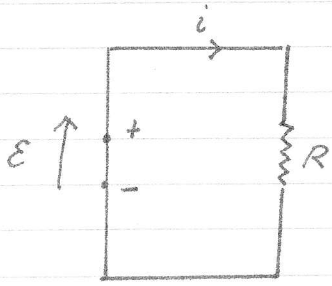
Chapter 27

This Time: DC Circuits

Chapter 27 Direct Current Circuits

1. EMF $\mathcal{E}$

$\mathcal{E}$ does work on charge carrier



$$\mathcal{E} = \frac{dW}{dq}$$

$$dW = dq\,\mathcal{E}$$

↑ work on the charge dq

$$\underbrace{dW}_{i^2R\,dt} = \mathcal{E}\,dq = \mathcal{E}\,i\,dt$$

energy conservation

$$\Rightarrow \quad \mathcal{E} = iR.$$

2. Potential method

$$V_a + \mathcal{E} - iR = V_a \qquad dq\,V_a + \underset{\substack{\parallel \\ dW}}{dq\,\mathcal{E}} - dq\,iR = dq\,V_a$$

$$dW = i^2R\,dt$$

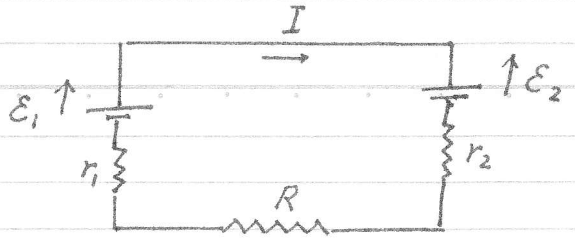
$$\Rightarrow \quad \mathcal{E} = iR$$

Loop rule: The algebraic sum of the changes in potential encountered in a complete traversal of any loop of circuit must be zero (Kirchkoff law)

Resistence rule
move through a resistence in the direction of the current the change in potential is $-iR$; in the opposite direction it is iR.

EMF rule: in the direction of EMF arrow $+\mathcal{E}$
in the opposite direction of the EMF $-\mathcal{E}$

3. Example

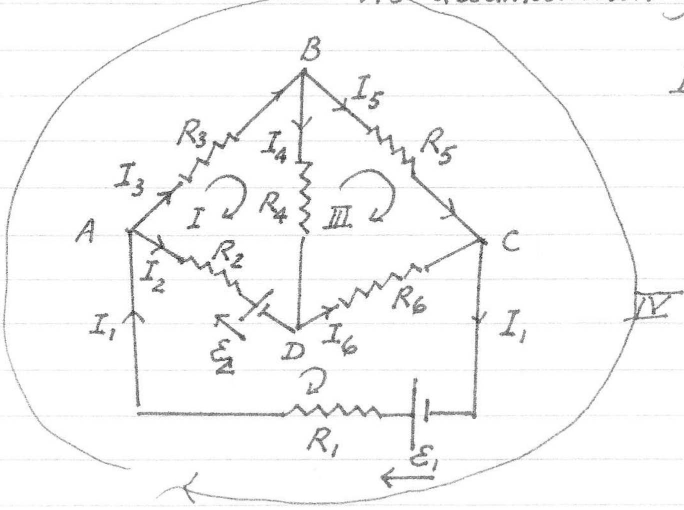


$$-\ \mathcal{E}_2 - ir_2 - iR - ir_1 + \mathcal{E}_1 = 0$$
$$-\ iR_{equ}.$$
$$R_{equ} = r_1 + r_2 + R$$

Single loop probtem

Resistences in series.

4. Junction rule: The sum of the currents entering any junction must be equal to the sum of the currents leaving the junction.
↓
~~no~~ accumulation of charge.



$I_1, I_2, I_3, I_4, I_5, I_6$

Junction equation

A: $I_2 + I_3 = I_1$

B: $I_3 = I_4 + I_5$

C: $I_5 + I_6 = I_1$

D: $I_4 + I_2 = I_6$

Loop rule

I: $-I_3 R_3 - I_4 R_4 + \mathcal{E}_2 + I_2 R_2 = 0$
II: $-I_5 R_5 + I_6 R_6 + I_4 R_4 = 0$
III: $\mathcal{E}_1 - I_1 R_1 - I_2 R_2 - \mathcal{E}_2 - I_6 R_6 = 0$

$$\mathcal{E}_1 - I_1 R_1 - I_3 R_3 - I_5 R_5 = 0$$

There are six unknowns
are seven unknowns $\Rightarrow$ not all loop equations are independent.

Choose three of these four equations, that are independent, with the junction equations $\Rightarrow$ solve for $I_1, \cdots I_6$

5. Resistances in parallel

See notes for the two examples.

6. R-C circuit

$$\frac{q(t)}{C} + R\frac{dq(t)}{dt} = \mathcal{E} \qquad t=0,\ q=0$$

$$R\frac{dq}{dt} = \mathcal{E} - \frac{q}{C}$$

$$-RC\frac{dq}{dt} = q - C\mathcal{E}$$

$$q' = q - C\mathcal{E}$$

$$\frac{dq'}{dt} = -\frac{1}{RC}q'$$

$$\frac{dq'}{q'} = -\frac{1}{RC}dt$$

$$\ln q' = -\frac{1}{RC}(t+\alpha)$$

$$q' = Ae^{-\frac{1}{RC}t}$$

$$= e^{-\frac{1}{RC}t}\underset{\uparrow\ A}{e^{-\frac{1}{RC}\alpha'}}$$

$$q = C\mathcal{E} + Ae^{-\frac{1}{RC}t}$$

$$t=0,\ q=0 \Rightarrow C\mathcal{E} + A = 0$$

$$q(t) = C\mathcal{E} - C\mathcal{E}e^{-\frac{1}{RC}t}$$

$$I(t) = \frac{\mathcal{E}}{R}e^{-\frac{1}{RC}t}.$$ charging the capacitor

分類：
編號： 28-4
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See note for discharge of capacitor

Direct Current Circuit

7.1 Introduction

Electrical circuits connect power supplies to *loads* such as resistors, motors, heaters, or lamps. The connection between the supply and the load is made by soldering with wires that are often called *leads*, or with many kinds of connectors and terminals. Energy is delivered from the source to the user on demand at the flick of a switch. Sometimes many circuit elements are connected to the same lead, which is the called a *common lead* for those elements. Various parts of the circuits are called circuit elements, which can be in series or in parallel, as we have already seen in the case of capacitors.

Elements are said to be in *parallel* when they are connected across the same potential difference (see Figure 7.1.1a).

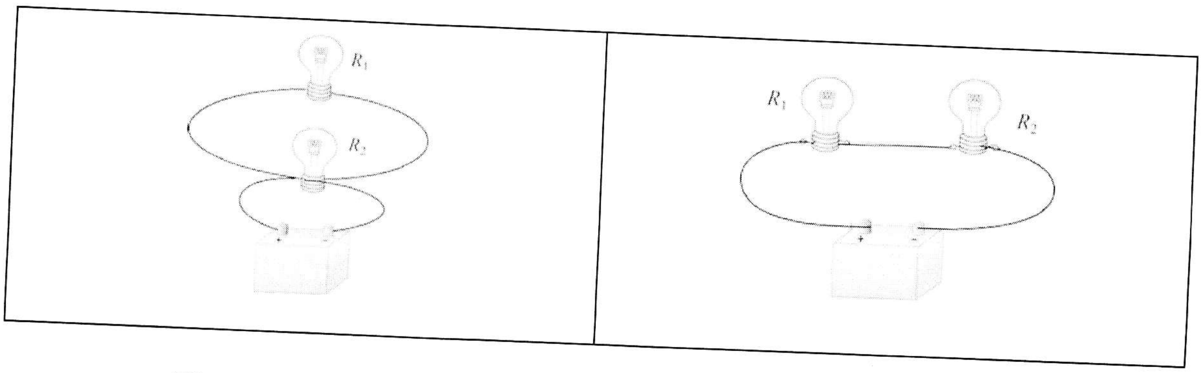
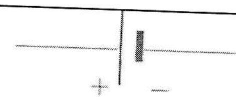
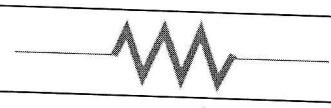
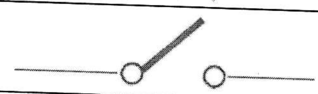


Figure 7.1.1 Elements connected (a) in parallel, and (b) in series.

Generally, loads are connected in parallel across the power supply. On the other hand, when the elements are connected one after another, so that the current passes through each element without any branches, the elements are in *series* (see Figure 7.1.1b).

There are pictorial diagrams that show wires and components roughly as they appear, and schematic diagrams that use conventional symbols, somewhat like road maps. Some frequently used symbols are shown below:

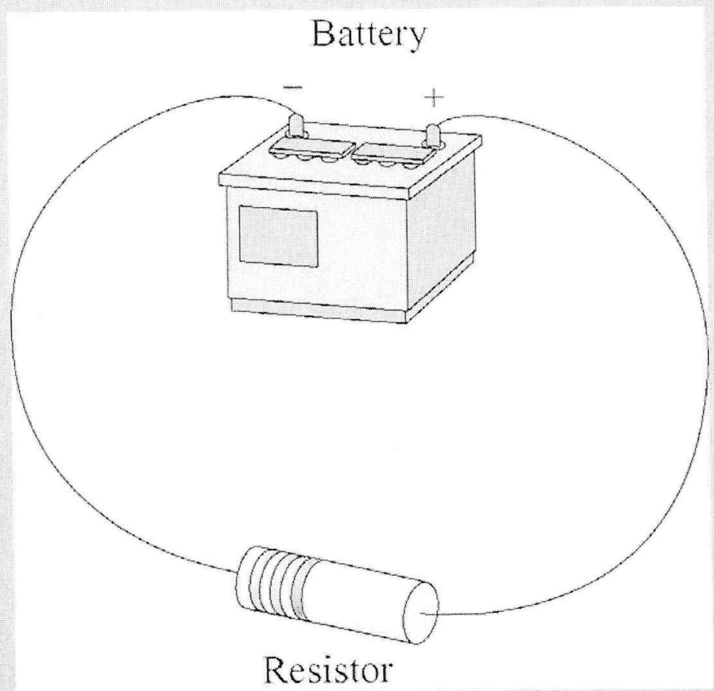
Voltage Source	
Resistor	
Switch	

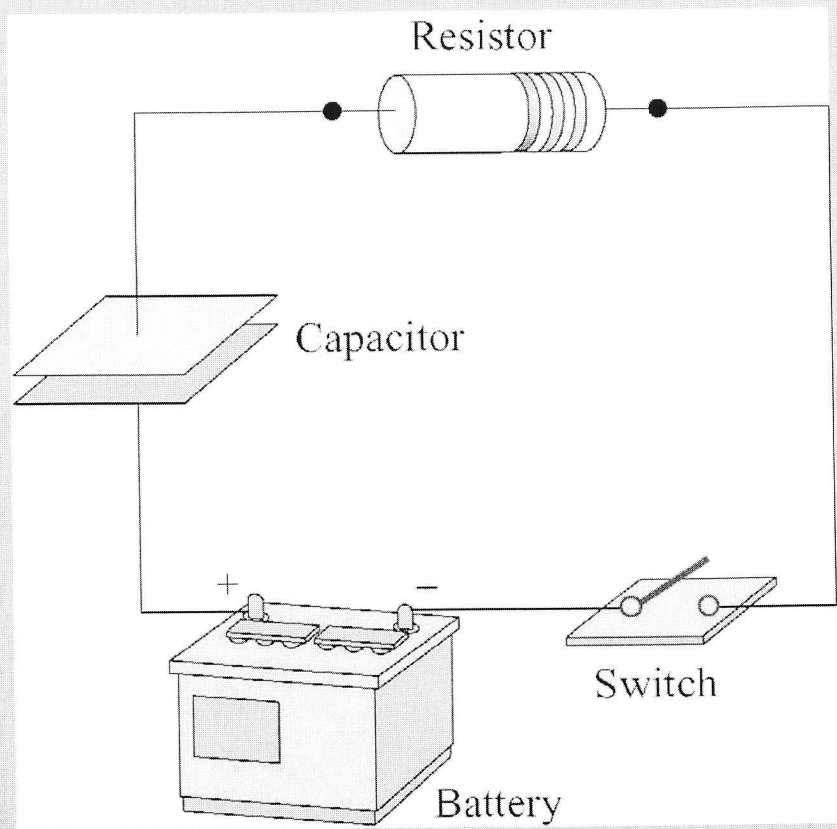
Often there is a switch in series; when the switch is open the load is disconnected; when the switch is closed, the load is connected.

One can have closed circuits, through which current flows, or open circuits in which there are no currents. Usually by accident, wires may touch, causing a *short circuit*. Most of the current flows through the short, very little will flow through the load. This may burn out a piece of electrical equipment such as a transformer. To prevent damage, a fuse or circuit breaker is put in series. When there is a short the fuse blows, or the breaker opens.

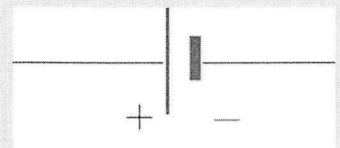
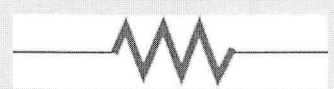
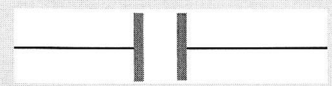
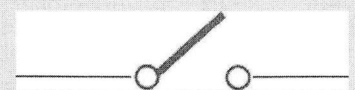
In electrical circuits, a point (or some common lead) is chosen as the *ground*. This point is assigned an arbitrary voltage, usually zero, and the voltage V at any point in the circuit is defined as the voltage difference between that point and ground.

Examples of Circuits





Symbols for Circuit Elements

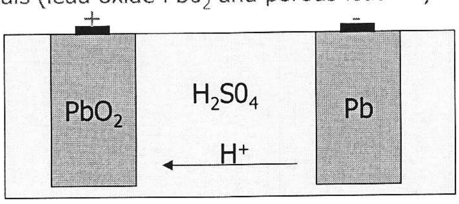
Battery	
Resistor	
Capacitor	
Switch	

EMF: Electromotive force

- What makes charges flow in circuits?
  - Potential difference ΔV
  - Source of charges
- This is what the EMF provides
  - NB: EMF=Electromotive force but it's <u>not</u> a force!!!
- Example of EMF: battery
  - Device that maintains separation of charges between 2 electrodes
  - Current flows inside via electrochemical reactions that produce ΔV

Car Battery

- Two terminals (lead oxide PbO_2 and porous lead Pb) in sulfuric acid (H_2SO_4)



- When immersed in acid, Pb provides free electrons:

$$Pb + HSO_4^- \rightarrow PbSO_4 + H^+ + 2e^-$$

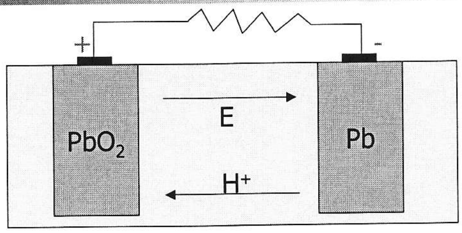
- At the lead oxide electron, this reaction is energetically favored:

$$PbO_2 + 3H^+ + HSO_4^- + 2e^- \rightarrow PbSO_4 + 2\,H_2O$$

- If it is possible for both e^- and H^+ to travel from one terminal to the other:

$$Pb + PbO_2 + H_2SO_4 \rightarrow 2PbSO_4 + 2\,H_2O \qquad +4.4\ eV$$

Car Battery (2)



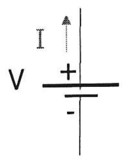
- When terminals are not connected: no flow of e-
- **E** in battery does not allow flow of H+ → inhibits reaction
- When terminals are connected: electrons start flowing freely
- Electric field is reduced → H^+ can flow → reaction occurs

EMF of battery: ϕ(+ terminal) – ϕ(- terminal): ΔV available to drive circuit

$$EMF = \int_{-\text{terminal}}^{+\text{terminal}} \vec{E} \cdot d\vec{s}$$

Convention

- We indicate EMF with this symbol:
  - Long side: + terminal
  - Short side: - terminal

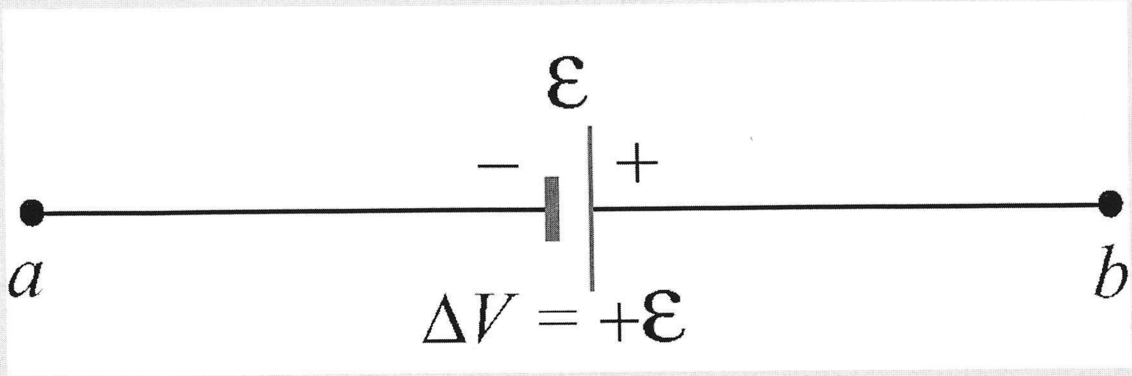


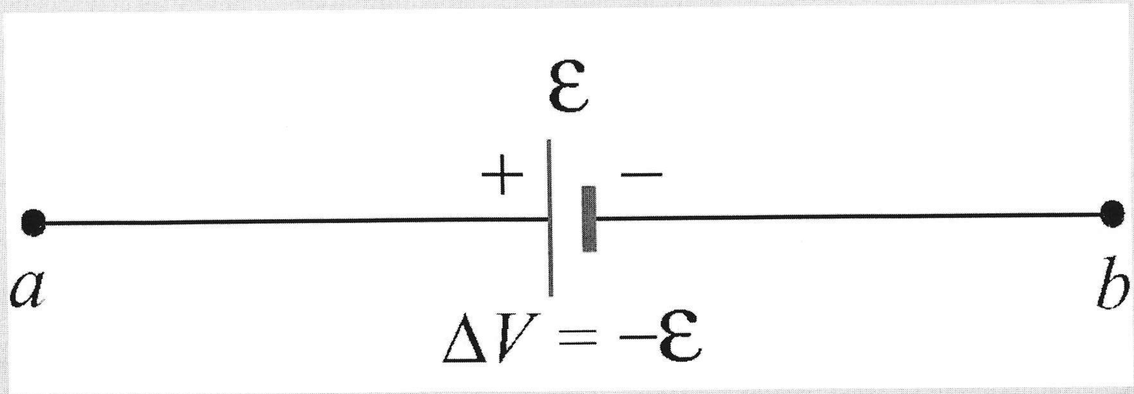
- The current flows from + to –
  - Counterintuitive if you think about it in terms of electrons…

Sign Conventions - Battery

Moving from the negative to positive terminal of a battery **increases** your potential

$$\Delta V = V_b - V_a$$



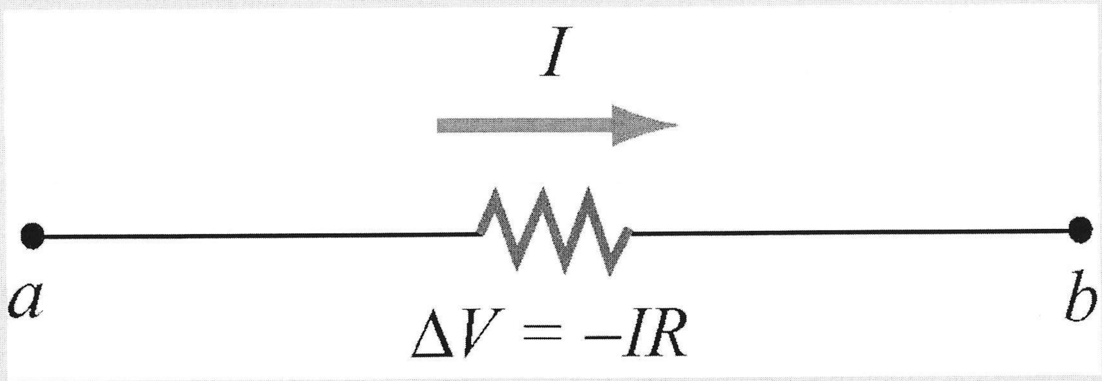


Think:

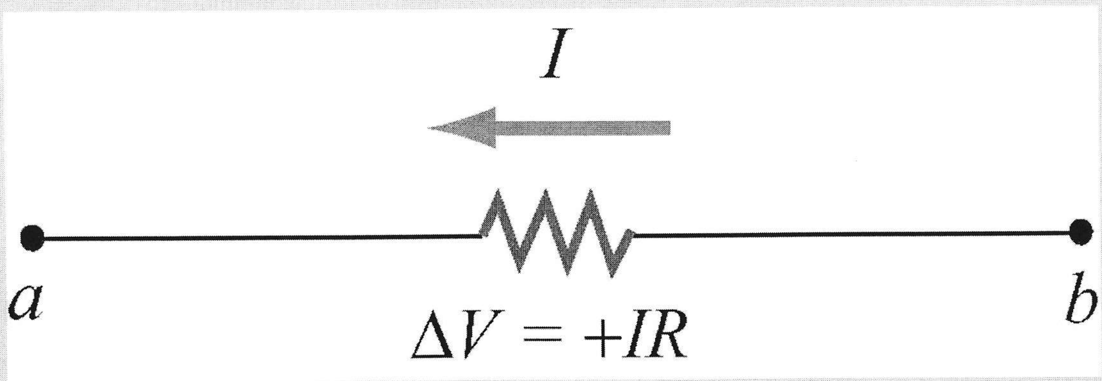
Ski Lift

Sign Conventions - Resistor

Moving across a resistor in the direction of current **decreases** your potential



$$\Delta V = V_b - V_a$$



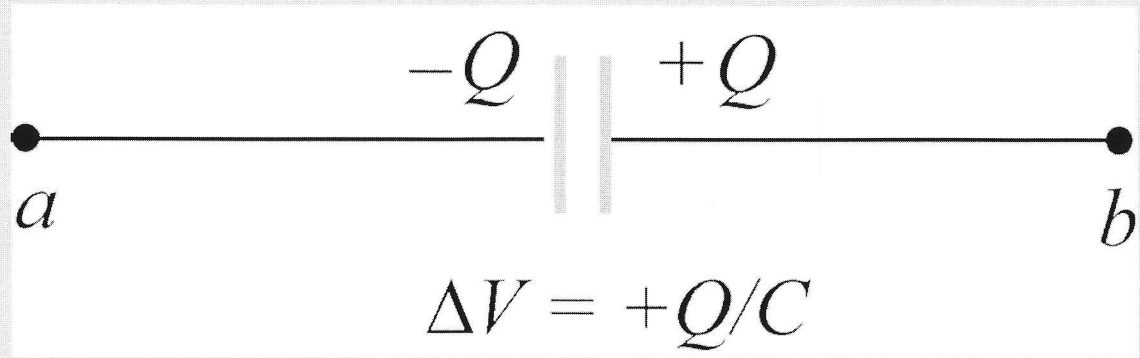
Think:

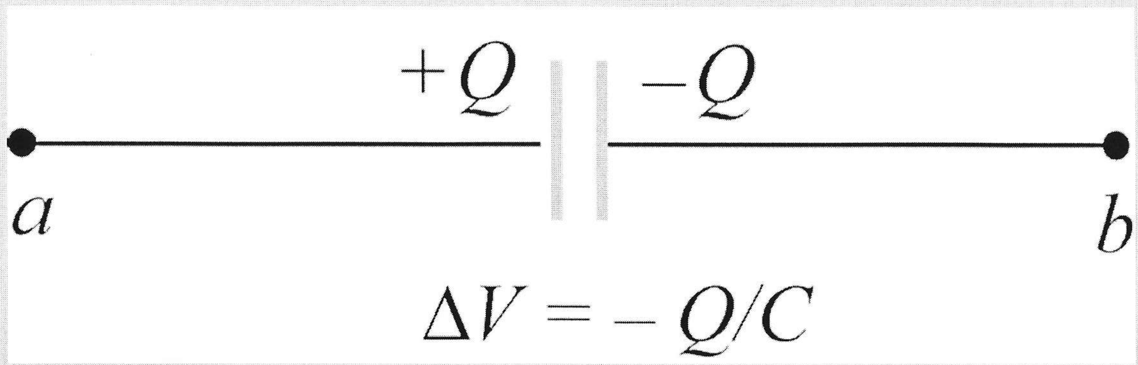
Ski Slope

Sign Conventions - Capacitor

Moving across a capacitor from the negatively to positively charged plate **increases** your potential

$$\Delta V = V_b - V_a$$





Think:

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7.2 Electromotive Force

In the last Chapter, we have shown that electrical energy must be supplied to maintain a constant current in a closed circuit. The source of energy is commonly referred to as the electromotive force, or emf (symbol ε). Batteries, solar cells and thermocouples are some examples of emf source. They can be thought of as a "charge pump" that moves charges from lower potential to the higher one. Mathematically emf is defined as

$$\varepsilon \equiv \frac{dW}{dq} \tag{7.2.1}$$

which is the work done to move a unit charge in the direction of higher potential. The SI unit for ε is the volt (V).

Consider a simple circuit consisting of a battery as the emf source and a resistor of resistance R, as shown in Figure 7.2.1.

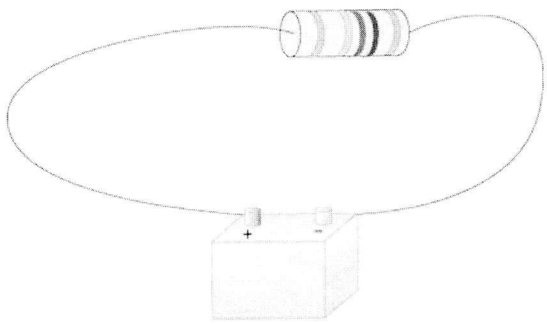


Figure 7.2.1 A simple circuit consisting of a battery and a resistor

Assuming that the battery has no internal resistance, the potential difference ΔV (or terminal voltage) between the positive and the negative terminals of the battery is equal to the emf ε. To drive the current around the circuit, the battery undergoes a discharging process which converts chemical energy to emf (recall that the dimensions of emf are the same as energy per charge). The current I can be found by noting that no work is done in moving a charge q around a closed loop due to the conservative nature of the electrostatic force:

$$W = -q\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0 \tag{7.2.2}$$

Let point a in Figure 7.2.2 be the starting point.

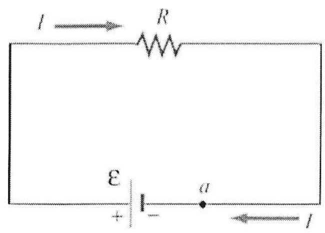


Figure 7.2.2

When crossing from the negative to the positive terminal, the potential increases by ε. On the other hand, as we cross the resistor, the potential decreases by an amount IR, and the potential energy is converted into thermal energy in the resistor. Assuming that the connecting wire carries no resistance, upon completing the loop, the net change in potential difference is zero,

$$\varepsilon - IR = 0 \tag{7.2.3}$$

which implies

$$I = \frac{\varepsilon}{R} \tag{7.2.4}$$

However, a real battery always carries an internal resistance r (Figure 7.2.3a),

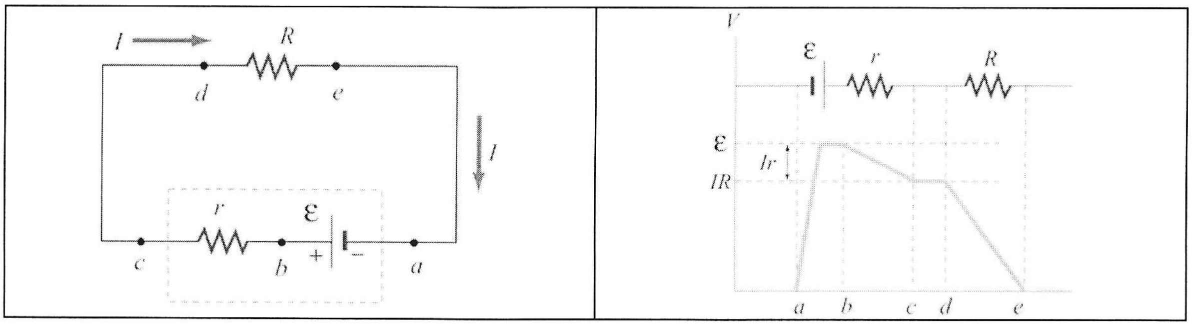


Figure 7.2.3 (a) Circuit with an emf source having an internal resistance r and a resistor of resistance R. (b) Change in electric potential around the circuit.

and the potential difference across the battery terminals becomes

$$\Delta V = \varepsilon - Ir \tag{7.2.5}$$

Since there is no net change in potential difference around a closed loop, we have

$$\varepsilon - Ir - IR = 0 \tag{7.2.6}$$

$$I = \frac{\varepsilon}{R + r} \tag{7.2.7}$$

Figure 7.2.3(b) depicts the change in electric potential as we traverse the circuit clockwise. From the Figure, we see that the highest voltage is immediately after the battery. The voltage drops as each resistor is crossed. Note that the voltage is essentially constant along the wires. This is because the wires have a negligibly small resistance compared to the resistors.

For a source with emf ε, the power or the rate at which energy is delivered is

$$P = I\varepsilon = I(IR + Ir) = I^2 R + I^2 r \tag{7.2.8}$$

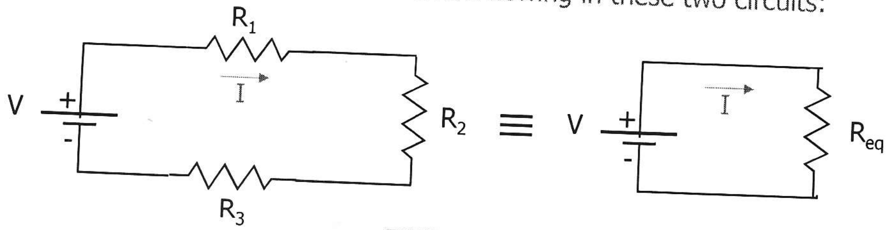
That the power of the source emf is equal to the sum of the power dissipated in both the internal and load resistance is required by energy conservation.

Resistors in series

- We implicitly derived an important result. We wrote:

$$V-\sum_{i=1,3} V_i = V - I\sum_{i=1,3} R_i = 0 \;\Rightarrow\; I = \frac{V}{R_1+R_2+R_3}$$

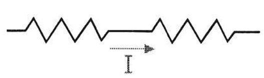
- What does it mean? Same current flowing in these two circuits:



$$R_{eq} = \sum_i R_i$$

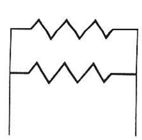
Resistors in parallel vs. in series

- Resistors in series:
  - The current flowing is one → add resistors → make the path harder → I decreases → R_{eq} increases → R_{eq} larger than any single resistor



$$R_{eq} = \sum_i R_i$$

- Resistors in parallel:
  - The current flows is many resistors → add resistors → make path easier → I increases → R_{eq} is smaller than any single resistor



$$\frac{1}{R_{eq}} = \sum_i \frac{1}{R_i}$$

7.3 Resistors in Series and in Parallel

The two resistors R_1 and R_2 in Figure 7.3.1 are connected in series to a voltage source ΔV. By current conservation, the same current I is flowing through each resistor.

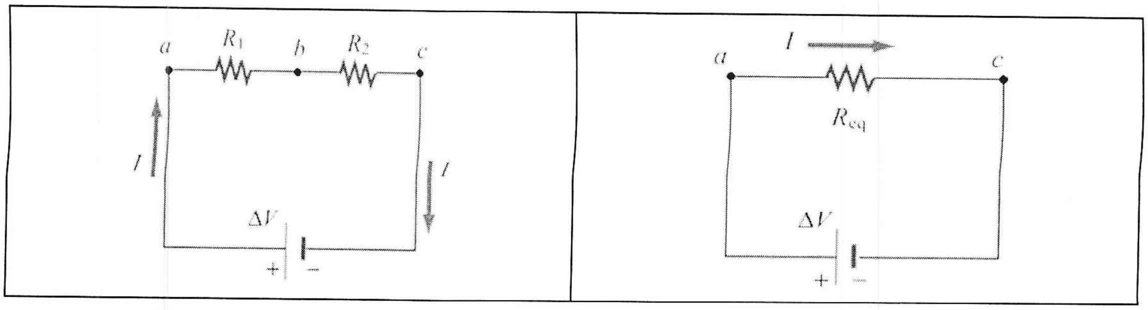


Figure 7.3.1 (a) Resistors in series. (b) Equivalent circuit.

The total voltage drop from a to c across both elements is the sum of the voltage drops across the individual resistors:

$$\Delta V = I R_1 + I R_2 = I(R_1 + R_2) \tag{7.3.1}$$

The two resistors in series can be replaced by one equivalent resistor R_{eq} (Figure 7.3.1b) with the identical voltage drop $\Delta V = I R_{eq}$ which implies that

$$R_{eq} = R_1 + R_2 \tag{7.3.2}$$

The above argument can be extended to N resistors placed in series. The equivalent resistance is just the sum of the original resistances,

$$R_{eq} = R_1 + R_2 + \cdots = \sum_{i=1}^{N} R_i \tag{7.3.3}$$

Notice that if one resistance R_1 is much larger than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the largest resistor R_1.

Next let's consider two resistors R_1 and R_2 that are connected in parallel across a voltage source ΔV (Figure 7.3.2a).

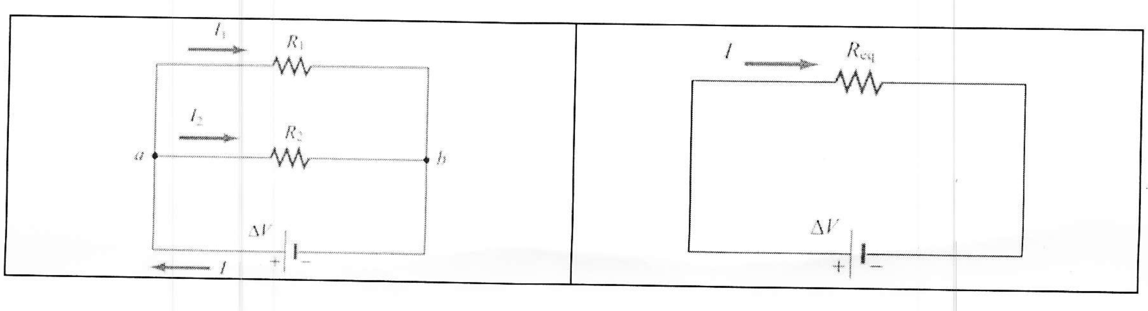


Figure 7.3.2 (a) Two resistors in parallel. (b) Equivalent resistance

By current conservation, the current I that passes through the voltage source must divide into a current I_1 that passes through resistor R_1 and a current I_2 that passes through resistor R_2. Each resistor individually satisfies Ohm's law, $\Delta V_1 = I_1 R_1$ and $\Delta V_2 = I_2 R_2$. However, the potential across the resistors are the same, $\Delta V_1 = \Delta V_2 = \Delta V$. Current conservation then implies

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \Delta V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{7.3.4}$$

The two resistors in parallel can be replaced by one equivalent resistor R_{eq} with $\Delta V = IR_{\text{eq}}$ (Figure 7.3.2b). Comparing these results, the equivalent resistance for two resistors that are connected in parallel is given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{7.3.5}$$

This result easily generalizes to N resistors connected in parallel

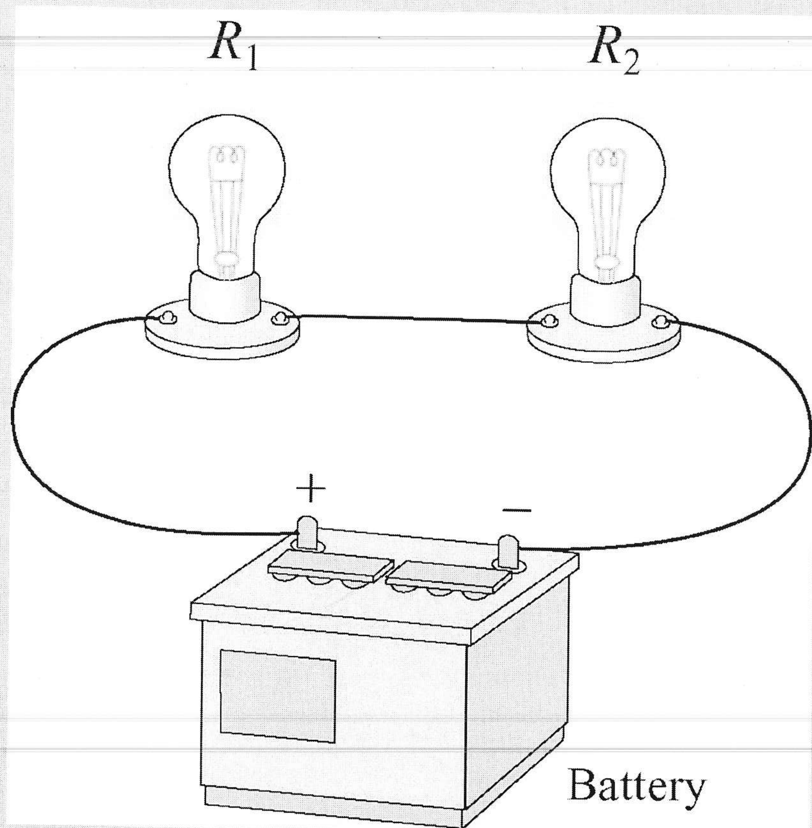
$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i} \tag{7.3.6}$$

When one resistance R_1 is much smaller than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the smallest resistor R_1. In the case of two resistors,

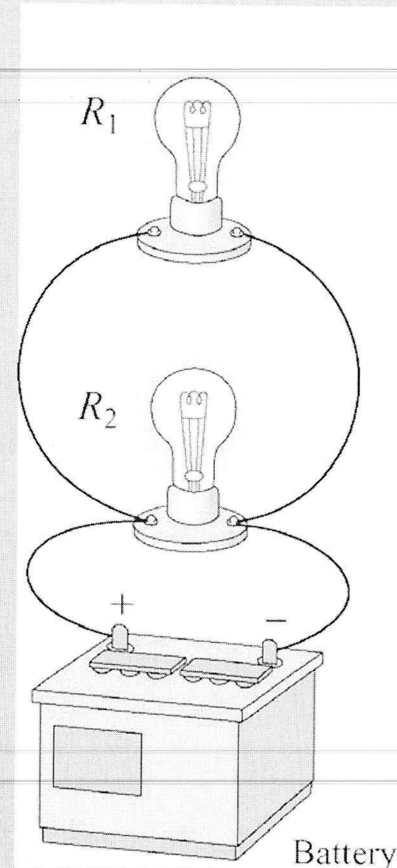
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \approx \frac{R_1 R_2}{R_2} = R_1$$

This means that almost all of the current that enters the node point will pass through the branch containing the smallest resistance. So, when a short develops across a circuit, all of the current passes through this path of nearly zero resistance.

Series vs. Parallel



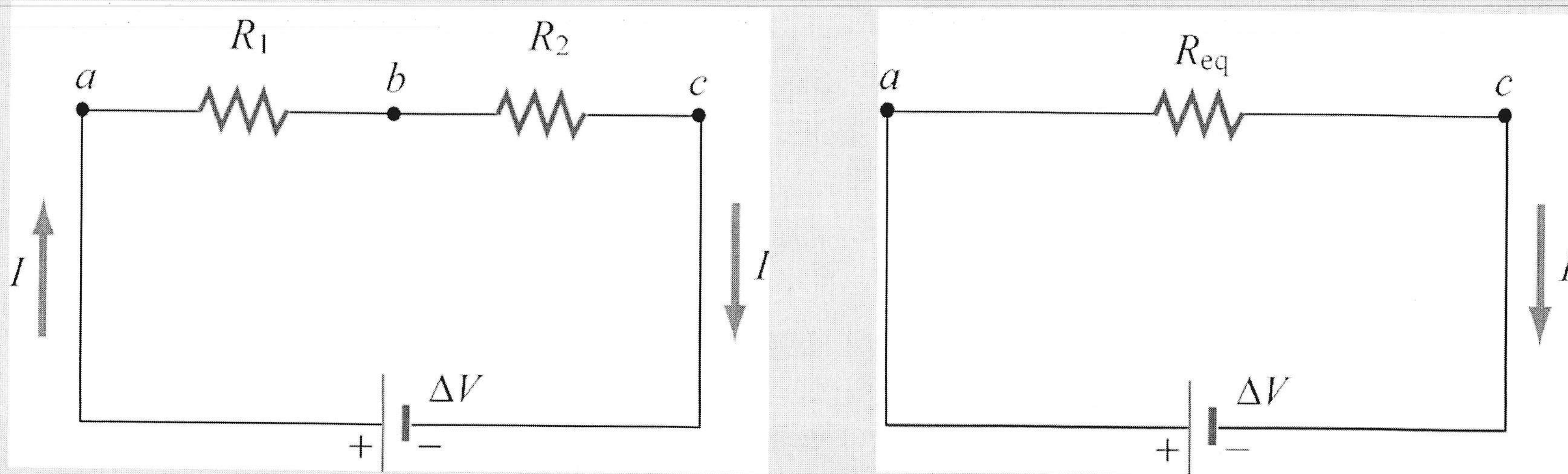
Series



Parallel

Resistors In Series

The same current *I* must flow through both resistors

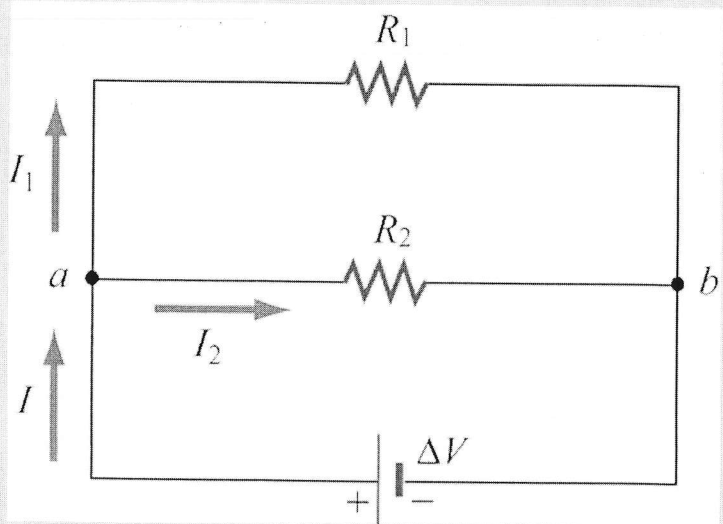


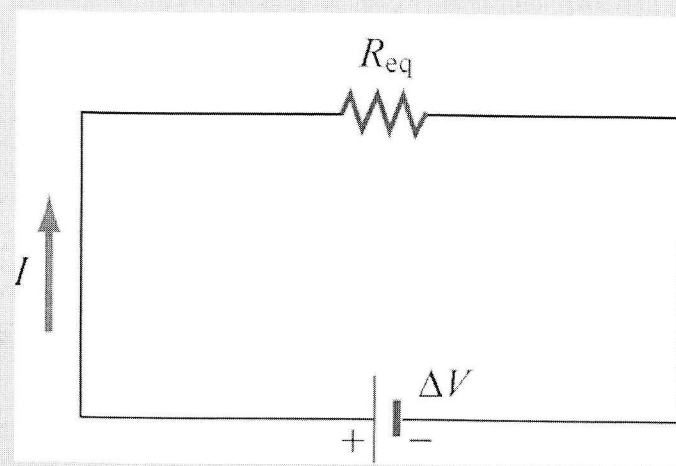
$$\Delta V = I\,R_1 + I\,R_2 = I(R_1 + R_2) = I\,R_{eq}$$

$$\boxed{R_{eq} = R_1 + R_2}$$

Resistors In Parallel

Voltage drop across the resistors must be the same





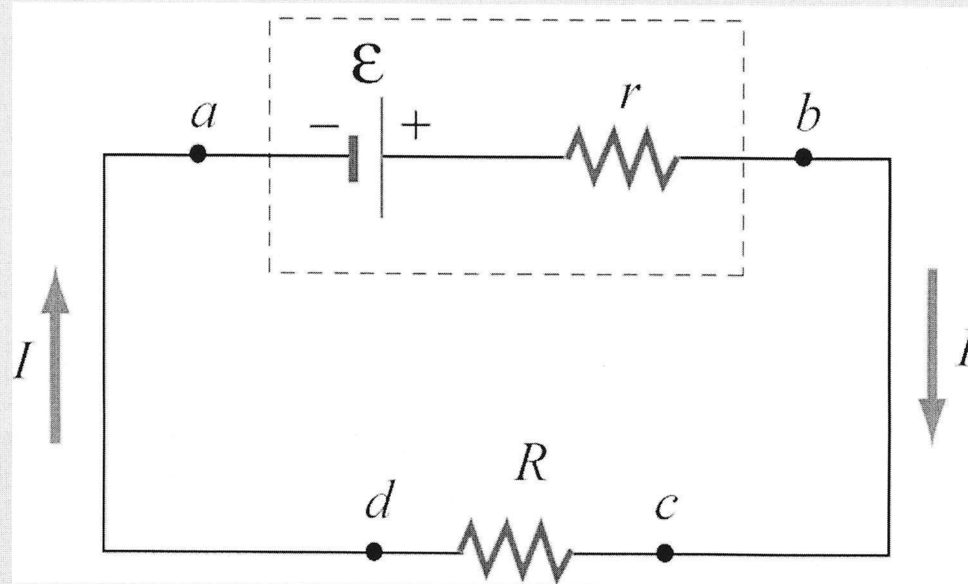
$$\Delta V = \Delta V_1 = \Delta V_2 = I_1 R_1 = I_2 R_2 = I R_{eq}$$

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \frac{\Delta V}{R_{eq}}$$

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

Internal Resistance

Real batteries have an internal resistance, r, which is small but non-zero



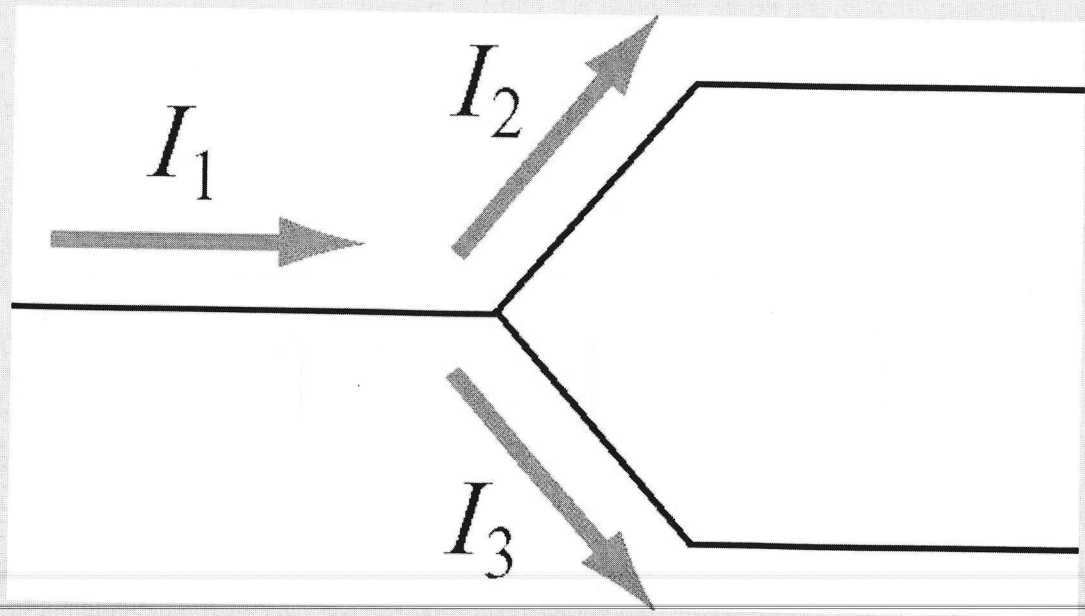
Terminal voltage: $\Delta V = V_b - V_a = \mathcal{E} - I\,r$

(Even if you short the leads you don't get infinite current)

Kirchhoff’s Loop Rules

Kirchhoff's Rules

1. Sum of currents entering any junction in a circuit must equal sum of currents leaving that junction.

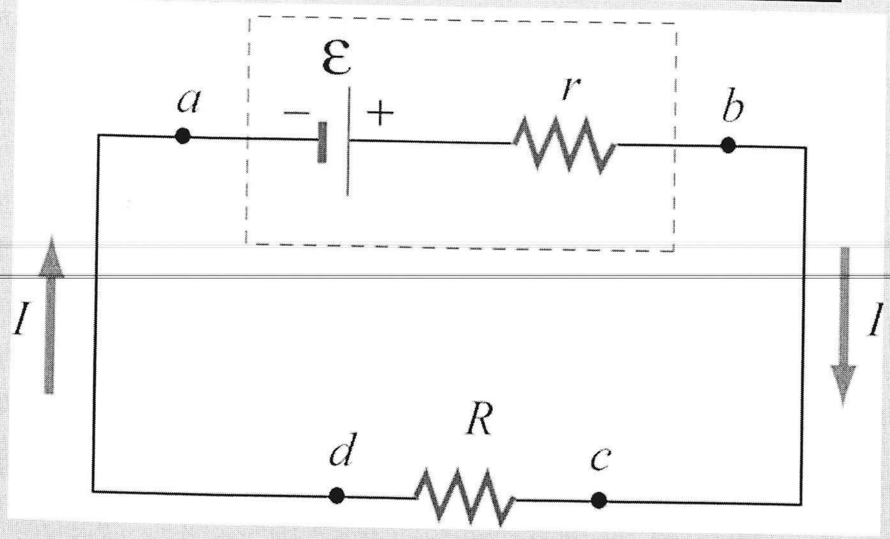


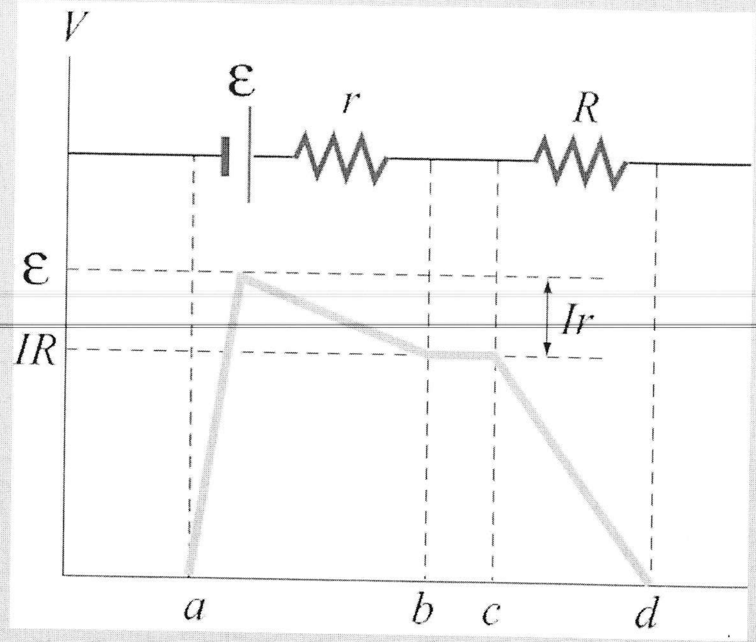
$$I_1 = I_2 + I_3$$

Kirchhoff's Rules

2. Sum of potential differences across all elements around any closed circuit loop must be zero.

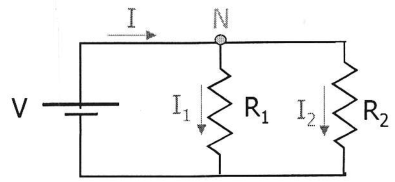
$$\Delta V = -\oint_{\text{Closed Path}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$$





Kirchhoff's first rule

- Let's now connect resistors in parallel:



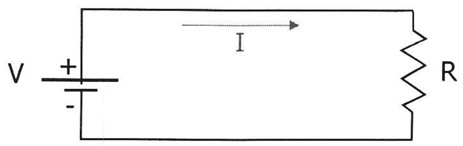
- At the node N the current I divides up into 2 pieces: I_1 and I_2

<u>Kirchhoff's first law:</u>
At any node, sum of the currents in = sum of the currents out

- In other words: there is no accumulation of charges in the circuit

Kirchhoff' second rule

- Close a battery on a resistor: simplest circuit!



- How much current flows in the circuit? Ohm's law: $I = \frac{V}{R}$
- When the current flows in a resistor there is a voltage drop $\Delta V = -IR$

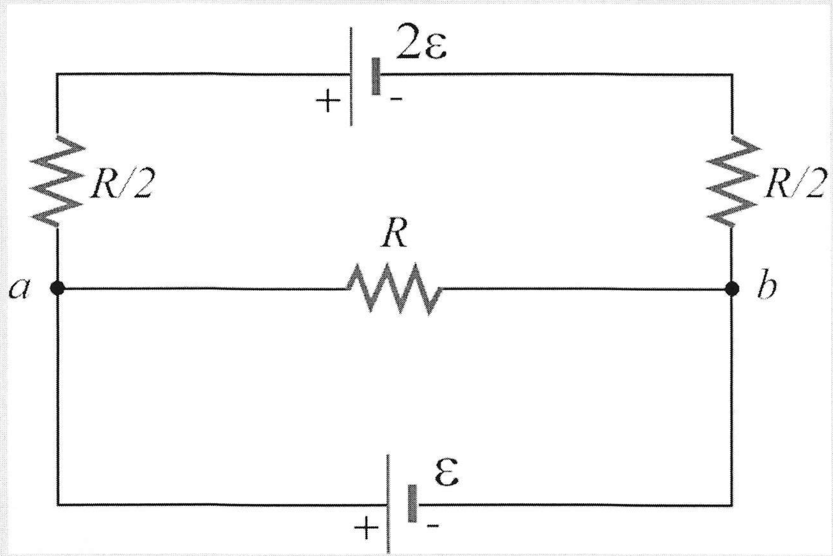
<u>Kirchhoff's second law:</u>
Around any closed loops, the sum of EMF and potential drops is 0

- Equivalent to say that Electrostatic filed is conservative: $\oint \vec{E} \cdot d\vec{s} = 0$

Steps of Solving Circuit Problem

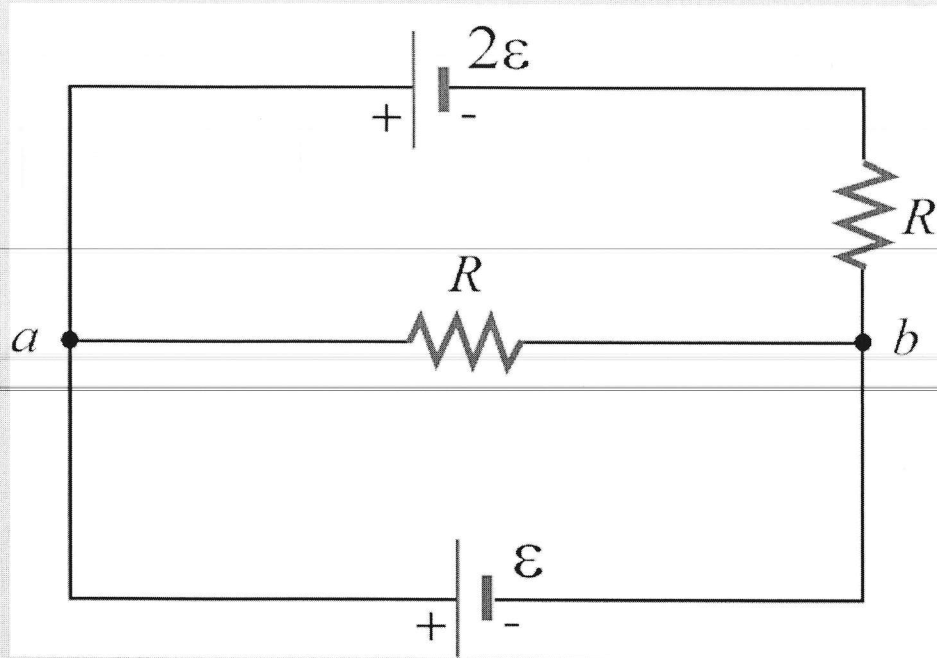
1. Straighten out circuit (make squares)
2. Simplify resistors in series/parallel
3. Assign current loops (arbitrary)
4. Write loop equations (1 per loop)
5. Solve

Example: Simple Circuit

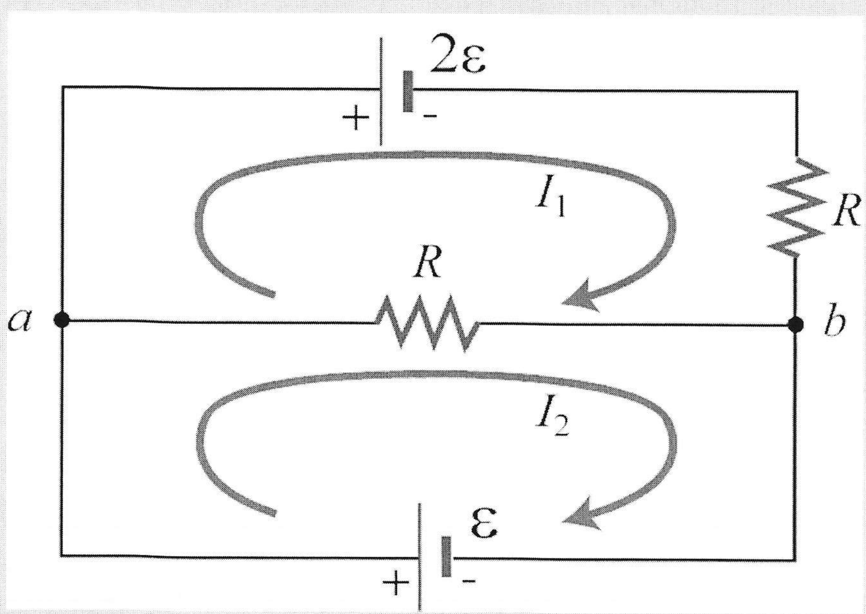


You can simplify resistors in series (but don't need to)

What is current through the bottom battery?



Example: Simple Circuit



Start at a in both loops

Walk in direction of current

$$-2\varepsilon - I_1 R - \left(I_1 - I_2\right) R = 0$$

$$-\left(I_2 - I_1\right) R + \varepsilon = 0$$

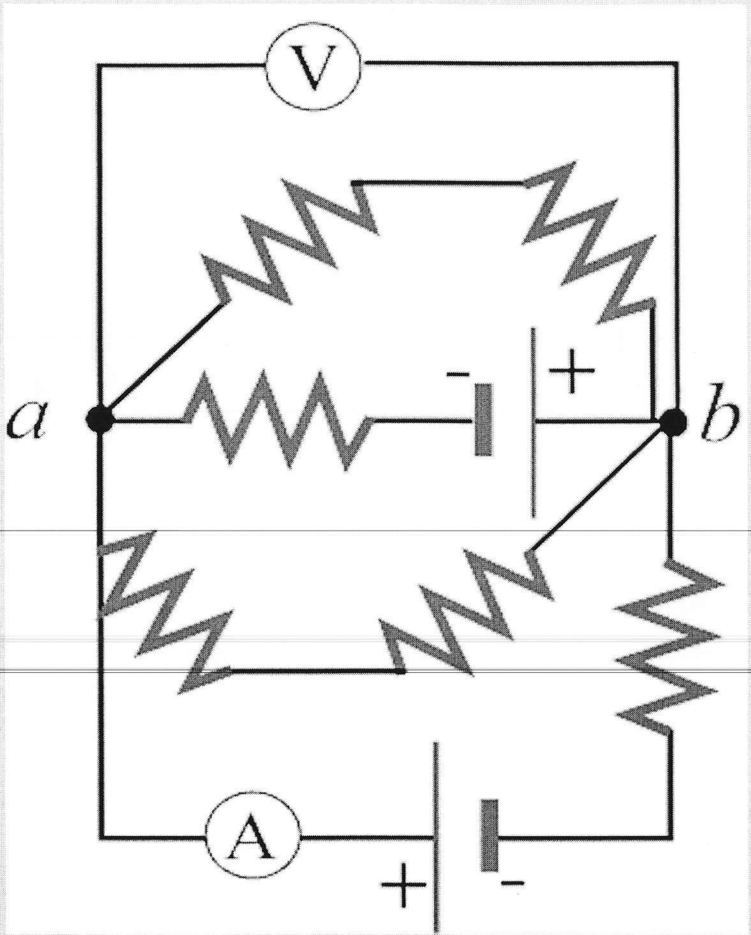
Add these: $-2\varepsilon - I_1 R + \varepsilon = 0 \rightarrow I_1 = \dfrac{-\varepsilon}{R}$

We wanted I_2: $\left(I_2 - I_1\right) R = \varepsilon \rightarrow I_2 = \dfrac{\varepsilon}{R} + I_1$

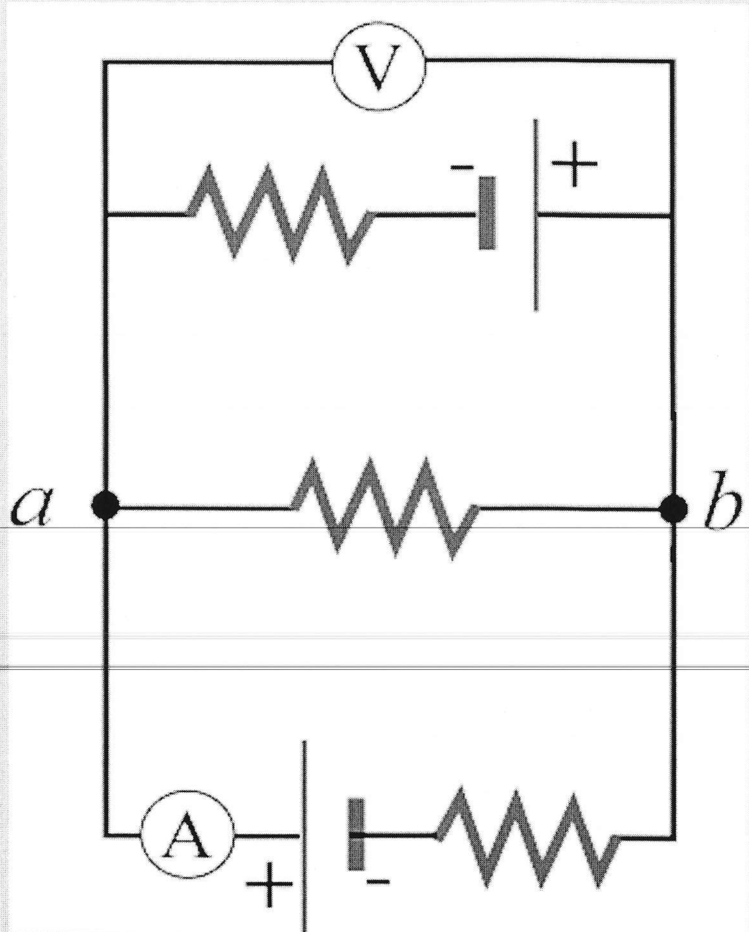
$$\boxed{I_2 = 0}$$

Group Problem: Circuit

Find meters' values. All resistors are *R*, batteries are $\mathcal{E}$



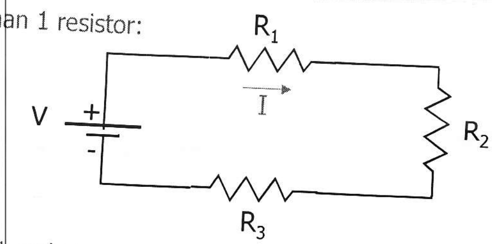
HARDER



EASIER

Solving circuits

- If we have more than 1 resistor:

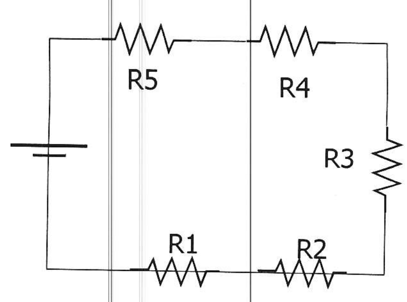


- Solve the circuit: determine currents and voltages everywhere
- What we know:
  - Current flowing in the circuit must be the same everywhere, or Q would accumulate somewhere
  - Voltage drop in the i^{th} resistor: $\Delta V_i = -IR_i$
  - Second Kirchhoff rule: $V - \sum_i V_i = 0$

$$V - \sum_{i=1,3} V_i = V - I \sum_{i=1,3} R_i = 0 \Rightarrow I = \frac{V}{R1 + R2 + R3}$$

Application (F13)

- What is the resistance of electrical components?



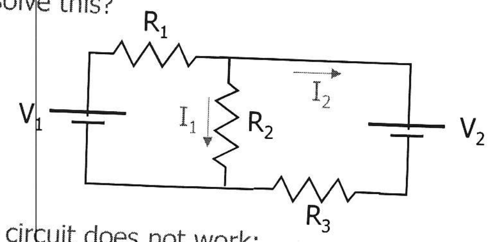
Elements of the circuit:

- Saline solution
- Resistor
- Diod
- light bulb
-Fluoreschent light

- How to measure knowing the current = 135 mV?
- You are given a voltmeter!

Slightly harder circuits

- How do we solve this?



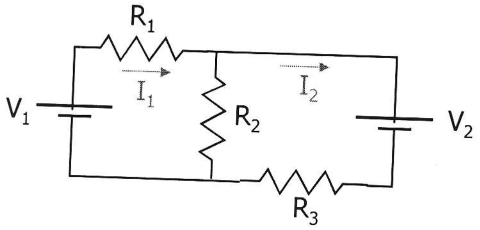
- Reducing the circuit does not work:
  - Series and parallels won't work
    - Because of second EMF
- But Kirchhoff still holds so:
  - Apply First Kirchhoff law to each node
  - Apply Second Kirchhoff law to each loop

Slightly harder circuits (2)

- Solution:
  - Left loop:
    - $V_1 - I_1R_1 - (I_1 - I_2)R_2 = 0$
  - Right loop:
    - $-V_2 - I_2R_3 - (I_2 - I_1)R_2 = 0$
  - Node:
    - I(in R2)=I1-I2
- Solving the system:

$$I1 = \frac{V1R3 + (V1 - V2)R2}{R1R2 + R1R3 + R2R3}$$

$$I2 = \frac{(V1 - V2)R2 + V2R1}{R1R2 + R1R3 + R2R3}$$



How to go through a loop?
- Assign current direction (arbitrary)
- Choose a path (clockwise or ACW)
- EMF: >0 when -→+ and <0 when +→-
- V always drops on R

Power

Electrical Power

Power is change in energy per unit time

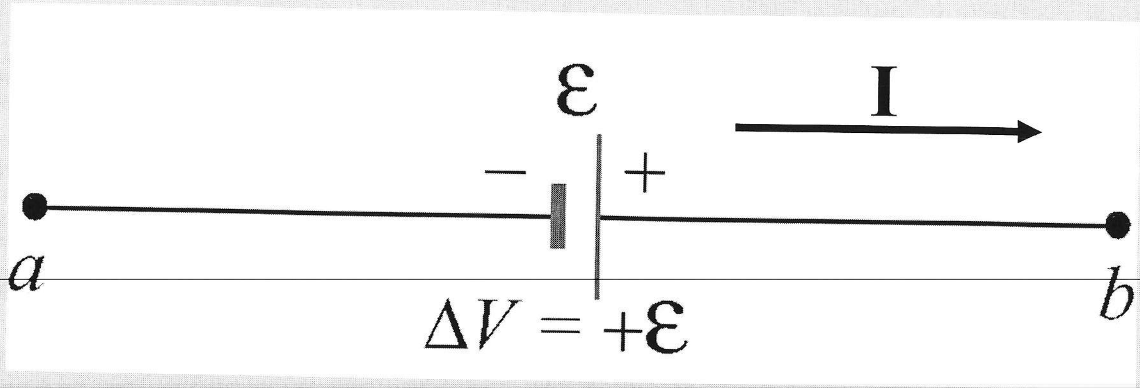
So power to move current through circuit elements:

$$P = \frac{d}{dt}U = \frac{d}{dt}\left(q\Delta V\right) = \frac{dq}{dt}\Delta V$$

$$\boxed{P = I\,\Delta V}$$

Power - Battery

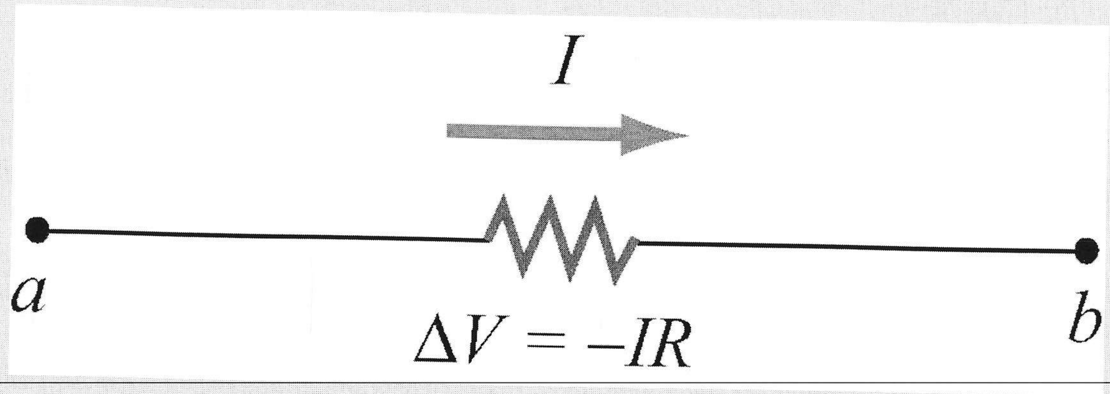
Moving from the negative to positive terminal of a battery **increases** your potential. If current flows in that direction the battery **supplies** power



$$P_{\text{supplied}} = I\,\Delta V = I\,\varepsilon$$

Power - Resistor

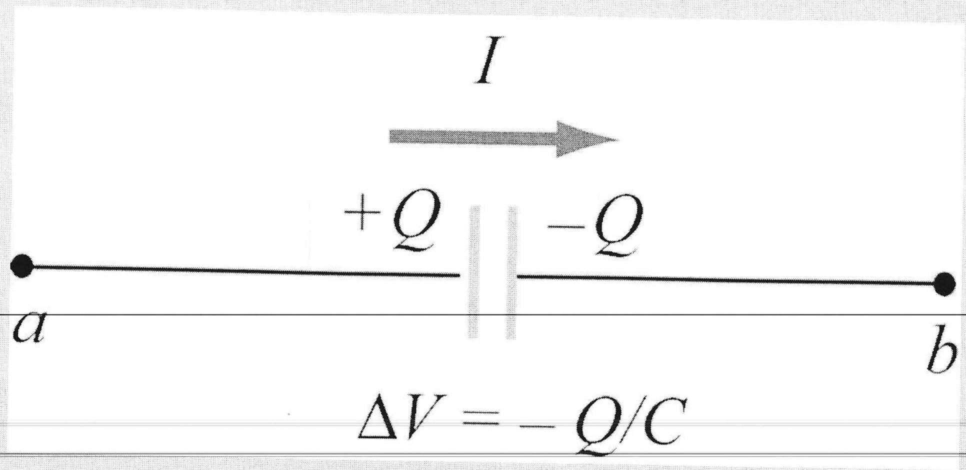
Moving across a resistor in the direction of current **decreases** your potential. Resistors always **dissipate** power



$$P_{\text{dissipated}} = I\,\Delta V = I^2 R = \frac{\Delta V^2}{R}$$

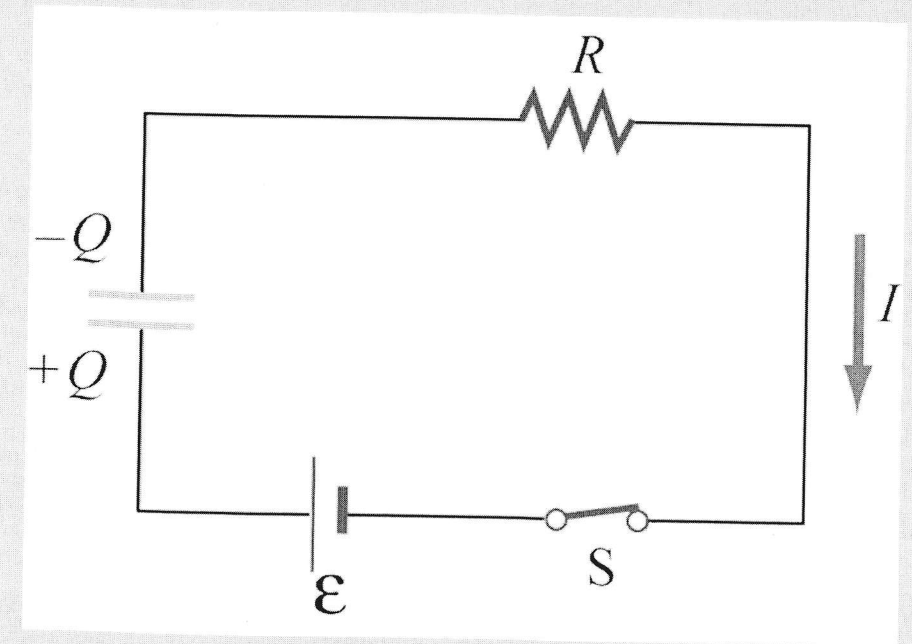
Power - Capacitor

Moving across a capacitor from the positive to negative plate **decreases** your potential. If current flows in that direction the capacitor **absorbs** power (stores charge)



$$P_{\text{absorbed}} = I\,\Delta V = \frac{dQ}{dt}\frac{Q}{C} = \frac{d}{dt}\frac{Q^2}{2C} = \frac{dU}{dt}$$

Energy Balance



$$\mathcal{E} - \frac{Q}{C} - IR = 0$$

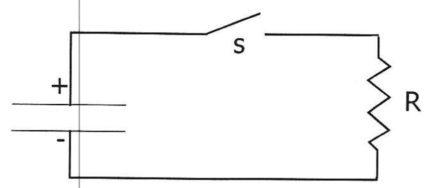
Multiplying by *I:*

$$\mathcal{E}I = I^2 R + \frac{Q}{C}\frac{dQ}{dt} = I^2 R + \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}\right)$$

(power delivered by battery) = (power dissipated through resistor)
+ (power absorbed by the capacitor)

Capacitors in circuits

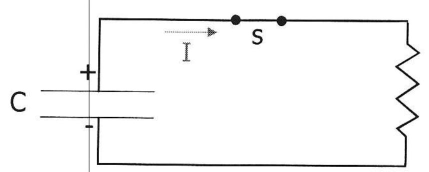
- A new way of looking at problems:
  - Until now: charges at rest or constant currents
  - When capacitors present: currents vary over time



- Consider the following situation:
  - A capacitor C with charge Q_0 → $V_0=Q_0/C$
  - A resistor R in series connected by switch s
- What happens when switch s is closed?

Discharging capacitors: qualitative

- Before switch s is closed:
  - Difference in potential between C plates: V_0
  - No current circulating in the circuit (open)



- After switch s is closed:
  - Difference in potential between capacitor plates will induce current I
  - As I flows, charge difference on capacitor decreases → V_C decreases → I decreases over time

Discharging capacitors: quantitative

- Apply second Kirchhoff's law:
  - EMF supplied by capacitor C: V=Q/C
    - NB: this is true at any moment in time → Q(t) → V(t)
  - Voltage drop on the resistor: -IR

$$\frac{Q}{C} - IR = 0$$

- Not useful in this form since I=I(Q)
  - I=-dQ/dt (- sign because C is losing charge)

$$\frac{Q}{C} + \frac{dQ}{dt}R = 0$$

- Easy integral yields to exponential decay of the charge:

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

How to integrate RC circuits

To solve $\frac{Q}{C} + \frac{dQ}{dt}R = 0$, rewrite as: $\frac{dQ}{Q} = -\frac{dt}{RC}$

Integrate both sides:

$$\int_{Q_0}^{Q(t)} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC}$$

$$\ln \frac{Q(t)}{Q_0} = -\frac{t}{RC}$$

$$\boxed{Q(t) = Q_0 e^{-\frac{t}{RC}}}$$

NB: τ=RC is called "decay constant" of the circuit

Solution of RC circuit

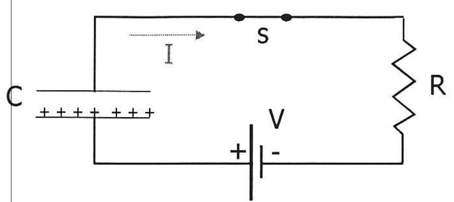
- Solution: $Q(t) = Q_0 e^{-\frac{t}{RC}}$
  - Exponential decay of charge stored in capacitor
  - τ=RC is called "decay constant" of the circuit
  - After a time RC, the charge decreased by 1/e w.r.t. original value
  - Units of RC:
    - cgs: [R]= statvolt s /esu; [C]=esu/statvolt → [RC]=s
    - SI: [R]=V/A; [C]=C/V; A=C/s → [RC]=s
- Derive the current:

$$I(t) = -\frac{dQ}{dt} = -Q_0 \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = \frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

  - Same exponential decay as for Q(t)

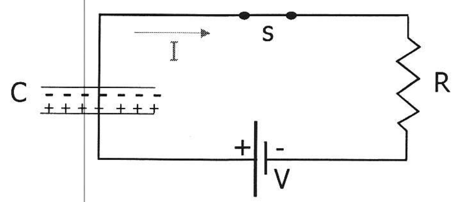
Charging capacitors

- Now 3 elements in circuit: EMF, capacitor and resistor
  - Capacitor starts uncharged



- What happens when switch s is closed?
  - When s is closed, current will suddenly flow and C will charge
  - As C charges, E opposite to EMF builds up and slows down current
  - I(t) stops when V_C reaches V

Charging capacitor: solve the circuit



- Solve using Kirchhoff's second law: $V - \frac{Q}{C} - IR = 0$
  - I(t)=+dQ/dt
  - NB: + because the capacitor is now charging!
- First order differential equation $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$
- Solution: $Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$

Details of integration

To solve $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$, rewrite as: $\frac{dQ}{dt} = -\frac{(Q - CV)}{RC}$

Setting: $Q' = Q - CV$

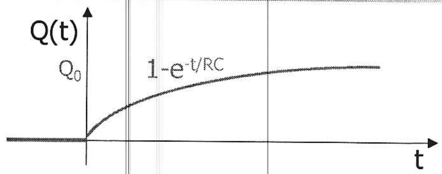
$$\Rightarrow \frac{dQ'}{Q'} = -\frac{dt}{RC}$$

Integrating between t=0 and t:

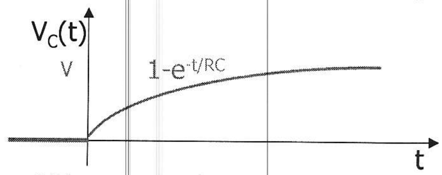
$$\int_{Q=0}^{Q=Q(t)} \frac{dQ'}{Q'} = -\int_{t=0}^{t=t} \frac{dt}{RC} \Rightarrow \ln\frac{Q(t)\text{-}CV}{\text{-}CV} = -\frac{t}{RC} \Rightarrow \frac{Q(t)\text{-}CV}{CV} = -e^{-\frac{t}{RC}}$$

$$\boxed{Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)}$$

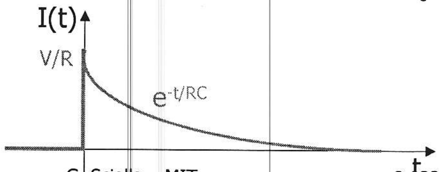
Graphical solution



$$Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$



$$V_C(t) = Q(t)/C = V\left(1 - e^{-\frac{t}{RC}}\right)$$



$$I(t) = \frac{dQ(t)}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}$$

Important comments

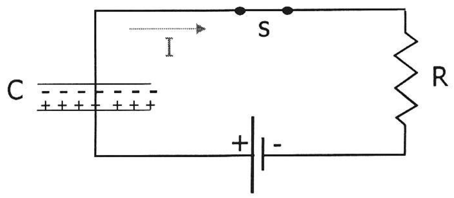
- Solution of RC circuit: $V_C(t) = V\left(1 - e^{-\frac{t}{RC}}\right)$; $I(t) = \frac{V}{R}e^{-\frac{t}{RC}}$
- Are Kirchhoff's laws valid at any moment in time?

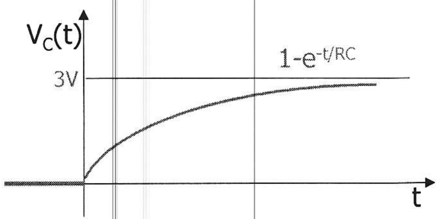
$$V - \frac{Q}{C} - IR = V - V\left(1 - e^{-\frac{t}{RC}}\right) - R\frac{V}{R}e^{-\frac{t}{RC}} = 0 \quad \text{OK!}$$

- Asymptotic behavior of the capacitor:
  - At t=0: I=V/R as if C were a short circuit
  - At t=infinity, I=0 as if C were an open circuit
- Conclusion: no need to solve the differential equation!
  - Solution is an exponential with time constant RC
  - Asymptotic behavior of C gives initial/final values for V(t) and I(t)

Time constant of RC circuit (E9)

- Simple RC circuit with
  - $V_{EMF} = 3$ V
  - C = 1.3 F
  - R = 11.7 Ω
- Questions:
  - What are V_C and I?
  - Verify that time constant is RC





$$V_C(t) = V_{EMF}\left(1-e^{-\frac{t}{RC}}\right)$$

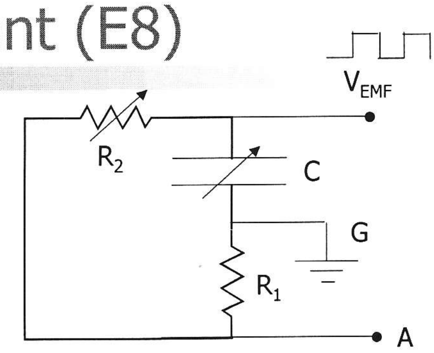
$RC = 15.2s$

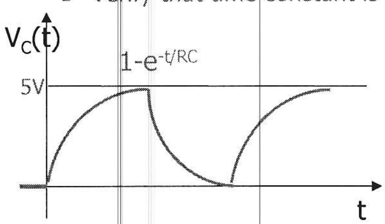
If formula is correct ⇒

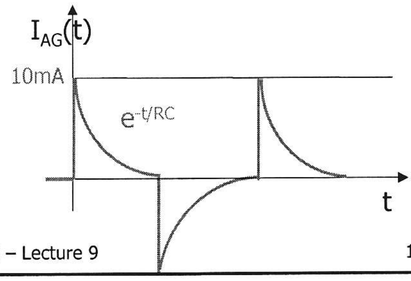
$V_C = V_{EMV}(1-1/e) = 1.9$ V when t=15.2

Verify time constant (E8)

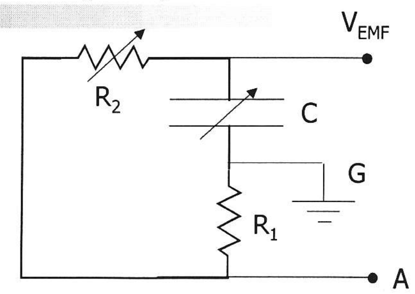
- RC circuit with
  - V_{EMF} = squared 5 V pulses
  - Variable C initially = 0.3 μF
  - Variable R_2 initially = 400 Ω
  - R_1 = 100 Ω
- Display on scope V_C and $I(R_1)$
  - Verify that time constant is RC







Verify time constant (E8)



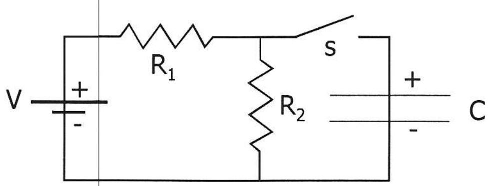
- RC circuit with
  - V_{EMF} = squared 5 V pulses
  - Variable C initially = 0.3 μF
  - Variable R_2 initially = 400 Ω
  - R_1 = 100 Ω

Assuming τ=RC...

- What happens when we double C?
  - τ_1=RC'=2RC=2τ_0 → V (I_{AG}) raises (falls) twice as fast
- How should we change R2 to have the same effect?
  - R'=2R=2(R_1+R_2') → R2': 400 → 900 Ω

More complicated RC circuits

- What if the RC circuit is more than just a series of R and C?
- Consider the following circuit:



- Calculate Q(t) on the capacitor
- Solution:
  - Kirckhoff's laws will solve it: TEDIOUS!
  - Use Thevenin's Theorem

7.7 Summary

- The equivalent resistance of a set of resistors connected in series:

$$R_{eq} = R_1 + R_2 + R_3 + \cdots = \sum_{i=1}^{N} R_i$$

- The equivalent resistance of a set of resistors connected in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i}$$

- **Kirchhoff's rules**:

(1) The sum of the currents flowing into a junction is equal to the sum of the currents flowing out of the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

(2) The algebraic sum of the changes in electric potential in a closed-circuit loop is zero.

$$\sum_{\text{closed loop}} \Delta V = 0$$

- In a charging capacitor, the charges and the current as a function of time are

$$q(t) = Q\left(1 - e^{-\frac{t}{RC}}\right), \qquad I(t) = \left(\frac{\varepsilon}{R}\right) e^{-t/RC}$$

- In a discharging capacitor, the charges and the current as a function of time are

$$q(t) = Q e^{-t/RC}, \quad I(t) = \left(\frac{Q}{RC}\right) e^{-t/RC}$$