Chapter 28

The Effects of Magnetic Field.

分類：
編號：
總號：

Magnetostatics
Outline
Introduction

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$ Lorentz equation

Definition of $\vec{E}$

Charge
ρ
Q

Stationary charge
⇓
constant electric fields
electrostatics

$$\vec{E} = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r}$$

· Coulomb law

·· $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

··· $\nabla \times \vec{E} = 0$

·· $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{enc}$
↓
Gauss's law

$\vec{E} = -\nabla V$
↓
scalar potential

Electric dipole $\vec{p}$

Multipole expansion of Scalar potential

Boundary conditions

Definition of $\vec{B}$

Current
$\vec{J}$
I

Stationary currents
⇓
constant magnetic field
magnetostatics

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}' \times \hat{s}}{s^2}$$

·Biot - Savart law.

·· $\nabla \times \vec{B} = \mu_0 \vec{J}$

··· $\nabla \cdot \vec{B} = 0$

·· $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

Ampére's law

$\vec{B} = \nabla \times \vec{A}$
↓
vector potential

Magnetic dipole $\vec{\mu}$

Multipole expansion of Vector potential.

Boundary conditions.

A New Topic: Magnetic Fields

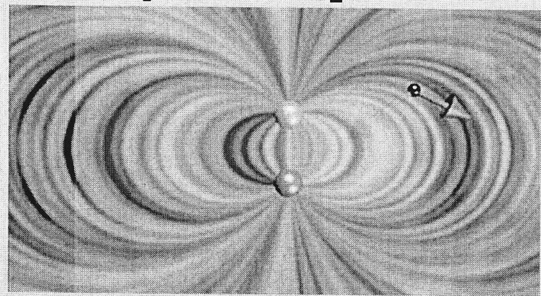
Gravitational – Electric Fields

	Mass m	Charge q (±)
Create:	$\vec{\mathbf{g}} = -G\frac{m}{r^2}\hat{\mathbf{r}}$	$\vec{\mathbf{E}} = k_e\frac{q}{r^2}\hat{\mathbf{r}}$
Feel:	$\vec{\mathbf{F}}_g = m\vec{\mathbf{g}}$	$\vec{\mathbf{F}}_E = q\vec{\mathbf{E}}$

Also saw…

Dipole **p**

Create:

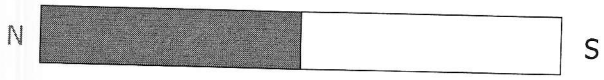


Feel:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$$

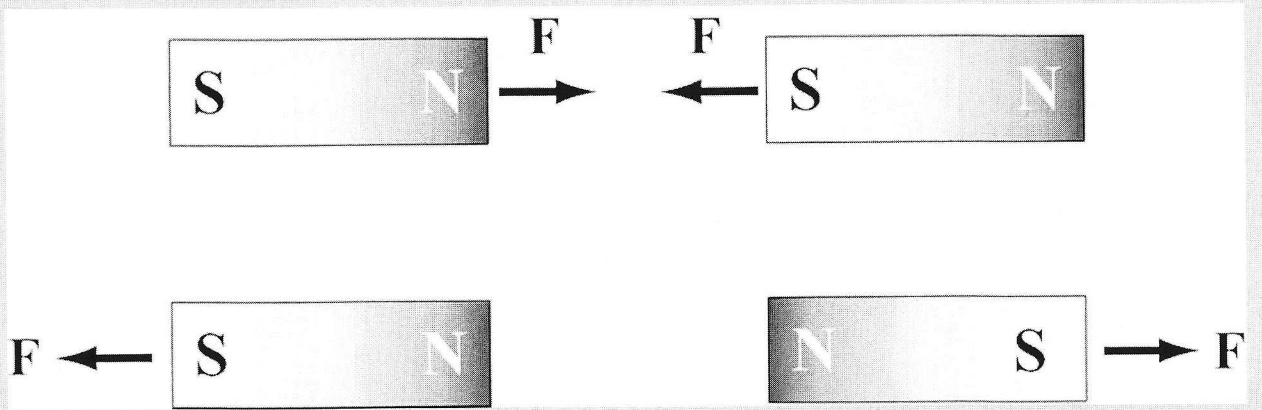
The Origins of Magnetism

- Ancient Greeks noticed that a piece of a mineral magnetite (an oxide of iron) had very special properties:
  - Could attract a piece of iron, but no effect on Au, Ag, Cu, etc
  - Can attract or repel piece of magnetite depending on relative orientation
- By the 12th century people could build a magnetic compass
  - A small magnetic needle is suspended so it can pivot around vertical axis
  - The needle will always come to rest with one end pointing North
  - By definition we call that end "North" and the other "South"



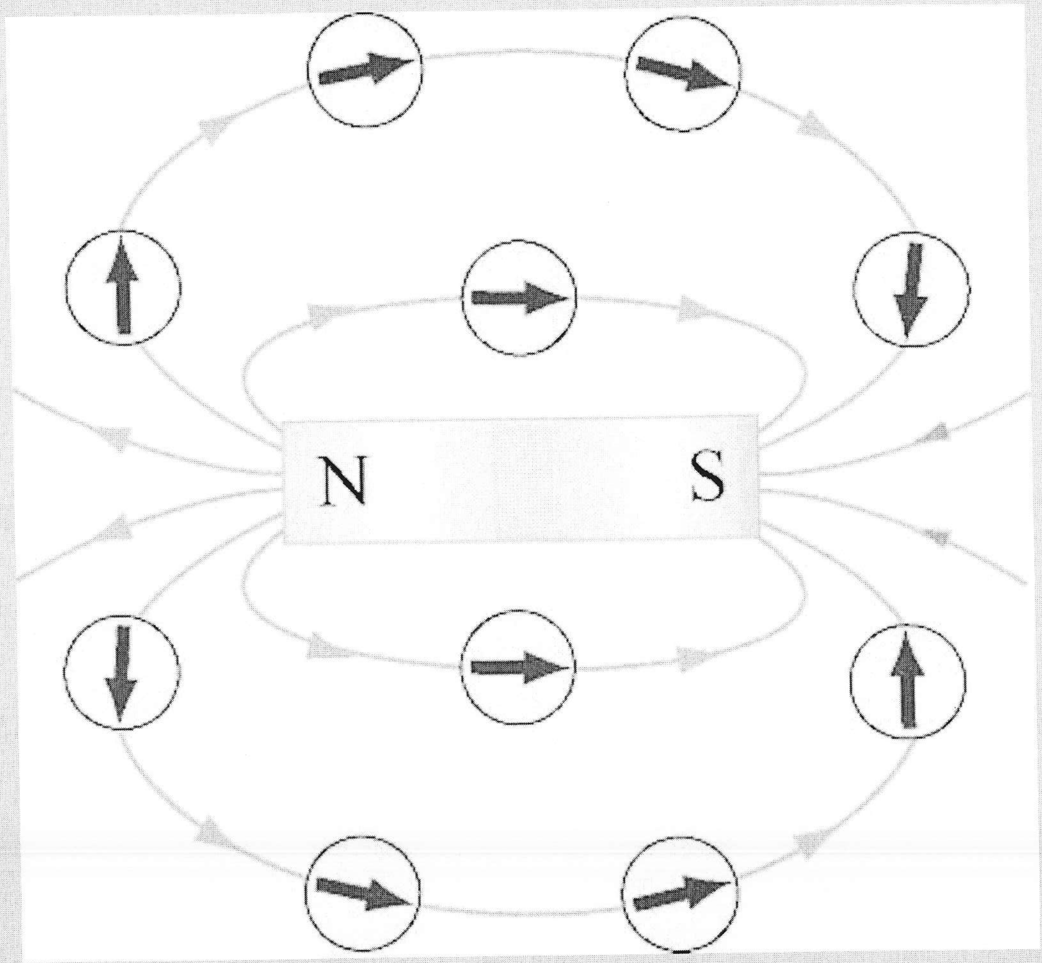
  - Like poles repel, unlike poles attract: demo
  - North and South cannot be separated in a magnet: demo
  - Magnetic forces can be pretty strong! Demo G3: nail on a sting

Magnetism – Bar Magnet



Like poles repel, opposite poles attract

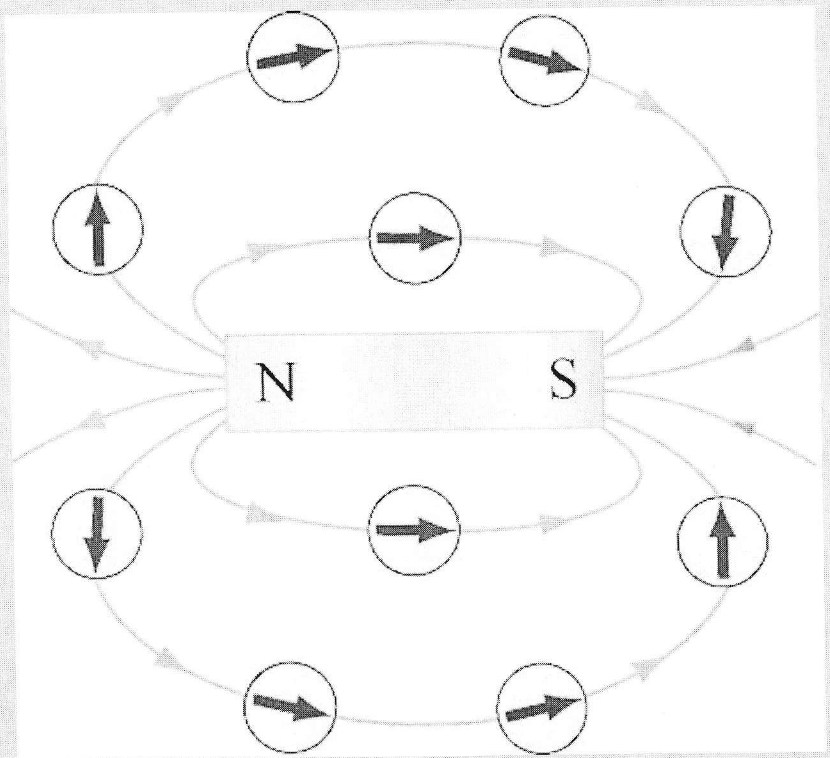
Magnetic Field of Bar Magnet



(1) A magnet has two poles, North (N) and South (S)

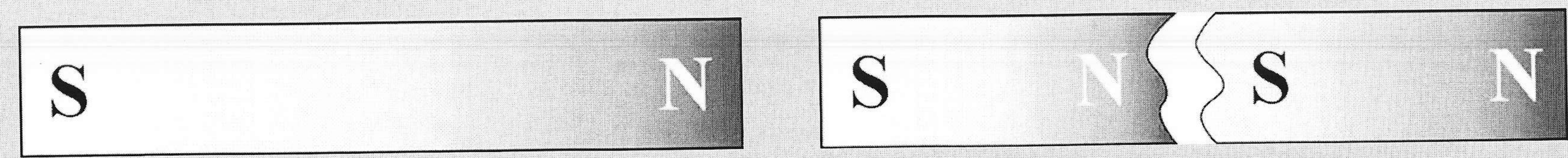
(2) Magnetic field lines leave from N, end at S

Bar Magnets Are Dipoles!



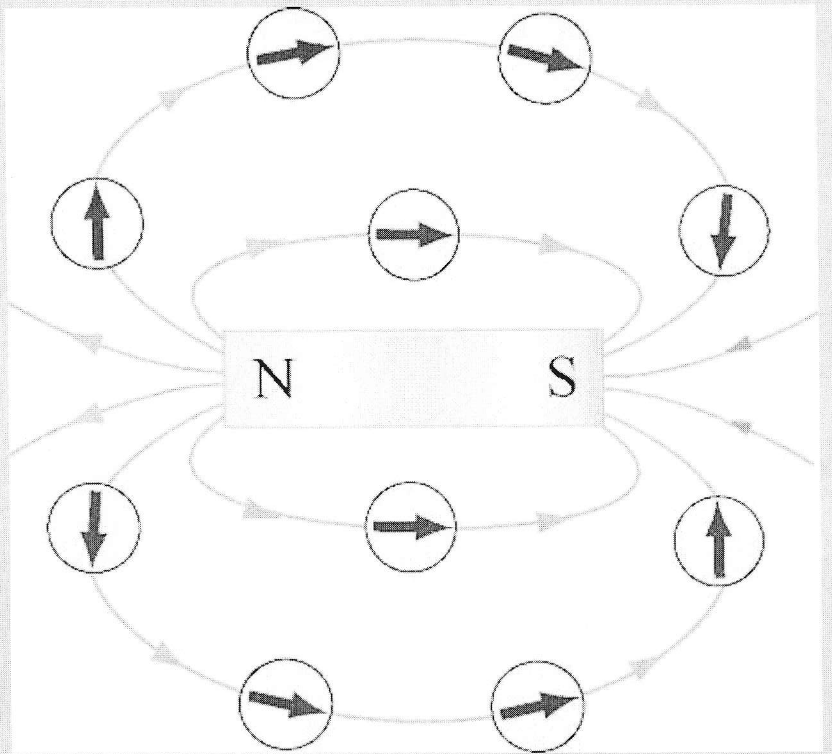
- Create Dipole Field
- Rotate to orient with Field

Is there magnetic "mass" or magnetic "charge?"



NO! Magnetic monopoles do not exist in isolation

Bar Magnets Are Dipoles!



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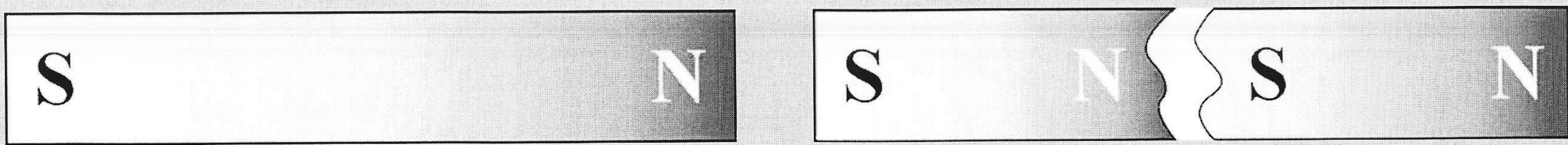
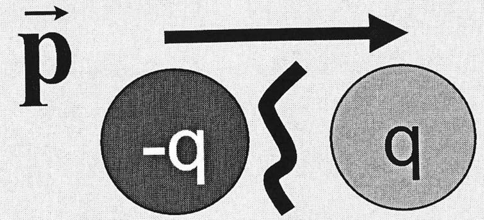


NO! Magnetic monopoles do not exist in isolation

Magnetic Monopoles?

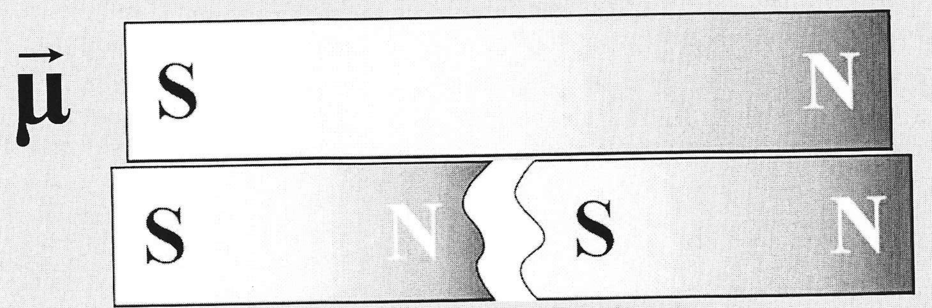
Electric Dipole



When cut:

2 monopoles (charges)

Magnetic Dipole



When cut: 2 dipoles

Magnetic monopoles do not exist in isolation

Another Maxwell's Equation! (2 of 4)

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Gauss's Law

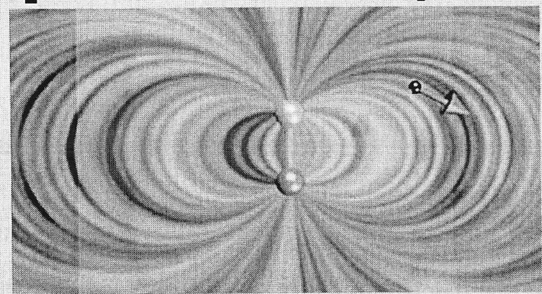
$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

Magnetic Gauss's Law

Fields: Grav., Electric, Magnetic

	Mass m	Charge q (±)	No Magnetic Monopoles!
Create:	$\vec{\mathbf{g}} = -G\frac{m}{r^2}\hat{\mathbf{r}}$	$\vec{\mathbf{E}} = k_e\frac{q}{r^2}\hat{\mathbf{r}}$	
Feel:	$\vec{\mathbf{F}}_g = m\vec{\mathbf{g}}$	$\vec{\mathbf{F}}_E = q\vec{\mathbf{E}}$	

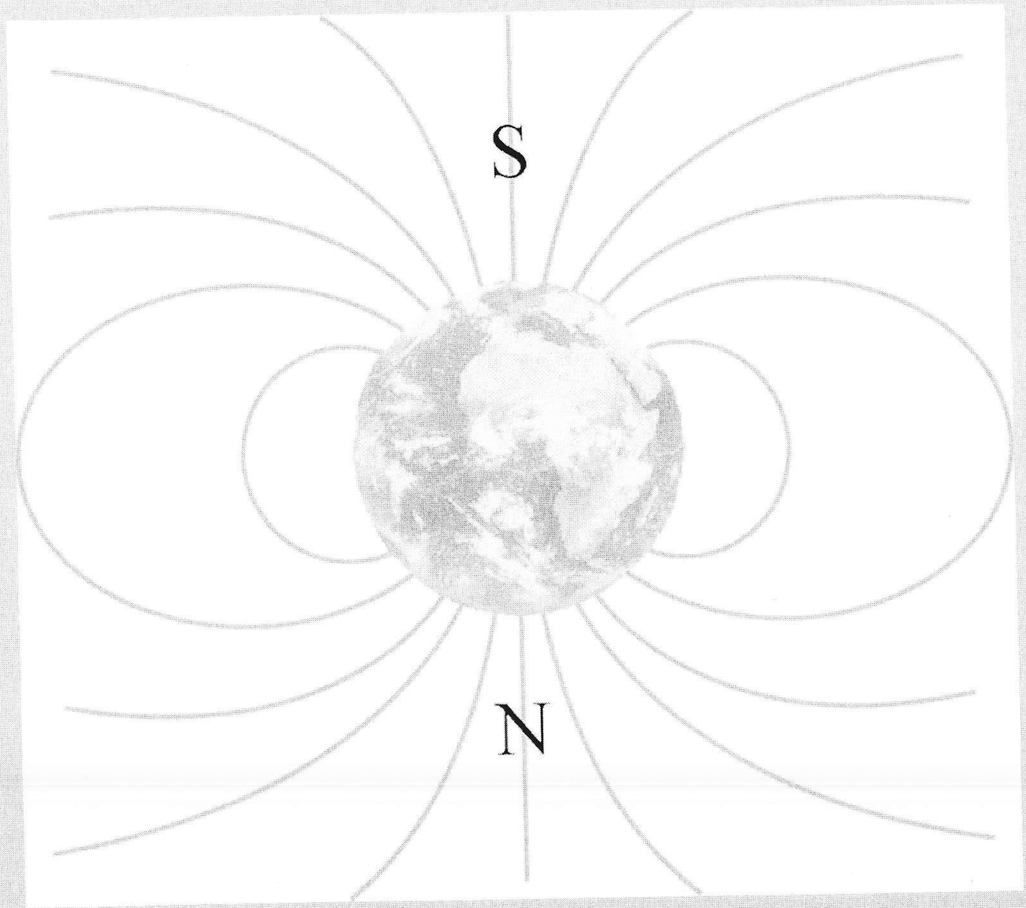
Also saw… Dipole **p** Dipole μ

Create: $\vec{\mathbf{E}} \rightarrow$  $\leftarrow \vec{\mathbf{B}}$

Feel: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$ $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$

What else is magnetic?

Magnetic Field of the Earth

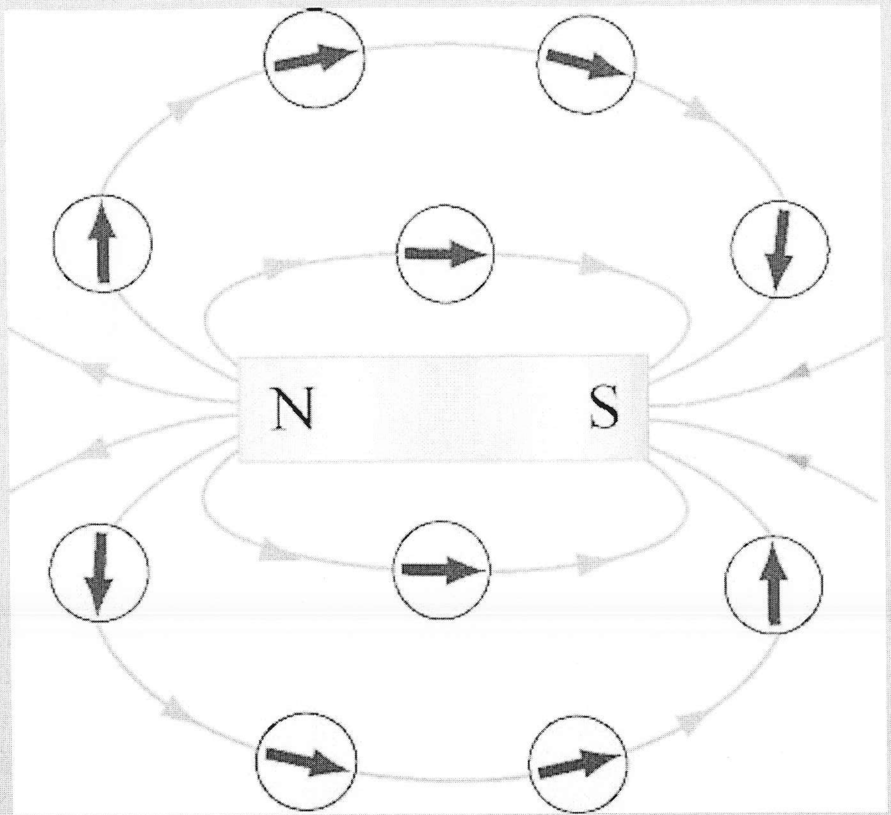


Also a magnetic dipole!

North magnetic pole located in southern hemisphere

Magnetic Field B Thus Far…

Bar Magnets (Magnetic Dipoles)…



- **Create:** Dipole Field
  - **Feel**: Orient with Field

Does anything else create or feel a magnetic field?

1819 Oersted : wire carrying electric currents produced deflections of permanent magnetic dipoles placed in their neighborhood

↓

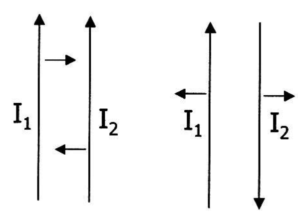
currents → source of magnetic flux density $\vec{B}$

The big step forward

- In 1820 Oersted realized that current flowing in a wire made the needle of a compass swing
  - The direction depends on the direction of the current

BIG discovery: proves that Electricity and Magnetism are related!

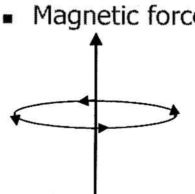
- Soon after, Ampere's experiment with parallel wires carrying current
  - If currents are parallel, wires attract
  - If anti-parallel, wires repel
  - No force on a stationary charge nearby...
  - NB: wires are overall neutral!
  - Demo



G. Sciolla – MIT 8.022 – Lecture 10 3

Magnetic force between currents

- More refined observations followed:
  - $F \sim I_1 I_2$ → F is proportional to velocity of charges in motion
  - Direction of F is perpendicular to velocity
- Interpretation
  - Some field (magnetic field B) is created by the charges in motion
  - Magnetic force is proportional to cross product **v** x **B**



$$\vec{F} = q \frac{\vec{v}}{c} \times \vec{B}$$

- Direction of **B**: **B** curls around the current (right hand rule)
- Iron fillings can be used to visualize B field lines: demo G2

G. Sciolla – MIT NB: this is an empirical law so far 4

Force on Charge Particle in a Magnetic
Current
Field.

↓

Topics in this
Chapter

Current Produces Magnetic Field

↓

Topics for next
Chapter

Later on, we will show that bar
magnets can be treated
as a magnetic dipole

Lorentz equation

$$F_{mag} = q\vec{V}\times\vec{B}$$

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

↳ Lorentz force acting on a charge moving in $\vec{E}$ and $\vec{B}$ field.

Determination of $\vec{B}$ (Using Lorentz equation to define $\vec{B}$)

First take out the electric field.

↓

force acting on the charge when it is not moving.

At any point, there is certain direction of motion in which no force is acting on the moving charge. (Note: electric field has been taken out). This direction is identified as the direction of $\vec{B}$ field.

With $\vec{v}$, q and the direction of $\vec{B}$ known, the magnitude of $\vec{B}$ is determined by

$$F = qvB\cos\alpha$$

α is the angle between $\vec{v}$ and $\vec{B}$

Unit of $\vec{B}$: $B \propto \frac{Kg\,m/sec^2}{C\cdot m/sec} = Kg\,C^{-1}sec^{-1} = $ Telsa $= 10$ KGa

$\vec{F}_{mag} \perp \vec{v} \Rightarrow \vec{F}\cdot\vec{v} = 0$

↓

power exerted on the charge particle by the magnetic force

$\vec{F}_{mag} \perp \vec{B}$

Magnetostatics:

(1) Steady current

↓

I must be maintained indefinitely

↓

cannot be charge piled up at any point

↓

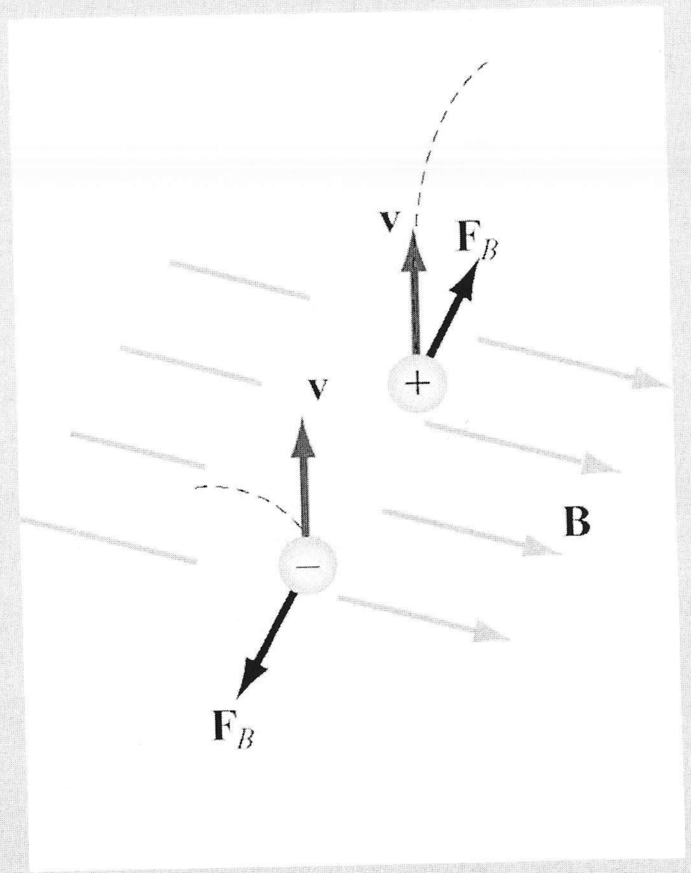
$$\frac{\partial\rho}{\partial t} = 0$$

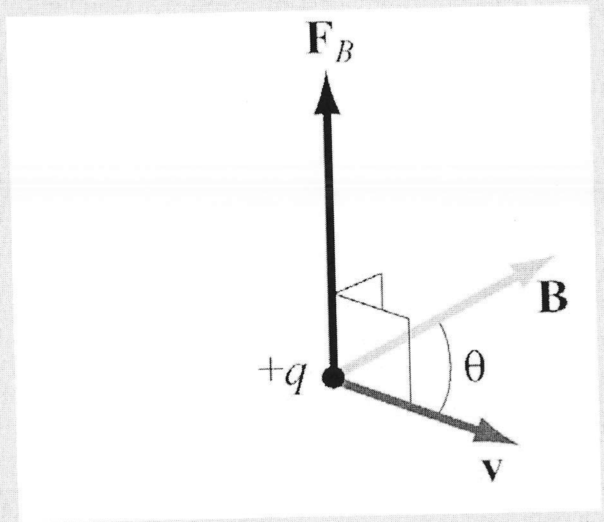
↓ from continuity equation

$$\nabla\cdot\vec{J} = 0$$

(2) No magnetic material, no moving material in the field.

Moving Charges Feel Magnetic Force

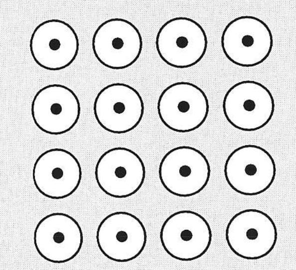




$$\vec{\mathbf{F}}_B = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

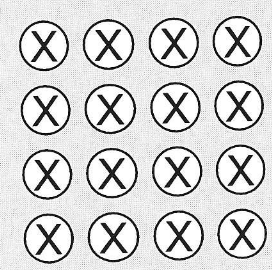
Magnetic force perpendicular both to:
Velocity **v** of charge and magnetic field **B**

Notation Demonstration



OUT of page

"Arrow Head"



INTO page

"Arrow Tail"

Magnetic Field B: Units

Since $\vec{\mathbf{F}}_B = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}}$

$$\text{B Units} = \frac{\text{newton}}{(\text{coulomb})(\text{meter/second})} = 1\frac{\text{N}}{\text{C}\cdot\text{m/s}} = 1\frac{\text{N}}{\text{A}\cdot\text{m}}$$

This is called 1 Tesla (T)

$$1\ \text{T} = 10^4\ \text{Gauss (G)}$$

Putting it Together: Lorentz Force

Charges Feel…

$$\vec{\mathbf{F}}_E = q\vec{\mathbf{E}}$$

Electric Fields

$$\vec{\mathbf{F}}_B = q\,\vec{\mathbf{v}}\times\vec{\mathbf{B}}$$

Magnetic Fields

$$\boxed{\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}}\times\vec{\mathbf{B}}\right)}$$

This is the final word on the force on a charge

Chapter 29 Magnetic Fields

1. $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
   ↓
   Lorentz equation.

2. Definition of magnetic field $\vec{B}$
   ↑
   Operational, See P. 2 of the note., unit see 1A

3. Charged particle in a constant magnetic field.
   Discovery of electron, Hall effect, see P. 6 of the note.
   $\vec{B} = (0, 0, B)$
   ↳ constant
   Constant magnetic field along the z axis

   Equation of motion

$$q\,\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$m \frac{dv_x}{dt} = B v_y$$

$$m \frac{dv_z}{dt} = - B v_x$$

$$m \frac{dv_z}{dt} = 0$$

Initial condition: $\vec{v}(0)$; $\vec{r}(0)$

$$z(t) = z(0) + v_z(0)\,t \quad , \quad v_z = v_z(0)$$

↓ constant velocity motion.

$$m v_x \frac{dv_x}{dt} = qB\, v_x v_y$$

$$m v_y \frac{dv_y}{dt} = -qB\, v_x v_y$$

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = 0$$

↑ $v_x^2 + v_y^2$

$$\Rightarrow \quad v_x^2 + v_y^2 = v_x^2(0) + v_y^2(0) = v_0^2$$

↳ constant.

$$m\frac{d^2 v_x}{dt^2} = B\frac{dv_y}{dt} = B\cdot q\left(\frac{-B}{m}\right)v_x$$

$$\Rightarrow \frac{d^2 v_x}{dt^2} = -\frac{B^2 q^2}{m^2}v_x = -\omega^2 v_x$$

↕ similiar to simple harmonic oscillation problem.

$\omega = \frac{qB}{m}$

$$v_x(t) = v_0 \sin(\omega t + \alpha)$$

$$v_y(t) = v_0 \cos(\omega t + \alpha)$$

$$v_0 = [v_x^2(0) + v_y^2(0)]^{1/2}$$

$$m\frac{dv_x}{dt} = v_0\cos(\omega t+\alpha)\omega m = qB\underbrace{v_0\cos(\omega t+\alpha)}_{v_y}$$

$$m\frac{dv_y}{dt} = -v_0\sin(\omega t+\alpha)\omega m = -qB\,v_x$$

α is determined by

$$v_x(0) = v_0\sin\alpha$$

$$v_y(0) = t\,v_0\cos\alpha$$

$$\frac{v_x(0)}{v_y(0)} = \tan\alpha$$

$$\frac{dx}{dt} = v_0\sin(\omega t+\alpha)$$

$$\int_{x_0}^{x} dx = v_0\int_0^t \sin(\omega t'+\alpha)\,dt'$$

$$x - x_0 = -v_0\frac{1}{\omega}[\cos(\omega t'+\alpha)]\Big|_0^t$$

$$= \frac{v_0}{\omega}[-\cos(\omega t+\alpha) + \cos\alpha]$$

$$x - \underbrace{x_0 - \frac{v_0}{\omega}\cos\alpha}_{x_c} = -\frac{v_0}{\omega}\cos(\omega t+\alpha)$$

$$\frac{dy}{dt} = v_0\cos(\omega t+\alpha)$$

$$\int_{y_0}^{y} dy = v_0\int_0^t \cos(\omega t'+\alpha)\,dt'$$

$$y - y_0 = \frac{v_0}{\omega}[\sin(\omega t'+\alpha)]\Big|_0^t$$

$$= \frac{v_0}{\omega}[\sin(\omega t+\alpha) - \sin\alpha$$

$$y - \underbrace{y_0 + \frac{v_0}{\omega}\sin\alpha}_{y_c} = \frac{v_0}{\omega}\sin(\omega t+\alpha)$$

$$(x-x_c)^2+(y-y_c)^2=\left(\frac{v_0}{\omega}\right)^2$$

↓

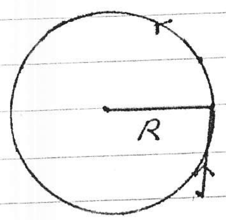
circular motion → motion in the x-y plane.
with center at (x_c, y_c)
↳ determined by the initial conditions

with radius $R=\frac{v_0}{\omega}=\frac{mv_0}{qB}$

$$p = qBR$$

Together with the motion along z direction
⇒ helical motion.

4. Given the fact that the motion is circular
velocity is along the x-y plane
↓
$\vec{v}\perp\vec{B}$



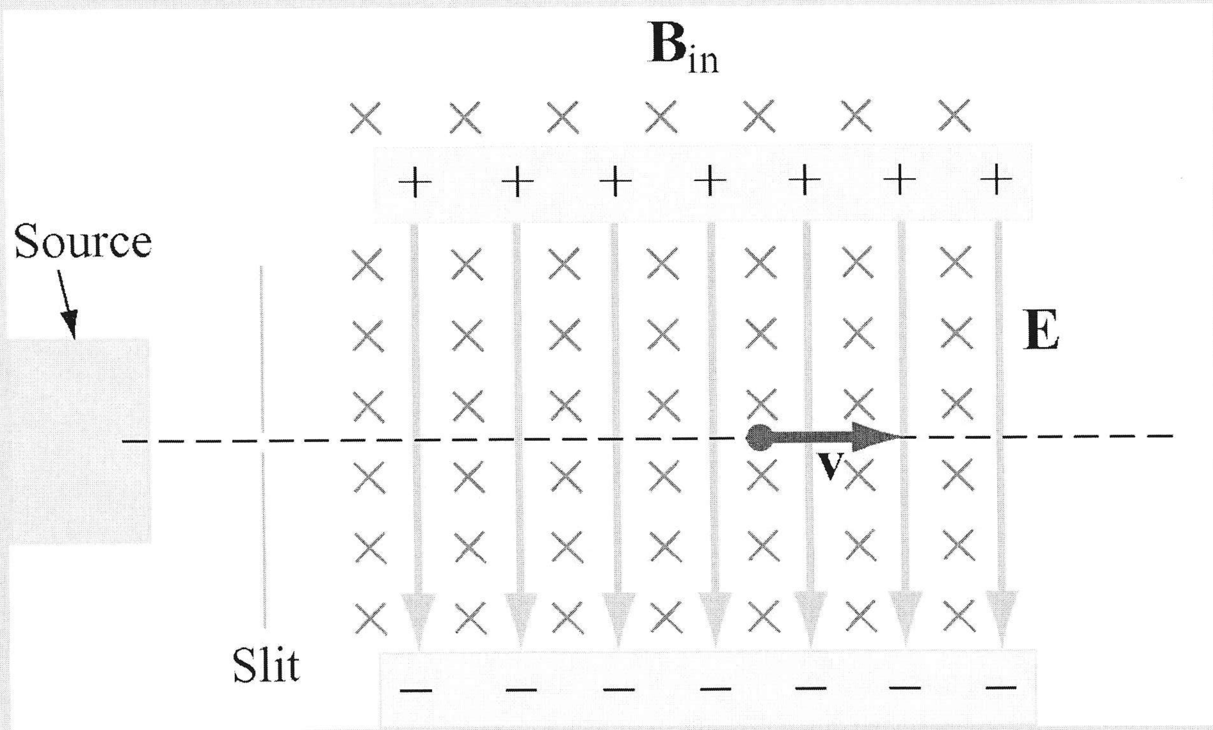
$\vec{v}$ along y direction
$\vec{B}$ along z direction
$q=-e$
$\vec{F}=-q\,\vec{v}\times\vec{B}$ → along $-x$

$$qv_0B = m\frac{v_0^2}{R}$$

$$p = qBR$$

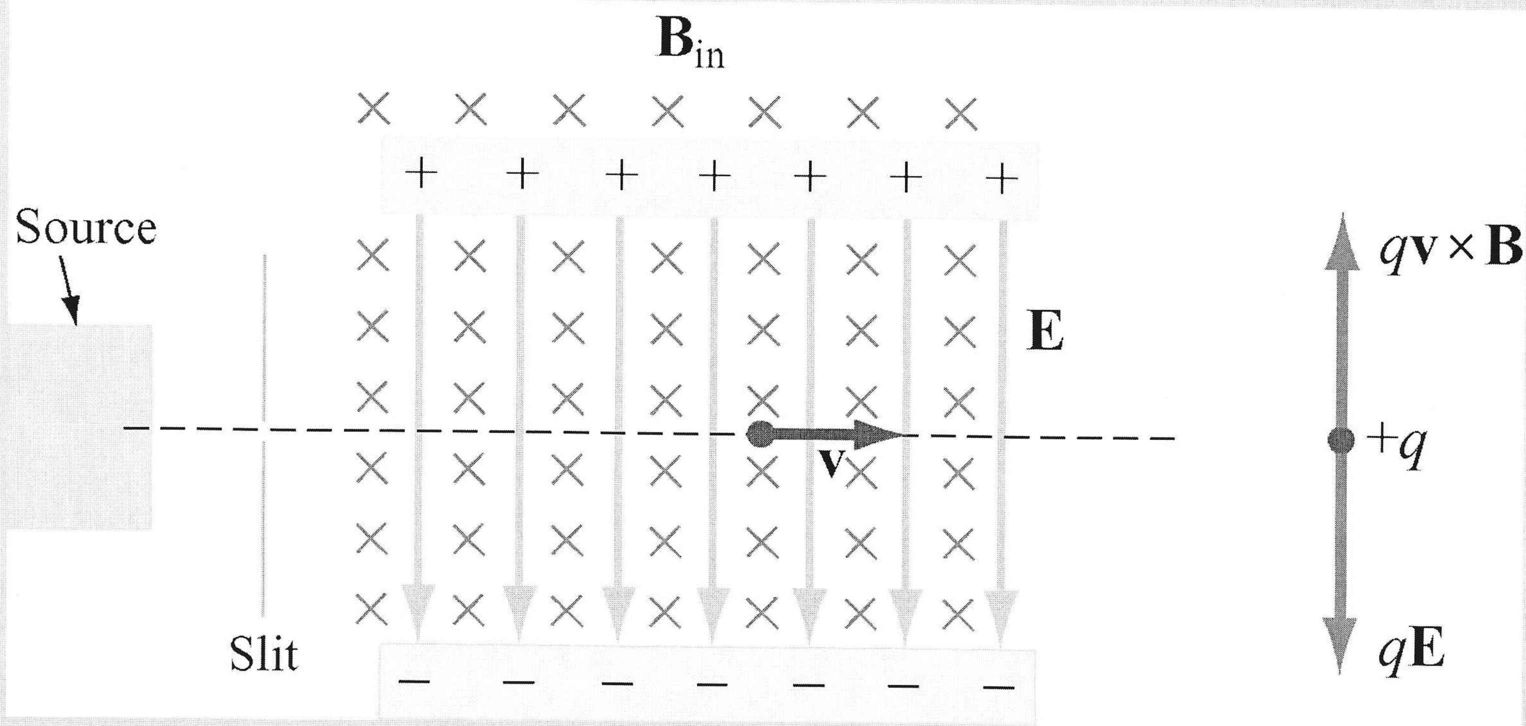
The same result as before.

Application: Velocity Selector



What happens here?

Velocity Selector



Particle moves in a straight line when

$$\vec{\mathbf{F}}_{net} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) = 0 \Rightarrow \ v = \frac{E}{B}$$

Lorentz force

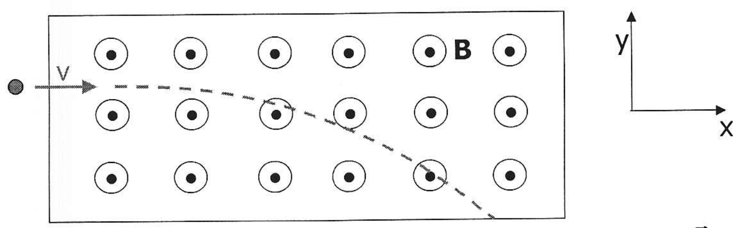
- When a charged particle moves in electric (E) and magnetic (B) fields it feels a force ($F_{Lorentz}$):

$$\vec{F}_{Lorentz} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

  - The above formula defines the magnetic field B
  - Units of B in cgs:
    - [B] = [F]/[q] = dyne/esu = Gauss (G)
    - NB: [B] = [E]
  - Units of B in SI: $\vec{F}_{Lorentz} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$
    - [B] = [F]/[q v] = N s /(m C) = Tesla (T)
  - Conversion: 1 T = 10^4 G

Trajectory in magnetic fields

- A particle of charge q and mass m moves with velocity **v** // +**x** axis in a magnetic field **B** // +**z** axis (out of the page):

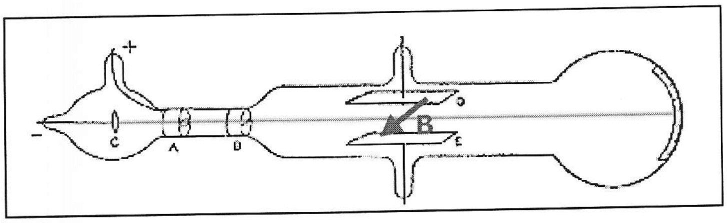


- What is the trajectory of q in the magnetic field? $\vec{F} = q\frac{\vec{v}}{c}\times\vec{B}$
  - v, B and F (a) are always perpendicular → circular motion!

$$F_{Lorentz} = F_{centripetal} \Rightarrow \frac{qvB}{c} = \frac{mv^2}{R} \Rightarrow \boxed{R = \frac{mvc}{qB}}$$

Deflection of electron beam by B

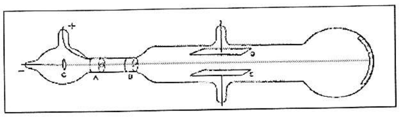
- An electron beam is produced by a cathode in a vacuum tube
  - Velocity of electrons: $\mathbf{v_e}$
- Magnetic field **B** perpendicular to $\mathbf{v_e}$ is produced by current in a wire or by permanent magnet
- What do we expect to happen?
  - Electrons curve according to Lorentz force (Demo G5, G6 TV)

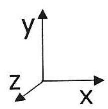


J.J. Thompson's experiment

- Discovery of electrons and measurement of e/m_e in 1897
- The idea:
  - A beam of "cathode rays" crosses a region with E and B present
  - Choosing v_e//x axis, B//z axis, E//y axis → $F_{Lorentz}$ // $F_{Electric}$
  - E and B can be adjusted so $F_{Magnetic} = -F_{Electric}$ so that e will go straight

$$\vec{F}_{Lorentz} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

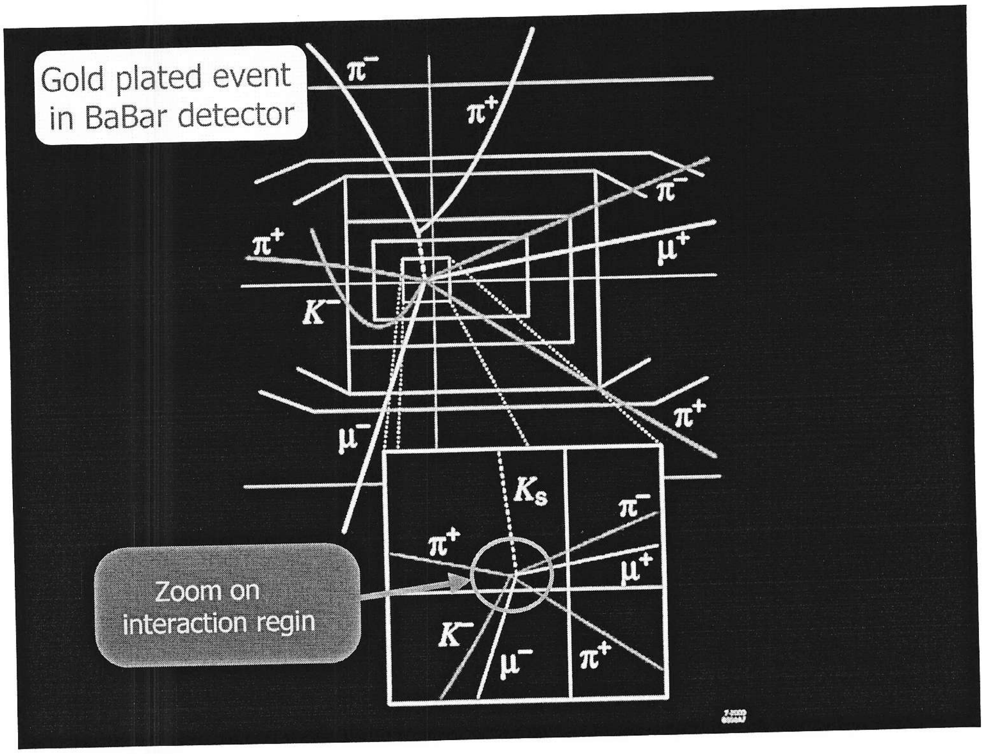




- Electric field alone causes a shift: $\Delta y = -\dfrac{qEL^2}{2mv^2}$
- Now turn on B and set it to cancel the shift due to E: $v = c\dfrac{E}{B}$
- Substituting this in the previous equation gives: $\dfrac{e}{m_e} = \dfrac{q}{m} = \dfrac{2\Delta y c^2 E}{B^2 L^2}$

Application in modern physics

- Tracking detectors in modern particle physics
- The problem
  - High energy collisions between elementary particles (such as e^+e^-) produce many particles (protons, electrons, pions, muons,...)
  - How can we "see" these particles?
    - Build detectors that can "visualize" the trajectory of charged particles using the fact that particles ionize the material they cross
  - How can I measure the properties of these particles?
    - E.g.: measure momentum, energy, mass, etc.
    - Immerse the detector in a very strong magnetic field B ~ 2 T
    - Charged particles will curve according to $R = \frac{mvc}{qB}$
      - Direction measures the charge
      - Radius of curvature measures momentum p=mv



Magnetic force and work

- Moving a charge in an electric field E requires work:

$$W_{12} = -q \int_1^2 \vec{E} \bullet d\vec{s}$$

- How much work does it take to move a charge in a magnetic field?

$$dW = \vec{F} \bullet d\vec{s} = \vec{F} \bullet \vec{v} dt = \frac{q}{c} (\vec{v} \times \vec{B}) \bullet \vec{v} dt = 0$$

→ No work is needed to move a particle in a magnetic field because v and F are always perpendicular!

Unit for B field.

F   q v B

Newton   C · M/sec   Telsa

= Ampere · M

1 Telsa = $\frac{\text{Newton}}{\text{Ampere meter}}$

↓

10^4 Gauss

Discovery of electron

$F_e = qE$ $a_y = \frac{qE}{m}$ E   y direction

$y = \frac{1}{2} a t^2$ $t = \frac{L}{v}$

$= \frac{1}{2} \frac{qE}{m} \frac{L^2}{v^2}$

$F_e = F_m$

$\vec{v}$ along x-direction
$\vec{B}$ along z direction
$q\vec{v} \times \vec{B} = -qvB$
-y direction

$qE = qvB$

$v = \frac{E}{B}$ application: velocity selector.

$y = \frac{1}{2} \frac{qE}{m} \frac{L^2}{E^2/B^2}$

$\frac{m}{q} = \frac{B^2 L^2}{2yE}$

1897   J. J. Thomson   (1906)

Millikan ~1909   measure q
1923

⇒ electron charge
electron mass.

Hall effect 1879

See P. 6 of notes

⊕ going downward $v \to -\hat{y}$
B going in $-\hat{z}$

$\vec{v} \times \vec{B}$ $\hat{x}$ ⊕ charge

V_b $\longrightarrow F$ V_a
$-$ $+$
at the left ↓ at the right

$V_a > V_b$

⊖ going upward $v \to y$
B $-\hat{z}$

$\vec{v} \times \vec{B}$ $\vec{F}$ on $-$charge along x direction

$+$ $-$
V_b V_a

$V_b > V_a$

Equilibrium $E_H = v_d B = \frac{j}{ne} B$

$\Rightarrow$ $n = \frac{jB}{e E_H}$
↓
number of charged carrier per unit volume.

1985 Klaus von Klitzing integral Quantum Hall effect

1998 Stormer, Laughing, Tsui 崔琪 fractional Quantum Hall effect.

To calculate the force exerted on the wire, consider a segment of wire of length ℓ and cross-sectional area A, as shown in Figure 8.3.2. The magnetic field points into the page, and is represented with crosses (X).

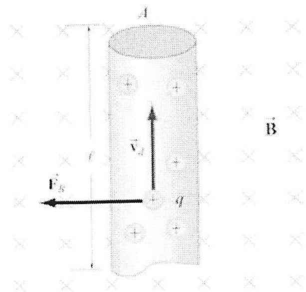


Figure 8.3.2 Magnetic force on a conducting wire

The charges move at an average drift velocity $\vec{\mathbf{v}}_d$. Since the total amount of charge in this segment is $Q_{\text{tot}} = q(nA\ell)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

$$\vec{\mathbf{F}}_B = Q_{\text{tot}}\vec{\mathbf{v}}_d \times \vec{\mathbf{B}} = qnA\ell(\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}) = I(\vec{\ell} \times \vec{\mathbf{B}}) \qquad (8.3.1)$$

where $I = nqv_dA$, and $\vec{\ell}$ is a *length vector* with a magnitude ℓ and directed along the direction of the electric current.

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{\mathbf{s}}$ (Figure 8.3.3).

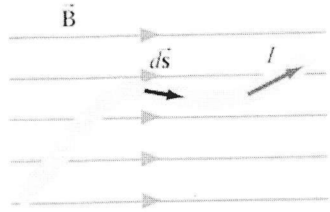


Figure 8.3.3 Current-carrying wire placed in a magnetic field

The magnetic force acting on the segment is

$$d\vec{\mathbf{F}}_B = Id\,\vec{\mathbf{s}} \times \vec{\mathbf{B}} \qquad (8.3.2)$$

Thus, the total force is

$$\boxed{\vec{\mathbf{F}}_B = I\int_a^b d\,\vec{\mathbf{s}} \times \vec{\mathbf{B}}} \qquad (8.3.3)$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$, as shown in Figure 8.3.4.

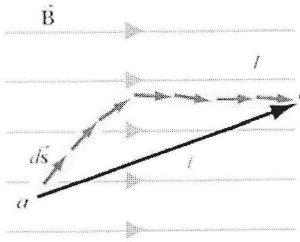


Figure 8.3.4 A curved wire carrying a current *I*.

Using Eq. (8.3.3), the magnetic force on the wire is given by

$$\vec{\mathbf{F}}_B = I\left(\int_a^b d\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} = I\vec{\ell}\times\vec{\mathbf{B}} \qquad (8.3.4)$$

where $\vec{\ell}$ is the length vector directed from a to b. However, if the wire forms a closed loop of arbitrary shape (Figure 8.3.5), then the force on the loop becomes

$$\vec{\mathbf{F}}_B = I\left(\oint d\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} \qquad (8.3.5)$$

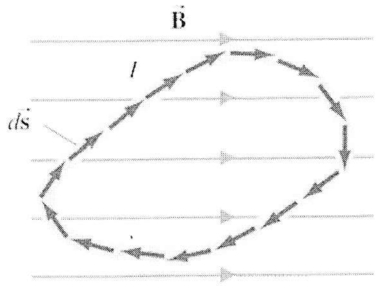


Figure 8.3.5 A closed loop carrying a current *I* in a uniform magnetic field.

Since the set of differential length elements $d\vec{\mathbf{s}}$ form a closed polygon, and their vector sum is zero, i.e., $\oint d\vec{\mathbf{s}} = 0$. The net magnetic force on a closed loop is $\vec{\mathbf{F}}_B = \vec{0}$.

Example 8.1: Magnetic Force on a Semi-Circular Loop

Consider a closed semi-circular loop lying in the xy plane carrying a current I in the counterclockwise direction, as shown in Figure 8.3.6.

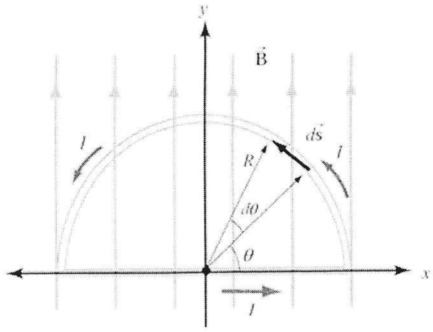


Figure 8.3.6 Semi-circular loop carrying a current I

A uniform magnetic field pointing in the $+y$ direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.

Solution:

Let $\vec{\mathbf{B}} = B\hat{\mathbf{j}}$ and $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ the forces acting on the straight segment and the semicircular parts, respectively. Using Eq. (8.3.3) and noting that the length of the straight segment is $2R$, the magnetic force is

$$\vec{\mathbf{F}}_1 = I(2R\hat{\mathbf{i}})\times(B\hat{\mathbf{j}}) = 2IRB\hat{\mathbf{k}}$$

where $\hat{\mathbf{k}}$ is directed out of the page.

To evaluate $\vec{\mathbf{F}}_2$, we first note that the differential length element $d\vec{\mathbf{s}}$ on the semicircle can be written as $d\vec{\mathbf{s}} = ds\,\hat{\mathbf{\theta}} = Rd\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$. The force acting on the length element $d\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{F}}_2 = Id\vec{\mathbf{s}}\times\vec{\mathbf{B}} = IR\,d\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{j}}) = -IBR\sin\theta d\theta\;\hat{\mathbf{k}}$$

Here we see that $d\vec{\mathbf{F}}_2$ points into the page. Integrating over the entire semi-circular arc, we have

$$\vec{\mathbf{F}}_2 = -IBR\hat{\mathbf{k}}\int_0^{\pi}\sin\theta d\theta = -2IBR\hat{\mathbf{k}}$$

Thus, the net force acting on the semi-circular wire is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \vec{\mathbf{0}}$$

This is consistent from our previous claim that the net magnetic force acting on a closed current-carrying loop must be zero.

8.4 Torque on a Current Loop

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{i}}$ which runs parallel to the plane of the loop, as shown in Figure 8.4.1(a)?

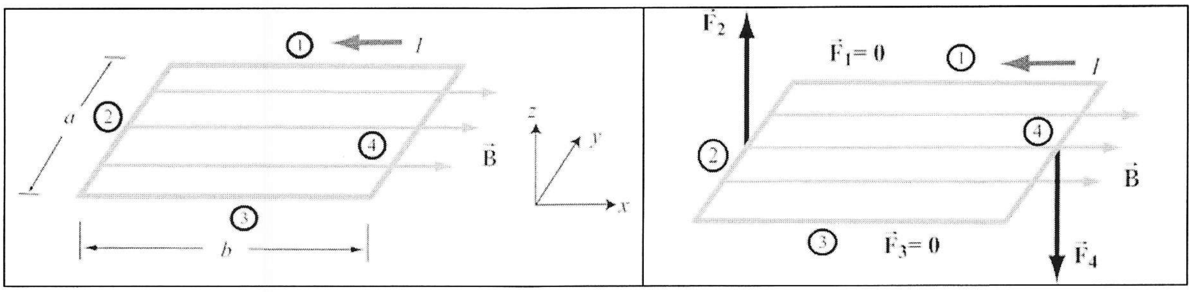


Figure 8.4.1 (a) A rectangular current loop placed in a uniform magnetic field. (b) The magnetic forces acting on sides 2 and 4.

From Eq. 8.4.1, we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{\ell}_1 = -b\hat{\mathbf{i}}$ and $\vec{\ell}_3 = b\hat{\mathbf{i}}$ are parallel and anti-parallel to $\vec{\mathbf{B}}$ and their cross products vanish. On the other hand, the magnetic forces acting on segments 2 and 4 are non-vanishing:

$$\begin{cases} \vec{\mathbf{F}}_2 = I(-a\,\hat{\mathbf{j}})\times(B\hat{\mathbf{i}}) = IaB\hat{\mathbf{k}} \\ \vec{\mathbf{F}}_4 = I(a\,\hat{\mathbf{j}})\times(B\hat{\mathbf{i}}) = -IaB\hat{\mathbf{k}} \end{cases} \tag{8.4.1}$$

with $\vec{\mathbf{F}}_2$ pointing out of the page and $\vec{\mathbf{F}}_4$ into the page. Thus, the net force on the rectangular loop is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}} \tag{8.4.2}$$

as expected. Even though the net force on the loop vanishes, the forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ will produce a torque which causes the loop to rotate about the y-axis (Figure 8.4.2). The torque with respect to the center of the loop is

$$\begin{aligned} \vec{\boldsymbol{\tau}} &= \left(-\frac{b}{2}\hat{\mathbf{i}}\right)\times\vec{\mathbf{F}}_2 + \left(\frac{b}{2}\hat{\mathbf{i}}\right)\times\vec{\mathbf{F}}_4 = \left(-\frac{b}{2}\hat{\mathbf{i}}\right)\times\left(IaB\hat{\mathbf{k}}\right) + \left(\frac{b}{2}\hat{\mathbf{i}}\right)\times\left(-IaB\hat{\mathbf{k}}\right) \\ &= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{\mathbf{j}} = IabB\hat{\mathbf{j}} = IAB\hat{\mathbf{j}} \end{aligned} \tag{8.4.3}$$

where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the *y*-axis. It is convenient to introduce the area vector $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{\mathbf{n}}$ is set by the conventional right-hand rule. In our case, we have $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$. The above expression for torque can then be rewritten as

$$\vec{\boldsymbol{\tau}} = I\vec{\mathbf{A}} \times \vec{\mathbf{B}} \tag{8.4.4}$$

Notice that the magnitude of the torque is at a maximum when $\vec{\mathbf{B}}$ is parallel to the plane of the loop (or perpendicular to $\vec{\mathbf{A}}$).

Consider now the more general situation where the loop (or the area vector $\vec{\mathbf{A}}$) makes an angle θ with respect to the magnetic field.

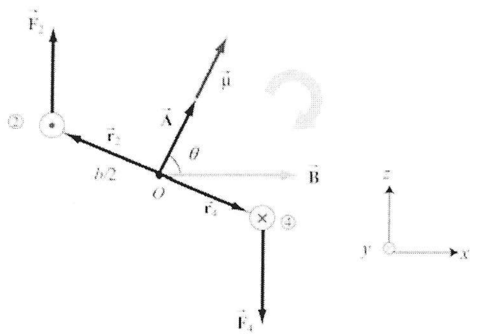


Figure 8.4.2 Rotation of a rectangular current loop

From Figure 8.4.2, the lever arms and can be expressed as:

$$\vec{\mathbf{r}}_2 = \frac{b}{2}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}\right) = -\vec{\mathbf{r}}_4 \tag{8.4.5}$$

and the net torque becomes

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2 + \vec{\mathbf{r}}_4 \times \vec{\mathbf{F}}_4 = 2\vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2 = 2\cdot\frac{b}{2}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}\right)\times\left(IaB\,\hat{\mathbf{k}}\right) \\ &= IabB\sin\theta\,\hat{\mathbf{j}} = I\vec{\mathbf{A}} \times \vec{\mathbf{B}}\end{aligned} \tag{8.4.6}$$

For a loop consisting of N turns, the magnitude of the toque is

$$\tau = NIAB\sin\theta \tag{8.4.7}$$

The quantity $NI\vec{\mathbf{A}}$ is called the magnetic dipole moment $\vec{\boldsymbol{\mu}}$:

$$\boxed{\vec{\boldsymbol{\mu}} = NI\vec{\mathbf{A}}} \tag{8.4.8}$$

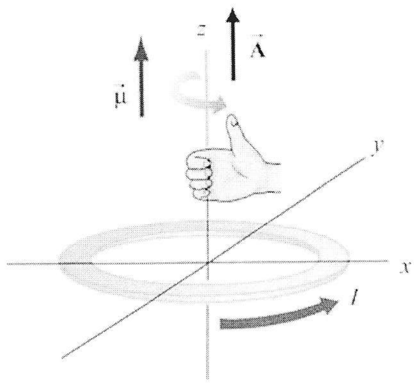


Figure 8.4.3 Right-hand rule for determining the direction of $\vec{\boldsymbol{\mu}}$

The direction of $\vec{\boldsymbol{\mu}}$ is the same as the area vector $\vec{\mathbf{A}}$ (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure 8.4.3). The SI unit for the magnetic dipole moment is ampere-meter^2 $(\mathrm{A}\cdot\mathrm{m}^2)$. Using the expression for $\vec{\boldsymbol{\mu}}$, the torque exerted on a current-carrying loop can be rewritten as

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}} \tag{8.4.9}$$

The above equation is analogous to $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$ in Eq. (2.8.3), the torque exerted on an electric dipole moment $\vec{\mathbf{p}}$ in the presence of an electric field $\vec{\mathbf{E}}$. Recalling that the potential energy for an electric dipole is $U = -\vec{\mathbf{p}}\cdot\vec{\mathbf{E}}$ [see Eq. (2.8.7)], a similar form is expected for the magnetic case. The work done by an external agent to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$\begin{aligned} W_{\text{ext}} &= \int_{\theta_0}^{\theta} \tau\, d\theta' = \int_{\theta_0}^{\theta} (\mu B \sin\theta')\, d\theta' = \mu B\left(\cos\theta_0 - \cos\theta\right) \\ &= \Delta U = U - U_0 \end{aligned} \tag{8.4.10}$$

Once again, $W_{\text{ext}} = -W$, where W is the work done by the magnetic field. Choosing $U_0 = 0$ at $\theta_0 = \pi/2$, the dipole in the presence of an external field then has a potential energy of

$$U = -\mu B\cos\theta = -\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}} \tag{8.4.11}$$

The configuration is at a stable equilibrium when $\vec{\boldsymbol{\mu}}$ is aligned parallel to $\vec{\mathbf{B}}$, making U a minimum with $U_{\min} = -\mu B$. On the other hand, when $\vec{\boldsymbol{\mu}}$ and $\vec{\mathbf{B}}$ are anti-parallel, $U_{\max} = +\mu B$ is a maximum and the system is unstable.

8.4.1 Magnetic force on a dipole

As we have shown above, the force experienced by a current-carrying rectangular loop (i.e., a magnetic dipole) placed in a uniform magnetic field is zero. What happens if the magnetic field is non-uniform? In this case, there will be a net force acting on the dipole.

Consider the situation where a small dipole $\vec{\boldsymbol{\mu}}$ is placed along the symmetric axis of a bar magnet, as shown in Figure 8.4.4.

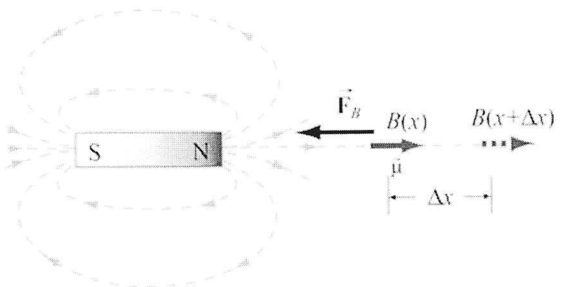


Figure 8.4.4 A magnetic dipole near a bar magnet.

The dipole experiences an attractive force by the bar magnet whose magnetic field is non-uniform in space. Thus, an external force must be applied to move the dipole to the right. The amount of force F_{ext} exerted by an external agent to move the dipole by a distance Δx is given by

$$F_{\text{ext}}\Delta x = W_{\text{ext}} = \Delta U = -\mu B(x+\Delta x) + \mu B(x) = -\mu[B(x+\Delta x) - B(x)] \qquad (8.4.12)$$

where we have used Eq. (8.4.11). For small Δx, the external force may be obtained as

$$F_{\text{ext}} = -\mu\frac{[B(x+\Delta x) - B(x)]}{\Delta x} = -\mu\frac{dB}{dx} \qquad (8.4.13)$$

which is a positive quantity since $dB/dx < 0$, i.e., the magnetic field decreases with increasing x. This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have

$$F_B = \mu\frac{dB}{dx} = \frac{d}{dx}(\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}}) \qquad (8.4.14)$$

More generally, the magnetic force experienced by a dipole $\vec{\boldsymbol{\mu}}$ placed in a non-uniform magnetic field $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}}) \qquad (8.4.15)$$

where

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{8.4.16}$$

is the gradient operator.

Charged Particles in a Uniform Magnetic Field

If a particle of mass m moves in a circle of radius r at a constant speed v, what acts on the particle is a radial force of magnitude $F = mv^2 / r$ that always points toward the center and is perpendicular to the velocity of the particle.

In Section 8.2, we have also shown that the magnetic force $\vec{\mathbf{F}}_B$ always points in the direction perpendicular to the velocity $\vec{\mathbf{v}}$ of the charged particle and the magnetic field $\vec{\mathbf{B}}$. Since $\vec{\mathbf{F}}_B$ can do not work, it can only change the direction of $\vec{\mathbf{v}}$ but not its magnitude. What would happen if a charged particle moves through a uniform magnetic field $\vec{\mathbf{B}}$ with its initial velocity $\vec{\mathbf{v}}$ at a right angle to $\vec{\mathbf{B}}$? For simplicity, let the charge be $+q$ and the direction of $\vec{\mathbf{B}}$ be into the page. It turns out that $\vec{\mathbf{F}}_B$ will play the role of a centripetal force and the charged particle will move in a circular path in a counterclockwise direction, as shown in Figure 8.5.1.

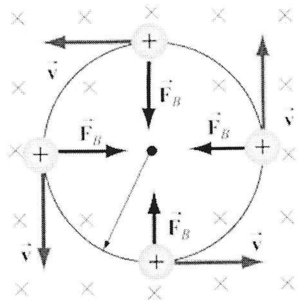


Figure 8.5.1 Path of a charge particle moving in a uniform $\vec{\mathbf{B}}$ field with velocity $\vec{\mathbf{v}}$ initially perpendicular to $\vec{\mathbf{B}}$.

With

$$qvB = \frac{mv^2}{r} \tag{8.5.1}$$

the radius of the circle is found to be

$$\boxed{r = \frac{mv}{qB}} \tag{8.5.2}$$

The period T (time required for one complete revolution) is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB} \tag{8.5.3}$$

Similarly, the angular speed (cyclotron frequency) ω of the particle can be obtained as

$$\boxed{\omega = 2\pi f = \frac{v}{r} = \frac{qB}{m}} \tag{8.5.4}$$

If the initial velocity of the charged particle has a component parallel to the magnetic field $\vec{\mathbf{B}}$, instead of a circle, the resulting trajectory will be a helical path, as shown in Figure 8.5.2:

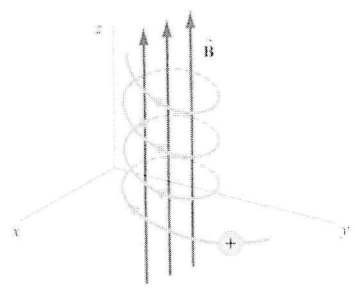


Figure 8.5.2 Helical path of a charged particle in an external magnetic field. The velocity of the particle has a non-zero component along the direction of $\vec{\mathbf{B}}$.

8.6.1 Velocity Selector

In the presence of both electric field $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$, the total force on a charged particle is

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \tag{8.6.1}$$

This is known as the Lorentz force. By combining the two fields, particles which move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons. In Figure 8.6.1 the schematic diagram of Thomson's apparatus is depicted.

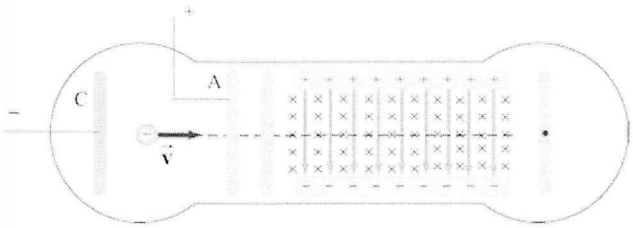


Figure 8.6.1 Thomson's apparatus

The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be $V_A - V_C = \Delta V$. The change in potential energy is equal to the external work done in accelerating the electrons: $\Delta U = W_{ext} = q\Delta V = -e\Delta V$. By energy conservation, the kinetic energy gained is $\Delta K = -\Delta U = mv^2/2$. Thus, the speed of the electrons is given by

$$v = \sqrt{\frac{2e\Delta V}{m}} \tag{8.6.2}$$

If the electrons further pass through a region where there exists a downward uniform electric field, the electrons, being negatively charged, will be deflected upward. However, if in addition to the electric field, a magnetic field directed into the page is also applied, then the electrons will experience an additional downward magnetic force $-e\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. When the two forces exactly cancel, the electrons will move in a straight path. From Eq. 8.6.1, we see that when the condition for the cancellation of the two forces is given by $eE = evB$, which implies

$$v = \frac{E}{B} \tag{8.6.3}$$

In other words, only those particles with speed $v = E/B$ will be able to move in a straight line. Combining the two equations, we obtain

$$\boxed{\frac{e}{m} = \frac{E^2}{2(\Delta V)B^2}} \quad (8.6.4)$$

By measuring E, ΔV and B, the charge-to-mass ratio can be readily determined. The most precise measurement to date is $e/m = 1.758820174(71)\times 10^{11}$ C/kg.

8.6.2 Mass Spectrometer

Various methods can be used to measure the mass of an atom. One possibility is through the use of a mass spectrometer. The basic feature of a *Bainbridge* mass spectrometer is illustrated in Figure 8.6.2. A particle carrying a charge $+q$ is first sent through a velocity selector.

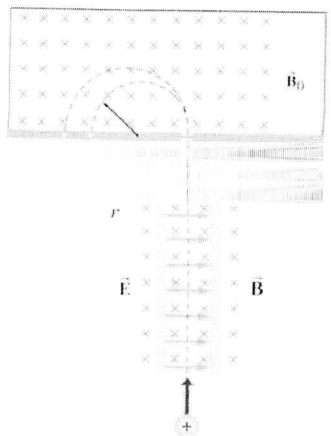


Figure 8.6.2 A Bainbridge mass spectrometer

The applied electric and magnetic fields satisfy the relation $E = vB$ so that the trajectory of the particle is a straight line. Upon entering a region where a second magnetic field $\vec{\mathbf{B}}_0$ pointing into the page has been applied, the particle will move in a circular path with radius r and eventually strike the photographic plate. Using Eq. 8.5.2, we have

$$r = \frac{mv}{qB_0} \quad (8.6.5)$$

Since $v = E/B$, the mass of the particle can be written as

$$m = \frac{qB_0 r}{v} = \frac{qB_0 Br}{E} \quad (8.6.6)$$

Ampere's law

- In electrostatics, the electric field E and its sources (charges) are related by Gauss's law:

$$\int_{Surface} \vec{E} \bullet d\vec{A} = 4\pi Q_{encl}$$

  - Why useful? When symmetry applies, E can be easily computed
- Similarly, in magnetism the magnetic field B and its sources (currents) are related by Ampere's law:

$$\oint_C \vec{B} \bullet d\vec{s} = \frac{4\pi}{c} I_{encl}$$

  - Why useful? When symmetry applies, E can be easily computed
  - NB: This is a line integral!

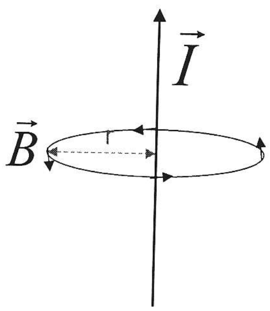
NB: no demonstration has been given so far for Ampere's law.

Application of Ampere's law:

B created by current in a wire

- Long, straight wire in which flows a current I
- Calculate magnetic field B created by I
- Solution:
  - Apply Ampere's law:

$$\oint_C \vec{B} \bullet d\vec{s} = B(r) 2\pi r = \frac{4\pi}{c} I_{encl} \Rightarrow \boxed{\vec{B} = \frac{2I}{cr}\hat{\varphi}}$$



  - Direction: right hand rule
- NB: $B_{wire} \sim 1/r$. Does this look familiar?
  - Remember E created by a line of charge: $E(r) = \frac{2\lambda}{r}$
  - Coincidence? Not at all...

Force between 2 wires

- Force on wire 1 due to magnetic field B created by wire 2:

$$\vec{F}_1 = \frac{I_1}{c} L\hat{n} \times \vec{B}_2$$

- Magnetic field created by wire 2: $\vec{B}_2 = \frac{2I_2}{cr}\hat{\varphi}$
- Total force F: $F = \frac{2I_2 I_1}{c^2 r} L$
- Usually we quote the force/unit length: $\frac{F}{L} = \frac{2I_2 I_1}{c^2 r}$
  - Direction? $\vec{F} \propto I_1 \times \hat{\varphi}_2$ Using right hand rule:
    - I_1 and I_2 parallel: attractive
    - I_1 and I_2 anti-parallel: repulsive

Can we test this experimentally? Demo G8, G9