Sources of Magnetic Fields

9.1 Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current I, the magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions, $d\vec{\mathbf{B}}$, from small segments of the wire $d\vec{\mathbf{s}}$, (Figure 9.1.1).

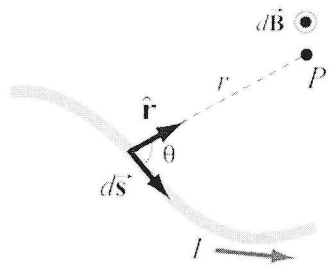


Figure 9.1.1 Magnetic field $d\vec{\mathbf{B}}$ at point P due to a current-carrying element $I\,d\vec{\mathbf{s}}$.

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $I\,d\vec{\mathbf{s}}$.

Let r denote as the distance form the current source to the field point P, and $\hat{\mathbf{r}}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{\mathbf{B}}$, from the current source, $Id\vec{\mathbf{s}}$,

$$\boxed{d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}} \tag{9.1.1}$$

where μ_0 is a constant called the *permeability of free space*:

$$\mu_0 = 4\pi\times10^{-7}\,\mathrm{T\cdot m/A} \tag{9.1.2}$$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element dq:

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} \tag{9.1.3}$$

Adding up these contributions to find the magnetic field at the point P requires integrating over the current source,

Biot - Savart Law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{s} \times \hat{r}}{r^2} \qquad d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}\hat{r}$$

$\downarrow$

remarkably similar

Major difference

current has direction

$\hat{r}$ is a vector

$d\vec{s}$ is a vector

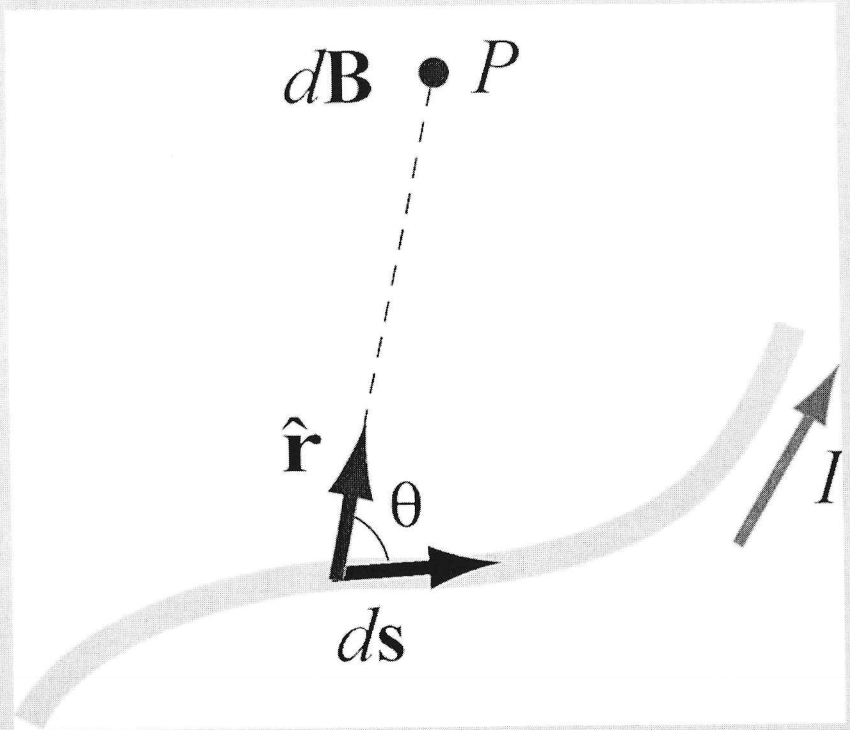
$\vec{B}$ is a vector (from Lorentz equation

$d\vec{s} \times \hat{r}$ is a cross product

$\downarrow$ review of the right hand rule

The Biot-Savart Law

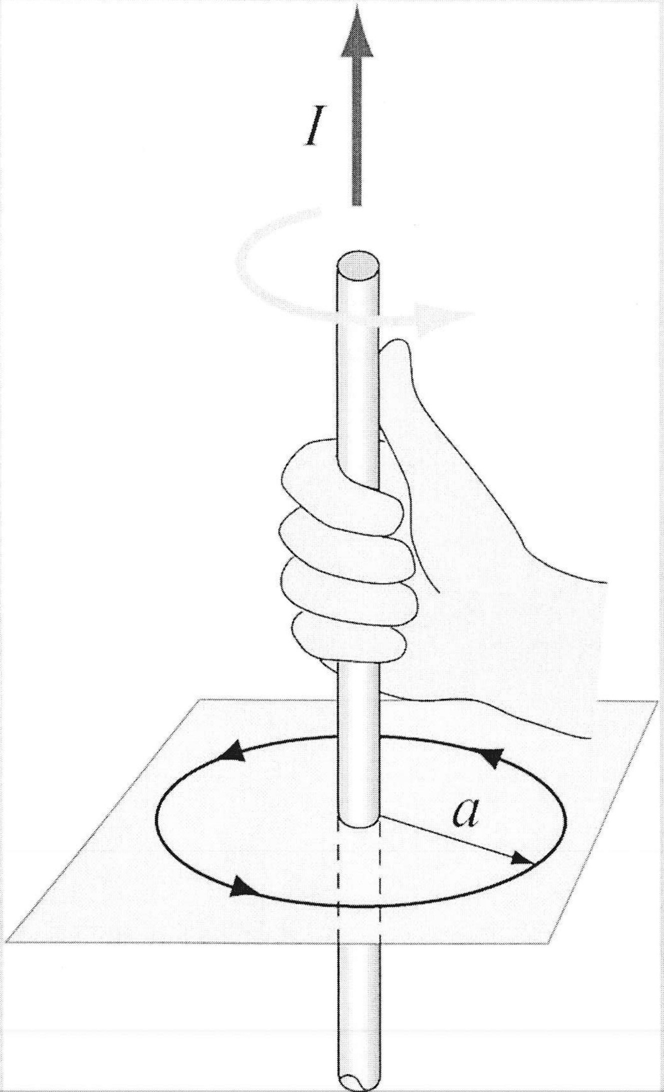
Current element of length ds carrying current I produces a magnetic field:

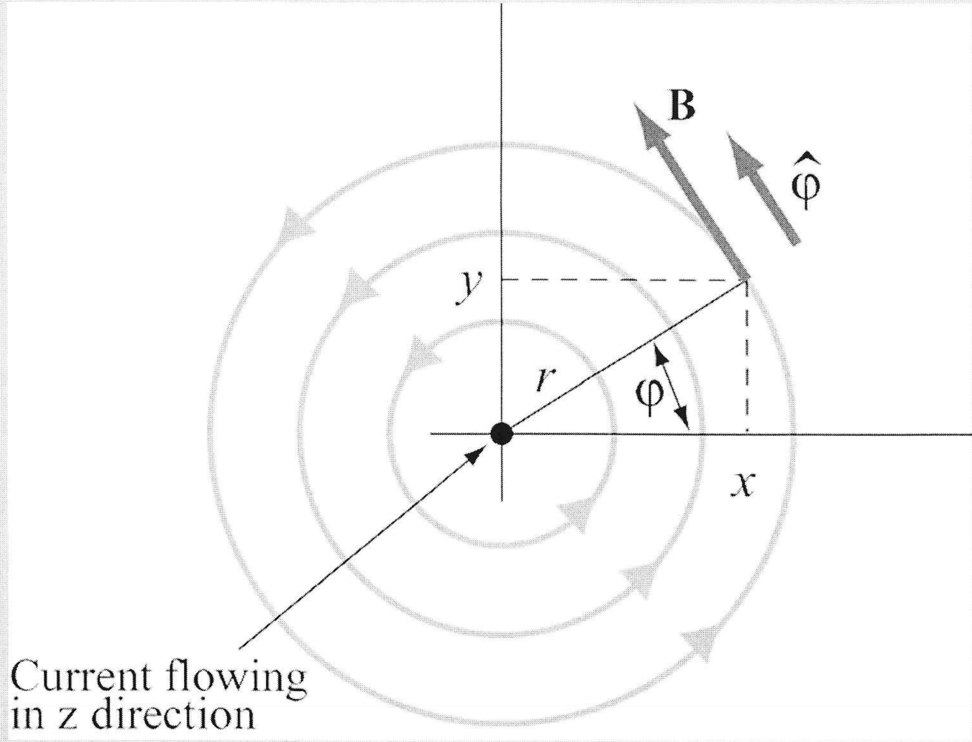


$$\boxed{\mathbf{d}\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}}$$

(http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/03-CurrentElement3d/03-cElement320.html)

The Right-Hand Rule #2

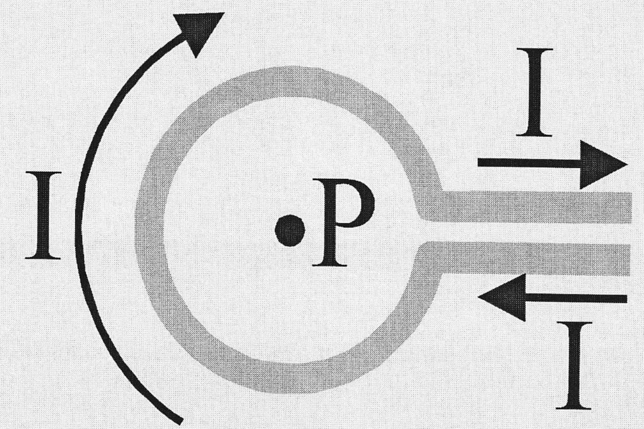




$$\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\varphi}}$$

Example : Coil of Radius *R*

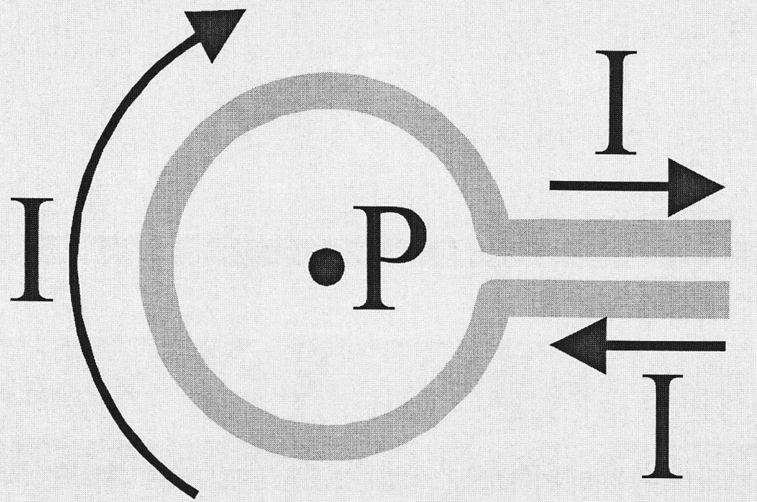
Consider a coil with radius *R* and current *I*



Find the magnetic field B at the center (P)

Example : Coil of Radius *R*

Consider a coil with radius *R* and current *I*

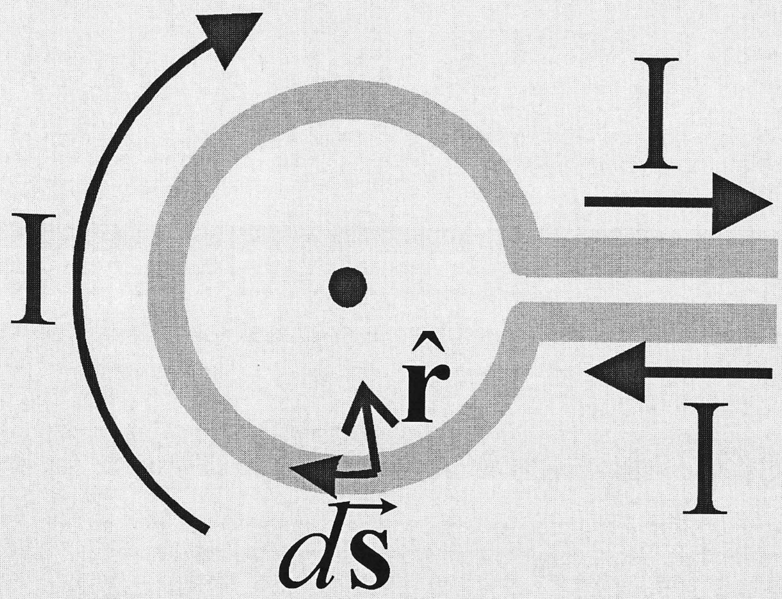


1) Think about it:
   - Legs contribute nothing
     I parallel to *r*
   - Ring makes field into page
2) Choose a ds
3) Pick your coordinates
4) Write Biot-Savart

Example : Coil of Radius *R*

In the circular part of the coil…

$$d\vec{\mathbf{s}} \perp \hat{\mathbf{r}} \quad \rightarrow \quad |\,d\vec{\mathbf{s}} \times \hat{\mathbf{r}}\,| = ds$$



Biot-Savart:

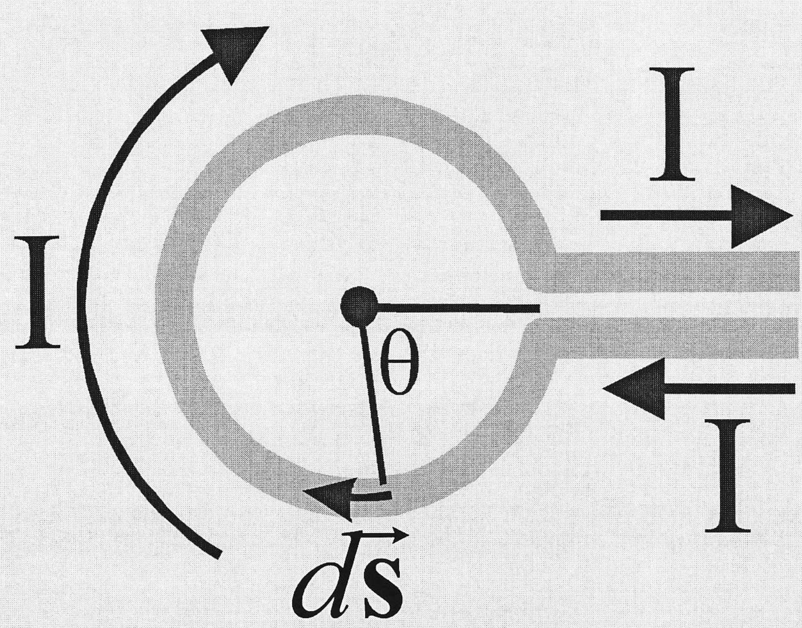
$$dB = \frac{\mu_0 I}{4\pi}\frac{\left|d\vec{\mathbf{s}} \times \hat{\mathbf{r}}\right|}{r^2} = \frac{\mu_0 I}{4\pi}\frac{ds}{r^2}$$

$$= \frac{\mu_0 I}{4\pi}\frac{R\,d\theta}{R^2}$$

$$= \frac{\mu_0 I}{4\pi}\frac{d\theta}{R}$$

Example : Coil of Radius *R*

Consider a coil with radius *R* and current *I*



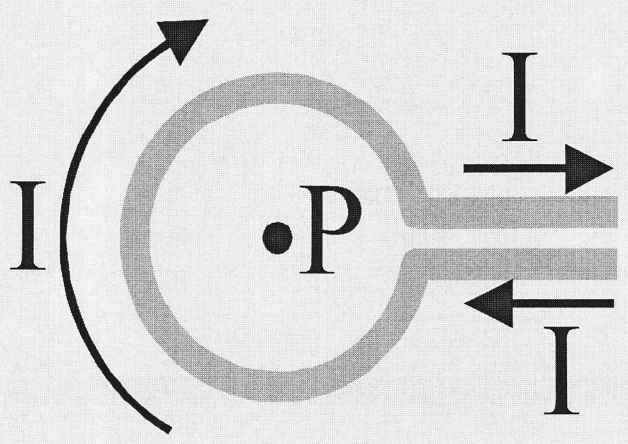
$$dB = \frac{\mu_0 I}{4\pi}\frac{d\theta}{R}$$

$$B = \int dB = \int_0^{2\pi} \frac{\mu_0 I}{4\pi}\frac{d\theta}{R}$$

$$= \frac{\mu_0 I}{4\pi R}\int_0^{2\pi} d\theta = \frac{\mu_0 I}{4\pi R}(2\pi)$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R} \text{ into page}$$

Example : Coil of Radius *R*



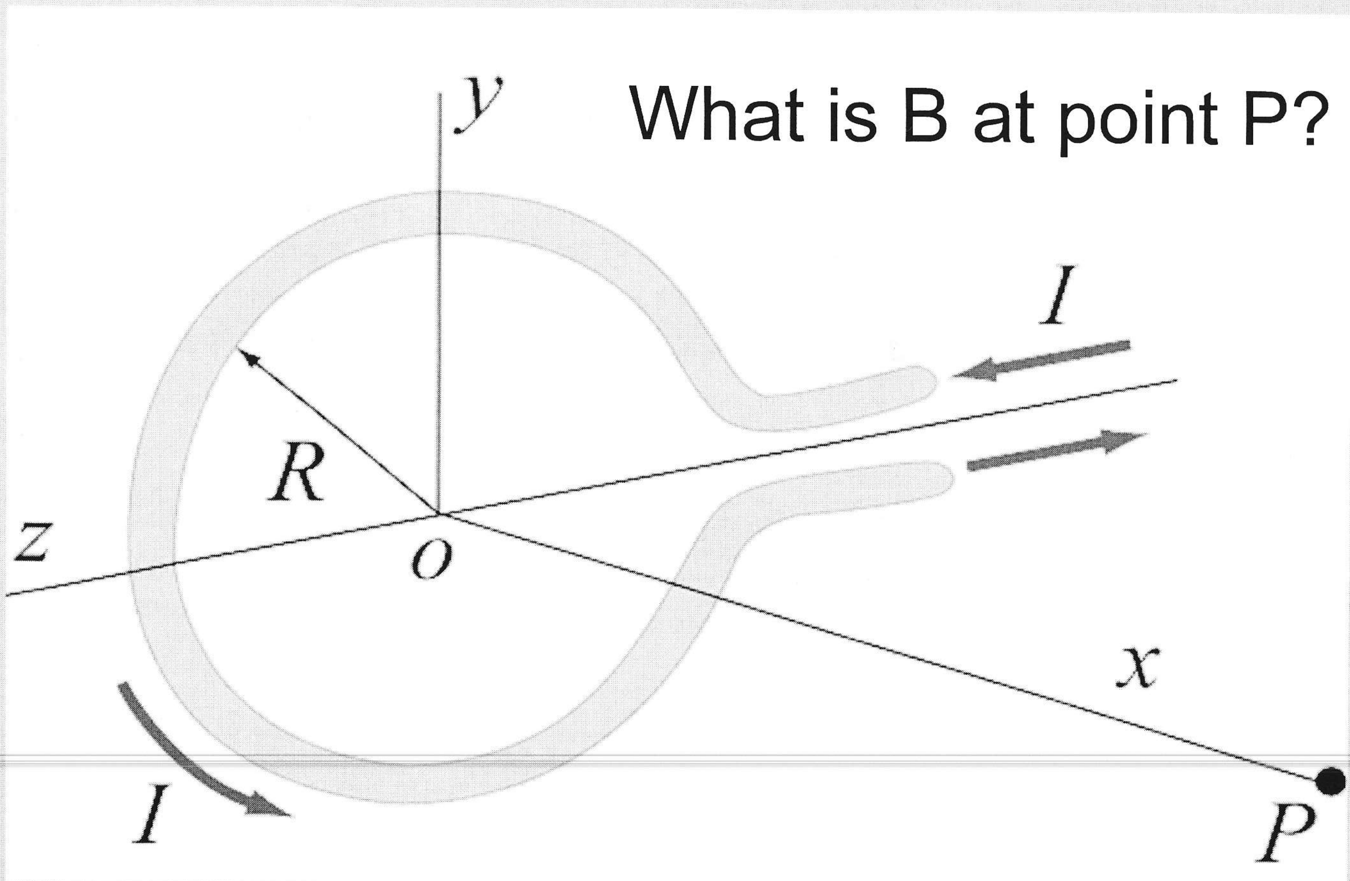
$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R} \text{ into page}$$

Notes:

- This is an EASY Biot-Savart problem:
    - No vectors involved
- This is what I would expect on exam

Group Problem: B Field from Coil of Radius R

Consider a coil with radius *R* and carrying a current *I*



WARNING:

This is much harder than what I just did! Why??

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第四節 必歐-沙伐定律

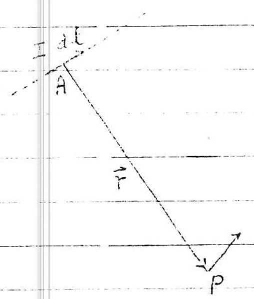
簡介 我們在此節中將討論必歐-沙伐定律及其應用
Biot-Savarat

基本觀念

厄司特之發現，當一電流通過導体時其附近之磁針會偏轉，也即是一個穩定的電流會在其周圍的空間中產生磁場

Ampere-Laplace

必歐-沙伐公式[1,2,3]
$$\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{\hat{u}_T \times \hat{u}_r}{r^2}\, dl$$

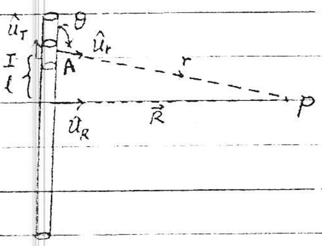


若電流元 I dl 位於 A 點，則它在 P 點所產生之磁場強度為

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{\hat{u}_T \times \hat{u}_r}{r^2}\, dl \qquad (1)$$

此處 $\hat{u}_T$, $\hat{u}_r$ 分別是沿電流及 $\vec{AP}$ 方向之單位向量，μ_0 是自由空間之磁導率

我們現在利用此一公式來計算一直線電流所產生之磁場



為了方便起見，我們將取電流的方向為 z 軸

A 點之座標為 (0, 0, z′) source point

P 點之座標為 (x, y, z) observation point

因此 $$\hat{u}_r = \frac{1}{\sqrt{x^2+y^2+(z-z')^2}}(x, y, z-z') \qquad (2)$$

$$\hat{u}_T = \hat{k}$$

$$\hat{u}_T \times \hat{u}_r = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ x & y & z-z' \end{vmatrix} \frac{1}{\sqrt{x^2+y^2+(z-z')^2}} \qquad (3)$$

$$= (-y\hat{i} + x\hat{j}) \frac{1}{\sqrt{x^2+y^2+(z-z')^2}} \qquad (4)$$

我們強調 $\hat{u}_T \times \hat{u}_r$ 之方向與 z′ 無關[4]

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令 $\hat{u}_\theta = \dfrac{1}{\sqrt{x^2+y^2}}(-y\hat{i}+x\hat{j})$，則 (5)

$$\hat{u}_T \times \hat{u}_r = \sin\theta\,\hat{u}_\theta \tag{6}$$

此處 θ 是 $\hat{u}_T$ 及 $\hat{u}_r$ 之間的夾角

因此此段 $I\,dl$ 在 P 點產生之磁場強度為

$$d\vec{B} = \frac{\mu_0}{4\pi} I \,\hat{u}_\theta \,\frac{\sin\theta}{r^2}\, dl \tag{7}$$

至此已成為一個積分之問題

由圖中可以看出

$$\frac{R}{r} = \sin\theta \iff r = R\sin\theta \tag{8}$$

$$\frac{R}{l} = -\tan\theta \iff l = -R\cot\theta \tag{9}$$

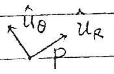
由 (9) 式得 $dl = R\csc^2\theta\, d\theta$ (10)

同時 $l = -\infty$ 對應於 $\theta = 0$，$l = +\infty$ 對應於 $\theta = \pi$

因此，由無限長導線在 P 點所產生之磁場為

$$\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi} I\,\hat{u}_\theta \int_{-\infty}^{+\infty} \frac{\sin\theta}{r^2}\, dl \\
&= \frac{\mu_0}{4\pi} I\,\hat{u}_\theta \int_0^{\pi} \frac{\sin\theta}{R^2\csc^2\theta}\, R\csc^2\theta\, d\theta \\
&= \frac{\mu_0}{4\pi} I\,\hat{u}_\theta \frac{1}{R}\left[-\cos\theta\right]_0^{\pi} \\
&= \frac{\mu_0}{2\pi}\frac{I}{R}\,\hat{u}_\theta
\end{aligned} \tag{11}$$

$\hat{u}_\theta \perp \vec{z}$，及 $\vec{R}$. 我們在通過 P 點垂直於 z 軸之平面上來看此問題



若我們沿此圓周作一線積分，由於 $\vec{B} \parallel d\vec{l}'$

$$\begin{aligned}
\oint \vec{B}\cdot d\vec{l}' &= \oint \frac{\mu_0}{2\pi}\frac{I}{R}\, dl \\
&= \frac{\mu_0 I}{2\pi R} \underbrace{\oint dl}_{2\pi R} \\
&= \mu_0 I
\end{aligned} \tag{12}$$

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(1) 此一公式是由實驗結果歸納而來的。

(2) 它也是与 r^2 成反比

(3) 嚴格講起來，此一公式只在穩定流時成立，但當電流甚慢地隨時間改變，輻射效應可略去不計，此一公式仍適用

(4) 電流之任一段在 P 點產生磁場之方向均相同

(5) 安培定律即是由此式推廣而得。

$$\vec{\mathbf{B}} = \int_{\text{wire}} d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{9.1.4}$$

The integral is a vector integral, which means that the expression for $\vec{\mathbf{B}}$ is really three integrals, one for each component of $\vec{\mathbf{B}}$. The vector nature of this integral appears in the cross product $I\,d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Interactive Simulation 9.1: Magnetic Field of a Current Element

Figure 9.1.2 is an interactive ShockWave display that shows the magnetic field of a current element from Eq. (9.1.1). This interactive display allows you to move the position of the observer about the source current element to see how moving that position changes the value of the magnetic field at the position of the observer.

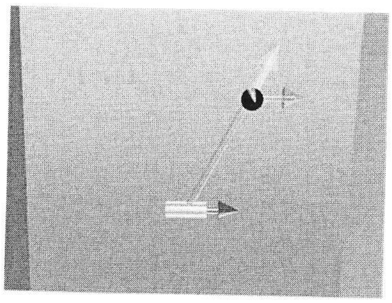

Figure 9.1.2 Magnetic field of a current element.

Example 9.1: Magnetic Field due to a Finite Straight Wire

A thin, straight wire carrying a current I is placed along the x-axis, as shown in Figure 9.1.3. Evaluate the magnetic field at point P. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point P.

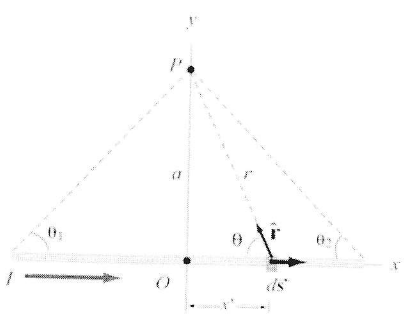

Figure 9.1.3 A thin straight wire carrying a current I.

Solution:

This is a typical example involving the use of the Biot-Savart law. We solve the problem using the methodology summarized in Section 9.10.

(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{\mathbf{s}} = dx'\hat{\mathbf{i}}$ carrying current I in the x-direction. The location of this source is represented by $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$.

(2) Field point (coordinates denoted with a subscript "P")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing P is $\vec{\mathbf{r}}_P = a\hat{\mathbf{j}}$.

(3) Relative position vector

The vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$ is a "relative" position vector which points from the source point to the field point. In this case, $\vec{\mathbf{r}} = a\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$, and the magnitude $r = |\vec{\mathbf{r}}| = \sqrt{a^2 + x'^2}$ is the distance from between the source and P. The corresponding unit vector is given by

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{a\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{i}}$$

(4) The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$

The cross product is given by

$$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = (dx'\hat{\mathbf{i}}) \times (-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (dx'\sin\theta)\hat{\mathbf{k}}$$

(5) Write down the contribution to the magnetic field due to $Id\vec{\mathbf{s}}$

The expression is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\,\sin\theta}{r^2}\hat{\mathbf{k}}$$

which shows that the magnetic field at P will point in the $+\hat{\mathbf{k}}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure 9.1.3, we have

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \quad \Rightarrow \quad dx = -a\csc^2\theta\, d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi}\frac{(-a\csc^2\theta\, d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we obtain

$$\boxed{B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)} \tag{9.1.5}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add!

Let's examine the following cases:

(i) In the symmetric case where $\theta_2 = -\theta_1$, the field point P is located along the perpendicular bisector. If the length of the rod is $2L$, then $\cos\theta_1 = L/\sqrt{L^2 + a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta_1 = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2 + a^2}} \tag{9.1.6}$$

(ii) The infinite length limit $L \to \infty$

This limit is obtained by choosing $(\theta_1, \theta_2) = (0, 0)$. The magnetic field at a distance a away becomes

$$\boxed{B = \frac{\mu_0 I}{2\pi a}} \tag{9.1.7}$$

Note that in this limit, the system possesses cylindrical symmetry, and the magnetic field lines are circular, as shown in Figure 9.1.4.

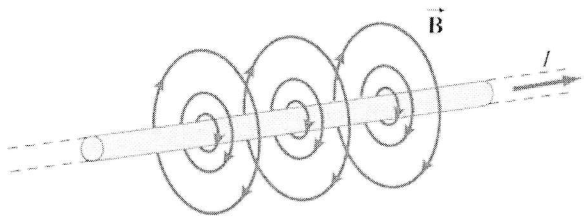

Figure 9.1.4 Magnetic field lines due to an infinite wire carrying current *I*.

In fact, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule (Figure 9.1.5).

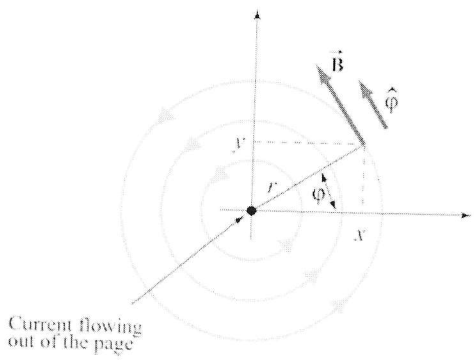

Figure 9.1.5 Direction of the magnetic field due to an infinite straight wire

If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field. In cylindrical coordinates (r, φ, z) where the unit vectors are related by $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{z}}$, if the current flows in the $+z$-direction, then, using the Biot-Savart law, the magnetic field must point in the φ-direction.

Example 9.2: Magnetic Field due to a Circular Current Loop

A circular loop of radius R in the xy plane carries a steady current I, as shown in Figure 9.1.6.

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z \hat{\mathbf{k}}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e., $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$

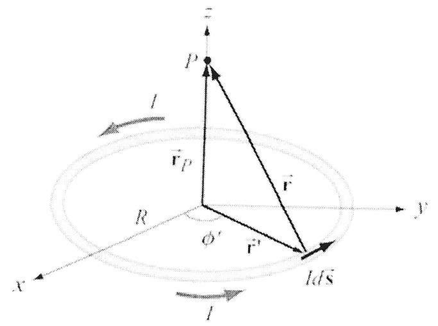

Figure 9.1.6 Magnetic field due to a circular loop carrying a steady current.

Solution:

(a) This is another example that involves the application of the Biot-Savart law. Again let's find the magnetic field by applying the same methodology used in Example 9.1.

(1) Source point

In Cartesian coordinates, the differential current element located at $\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ can be written as $Id\vec{\mathbf{s}} = I(d\vec{\mathbf{r}}'/d\phi')d\phi' = IRd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$.

(2) Field point

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$.

(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$

The relative position vector is given by

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = -R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{9.1.8}$$

and its magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + (-R\sin\phi')^2 + z^2} = \sqrt{R^2 + z^2} \tag{9.1.9}$$

is the distance between the differential current element and P. Thus, the corresponding unit vector from $Id\vec{\mathbf{s}}$ to P can be written as

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')$ can be simplified as

$$\begin{aligned}d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,') &= R\,d\phi'\left(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}}\right)\times[-R\cos\phi'\hat{\mathbf{i}}-R\sin\phi'\hat{\mathbf{j}}+z\hat{\mathbf{k}}]\\ &= R\,d\phi'[z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}}]\end{aligned} \tag{9.1.10}$$

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$\begin{aligned}d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}=\frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}=\frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')}{|\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,'|^3}\\ &= \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}d\phi'\end{aligned} \tag{9.1.11}$$

(6) Carrying out the integration

Using the result obtained above, the magnetic field at P is

$$\vec{\mathbf{B}}=\frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}d\phi' \tag{9.1.12}$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero:

$$B_x=\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\cos\phi'\,d\phi'=\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\sin\phi'\Big|_0^{2\pi}=0 \tag{9.1.13}$$

$$B_y=\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\phi'\,d\phi'=-\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\cos\phi'\Big|_0^{2\pi}=0 \tag{9.1.14}$$

On the other hand, the z component is

$$B_z=\frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi}d\phi'=\frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}}=\frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \tag{9.1.15}$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments.

The behavior of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown in Figure 9.1.7:

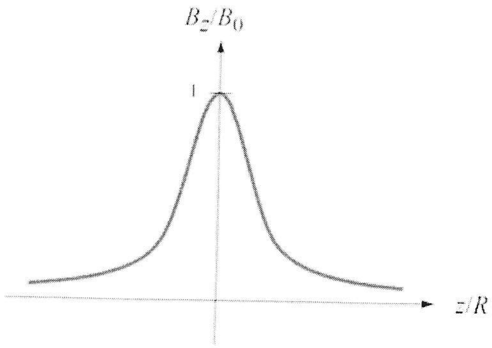

Figure 9.1.7 The ratio of the magnetic field, B_z / B_0, as a function of z / R

(b) If we place a magnetic dipole $\vec{\mu} = \mu_z \hat{\mathbf{k}}$ at the point P, as discussed in Chapter 8, due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z}{dz} \right) \hat{\mathbf{k}} \tag{9.1.16}$$

Upon differentiating Eq. (9.1.15) and substituting into Eq. (9.1.16), we obtain

$$\vec{\mathbf{F}}_B = -\frac{3\mu_z \mu_0 I R^2 z}{2(R^2 + z^2)^{5/2}} \hat{\mathbf{k}} \tag{9.1.17}$$

Thus, the dipole is attracted toward the current-carrying ring. On the other hand, if the direction of the dipole is reversed, $\vec{\mu} = -\mu_z \hat{\mathbf{k}}$, the resulting force will be repulsive.

9.1.1 Magnetic Field of a Moving Point Charge

Suppose we have an infinitesimal current element in the form of a cylinder of cross-sectional area A and length ds consisting of n charge carriers per unit volume, all moving at a common velocity $\vec{\mathbf{v}}$ along the axis of the cylinder. Let I be the current in the element, which we define as the amount of charge passing through any cross-section of the cylinder per unit time. From Chapter 6, we see that the current I can be written as

$$n A q |\vec{\mathbf{v}}| = I \tag{9.1.18}$$

The total number of charge carriers in the current element is simply $dN = n A\, ds$, so that using Eq. (9.1.1), the magnetic field $d\vec{\mathbf{B}}$ due to the dN charge carriers is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{(nAq\,|\,\vec{\mathbf{v}}\,|)\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{(n\,A\,ds)q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{(dN)q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} \tag{9.1.19}$$

where r is the distance between the charge and the field point P at which the field is being measured, the unit vector $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ points *from* the source of the field (the charge) *to P*. The differential length vector $d\vec{\mathbf{s}}$ is defined to be parallel to $\vec{\mathbf{v}}$. In case of a single charge, $dN = 1$, the above equation becomes

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} \tag{9.1.20}$$

Note, however, that since a point charge does not constitute a steady current, the above equation strictly speaking only holds in the non-relativistic limit where $v \ll c$, the speed of light, so that the effect of "retardation" can be ignored.

The result may be readily extended to a collection of N point charges, each moving with a different velocity. Let the ith charge q_i be located at (x_i, y_i, z_i) and moving with velocity $\vec{\mathbf{v}}_i$. Using the superposition principle, the magnetic field at P can be obtained as:

$$\vec{\mathbf{B}} = \sum_{i=1}^{N}\frac{\mu_0}{4\pi}q_i\vec{\mathbf{v}}_i \times \left[\frac{(x-x_i)\hat{\mathbf{i}}+(y-y_i)\hat{\mathbf{j}}+(z-z_i)\hat{\mathbf{k}}}{\left[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right]^{3/2}}\right] \tag{9.1.21}$$

Animation 9.1: Magnetic Field of a Moving Charge

Figure 9.1.8 shows one frame of the animations of the magnetic field of a moving positive and negative point charge, assuming the speed of the charge is small compared to the speed of light.

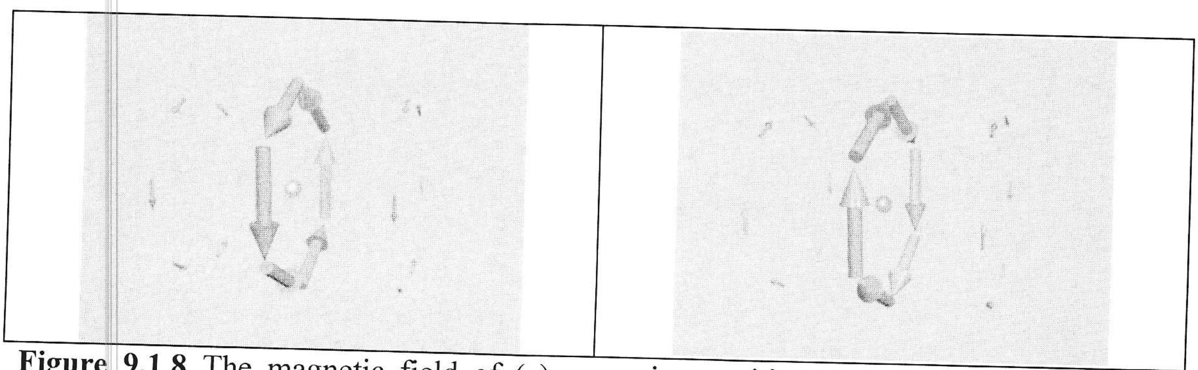

Figure 9.1.8 The magnetic field of (a) a moving positive charge, and (b) a moving negative charge, when the speed of the charge is small compared to the speed of light.

Animation 9.2: Magnetic Field of Several Charges Moving in a Circle

Suppose we want to calculate the magnetic fields of a number of charges moving on the circumference of a circle with equal spacing between the charges. To calculate this field we have to add up vectorially the magnetic fields of each of charges using Eq. (9.1.19).

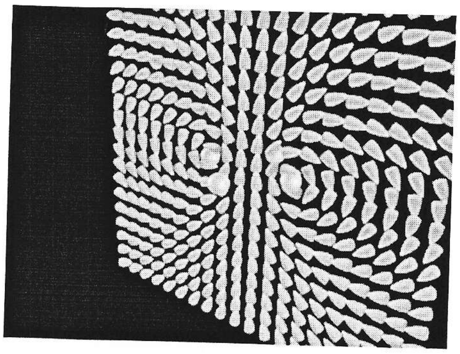

Figure 9.1.9 The magnetic field of four charges moving in a circle. We show the magnetic field vector directions in only one plane. The bullet-like icons indicate the direction of the magnetic field at that point in the array spanning the plane.

Figure 9.1.9 shows one frame of the animation when the number of moving charges is four. Other animations show the same situation for $N=1$, 2, and 8. When we get to eight charges, a characteristic pattern emerges--the magnetic dipole pattern. Far from the ring, the shape of the field lines is the same as the shape of the field lines for an electric dipole.

Interactive Simulation 9.2: Magnetic Field of a Ring of Moving Charges

Figure 9.1.10 shows a ShockWave display of the vectoral addition process for the case where we have 30 charges moving on a circle. The display in Figure 9.1.10 shows an observation point fixed on the axis of the ring. As the addition proceeds, we also show the resultant up to that point (large arrow in the display).

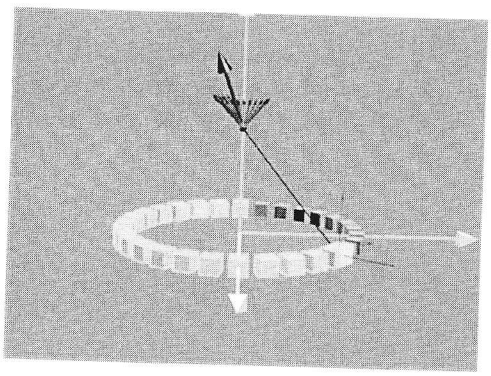

Figure 9.1.10 A ShockWave simulation of the use of the principle of superposition to find the magnetic field due to 30 moving charges moving in a circle at an observation point on the axis of the circle.

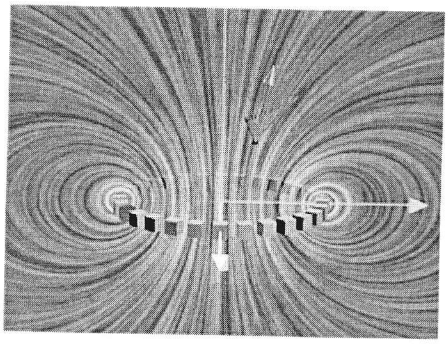

Figure 9.1.11 The magnetic field due to 30 charges moving in a circle at a given observation point. The position of the observation point can be varied to see how the magnetic field of the individual charges adds up to give the total field.

In Figure 9.1.11, we show an interactive ShockWave display that is similar to that in Figure 9.1.10, but now we can interact with the display to move the position of the observer about in space. To get a feel for the total magnetic field, we also show a "iron filings" representation of the magnetic field due to these charges. We can move the observation point about in space to see how the total field at various points arises from the individual contributions of the magnetic field of to each moving charge.

9.2 Force Between Two Parallel Wires

We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert force on each other.

Consider two parallel wires separated by a distance *a* and carrying currents I_1 and I_2 in the +*x*-direction, as shown in Figure 9.2.1.

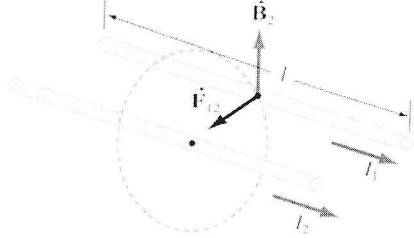

Figure 9.2.1 Force between two parallel wires

The magnetic force, $\vec{\mathbf{F}}_{12}$, exerted on wire 1 by wire 2 may be computed as follows: Using the result from the previous example, the magnetic field lines due to I_2 going in the +*x*-direction are circles concentric with wire 2, with the field $\vec{\mathbf{B}}_2$ pointing in the tangential direction. Thus, at an arbitrary point P on wire 1, we have $\vec{\mathbf{B}}_2 = -(\mu_0 I_2 / 2\pi a)\hat{\mathbf{j}}$, which points in the direction perpendicular to wire 1, as depicted in Figure 9.2.1. Therefore,

$$\vec{\mathbf{F}}_{12} = I_1 \vec{l} \times \vec{\mathbf{B}}_2 = I_1 \left(l\hat{\mathbf{i}} \right) \times \left(-\frac{\mu_0 I_2}{2\pi a} \hat{\mathbf{j}} \right) = -\frac{\mu_0 I_1 I_2 l}{2\pi a} \hat{\mathbf{k}} \tag{9.2.1}$$

Clearly $\vec{\mathbf{F}}_{12}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive.

Animation 9.3: Forces Between Current-Carrying Parallel Wires

Figures 9.2.2 shows parallel wires carrying current in the same and in opposite directions. In the first case, the magnetic field configuration is such as to produce an attraction between the wires. In the second case the magnetic field configuration is such as to produce a repulsion between the wires.

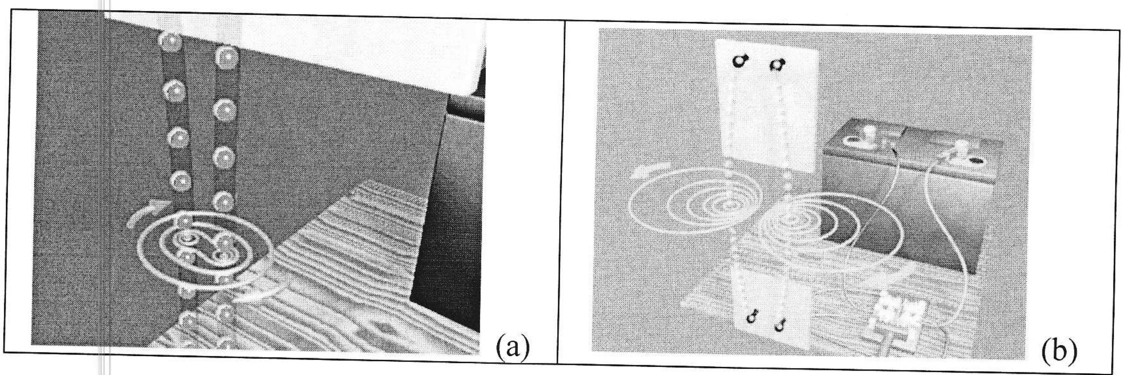

Figure 9.2.2 (a) The attraction between two wires carrying current in the same direction. The direction of current flow is represented by the motion of the orange spheres in the visualization. (b) The repulsion of two wires carrying current in opposite directions.

9.3 Ampere's Law

We have seen that moving charges or currents are the source of magnetism. This can be readily demonstrated by placing compass needles near a wire. As shown in Figure 9.3.1a, all compass needles point in the same direction in the absence of current. However, when $I \neq 0$, the needles will be deflected along the tangential direction of the circular path (Figure 9.3.1b).

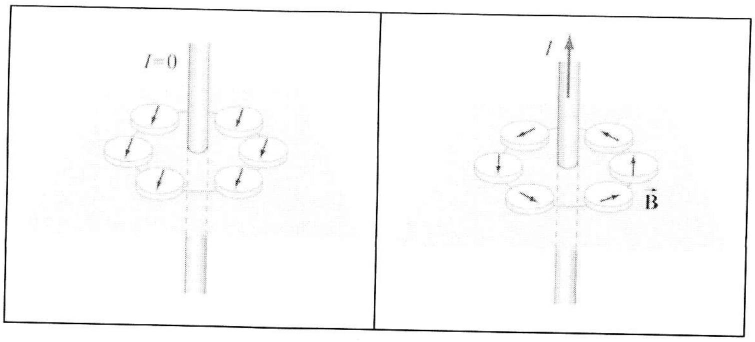

Figure 9.3.1 Deflection of compass needles near a current-carrying wire

Let us now divide a circular path of radius r into a large number of small length vectors $\Delta\vec{s} = \Delta s\,\hat{\boldsymbol{\varphi}}$, that point along the tangential direction with magnitude Δs (Figure 9.3.2).

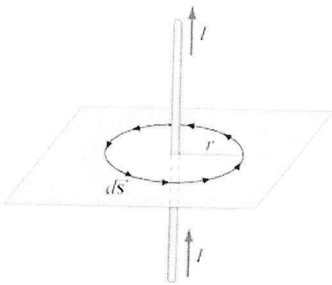

Figure 9.3.2 Amperian loop

In the limit $\Delta\vec{s} \to \vec{0}$, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\oint ds = \left(\frac{\mu_0 I}{2\pi r}\right)(2\pi r) = \mu_0 I \qquad (9.3.1)$$

The result above is obtained by choosing a closed path, or an "Amperian loop" that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, as that shown in Figure 9.3.3

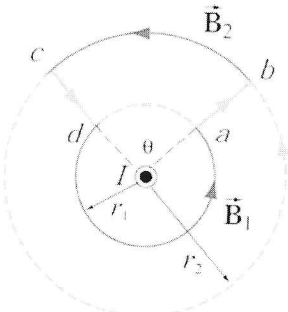

Figure 9.3.3 An Amperian loop involving two field lines

The line integral of the magnetic field around the contour *abcda* is

$$\begin{aligned}\oint_{abcda} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} &= \int_{ab} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{bc} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} \\ &= 0 + B_2(r_2\theta) + 0 + B_1[r_1(2\pi-\theta)]\end{aligned} \tag{9.3.2}$$

where the length of arc bc is $r_2\theta$, and $r_1(2\pi-\theta)$ for arc da. The first and the third integrals vanish since the magnetic field is perpendicular to the paths of integration. With $B_1 = \mu_0 I/2\pi r_1$ and $B_2 = \mu_0 I/2\pi r_2$, the above expression becomes

$$\oint_{abcda} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1}[r_1(2\pi-\theta)] = \frac{\mu_0 I}{2\pi}\theta + \frac{\mu_0 I}{2\pi}(2\pi-\theta) = \mu_0 I \tag{9.3.3}$$

We see that the same result is obtained whether the closed path involves one or two magnetic field lines.

As shown in Example 9.1, in cylindrical coordinates (r,φ,z) with current flowing in the $+z$-axis, the magnetic field is given by $\vec{\mathbf{B}} = (\mu_0 I/2\pi r)\hat{\boldsymbol{\varphi}}$. An arbitrary length element in the cylindrical coordinates can be written as

$$d\vec{\mathbf{s}} = dr\,\hat{\mathbf{r}} + r\,d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}} \tag{9.3.4}$$

which implies

$$\oint_{\text{closed path}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \oint_{\text{closed path}} \left(\frac{\mu_0 I}{2\pi r}\right) r\,d\varphi = \frac{\mu_0 I}{2\pi}\oint_{\text{closed path}} d\varphi = \frac{\mu_0 I}{2\pi}(2\pi) = \mu_0 I \tag{9.3.5}$$

In other words, the line integral of $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed Amperian loop is proportional to I_{enc}, the current encircled by the loop.

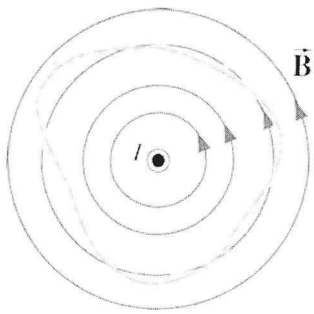

Figure 9.3.4 An Amperian loop of arbitrary shape.

The generalization to any closed loop of arbitrary shape (see for example, Figure 9.3.4) that involves many magnetic field lines is known as Ampere's law:

$$\boxed{\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}} \tag{9.3.6}$$

Ampere's law in magnetism is analogous to Gauss's law in electrostatics. In order to apply them, the system must possess certain symmetry. In the case of an infinite wire, the system possesses cylindrical symmetry and Ampere's law can be readily applied. However, when the length of the wire is finite, Biot-Savart law must be used instead.

Biot-Savart Law	$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$	general current source ex: finite wire
Ampere's law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$	current source has certain symmetry ex: infinite wire (cylindrical)

Ampere's law is applicable to the following current configurations:

1. Infinitely long straight wires carrying a steady current I (Example 9.3)

2. Infinitely large sheet of thickness b with a current density J (Example 9.4).

3. Infinite solenoid (Section 9.4).

4. Toroid (Example 9.5).

We shall examine all four configurations in detail.

Example 9.3: Field Inside and Outside a Current-Carrying Wire

Consider a long straight wire of radius R carrying a current I of uniform current density, as shown in Figure 9.3.5. Find the magnetic field everywhere.

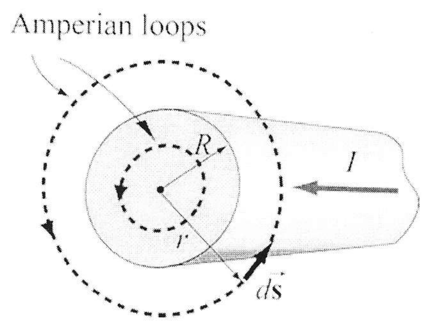

Figure 9.3.5 Amperian loops for calculating the $\vec{\mathbf{B}}$ field of a conducting wire of radius R.

Solution:

(i) Outside the wire where $r \geq R$, the Amperian loop (circle 1) completely encircles the current, i.e., $I_{\text{enc}} = I$. Applying Ampere's law yields

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 I$$

which implies

$$B = \frac{\mu_0 I}{2\pi r}$$

(ii) Inside the wire where $r < R$, the amount of current encircled by the Amperian loop (circle 2) is proportional to the area enclosed, i.e.,

$$I_{\text{enc}} = \left(\frac{\pi r^2}{\pi R^2}\right) I$$

Thus, we have

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r) = \mu_0 I \left(\frac{\pi r^2}{\pi R^2}\right) \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 I r}{2\pi R^2}}$$

We see that the magnetic field is zero at the center of the wire and increases linearly with r until $r=R$. Outside the wire, the field falls off as $1/r$. The qualitative behavior of the field is depicted in Figure 9.3.6 below:

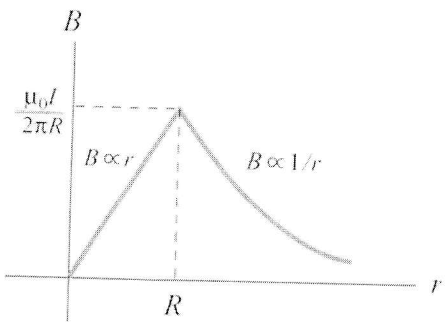

Figure 9.3.6 Magnetic field of a conducting wire of radius R carrying a steady current I.

Example 9.4: Magnetic Field Due to an Infinite Current Sheet

Consider an infinitely large sheet of thickness b lying in the xy plane with a uniform current density $\vec{\mathbf{J}} = J_0 \hat{\mathbf{i}}$. Find the magnetic field everywhere.

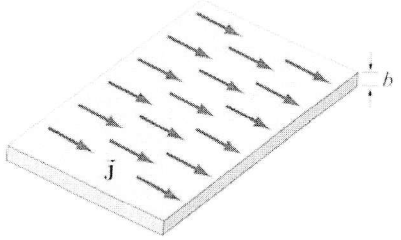

Figure 9.3.7 An infinite sheet with current density $\vec{\mathbf{J}} = J_0\hat{\mathbf{i}}$.

Solution:

We may think of the current sheet as a set of parallel wires carrying currents in the $+x$-direction. From Figure 9.3.8, we see that magnetic field at a point P above the plane points in the $-y$-direction. The z-component vanishes after adding up the contributions from all wires. Similarly, we may show that the magnetic field at a point below the plane points in the $+y$-direction.

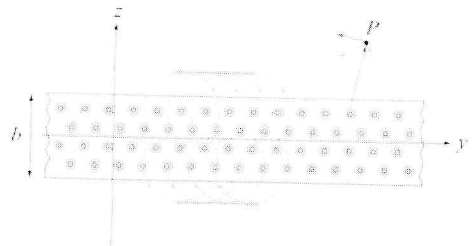

Figure 9.3.8 Magnetic field of a current sheet

We may now apply Ampere's law to find the magnetic field due to the current sheet. The Amperian loops are shown in Figure 9.3.9.

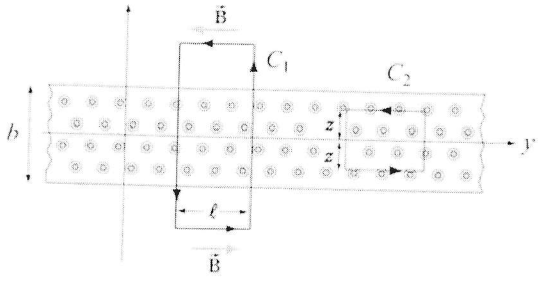

Figure 9.3.9 Amperian loops for the current sheets

For the field outside, we integrate along path C_1. The amount of current enclosed by C_1 is

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(b\ell) \tag{9.3.7}$$

Applying Ampere's law leads to

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0(J_0 b\ell) \tag{9.3.8}$$

or $B = \mu_0 J_0 b/2$. Note that the magnetic field outside the sheet is constant, independent of the distance from the sheet. Next we find the magnetic field inside the sheet. The amount of current enclosed by path C_2 is

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(2|z|\ell) \tag{9.3.9}$$

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0 J_0(2|z|\ell) \tag{9.3.10}$$

or $B = \mu_0 J_0 |z|$. At $z = 0$, the magnetic field vanishes, as required by symmetry. The results can be summarized using the unit-vector notation as

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 J_0 b}{2}\hat{\mathbf{j}}, & z > b/2 \\ -\mu_0 J_0 z\,\hat{\mathbf{j}}, & -b/2 < z < b/2 \\ \dfrac{\mu_0 J_0 b}{2}\hat{\mathbf{j}}, & z < -b/2 \end{cases} \tag{9.3.11}$$

Let's now consider the limit where the sheet is infinitesimally thin, with $b \to 0$. In this case, instead of current density $\vec{\mathbf{J}} = J_0\hat{\mathbf{i}}$, we have surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$, where $K = J_0 b$. Note that the dimension of K is current/length. In this limit, the magnetic field becomes

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > 0 \\ \dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < 0 \end{cases} \tag{9.3.12}$$

9.4 Solenoid

A solenoid is a long coil of wire tightly wound in the helical form. Figure 9.4.1 shows the magnetic field lines of a solenoid carrying a steady current I. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform,

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(b\ell) \tag{9.3.7}$$

Applying Ampere's law leads to

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0(J_0 b\ell) \tag{9.3.8}$$

or $B = \mu_0 J_0 b/2$. Note that the magnetic field outside the sheet is constant, independent of the distance from the sheet. Next we find the magnetic field inside the sheet. The amount of current enclosed by path C_2 is

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(2|z|\ell) \tag{9.3.9}$$

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0 J_0(2|z|\ell) \tag{9.3.10}$$

or $B = \mu_0 J_0 |z|$. At $z = 0$, the magnetic field vanishes, as required by symmetry. The results can be summarized using the unit-vector notation as

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 J_0 b}{2}\hat{\mathbf{j}}, & z > b/2 \\ -\mu_0 J_0 z\,\hat{\mathbf{j}}, & -b/2 < z < b/2 \\ \dfrac{\mu_0 J_0 b}{2}\hat{\mathbf{j}}, & z < -b/2 \end{cases} \tag{9.3.11}$$

Let's now consider the limit where the sheet is infinitesimally thin, with $b \to 0$. In this case, instead of current density $\vec{\mathbf{J}} = J_0\hat{\mathbf{i}}$, we have surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$, where $K = J_0 b$. Note that the dimension of K is current/length. In this limit, the magnetic field becomes

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > 0 \\ \dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < 0 \end{cases} \tag{9.3.12}$$

9.4 Solenoid

A solenoid is a long coil of wire tightly wound in the helical form. Figure 9.4.1 shows the magnetic field lines of a solenoid carrying a steady current I. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter. For an "ideal" solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.

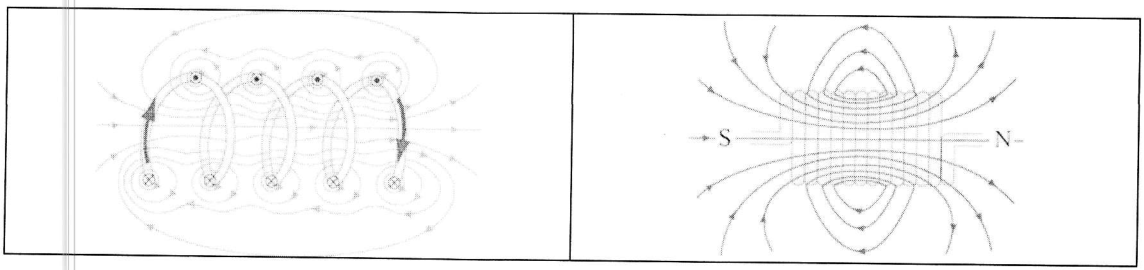

Figure 9.4.1 Magnetic field lines of a solenoid

We can use Ampere's law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure 9.4.2. To compute $\vec{\mathbf{B}}$, we consider a rectangular path of length l and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{\mathbf{B}}$ along this loop is

$$\begin{aligned}\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} &= \int_1 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_2 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_3 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_4 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} \\ &= 0 \quad + \quad 0 \quad + \quad Bl \quad + \quad 0\end{aligned} \tag{9.4.1}$$

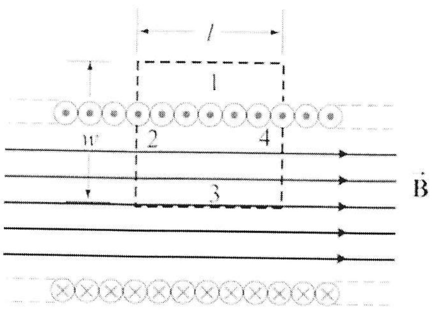

Figure 9.4.2 Amperian loop for calculating the magnetic field of an ideal solenoid.

In the above, the contributions along sides 2 and 4 are zero because $\vec{\mathbf{B}}$ is perpendicular to $d\vec{\mathbf{s}}$. In addition, $\vec{\mathbf{B}}=\vec{\mathbf{0}}$ along side 1 because the magnetic field is non-zero only inside the solenoid. On the other hand, the total current enclosed by the Amperian loop is $I_{\text{enc}} = NI$, where N is the total number of turns. Applying Ampere's law yields

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl = \mu_0 NI \tag{9.4.2}$$

or

$$B = \frac{\mu_0 N I}{l} = \mu_0 n I \qquad (9.4.3)$$

where $n = N/l$ represents the number of turns per unit length., In terms of the surface current, or current per unit length $K = nI$, the magnetic field can also be written as,

$$B = \mu_0 K \qquad (9.4.4)$$

What happens if the length of the solenoid is finite? To find the magnetic field due to a finite solenoid, we shall approximate the solenoid as consisting of a large number of circular loops stacking together. Using the result obtained in Example 9.2, the magnetic field at a point P on the z axis may be calculated as follows: Take a cross section of tightly packed loops located at z' with a thickness dz', as shown in Figure 9.4.3

The amount of current flowing through is proportional to the thickness of the cross section and is given by $dI = I(ndz') = I(N/l)dz'$, where $n = N/l$ is the number of turns per unit length.

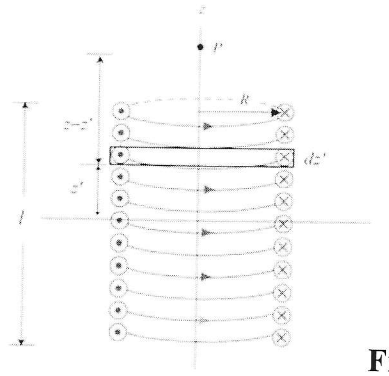

Figure 9.4.3 Finite Solenoid

The contribution to the magnetic field at P due to this subset of loops is

$$dB_z = \frac{\mu_0 R^2}{2[(z-z')^2 + R^2]^{3/2}} dI = \frac{\mu_0 R^2}{2[(z-z')^2 + R^2]^{3/2}} (nIdz') \qquad (9.4.5)$$

Integrating over the entire length of the solenoid, we obtain

$$\begin{aligned} B_z &= \frac{\mu_0 n I R^2}{2} \int_{-l/2}^{l/2} \frac{dz'}{[(z-z')^2 + R^2]^{3/2}} = \frac{\mu_0 n I R^2}{2} \left. \frac{z'-z}{R^2\sqrt{(z-z')^2 + R^2}} \right|_{-l/2}^{l/2} \\ &= \frac{\mu_0 n I}{2} \left[\frac{(l/2)-z}{\sqrt{(z-l/2)^2 + R^2}} + \frac{(l/2)+z}{\sqrt{(z+l/2)^2 + R^2}} \right] \end{aligned} \qquad (9.4.6)$$

A plot of B_z / B_0, where $B_0 = \mu_0 n I$ is the magnetic field of an infinite solenoid, as a function of z/R is shown in Figure 9.4.4 for $l = 10R$ and $l = 20R$.

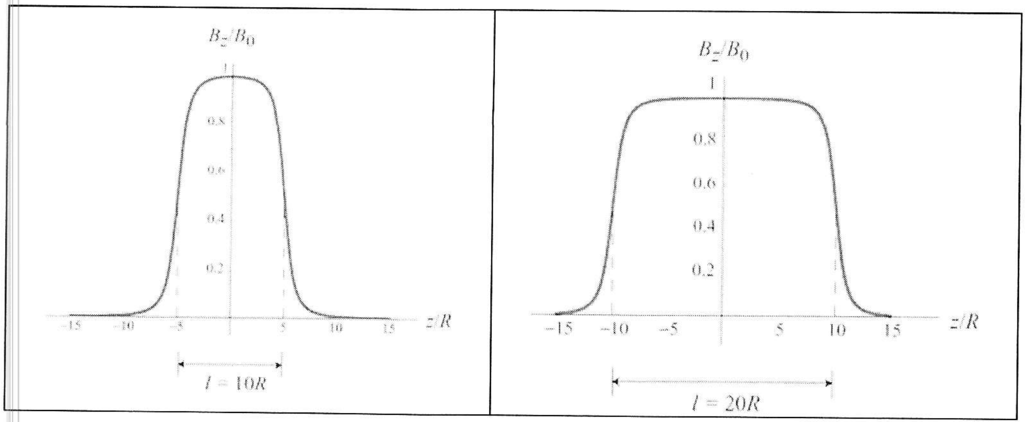

Figure 9.4.4 Magnetic field of a finite solenoid for (a) $l = 10R$, and (b) $l = 20R$.

Notice that the value of the magnetic field in the region $|z| < l/2$ is nearly uniform and approximately equal to B_0.

Examaple 9.5: Toroid

Consider a toroid which consists of N turns, as shown in Figure 9.4.5. Find the magnetic field everywhere.

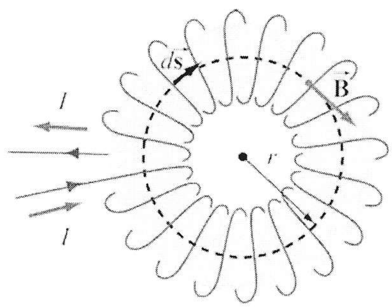

Figure 9.4.5 A toroid with N turns

Solutions:

One can think of a toroid as a solenoid wrapped around with its ends connected. Thus, the magnetic field is completely confined inside the toroid and the field points in the azimuthal direction (clockwise due to the way the current flows, as shown in Figure 9.4.5.)

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint B\,ds = B\oint ds = B(2\pi r) = \mu_0 NI \tag{9.4.7}$$

or

$$B = \frac{\mu_0 NI}{2\pi r} \tag{9.4.8}$$

where r is the distance measured from the center of the toroid.. Unlike the magnetic field of a solenoid, the magnetic field inside the toroid is non-uniform and decreases as $1/r$.

9.5 Magnetic Field of a Dipole

Let a magnetic dipole moment vector $\vec{\boldsymbol{\mu}} = -\mu\hat{\mathbf{k}}$ be placed at the origin (*e.g.*, center of the Earth) in the yz plane. What is the magnetic field at a point (*e.g.*, MIT) a distance r away from the origin?

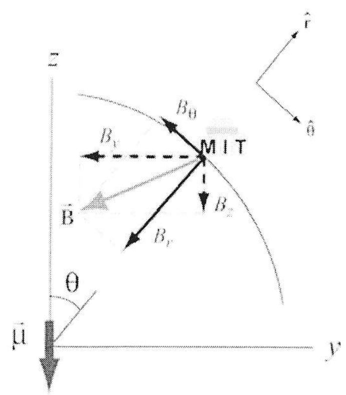

Figure 9.5.1 Earth's magnetic field components

In Figure 9.5.1 we show the magnetic field at MIT due to the dipole. The y- and z-components of the magnetic field are given by

$$B_y = -\frac{\mu_0}{4\pi}\frac{3\mu}{r^3}\sin\theta\cos\theta, \qquad B_z = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(3\cos^2\theta - 1) \tag{9.5.1}$$

Readers are referred to Section 9.8 for the detail of the derivation.

In spherical coordinates (r, θ, ϕ), the radial and the polar components of the magnetic field can be written as

$$B_r = B_y \sin\theta + B_z \cos\theta = -\frac{\mu_0}{4\pi}\frac{2\mu}{r^3}\cos\theta \tag{9.5.2}$$

and

$$B_\theta = B_y \cos\theta - B_z \sin\theta = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}\sin\theta \tag{9.5.3}$$

respectively. Thus, the magnetic field at MIT due to the dipole becomes

$$\vec{\mathbf{B}} = B_\theta\,\hat{\boldsymbol{\theta}} + B_r\,\hat{\mathbf{r}} = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(\sin\theta\,\hat{\boldsymbol{\theta}} + 2\cos\theta\,\hat{\mathbf{r}}) \tag{9.5.4}$$

Notice the similarity between the above expression and the electric field due to an electric dipole $\vec{\mathbf{p}}$ (see Solved Problem 2.13.6):

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(\sin\theta\,\hat{\boldsymbol{\theta}} + 2\cos\theta\,\hat{\mathbf{r}})$$

The negative sign in Eq. (9.5.4) is due to the fact that the magnetic dipole points in the $-z$-direction. In general, the magnetic field due to a dipole moment $\vec{\boldsymbol{\mu}}$ can be written as

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{\boldsymbol{\mu}}}{r^3} \tag{9.5.5}$$

The ratio of the radial and the polar components is given by

$$\frac{B_r}{B_\theta} = \frac{-\dfrac{\mu_0}{4\pi}\dfrac{2\mu}{r^3}\cos\theta}{-\dfrac{\mu_0}{4\pi}\dfrac{\mu}{r^3}\sin\theta} = 2\cot\theta \tag{9.5.6}$$

9.5.1 Earth's Magnetic Field at MIT

The Earth's field behaves as if there were a bar magnet in it. In Figure 9.5.2 an imaginary magnet is drawn inside the Earth oriented to produce a magnetic field like that of the Earth's magnetic field. Note the South pole of such a magnet in the northern hemisphere in order to attract the North pole of a compass.

It is most natural to represent the location of a point P on the surface of the Earth using the spherical coordinates (r,θ,ϕ), where r is the distance from the center of the Earth, θ is the polar angle from the z-axis, with $0 \le \theta \le \pi$, and ϕ is the azimuthal angle in the xy plane, measured from the x-axis, with $0 \le \phi \le 2\pi$ (See Figure 9.5.3.) With the distance fixed at $r = r_E$, the radius of the Earth, the point P is parameterized by the two angles θ and ϕ.

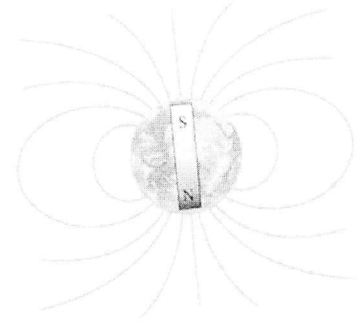

Figure 9.5.2 Magnetic field of the Earth

In practice, a location on Earth is described by two numbers – latitude and longitude. How are they related to θ and ϕ? The latitude of a point, denoted as δ, is a measure of the elevation from the plane of the equator. Thus, it is related to θ (commonly referred to as the colatitude) by $\delta = 90° - \theta$. Using this definition, the equator has latitude $0°$, and the north and the south poles have latitude $\pm 90°$, respectively.

The longitude of a location is simply represented by the azimuthal angle ϕ in the spherical coordinates. Lines of constant longitude are generally referred to as *meridians*. The value of longitude depends on where the counting begins. For historical reasons, the meridian passing through the Royal Astronomical Observatory in Greenwich, UK, is chosen as the "prime meridian" with zero longitude.

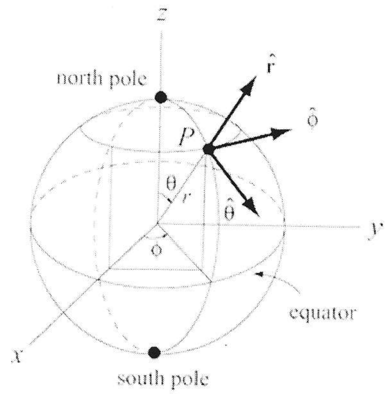

Figure 9.5.3 Locating a point P on the surface of the Earth using spherical coordinates.

Let the z-axis be the Earth's rotation axis, and the x-axis passes through the prime meridian. The corresponding magnetic dipole moment of the Earth can be written as

$$\begin{aligned}\vec{\boldsymbol{\mu}}_E &= \mu_E(\sin\theta_0\cos\phi_0\,\hat{\mathbf{i}} + \sin\theta_0\sin\phi_0\,\hat{\mathbf{j}} + \cos\theta_0\,\hat{\mathbf{k}}) \\ &= \mu_E(-0.062\,\hat{\mathbf{i}} + 0.18\,\hat{\mathbf{j}} - 0.98\,\hat{\mathbf{k}})\end{aligned} \tag{9.5.7}$$

Next Lectures

Vector Calculus

Magnetism in Matter

Ampere's law

Vector potential

"Static" Maxwell's Equation

Next Week.