperature:
    - ρ depends upon collisional processes → when T increases → more collisions → ρ increases
  - Very low temperature:
    - Mean free path dominated by impurities or defects in the material → ~ constant with temperature.
    - With sufficient purity, some metals become superconductors

Why Does Current Flow?

Instead of thinking of Electric Field, think of potential difference across the conductor

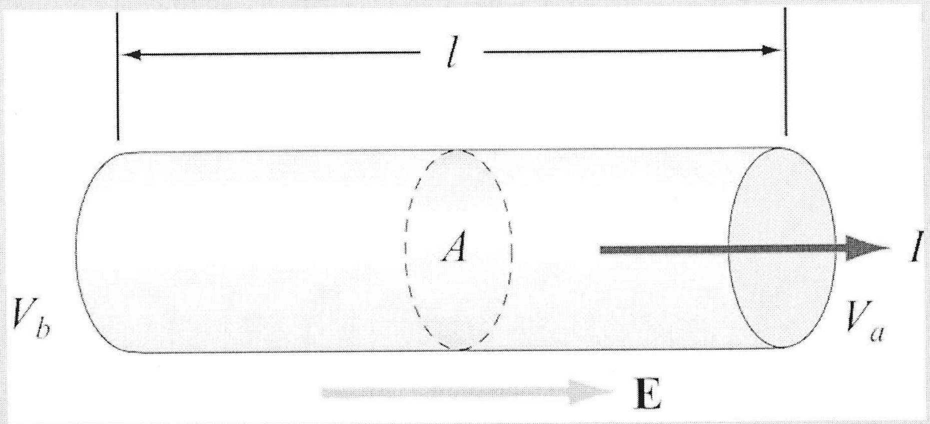


$$\Delta V = V_b - V_a$$

Ohm's Law

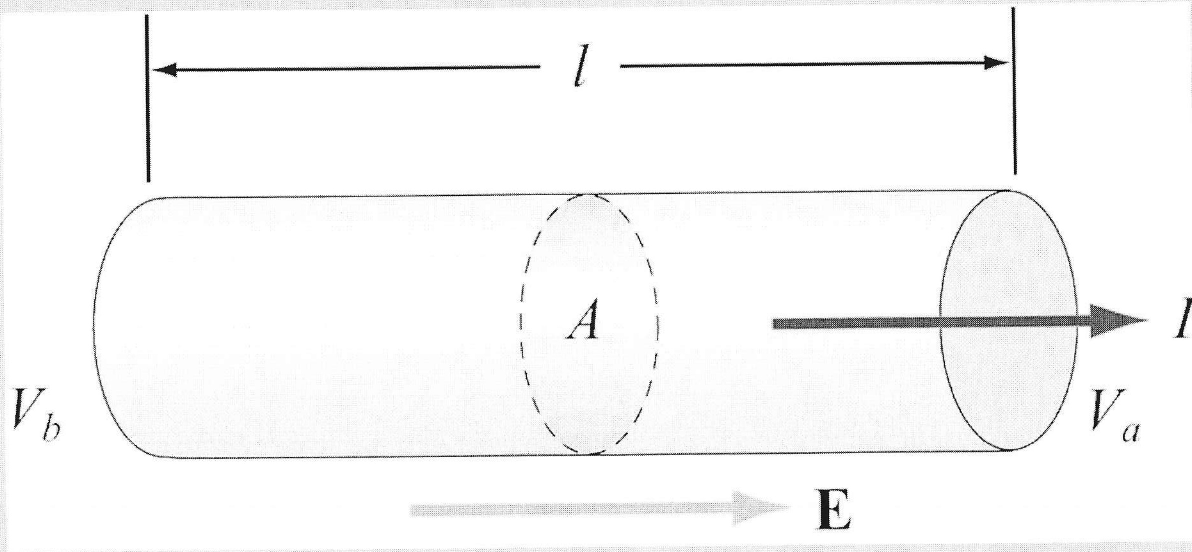
What is relationship between ΔV and current?

$$\Delta V = V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E\ell$$



$$\left.\begin{aligned} J &= \frac{E}{\rho} = \frac{\Delta V / \ell}{\rho} \\ J &= \frac{I}{A} \end{aligned}\right\} \Rightarrow \Delta V = I\left(\frac{\rho \ell}{A}\right) \equiv IR$$

Ohm's Law



$$\Delta V = IR$$

$$R = \frac{\rho \ell}{A}$$

R has units of Ohms (Ω) = Volts/Amp

More Useful Form of Ohm's Law

To obtain a more useful form of Ohm's law for practical applications, consider a segment of straight wire of length l and cross-sectional area A, as shown in Figure 6.2.1.

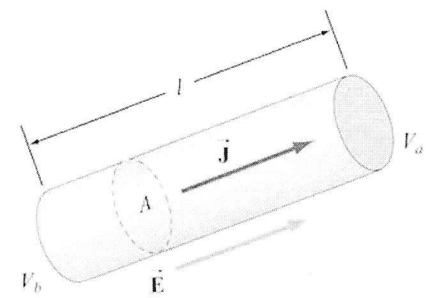


Figure 6.2.1 A uniform conductor of length l and potential difference $\Delta V = V_b - V_a$.

Suppose a potential difference $\Delta V = V_b - V_a$ is applied between the ends of the wire, creating an electric field $\vec{\mathbf{E}}$ and a current I. Assuming $\vec{\mathbf{E}}$ to be uniform, we then have

$$\Delta V = V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = El \tag{6.2.3}$$

The current density can then be written as

$$J = \sigma E = \sigma \left(\frac{\Delta V}{l} \right) \tag{6.2.4}$$

With $J = I / A$, the potential difference becomes

$$\Delta V = \frac{l}{\sigma} J = \left(\frac{l}{\sigma A} \right) I = RI \tag{6.2.5}$$

where

$$R = \frac{\Delta V}{I} = \frac{l}{\sigma A} \tag{6.2.6}$$

is the resistance of the conductor. The equation

$$\Delta V = IR \tag{6.2.7}$$

is the "macroscopic" version of the Ohm's law. The SI unit of R is the ohm (Ω, Greek letter Omega), where

$$1\,\Omega \equiv \frac{1\text{ V}}{1\text{ A}} \tag{6.2.8}$$

Once again, a material that obeys the above relation is ohmic, and non-ohmic if the relation is not obeyed. Most metals, with good conductivity and low resistivity, are ohmic. We shall focus mainly on ohmic materials.

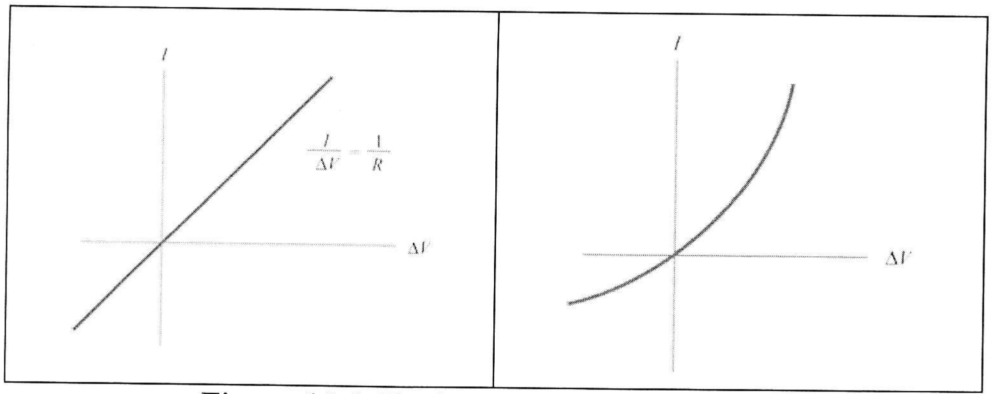


Figure 6.2.2 Ohmic vs. Non-ohmic behavior.

The resistivity ρ of a material is defined as the reciprocal of conductivity,

$$\rho = \frac{1}{\sigma} = \frac{m_e}{ne^2\tau} \tag{6.2.9}$$

From the above equations, we see that ρ can be related to the resistance R of an object by

$$\rho = \frac{E}{J} = \frac{\Delta V / l}{I / A} = \frac{RA}{l}$$

or

$$R = \frac{\rho l}{A} \tag{6.2.10}$$

The resistivity of a material actually varies with temperature T. For metals, the variation is linear over a large range of T:

$$\rho = \rho_0\left[1 + \alpha(T - T_0)\right] \tag{6.2.11}$$

where α is the *temperature coefficient of resistivity*. Typical values of ρ, σ and α (at $20°\text{C}$) for different types of materials are given in the Table below.

Resistance of a Hollow Cylinder

Consider a hollow cylinder of length L and inner radius a and outer radius b, as shown in Figure 6.5.3. The material has resistivity ρ.

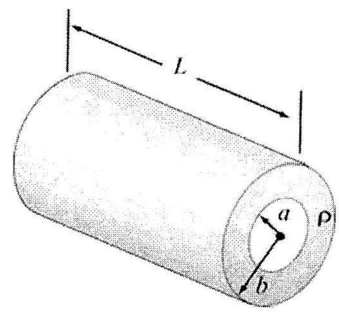


Figure 6.5.3 A hollow cylinder.

(a) Suppose a potential difference is applied between the ends of the cylinder and produces a current flowing parallel to the axis. What is the resistance measured?

(b) If instead the potential difference is applied between the inner and outer surfaces so that current flows radially outward, what is the resistance measured?

Solution:

(a) When a potential difference is applied between the ends of the cylinder, current flows parallel to the axis. In this case, the cross-sectional area is $A = \pi(b^2 - a^2)$, and the resistance is given by

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi(b^2 - a^2)}$$

(b) Consider a differential element which is made up of a thin cylinder of inner radius r and outer radius $r + dr$ and length L. Its contribution to the resistance of the system is given by

$$dR = \frac{\rho\, dl}{A} = \frac{\rho\, dr}{2\pi r L}$$

where $A = 2\pi r L$ is the area normal to the direction of current flow. The total resistance of the system becomes

$$R = \int_a^b \frac{\rho\, dr}{2\pi r L} = \frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)$$

Electric Energy and Power

Consider a circuit consisting of a battery and a resistor with resistance R (Figure 6.3.1). Let the potential difference between two points a and b be $\Delta V = V_b - V_a > 0$. If a charge Δq is moved from a through the battery, its electric potential energy is increased by $\Delta U = \Delta q\,\Delta V$. On the other hand, as the charge moves across the resistor, the potential energy is decreased due to collisions with atoms in the resistor. If we neglect the internal resistance of the battery and the connecting wires, upon returning to a the potential energy of Δq remains unchanged.

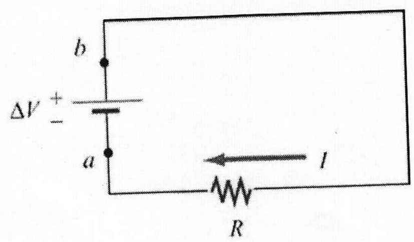
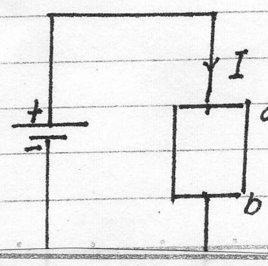


Figure 6.3.1 A circuit consisting of a battery and a resistor of resistance R.

5. Power in electric circuit



If $V_a > V_b$ dq $(a \to b)$

Electric potential energy decreases by $dq\,V_{ab} = dq\,V$

$dU = dq\,V = I\,dt\,V$

$P = IV$
↳ energy transfer rate

black box ↔ resistance

$V = IR$

$\Rightarrow P = I^2 R = \frac{V^2}{R}$

energy transfer from electric potential energy to heat energy → Joule heating.

6. Temperature dependence of resistivity

$\rho - \rho_0 = \rho_0 \alpha (T - T_0)$

↓ temperature coefficient of resistivity.

Material	Resistivity ρ $(\Omega \cdot m)$	Conductivity σ $(\Omega \cdot m)^{-1}$	Temperature Coefficient α $(°C)^{-1}$
Elements Silver	1.59×10^{-8}	6.29×10^{7}	0.0038
Copper	1.72×10^{-8}	5.81×10^{7}	0.0039
Aluminum	2.82×10^{-8}	3.55×10^{7}	0.0039
Tungsten	5.6×10^{-8}	1.8×10^{7}	0.0045
Iron	10.0×10^{-8}	1.0×10^{7}	0.0050
Platinum	10.6×10^{-8}	1.0×10^{7}	0.0039
Alloys Brass	7×10^{-8}	1.4×10^{7}	0.002
Manganin	44×10^{-8}	0.23×10^{7}	1.0×10^{-5}
Nichrome	100×10^{-8}	0.1×10^{7}	0.0004
Semiconductors Carbon (graphite)	3.5×10^{-5}	2.9×10^{4}	−0.0005
Germanium (pure)	0.46	2.2	−0.048
Silicon (pure)	640	1.6×10^{-3}	−0.075
Insulators Glass	$10^{10}-10^{14}$	$10^{-14}-10^{-10}$	
Sulfur	10^{15}	10^{-15}	
Quartz (fused)	75×10^{16}	1.33×10^{-18}	

6.3 Electrical Energy and Power

Summary

- The **electric current** is defined as:

$$I = \frac{dQ}{dt}$$

- The average current in a conductor is

$$I_{\text{avg}} = nqv_d A$$

where n is the number density of the charge carriers, q is the charge each carrier has, v_d is the **drift speed**, and A is the cross-sectional area.

- The **current density** J through the cross sectional area of the wire is

$$\vec{\mathbf{J}} = nq\vec{\mathbf{v}}_d$$

- Microscopic **Ohm's law**: the current density is proportional to the electric field, and the constant of proportionality is called **conductivity** σ:

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$$

- The reciprocal of conductivity σ is called **resistivity** ρ:

$$\rho = \frac{1}{\sigma}$$

- Macroscopic Ohm's law: The **resistance** R of a conductor is the ratio of the potential difference ΔV between the two ends of the conductor and the current I:

$$R = \frac{\Delta V}{I}$$

- Resistance is related to resistivity by

$$R = \frac{\rho l}{A}$$

where l is the length and A is the cross-sectional area of the conductor.

- The **drift velocity** of an electron in the conductor is

$$\vec{\mathbf{v}}_d = -\frac{e\vec{\mathbf{E}}}{m_e}\tau$$

where m_e is the mass of an electron, and τ is the average time between successive collisions.

- The resistivity of a metal is related to τ by

$$\rho = \frac{1}{\sigma} = \frac{m_e}{ne^2\tau}$$

- The temperature variation of resistivity of a conductor is

$$\rho = \rho_0\left[1 + \alpha\left(T - T_0\right)\right]$$

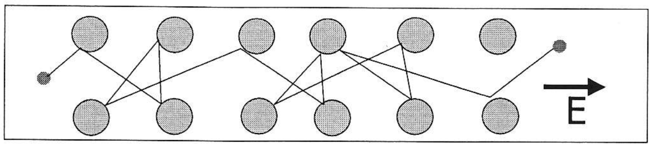
where α is the **temperature coefficient of resistivity**.

- **Power**, or rate at which energy is delivered to the resistor is

$$P = I\Delta V = I^2 R = \frac{(\Delta V)^2}{R}$$

Thoughts on Ohm's law

- Ohm's law in microscopic formulation: $\vec{J} = \sigma \vec{E}$
  - In plain English:
    - A constant electric field creates a steady current: $\vec{E} \propto \vec{v}$
    - Does this make sense? $\vec{F} = m\vec{a} \Rightarrow \vec{E} \propto \vec{a}$
- Charges are moving in an effectively viscous medium
  - As sky diver in free fall: first accelerate, then reach constant v
  - Why? Charges are accelerated by E but then bump into nuclei and are scattered → the average behavior is a uniform drift



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Motion of electrons in conductor

N electrons are moving in a material immersed in **E**

- Two components contribute to the momentum:
  - Random collision velocity u_0: $\vec{p}_{Random} = m\vec{u}_0$
  - Impulse due to electric field: $\vec{p}_E = q\vec{E}t$
- The average momentum is:

$$\langle p \rangle = m\langle u \rangle = \frac{1}{N}\sum_{i=1}^{N}(m\vec{u}_i + q\vec{E}t_i) = m\frac{1}{N}\sum_{i=1}^{N}\vec{u}_i + q\vec{E}\frac{1}{N}\sum_{i=1}^{N}t_i$$

- For large N: $\sum_{i=1}^{N}\vec{u}_i \to 0$ → $m\langle u \rangle = \frac{q\vec{E}}{N}\sum_{i=1}^{N}t_i \equiv q\vec{E}\tau$
- Where $\tau \equiv \frac{1}{N}\sum_{i=1}^{N}t_i$ is the average time between 2 collisions
  - Property of the material

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Conductivity

- From this derivation we can read off the conductivity

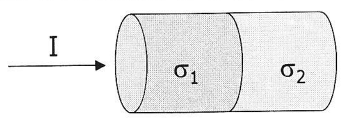
$$\begin{cases} \vec{J} = nq\langle \vec{u} \rangle \\ m\langle \vec{u} \rangle = q\vec{E}\tau \end{cases} \Rightarrow \vec{J} = nq\frac{q\vec{E}\tau}{m} = \sigma\vec{E} \Rightarrow \boxed{\sigma = \frac{nq^2\tau}{m}}$$

- For multiple carriers:

$$\sigma = \sum_{i=1}^{N} \frac{n_k q_k^{\ 2} \tau_k}{m_k}$$

What if σ is not constant?

- Cylindrical wire made of 2 conductors with conductivity σ_1 and σ_2



- What is the consequence?
  - Current flowing must be the same in the whole cylinder

$$I = A\sigma_1 E_1 = A\sigma_2 E_2$$

→ Electric fields are different in the 2 regions

→ E discontinuous → surface layer σ_q at the boundary

$$\sigma_q = \frac{E_{surface}}{4\pi} = \frac{E_2 - E_1}{4\pi} = \frac{I(\rho_2 - \rho_1)}{4\pi A}$$

When conductivity changes there is the possibility that some charge accumulates somewhere. This is necessary to maintain steady flow.