Fundamental Theorems for Vector Calculus

Theorem for Gradient

$$d\phi = \nabla\phi \cdot d\vec{r}$$

$$\int_{a \atop (P)}^{b} (\nabla\phi) \cdot d\vec{r} = \phi(b) - \phi(a)$$

Theorem for Divergence

(Gauss' theorem)
(Green's theorem)
(Divergence theorem)

$$\int_V (\nabla \cdot \vec{V})\, d\tau = \oint_S \vec{V} \cdot dS$$

Theorem for Curl

$$\int_S (\nabla \times \vec{V}) \cdot dS = \int_P \vec{V} \cdot d\vec{r}$$

Helmholtz theorem

$\vec{F}(\vec{r})$ is a vector field with

$$\nabla \cdot \vec{F} = D$$

scalar function

$$\nabla \times \vec{F} = \vec{C}$$

(For consistency $\vec{C}$ must be divergenceless because the divergence of a curl is always zero)

If (i) $D(\vec{r})$, $\vec{C}(\vec{r})$ are specified

(ii) $\vec{D}(\vec{r})$, $\vec{C}(\vec{r})$ go to zero faster than $\frac{1}{r^2}$ as $r \to \infty$

(iii) $\vec{F}(\vec{r})$ goes to zero as $r \to \infty$

then

$\vec{F}$ is given uniquely by

$$\vec{F} = -\nabla U + \nabla \times \vec{W}$$

$$U(\vec{r}) \equiv \frac{1}{4\pi} \int \frac{D(\vec{r}\,')}{\mathscr{r}} \, d\tau'$$

$$\vec{W}(\vec{r}\,') \equiv \frac{1}{4\pi} \int \frac{C(\vec{r}\,')}{\mathscr{r}} \, d\tau'$$

the integrals are over all of space and

$$\mathscr{r} = |\vec{r} - \vec{r}\,'|$$

Reference

David J. Griffiths "Introduction to Electrodynamics" Appendix B , P. 555

Scalar Potential

If the curl of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be written as the gradient of a **scalar potential** (V):

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V. \tag{1.103}$$

(The minus sign is purely conventional.) That's the essential burden of the following theorem:

Theorem 1: **Curl-less** (or "**irrotational**") **fields**. The following conditions are equivalent (that is, $\mathbf{F}$ satisfies one if and only if it satisfies all the others):
(a) $\nabla \times \mathbf{F} = 0$ everywhere.
(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points.
(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.
(d) $\mathbf{F}$ is the gradient of some scalar, $\mathbf{F} = -\nabla V$.

Vector Potential

If the divergence of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be expressed as the curl of a **vector potential** ($\mathbf{A}$):

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}. \tag{1.104}$$

That's the main conclusion of the following theorem:

Theorem 2: **Divergence-less** (or "**solenoidal**") **fields**. The following conditions are equivalent:
(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.
(b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface, for any given boundary line.
(c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.
(d) $\mathbf{F}$ is the curl of some vector, $\mathbf{F} = \nabla \times \mathbf{A}$.

The vector potential is not unique—the gradient of any scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

You should by now be able to prove all the connections in these theorems, save for the ones that say (a), (b), or (c) implies (d). Those are more subtle, and will come later. Incidentally, in *all* cases (*whatever* its curl and divergence may be) a vector field $\mathbf{F}$ can be written as the gradient of a scalar plus the curl of a vector:

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \quad \text{(always)}. \tag{1.105}$$

Stationary charges produce electric fields that are constant in time $\Rightarrow$ electrostatics

Summary $\rightarrow$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} = 0$$

$$\Updownarrow$$

$$\vec{E} = -\nabla \phi$$

(Follows for Coulomb's law and superposition principle)

$$\int \nabla \cdot \vec{E} \, dV = \frac{1}{\epsilon_0} \int \rho \, dV$$

$$\parallel$$

$$\int \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{inc}$$

$\downarrow$

Gauss law

$$\nabla \cdot (-\nabla \phi) = \frac{\rho}{\epsilon_0}$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

$\downarrow$

Laplacian

$\downarrow$ If the boundary condition is such that ρ goes to zero sufficiently rapidly at $r \to \infty$

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \, d\tau'}{|\vec{r} - \vec{r}'|} \qquad \vec{E} = -\nabla \phi$$

Summary; Electrostatic Boundary Conditions

In the typical electrostatic problem you are given a source charge distribution ρ, and you want to find the electric field $\mathbf{E}$ it produces. Unless the symmetry of the problem admits a solution by Gauss's law, it is generally to your advantage to calculate the potential first, as an intermediate step. These, then, are the three fundamental quantities of electrostatics: ρ, $\mathbf{E}$, and V. We have, in the course of our discussion, derived all six formulas interrelating them. These equations are neatly summarized in Fig. 2.35. We began with just two experimental observations: (1) the principle of superposition—a broad general rule applying to *all* electromagnetic forces, and (2) Coulomb's law—the fundamental law of electrostatics. From these, all else followed.

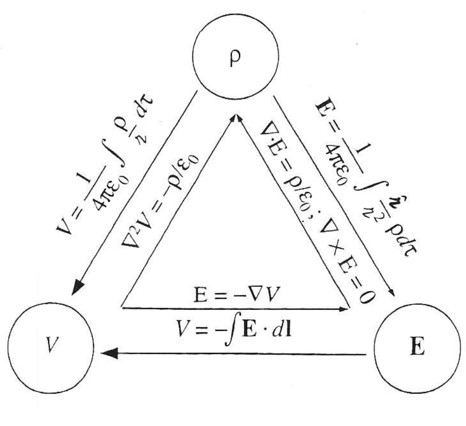


Figure 2.35

P. 87 of Griffith. Third Edition

Steady currents produce magnetic fields that are constant in time $\Rightarrow$

Ampere's Law $\Leftrightarrow$ Biot - Savart Laws

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 I_{enc}$$

$$\parallel$$

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$$

No magnetic pole

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\parallel$$

$$\int_V (\nabla \cdot \vec{B})\, d\tau = 0$$

$$\Rightarrow \nabla \cdot \vec{B} = 0 \qquad \Rightarrow \vec{B} = \nabla \times \vec{A}$$

[When a steady current flows in a wire, its magnitude I must be the same all along the wire

$\downarrow$

otherwise charge would be piling up somewhere

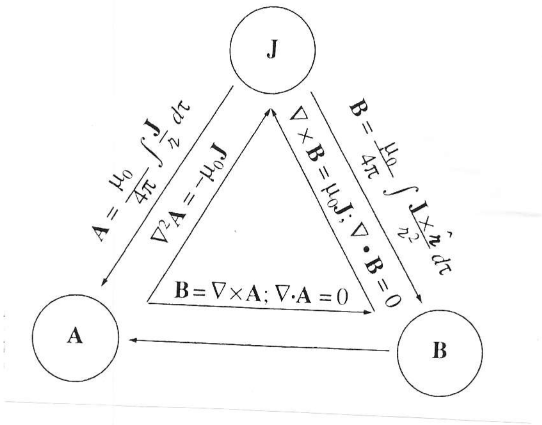
$\downarrow$

$$\nabla \cdot \vec{J} = 0$$

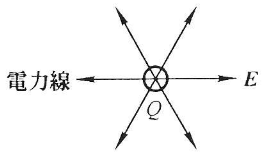
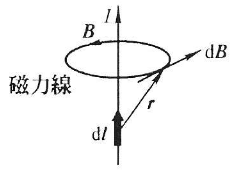
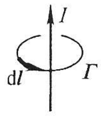
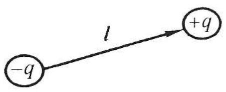
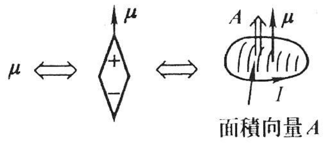
From the continity equation $\Rightarrow \frac{\partial \rho}{\partial t} = 0$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$

Summary of Magnostatics



P. 240 of Griffith.

	靜電學		靜磁學
(1)	有電荷 —— 有正和負，可獨立存在	(1)	沒磁荷 —— 正負極必須同時存在
(2)	Coulomb 電場 $\boldsymbol{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^3}\boldsymbol{r}$  電力線是有頭有尾的曲線(包含直線)。	(2)	Biot-Savart 定律： $\mathrm{d}\boldsymbol{B} = \dfrac{\mu_0}{4\pi}\dfrac{I\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3}$  磁力線是無頭無尾的封閉曲線。
(3)	Gauss 定律： $\oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = \dfrac{1}{\varepsilon_0}\sum_i q_i$ (S 內電荷)  S = Gauss 面	(3)	Gauss 定律： $\oint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$ 因爲沒磁單極 (正負極同量且同時存在) Ampere 定律： $\oint_\Gamma \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l} = \mu_0\sum_i I_i$($\Gamma$ 內電流)  Γ = 封閉曲線 $\mathrm{d}\boldsymbol{l}$ = 微小線段向量
(4)	電偶極矩 $\boldsymbol{P} \equiv q\,\boldsymbol{l}$ (電偶矩)  力矩 $\boldsymbol{\tau}_E = \boldsymbol{P}\times\boldsymbol{E}$ 取向能 $U_E = -\boldsymbol{P}\cdot\boldsymbol{E}$ $\boldsymbol{E}$ = 外電場	(4)	磁偶極矩 $\boldsymbol{\mu}$ (磁偶矩)  經典磁偶矩 $\boldsymbol{\mu} = I\boldsymbol{A}$ 電子磁偶矩 $\begin{cases}\boldsymbol{\mu}_l = g_l\dfrac{-e}{2m}\boldsymbol{L}\cdots\cdots \text{軌道} \\ \boldsymbol{\mu}_S = g_S\dfrac{-e}{2m}\boldsymbol{S}\cdots\cdots \text{自旋}\end{cases}$ $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{P}$ = 軌道角動量 $\boldsymbol{S}$ = 內稟角動量 (自旋) g_l (g_S) = 軌道 (自旋) 磁偶矩迴磁比 力矩 $\boldsymbol{\tau}_B = \boldsymbol{\mu}\times\boldsymbol{B}$，　$\boldsymbol{B}$ = 外磁場 取向能 $U_B = -\boldsymbol{\mu}\cdot\boldsymbol{B}$

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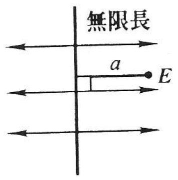
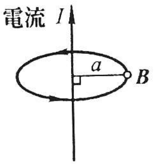
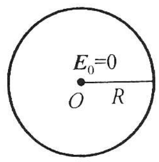
P.143

7

	靜電學		靜磁學
(5)	電偶極（electric dipole）場 $E_{\text{dipole}} \propto \frac{P}{r^3}$， $P = ql$ ←【**Ex.7－7**】	(5)	磁偶極場（magnetic dipole field） $B_{\text{dipole}} \propto \frac{\lvert \boldsymbol{\mu} \rvert}{r^3}$ ←【**Ex.7－34**】，式（7－41d）
(6)	橫切面積甚小的無限長帶電體 $E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{a}$ λ = 電荷線密度 	(6)	橫切面積甚小的無限長中性導線 $B = \frac{\mu_0}{4\pi}\frac{2I}{a}$ 
(7)	Coulomb力是二體力 且是縱向力	(7)	電流間的相互作用力是二體力 且是橫向力
(8)	半徑 R 的均勻圓狀帶電體 其在圓心的電場 $\boldsymbol{E}_0 = 0$【**Ex.7－7**】 	(8)	半徑 R 的穩定圓狀電流在圓心產生的磁場 B_0 $B_0 = \frac{\mu_0}{2R} I$【**Ex.7－32**】