9.8 Appendix 1: Magnetic Field off the Symmetry Axis of a Current Loop

In Example 9.2 we calculated the magnetic field due to a circular loop of radius R lying in the xy plane and carrying a steady current I, at a point P along the axis of symmetry. Let's see how the same technique can be extended to calculating the field at a point off the axis of symmetry in the yz plane.

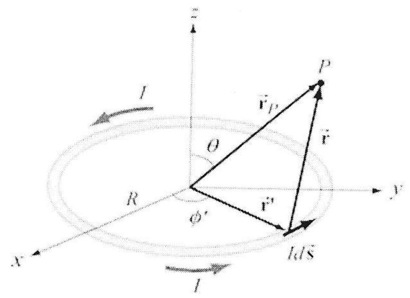


Figure 9.8.1 Calculating the magnetic field off the symmetry axis of a current loop.

Again, as shown in Example 9.1, the differential current element is

$$Id\vec{\mathbf{s}} = R\,d\phi'(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}})$$

and its position is described by $\vec{\mathbf{r}}' = R(\cos\phi'\,\hat{\mathbf{i}} + \sin\phi'\,\hat{\mathbf{j}})$. On the other hand, the field point P now lies in the yz plane with $\vec{\mathbf{r}}_P = y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, as shown in Figure 9.8.1. The corresponding relative position vector is

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = -R\cos\phi'\,\hat{\mathbf{i}} + (y - R\sin\phi')\,\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{9.8.1}$$

with a magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2} = \sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi} \tag{9.8.2}$$

and the unit vector

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|}$$

pointing from $Id\vec{\mathbf{s}}$ to P. The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ can be simplified as

$$\begin{aligned} d\vec{\mathbf{s}} \times \hat{\mathbf{r}} &= R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}}\right) \times [-R\cos\phi'\,\hat{\mathbf{i}} + (y - R\sin\phi')\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + (R - y\sin\phi')\,\hat{\mathbf{k}}] \end{aligned} \tag{9.8.3}$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3} = \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + (R - y\sin\phi')\hat{\mathbf{k}}}{\left(R^2 + y^2 + z^2 - 2yR\sin\phi'\right)^{3/2}}d\phi' \qquad (9.8.4)$$

Thus, magnetic field at P is

$$\vec{\mathbf{B}}(0, y, z) = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + (R - y\sin\phi')\hat{\mathbf{k}}}{\left(R^2 + y^2 + z^2 - 2yR\sin\phi'\right)^{3/2}}d\phi' \qquad (9.8.5)$$

The x-component of $\vec{\mathbf{B}}$ can be readily shown to be zero

$$B_x = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\cos\phi' d\phi'}{\left(R^2 + y^2 + z^2 - 2yR\sin\phi'\right)^{3/2}} = 0 \qquad (9.8.6)$$

by making a change of variable $w = R^2 + y^2 + z^2 - 2yR\sin\phi'$, followed by a straightforward integration. One may also invoke symmetry arguments to verify that B_x must vanish; namely, the contribution at ϕ' is cancelled by the contribution at $\pi - \phi'$. On the other hand, the y and the z components of $\vec{\mathbf{B}}$,

$$B_y = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\sin\phi' d\phi'}{\left(R^2 + y^2 + z^2 - 2yR\sin\phi'\right)^{3/2}} \qquad (9.8.7)$$

and

$$B_z = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{(R - y\sin\phi')d\phi'}{\left(R^2 + y^2 + z^2 - 2yR\sin\phi'\right)^{3/2}} \qquad (9.8.8)$$

involve *elliptic integrals* which can be evaluated numerically.

In the limit $y = 0$, the field point P is located along the z-axis, and we recover the results obtained in Example 9.2:

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}}\int_0^{2\pi}\sin\phi' d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}}\cos\phi'\Big|_0^{2\pi} = 0 \qquad (9.8.9)$$

and

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \tag{9.8.10}$$

Now, let's consider the "point-dipole" limit where $R \ll (y^2+z^2)^{1/2} = r$, i.e., the characteristic dimension of the current source is much smaller compared to the distance where the magnetic field is to be measured. In this limit, the denominator in the integrand can be expanded as

$$\begin{aligned}\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{-3/2} &= \frac{1}{r^3}\left[1+\frac{R^2-2yR\sin\phi'}{r^2}\right]^{-3/2} \\ &= \frac{1}{r^3}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)+\ldots\right]\end{aligned} \tag{9.8.11}$$

This leads to

$$\begin{aligned}B_y &\approx \frac{\mu_0 I}{4\pi}\frac{Rz}{r^3}\int_0^{2\pi}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)\right]\sin\phi' d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{3R^2yz}{r^5}\int_0^{2\pi}\sin^2\phi' d\phi' = \frac{\mu_0 I}{4\pi}\frac{3\pi R^2 yz}{r^5}\end{aligned} \tag{9.8.12}$$

and

$$\begin{aligned}B_z &\approx \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\int_0^{2\pi}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)\right](R-y\sin\phi')d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\int_0^{2\pi}\left[\left(R-\frac{3R^3}{2r^2}\right)-\left(1-\frac{9R^2}{2r^2}\right)\sin\phi' - \frac{3Ry^2}{r^2}\sin^2\phi'\right]d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\left[2\pi\left(R-\frac{3R^3}{2r^2}\right)-\frac{3\pi Ry^2}{r^2}\right] \\ &= \frac{\mu_0 I}{4\pi}\frac{\pi R^2}{r^3}\left[2-\frac{3y^2}{r^2}+\text{higher order terms}\right]\end{aligned} \tag{9.8.13}$$

The quantity $I(\pi R^2)$ may be identified as the magnetic dipole moment $\mu = IA$, where $A = \pi R^2$ is the area of the loop. Using spherical coordinates where $y = r\sin\theta$ and $z = r\cos\theta$, the above expressions may be rewritten as

$$B_y = \frac{\mu_0(I\pi R^2)}{4\pi}\frac{3(r\sin\theta)(r\cos\theta)}{r^5} = \frac{\mu_0}{4\pi}\frac{3\mu\sin\theta\cos\theta}{r^3} \tag{9.8.14}$$

and

$$B_z = \frac{\mu_0}{4\pi}\frac{(I\pi R^2)}{r^3}\left(2 - \frac{3r^2\sin^2\theta}{r^2}\right) = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(2 - 3\sin^2\theta) = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(3\cos^2\theta - 1) \qquad (9.8.15)$$

Thus, we see that the magnetic field at a point $r \gg R$ due to a current ring of radius R may be approximated by a small magnetic dipole moment placed at the origin (Figure 9.8.2).

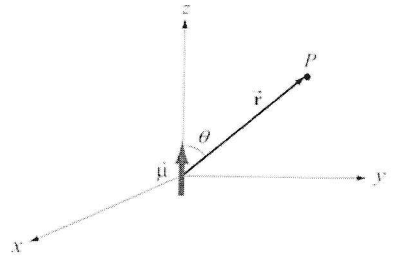


Figure 9.8.2 Magnetic dipole moment $\vec{\boldsymbol{\mu}} = \mu\hat{\mathbf{k}}$

The magnetic field lines due to a current loop and a dipole moment (small bar magnet) are depicted in Figure 9.8.3.

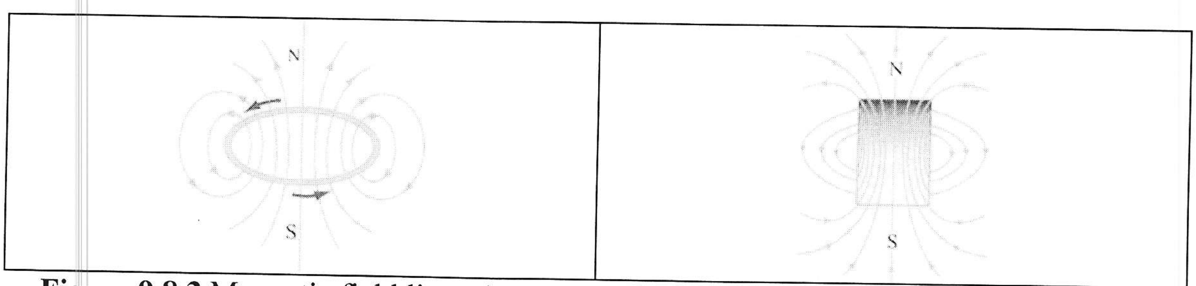


Figure 9.8.3 Magnetic field lines due to (a) a current loop, and (b) a small bar magnet.

The magnetic field at P can also be written in spherical coordinates

$$\vec{\mathbf{B}} = B_r\,\hat{\mathbf{r}} + B_\theta\,\hat{\boldsymbol{\theta}} \qquad (9.8.16)$$

The spherical components B_r and B_θ are related to the Cartesian components B_y and B_z by

$$B_r = B_y\sin\theta + B_z\cos\theta, \qquad B_\theta = B_y\cos\theta - B_z\sin\theta \qquad (9.8.17)$$

In addition, we have, for the unit vectors,

$$\hat{\mathbf{r}} = \sin\theta\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \qquad (9.8.18)$$

Using the above relations, the spherical components may be written as

$$B_r = \frac{\mu_0 I R^2 \cos\theta}{4\pi} \int_0^{2\pi} \frac{d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \tag{9.8.19}$$

and

$$B_\theta(r,\theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\left(r\sin\phi' - R\sin\theta\right) d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \tag{9.8.20}$$

In the limit where $R \ll r$, we obtain

$$B_r \approx \frac{\mu_0 I R^2 \cos\theta}{4\pi r^3} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi I R^2 \cos\theta}{r^3} = \frac{\mu_0}{4\pi} \frac{2\mu\cos\theta}{r^3} \tag{9.8.21}$$

and

$$\begin{aligned} B_\theta &= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\left(r\sin\phi' - R\sin\theta\right) d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \\ &\approx \frac{\mu_0 I R}{4\pi r^3} \int_0^{2\pi} \left[-R\sin\theta\left(1 - \frac{3R^2}{2r^2}\right) + \left(r - \frac{3R^2}{2r} - \frac{3R^2\sin^2\theta}{2r}\right)\sin\phi' + 3R\sin\theta\sin^2\phi' \right] d\phi' \\ &\approx \frac{\mu_0 I R}{4\pi r^3}\left(-2\pi R\sin\theta + 3\pi R\sin\theta\right) = \frac{\mu_0 (I\pi R^2)\sin\theta}{4\pi r^3} \\ &= \frac{\mu_0}{4\pi} \frac{\mu\sin\theta}{r^3} \end{aligned} \tag{9.8.22}$$

9.9 Appendix 2: Helmholtz Coils

Consider two N-turn circular coils of radius R, each perpendicular to the axis of symmetry, with their centers located at $z = \pm l/2$. There is a steady current I flowing in the same direction around each coil, as shown in Figure 9.9.1. Let's find the magnetic field $\vec{\mathbf{B}}$ on the axis at a distance z from the center of one coil.

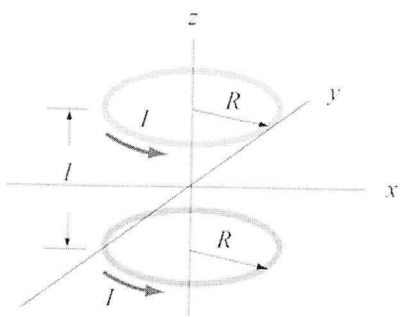


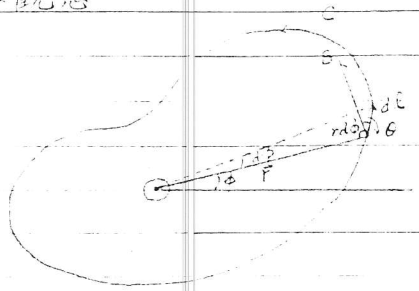
Figure 9.9.1 Helmholtz coils

分類：
編號： 1
總號：

第五節 安培定律

1 簡介 我們在此節中我們將討論安培定律及其應用

2 基本觀念



$$\vec B \cdot d\vec l = |\vec B|\,|d\vec l|\cos\theta \quad (1)$$

由上節得知

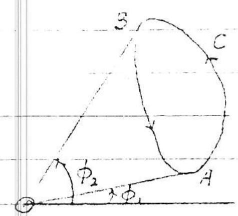
$$|\vec B| = \frac{\mu_0 I}{2\pi r} \quad (2)$$

由圖中可看出 $|d\vec l|\cos\theta = r\,d\phi$ (3)

將 (2),(3) 兩式得　　　I going outward

$$\vec B \cdot d\vec l = \frac{\mu_0 I}{2\pi}\,d\phi \quad (4)$$

$$\oint \vec B \cdot d\vec l = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I \quad (5)$$



若是線積分是沿如左圖之途徑則

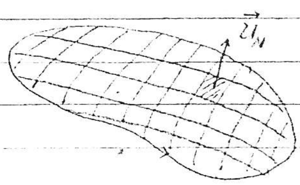
$$\oint \vec B \cdot d\vec l = \frac{\mu_0 I}{2\pi}\left[\int_{\phi_1}^{\phi_2} d\phi + \int_{\phi_2}^{\phi_1} d\phi\right]$$

$$= 0 \quad (6)$$

推廣此一結果得

$$\oint_c \vec B \cdot d\vec l = \mu_0 I_c \quad (7)$$

此處 I_c 是通過線積分途徑所包之表面

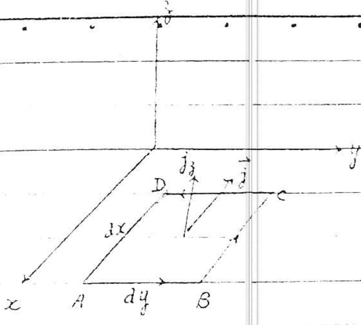


$$I = \int \vec J \cdot \hat u_N\, ds \quad (8)$$

因此

$$\oint \vec B \cdot d\vec l = \mu_0 \int \vec J \cdot \vec u_N\, dS \quad (9)$$

我們現在將此一公式寫成微分的形式



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot \hat{u}_N\, ds \tag{10}$$

$$A\to B \qquad \int_A^B \vec{B}\cdot d\vec{l} = B_y'\, dy \tag{11}$$

$$C\to D \qquad \int_C^D \vec{B}\cdot d\vec{l} = -B_y\, dy \tag{12}$$

兩者之和 $= (B_y' - B_y)\, dy$

$$= \frac{\partial B_y}{\partial x}\, dx\, dy \tag{13}$$

同理，另二邊線積分之和 $= -\frac{\partial B_x}{\partial y}\, dx\, dy$ (14)

因此第(10)式之左邊為 $\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dx\, dy$ (15)

第(10)式之右邊可寫成 $\mu_0 \vec{j}\cdot\vec{k}\, dx\, dy = \mu_0 j_z\, dx\, dy$ (16)

由第(15),(16)兩式得 $\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 j_z$ (17)

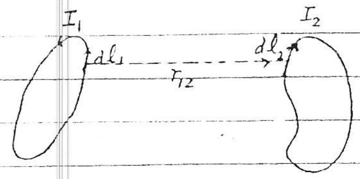
同理 $\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 j_x$ (18)

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 j_y \tag{19}$$

(17),(18),(19) 三式可寫成

$$\nabla\times\vec{B} = \mu_0 \vec{j} \tag{20}$$

我們現在討論兩個帶電流線圈間之磁力　　次序與書上有些出入



由 $I_1 d\vec{l}_1$ 在 $I_2 d\vec{l}_2$ 所產生之磁場依必歐－沙伐定律為

$$d\vec{B}_{12} = \frac{\mu_0}{4\pi} I_1 \frac{d\vec{l}_1\times\vec{r}_{12}}{r_{12}^3} \tag{21}$$

dl_2 段所受之力

$$d\vec{F}_{12} = I_2\, d\vec{l}_2 \times d\vec{B}_{12} \tag{22}$$

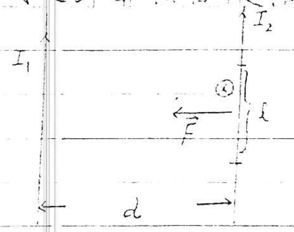
$$= \frac{\mu_0}{4\pi} I_1 I_2 \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{r_{12}^3} \qquad (23)$$

所以第一線圈作用在第二線圈上之力

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{c_1} \oint_{c_2} \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{r_{12}^3} \qquad (24)$$

此一公式之計算相當繁雜,我們將只討論兩個無窮長導線之磁力

vector 恆等式
第三定律



由 I_1 在 I_2 處所產生之磁場強度之大小為

$$B_{12} = \frac{\mu_0 I_1}{2\pi d}$$

其方向為指向紙內，長度為 l 之導線所受力為

$$\vec{F}_{12} = I \vec{l} \times \vec{B}_{12}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d} l \hat{u}$$

$\hat{u}$ 是沿第二導線指向第一導線之單位向量

$$\frac{\vec{F}_{12}}{l} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{u}$$

若 I_1, I_2 流向相同，此兩導線相吸，若 I_1, I_2 流向相反，則此兩導線相斥.

Lecture 5

Differential Calculus

Ordinary derivative
See the textbook.

Differentiation with respect to a scalar
Example : with respect to time

Gradient of a scalar field
The geometrical meaning of the gradient

"Del" operator
It will be shown that ∇ transforms as a vector
$\downarrow$
vector operator

Divergence $\nabla \cdot \vec{V}$

Curl $\nabla \times \vec{V}$

Laplacian operator ∇^2

We shall postpone our discussion after we discuss the integral calculus
Some useful theorem
Product Rules

Second Derivatives.

Vector Functions

If corresponding to each value of a scalar u we associate a vector $\vec{A}$, then is called a function of u denoted by $\vec{A}(u)$

In three dimensions we can write

$$\vec{A}(u) = A_1(u)\hat{i} + A_2(u)\hat{j} + A_3(u)\hat{k}$$

Example $\vec{r}(\underset{\hookrightarrow \text{time}}{t})$ position vector as a function of time

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

The function concept is easily extended.

If to each point (x, y, z) there corresponds a vector $\vec{A}$, then $\vec{A}$ is a function of (x, y, z) indicated by $\vec{A}(x, y, z) = A_1(x, y, z)\hat{i} + A_2(x, y, z)\hat{j} + A_3(x, y, z)\hat{k}$

$\vec{A}(x, y, z)$ defines a vector field since it associate a vector with each point of a region.

Example $\vec{V}(x, y, z)$
$\hookrightarrow$ velocity field.

$\vec{E}(x, y, z) = \vec{E}(\vec{r})$
$\hookrightarrow$ electric field

Similarly $\phi(x, y, z)$ defines a scalar field
$\Rightarrow$ it associates a scalar with each point of a region.

Example $T(x, y, z)$
$\hookrightarrow$ temperature field.

$V(x, y, z)$
$\hookrightarrow$ scalar potential field.

Limits, Continuity, and Derivatives of Vector Functions

Limits, continuity and derivatives of vector functions follow rules similar to scalar functions

· The vector function $\vec{A}(u)$ is said to be continuous at u_0 if for any given positive number ϵ, we can find a positive number δ such that $|\vec{A}(u)-\vec{A}(u_0)|<\epsilon$ whenever $|u-u_0|<\delta$

$$\Rightarrow \lim_{u\to u_0} \vec{A}(u) = \vec{A}(u_0)$$

· The derivative of $\vec{A}(u)$ is defined as

$$\frac{d\vec{A}}{du} = \lim_{\Delta u\to 0} \frac{\vec{A}(u+\Delta u)-\vec{A}(u)}{\Delta u}$$

provided the limit exists.

For $\vec{A}(u) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\frac{d\vec{A}}{du} = \frac{dA_1}{du}\hat{i} + \frac{dA_2}{du}\hat{j} + \frac{dA_3}{du}\hat{k}$$

Example $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\frac{d\vec{r}}{dt} = \vec{v}(t)$$

Higher derivatives such as $\frac{d^2\vec{A}}{du^2}$ $\left(\vec{a} = \frac{d^2\vec{r}}{dt^2}\right)$ etc can be similarly defined.

· If $\vec{A}(x,y,z) = A_1(x,y,z)\hat{i} + A_2(x,y,z)\hat{j} + A_3(x,y,z)\hat{k}$

$$d\vec{A} = \frac{\partial\vec{A}}{\partial x}dx + \frac{\partial\vec{A}}{\partial y}dy + \frac{\partial\vec{A}}{\partial z}dz$$

is the differential of $\vec{A}$

Example $\vec{r}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

· $$\frac{d}{du}(\phi\vec{A}) = \phi\frac{d\vec{A}}{du} + \frac{d\phi}{du}\vec{A}$$

$$\frac{\partial}{\partial y}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{\partial\vec{B}}{\partial y} + \frac{\partial\vec{A}}{\partial y}\cdot\vec{B}$$

$$\frac{\partial}{\partial z}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{\partial\vec{B}}{\partial z} + \frac{\partial\vec{A}}{\partial z}\times\vec{B}$$

[when cross products are involved the order may be important]

Gradient

Let $\varphi(x, y, z)$ be a scalar field

$\Rightarrow$ under rotation

$$\varphi'(\underset{x_1'}{x'}, \underset{x_2'}{y'}, \underset{x_3'}{z'}) = \varphi(\underset{x_1}{x}, \underset{x_2}{y}, \underset{x_3}{z})$$

$$\frac{\partial \varphi}{\partial x_j} = \sum_i \frac{\partial \varphi'}{\partial x_i'} \frac{\partial x_i'}{\partial x_j} = \sum_i \frac{\partial \varphi'}{\partial x_i'} a_{ij}$$

chain rule (transformation $x \to x'$)

$$\sum_j a_{kj} \frac{\partial \varphi}{\partial x_j} = \sum_j \sum_i \frac{\partial \varphi'}{\partial x_i'} \underline{a_{ij}} \, \underline{a_{kj}}$$

orthogonality condition

$$= \sum_i \frac{\partial \varphi'}{\partial x_i'} \delta_{ik}$$

$$= \frac{\partial \varphi'}{\partial x_k'}$$

Compare with $V_k' = \sum_j a_{kj} V_j$ (vector under rotation)

$\Rightarrow$ $\left\{ \frac{\partial \varphi}{\partial x_i} \right\}$ transform like a vector

$$\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \quad \text{is a vector}$$

$$= \nabla \varphi$$

$=$ gradient of a scalar field φ

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$\downarrow$

∇ (del) operator

$\downarrow$

it operates on a scalar field to obtain

$\nabla \varphi$ (gradient of φ)

$\hookrightarrow$ vector field.

All the relationships for ∇ (del) can be derived from the hybrid nature of del in terms of both the partial derivatives and its vector nature.

Example The gradient of a function of r

$$f(r) = f\left(\sqrt{x^2+y^2+z^2}\right)$$

$$\nabla f(r) = \frac{\partial f(r)}{\partial x} \hat{i} + \frac{\partial f(r)}{\partial y} \hat{j} + \frac{\partial f(r)}{\partial z} \hat{k}$$

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr}\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{2x}{2\cdot(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{r}$$

$$\Rightarrow \quad \frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr}\frac{x}{r}$$

Permuting coordinates to obtain the y and z derivatives

$$\Rightarrow \quad \nabla f = (x\hat{i} + y\hat{j} + z\hat{k})\frac{1}{r}\frac{df}{dr}$$

$$= \frac{\vec{r}}{r}\frac{df}{dr} = \hat{r}\frac{df}{dr}$$

This is a result used throughout the course.

$\varphi(x, y, z)$ is a scalar field

$\nabla\varphi(x,y,z) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$ is a vector field

$$|\nabla\varphi(x,y,z)| = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2}$$

Its direction is given by

$\dfrac{\nabla\varphi}{|\nabla\varphi|}$ a unit vector

Another way to introduce the gradient

$\varphi(x, y, z)$ is a scalar field

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= \underbrace{\left[\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right]}_{\nabla\varphi} \cdot \underbrace{[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]}_{d\vec{r}}$$

$$= |\nabla\varphi||d\vec{r}|\cos\theta$$

↳ angle between $\nabla\varphi$ and $d\vec{r}$

↳ infinitesimal displacement vector

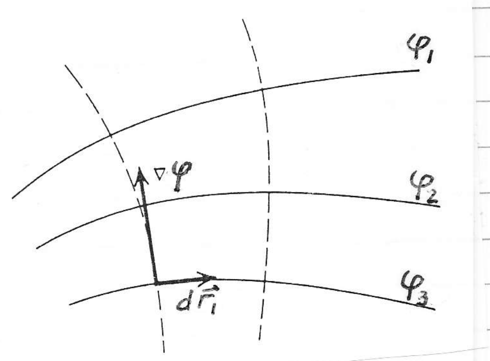
Meaning of the gradient:

In the following figure, surfaces are shown

$\downarrow$

corresponds to $\varphi = \varphi_1, \varphi_2, \cdots$

Now a displacement $d\vec{r}_1$ which takes us to a point on the same surface, i.e., does not take one to a point where φ has changed.



$\Rightarrow \quad d\varphi_1 = d\vec{r}_1 \cdot \nabla\varphi = 0$

$\downarrow$

$\nabla\varphi \perp d\vec{r}_1$

$\Rightarrow \quad \nabla\varphi$ is $\perp$ to a surface of constant φ

The surface $\varphi(x, y, z) = \varphi_1 =$ constant is known as the contour surface.

$\Rightarrow$ if $d\vec{r}$ is on the contour surface, then $d\varphi = 0$

$\Rightarrow \nabla\varphi \cdot d\vec{r} = 0$;

Consider displacements of constant magnitude $|d\vec{r}|$ from a given point but of varying directions such as $d\vec{r}_1, d\vec{r}_2, \cdots$

The change in φ resulting from any of these displacements is given by

$$d\varphi_i = |d\vec{r}|\,|\nabla u| \cos\theta_i$$

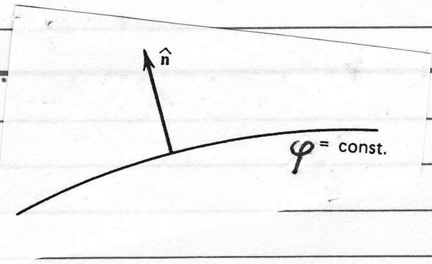
$\hookrightarrow$ angle between $\nabla\varphi$ and $d\vec{r}_i$

$d\varphi_i$ is maximum when $\theta_i = 0$

$\Rightarrow d\varphi_i \parallel d\vec{r}_i$

$\Rightarrow$ the direction of the gradient is also the direction in which the scalar has its maximum rate of change. and the magnitude is just that maximum rate of change of φ.

[A unit vector $\hat{n}$ that is $\perp$ to a given surface at a given point is called the normal vector and is illustrated for a surface $\varphi =$ constant in the following figure.



In this case

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$\hat{n}$ is in the direction in which φ is increasing.

We can show that ∇ is a vector operator
[Use similar method as in the beginning of this section]

Exercise Illustrate above statement by considering a rotation around z by an angle θ

$$x' = x\cos\theta + y\sin\theta \qquad\qquad x = x'\cos\theta - y'\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta \quad\Rightarrow\quad y = x'\sin\theta + y'\cos\theta$$

$$z' = z \qquad\qquad z = z'$$

$$\frac{\partial}{\partial x'} \longrightarrow \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'}$$

$$= \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial y'} \rightarrow \frac{\partial}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y'}$$

$$= -\sin\theta\,\frac{\partial}{\partial x} + \cos\theta\,\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial z'}$$

$$\hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{k}' = \hat{k}$$

Substitute above equation into the equation for ∇

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$$LHS = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)(\cos\theta\, \hat{i} + \sin\theta\, \hat{j}) + \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right)$$

$$\cdot(-\sin\theta\, \hat{i} + \cos\theta\, \hat{j}) + \frac{\partial}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = RHS$$

∇ operators on a vector field $\vec{V}(x, y, z)$

"Scalar product" $\nabla \cdot \vec{V}(x, y, z) \longrightarrow$ Divergence

"Vector product" $\nabla \times \vec{V}(x, y, z) \longrightarrow$ Curl

We shall first study the operation of ∇ before understanding the geometrical and physical meanings.

$\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ is a vector field

$$\nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

divergence of $\vec{V}$

$\nabla \cdot \vec{V}$ is a scalar field

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k}\right)$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

curl of $\vec{V}$

$\nabla \times \vec{V}$ is a vector field

The Laplacian operator $\nabla^2 = \nabla \cdot \nabla$

scalar operator

$\varphi(x, y, z)$ is a scalar field

$$\nabla^2 \varphi = \nabla \cdot \nabla \varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

It is again a scalar field.

Remembering $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$

$\vec{A} \to \nabla$, $\vec{B} \to \nabla$, $\vec{C} \to \vec{V}$

$\Rightarrow \quad \nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - (\nabla \cdot \nabla)\vec{A}$

$\Rightarrow \quad \nabla^2 \vec{V} \equiv \nabla(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V})$

use this equation to define $\nabla^2 \vec{V}$: Laplacian of a vector field.

Exercise: Show that in Carterian coordinate

$$\nabla^2 \vec{V} = \nabla^2 V_x \hat{i} + \nabla^2 V_y \hat{j} + \nabla^2 V_z \hat{k}$$

$$\nabla(\nabla\cdot\vec{V}) - \nabla\times(\nabla\times\vec{V}) = \nabla^2\vec{V}$$

The x component of $\nabla(\nabla\cdot\vec{V})$ in Cartesian coordinate

$$\frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_z}{\partial x\partial z}$$

The x component of $-\nabla\times(\nabla\times\vec{V})$ can be read off from

$$-\nabla\times(\nabla\times\vec{V}) = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} & -\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{vmatrix}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right)$$

$$= -\frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_z}{\partial x\partial z} + \frac{\partial^2 V_x}{\partial z^2}$$

$\Rightarrow$ thus the proof.

Exercise $\nabla\times(\nabla\varphi) = 0$

Exercise $\nabla\cdot(\nabla\times\vec{V}) = 0$

$$\nabla\cdot(\nabla\times\vec{V}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= 0$$

Product rule

- With two fields (f, g are scalar fields ; $\vec{A}$, $\vec{B}$ are vector fields)
  there are two ways to form a scalar fields, i.e.,

  $f g$, $\vec{A}\cdot\vec{B}$

From these two scalar fields, we can form two vector fields by taking the gradient

(1) $\nabla(fg) = f\,\nabla g + g\,\nabla f$

Look at the ith component

$LHS = \frac{\partial}{\partial x_i}(fg) = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i} = RHS$

(2) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

Let us prove it in long-hand, look at the first component

$LHS = \frac{\partial}{\partial x_1}(A_1B_1 + A_2B_2 + A_3B_3)$

$= \frac{\partial A_1}{\partial x_1}B_1 + A_1\frac{\partial B_1}{\partial x_1} + A_2\frac{\partial B_2}{\partial x_1} + \frac{\partial A_2}{\partial x_1}B_2 + A_3\frac{\partial B_3}{\partial x_1} + \frac{\partial A_3}{\partial x_1}B_3$

$$\vec{A}\times(\nabla\times\vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ \frac{\partial B_2}{\partial x_3}-\frac{\partial B_3}{\partial x_2} & \frac{\partial B_3}{\partial x_1}-\frac{\partial B_1}{\partial x_3} & \frac{\partial B_1}{\partial x_2}-\frac{\partial B_2}{\partial x_1} \end{vmatrix}$$

The first component is given by

$-A_2\overset{\checkmark}{\frac{\partial B_1}{\partial x_2}} + A_2\frac{\partial B_2}{\partial x_1} + A_3\frac{\partial B_3}{\partial x_1} - A_3\overset{\circ}{\frac{\partial B_1}{\partial x_3}}$

The first component of $\vec{B}\times(\nabla\times\vec{A})$

$-B_2\overset{\triangle}{\frac{\partial A_1}{\partial x_2}} + B_2\frac{\partial A_2}{\partial x_1} + B_3\frac{\partial A_3}{\partial x_1} - B_3\overset{+}{\frac{\partial A_1}{\partial x_3}}$

$(\vec{A}\cdot\nabla)\vec{B} = (A_1\frac{\partial}{\partial x_1} + A_2\frac{\partial}{\partial x_2} + A_3\frac{\partial}{\partial x_3})\vec{B}$

The first component of $(\vec{A}\cdot\nabla)\vec{B}$ is given by

$A_1\frac{\partial B_1}{\partial x_1} + A_2\overset{\checkmark}{\frac{\partial B_1}{\partial x_2}} + A_3\overset{\circ}{\frac{\partial B_1}{\partial x_3}}$

Same argument $\longrightarrow$ first component of $(\vec{B}\cdot\nabla)\vec{A}$

$B_1\frac{\partial A_1}{\partial x_1} + B_2\overset{\triangle}{\frac{\partial A_1}{\partial x_2}} + B_3\overset{+}{\frac{\partial A_1}{\partial x_3}}$

14 terms on the RHS, four terms with negative sign

If we can show that negative terms are all cancelled, then we should have 6 terms left

$\downarrow$

same number of terms as LHS

It can be readily show that they are identical to the 6 terms of LHS $\Rightarrow$ thus the proof.

- With two fields, there are two ways to form of vector: $f\vec{A}$, $\vec{A}\times\vec{B}$

With a vector field, we can form a scalar field by taking a divergence $\nabla\cdot(f\vec{A})$, $\nabla\cdot(\vec{A}\times\vec{B})$, or form a vector field by taking the curl $\nabla\times(f\vec{A})$, $\nabla\times(\vec{A}\times\vec{B})$

(3) $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

(4) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(5) $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$

(6) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

One can prove these vector identities using the same method used in proving (2). These identities are very useful and are listed inside the front cover of the text

Second Derivatives

With a field (scalar/vector), there are 5 possible types of second derivatives: $\nabla\cdot(\nabla f)$, $\nabla\times(\nabla f)$
$\nabla(\nabla\cdot\vec{V})$
$\nabla\cdot(\nabla\times\vec{V})$, $\nabla\times(\nabla\times\vec{V})$

(1) $\nabla\cdot(\nabla f) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\cdot(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z})$

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$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

(2) $\nabla \times (\nabla f) = 0$

(3) $\nabla(\underbrace{\nabla \cdot \vec{V}}_{\text{source}})$ $\longrightarrow$ seldom occurs in physics application

(4) $\nabla \cdot (\nabla \times \vec{V}) = 0$

(5) $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$

this equation defines $\nabla^2 \vec{V}$

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Lecture 6

· Integral Calculus.

Line integral.

Surface integral
Vector element of area
Volume integral   See the textbook.
Meaning of divergence and curl.

The Fundamental Theorem for Vector Integral Calculus

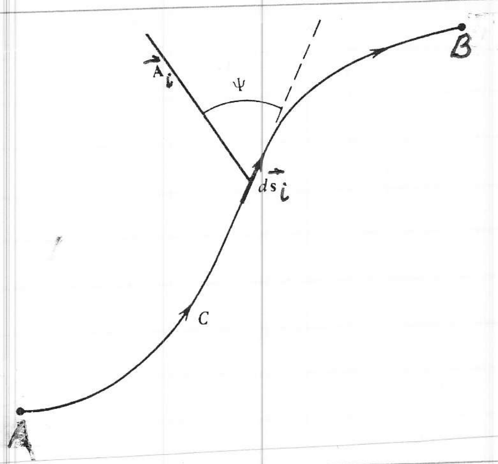
Integration by Parts

Line Integral

$\vec{A}_i \cdot d\vec{s}_i$ is a scalar product of

The definition of line integral between A, B along the path C is defined by

- dividing the path into N segments
- calculating $\lim_{N\to\infty} \sum_{i=1}^{N} \vec{A}_i \cdot d\vec{s}_i$



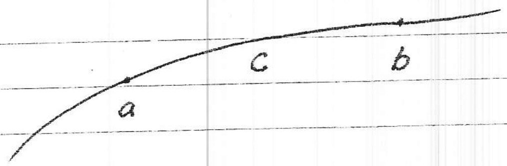
Relations for the calculation of a line integral.

The result depends on the end points a, b.

In general, it also depends on the path.

When the line integral is independent of path, it usually have important significance and implication.

Example : $W_{a\to b} = \int_{a \atop \text{along } C}^{b} \vec{F} \cdot d\vec{s}$



$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \quad ; \quad \vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\Rightarrow \quad W_{a\to b} = \int_{a \atop \text{along } C}^{b} (F_x\,dx + F_y\,dy + F_z\,dz)$$

If $W_{a\to b}$ is independent of the path, i.e.,

$$V_a - V_b = \int_a^b (F_x \cdot dx + F_y\,dy + F_z\,dz)$$

where V_a, V_b are function of end points a, b, respectively

then the force is said to be conservative

$$\int_a^b \vec{F} \cdot d\vec{r} \overset{\text{Newton's law}}{=} \int_a^b \frac{d\vec{p}}{dt} \cdot d\vec{r}$$

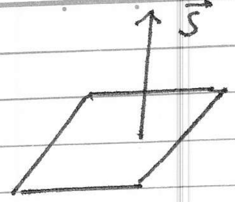
$$= \int_a^b d\vec{p} \cdot \frac{d\vec{r}}{dt} = \int_a^b d\vec{p} \cdot \frac{\vec{p}}{m}$$

$$= \frac{p_b^2}{2m} - \frac{p_a^2}{2m} = K_b - K_a \quad \text{work-energy relation}$$

$$\Rightarrow \quad V_a - V_b = K_b - K_a$$

$$\Rightarrow \quad \underbrace{V_a + K_a = V_b + K_b}_{\text{energy conservation.}}$$

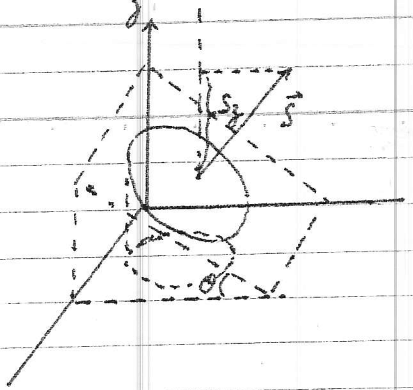
Vector Elements of Area



$\vec{S} \rightarrow$ surface element

$|\vec{S}|$ = area of the surface

$\hat{S}$ points outward perpendicular to the surface

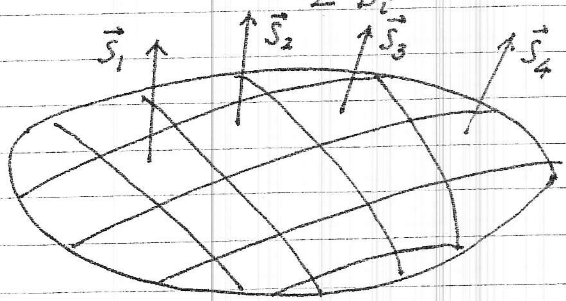


$S_z = |\vec{S}| \cos\theta$

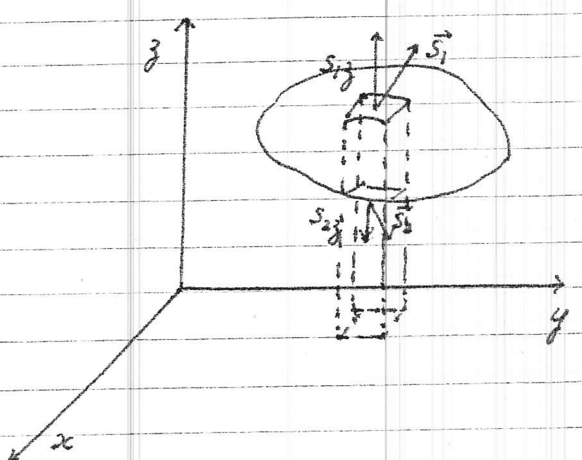
= area projection in the xy plane

If the surface is not a plane, then we shall cut the surface into small plane surface elements, the surface $\vec{S}$ is defined by $\vec{S} = \vec{S}_1 + \vec{S}_2 + \cdots + \vec{S}_N$

$= \sum \vec{S}_i$



$S_z = \sum S_{iz}$ = area projection in the xy plane



Theorem. The surface area vector $\vec{S}$ of a closed volume is zero

With the definition, it is obvious that $S_{1z} = -S_{2z}$

Add up all the contributions $\Rightarrow S_z = 0$

Use the same argument $\Rightarrow S_x = S_y = 0$

$\Rightarrow \vec{S} = 0$

Surface integral

$\vec{A}_i \cdot d\vec{S}_i$ is a scalar product between two vectors

$\quad\ \ \llcorner$ surface element

The definition of surface integral $\int_S \vec{A} \cdot d\vec{S}$ is defined

by (i) dividing the surface into N surface elements, (ii)

calculating $\lim_{N \to \infty} \sum_{i=1}^{N} \vec{A}_i \cdot d\vec{S}_i$

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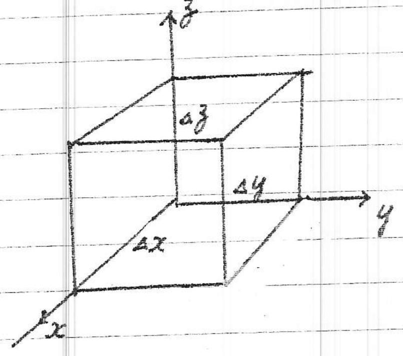
Lecture 7

Divergence $\nabla \cdot \vec{B}$



A differential volume element

in Cartesian coordinate

We want to calculate

$\int \vec{B} \cdot d\vec{s}$ over the closed surface.

Center of the cube (x_0, y_0, z_0)

$$\vec{B} = B_x(x,y,z)\,\hat{i} + B_y(x,y,z)\,\hat{j} + B_z(x,y,z)\,\hat{k}$$

$\int \vec{B} \cdot d\vec{s}$ through the front surface

defined by $d\vec{s} \rightarrow$ along the x axis

$|d\vec{s}| = \Delta y \Delta z$

at $x_0 + \Delta x/2$ plane

$$d\Phi_f = B_x\left(x_0 + \frac{\Delta x}{2}, y, z\right) \Delta y \Delta z$$

$$\simeq \left[B_x(x_0, y_0, z_0) + \left.\frac{\partial B_x}{\partial x}\right|_{(x_0, y_0, z_0)} \frac{\Delta x}{2} + \cdots \right] \Delta y \Delta z$$

in first order, the variation comes from $y - y_0$, $z - z_0$ averaged out

$$= \left[B_x(x_0, y_0, z_0) + \frac{\partial B_x}{\partial x} \cdot \frac{1}{2} \Delta x \right] \Delta y \Delta z$$

The contribution through the back surface, $d\vec{s}$ is now along $-x$

direction. The surface is on the $x - \frac{\Delta x}{2}$ plane

$$d\Phi_b \simeq -\left(B_x(x_0, y_0, z_0) - \frac{\partial B_x}{\partial x}\Big|_{(x_0,y_0,z_0)} \frac{1}{2}\Delta x\right)\Delta y \Delta z$$

$$d\Phi_f + d\Phi_B = \frac{\partial B_x}{\partial x}\Big|_{(x_0,y_0,z_0)} \Delta x \Delta y \Delta z$$

With the same method, the contribution from the other pairs of surface can be calculated

$$d\Phi_{tot} = \text{net outgoing flux of } \vec{B}$$

$$= \underbrace{\left(\frac{\partial B_x}{\partial x}\Big|_{(x_0,y_0,z_0)} + \frac{\partial B_y}{\partial y}\Big|_{(x_0,y_0,z_0)} + \frac{\partial B_z}{\partial z}\Big|_{(x_0,y_0,z_0)}\right)}_{\nabla\cdot\vec{B}|_{(x_0,y_0,z_0)}} \Delta x \Delta y \Delta z$$

$$\Rightarrow \nabla\cdot\vec{B}\big|_{x_0,y_0,z_0}$$

$$= \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right]\Big|_{(x_0,y_0,z_0)}$$

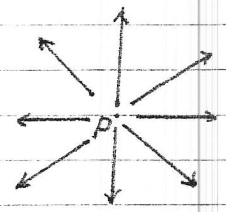
$$= \lim_{\Delta x \Delta y \Delta z \to 0} \frac{\oint \vec{B}\cdot d\vec{S}}{\Delta x \Delta y \Delta z}$$

$\nabla\cdot\vec{B} > 0 \Rightarrow$ positive net outgoing flux of $\vec{B}$ from (x_0, y_0, z_0)

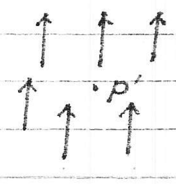
$\Rightarrow$ source

$\nabla\cdot\vec{B} < 0 \Rightarrow$ negative net outgoing flux of $\vec{B}$ from (x_0, y_0, z_0)

$\Rightarrow$ sink



The vector field spread out from point
↓
thus the name of divergence



The vector field has zero divergence at P'
↓
it is not spreading out at all.

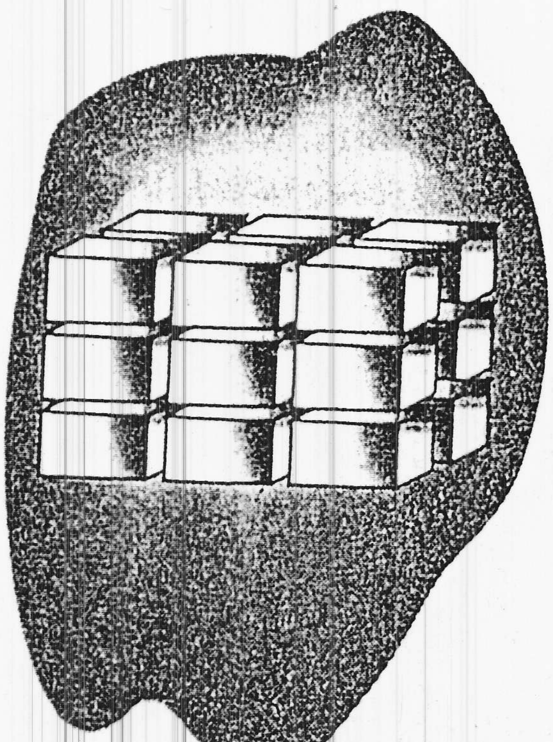
they cancel.

:nitesimal volume elements

boundary surface of the

dv

$\downarrow$



involve the value of $\vec{B}$ through out :he volume V

source

the

:e.

$\nabla \cdot \vec{B}$ is a scalar product of vector operator ∇ and vector field $\vec{B}$

$\Rightarrow$ it is a scalar

To extend the calculation to finite volume

(i) Divide the volume into adjoing infinitesimal volume elements

(ii) Note: at the common face, the fluxes are equal in magnitude but opposite in sign $\rightarrow$ they cancel.

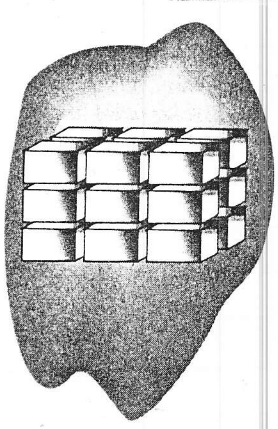
(iii) The sum of the flux from the infinitesimal volume elements is the flux emerging through the boundary surface of the combined volumes

$$\int_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B}\, dv$$

involve only the value of $\vec{B}$ on the surface area S

Divergence Theorem
The Gauss' theorem
(The Green's theorem)

involve the value of $\vec{B}$ through out the volume V

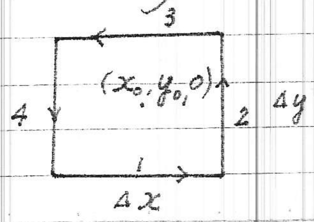


Total flux through the surface $\longleftrightarrow$ Total source inside the surface.

Curl $\nabla \times \vec{B}$

$\int_a^b \vec{B} \cdot d\vec{r}$ line integral

A vector field $\vec{B}$ is conservative $\Leftrightarrow \oint \vec{B} \cdot d\vec{r} = 0$



Want to calculate the loop integral

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First part (along $y = y_0 - \frac{\Delta y}{2}$)

$$\int_1 \vec{B}(x,y,0) \cdot dx\,\hat{i} = \int_1 B_x(x, y_0 - \frac{\Delta y}{2}, 0)\, dx$$

$$\simeq B_x(x_0, y_0, 0)\,\Delta x + \left.\frac{\partial B_x}{\partial y}\right|_{(x_0,y_0,0)} (-\frac{\Delta y}{2})\,\Delta x$$

$$+ \int \left.\frac{\partial B_x}{\partial x}\right|_{(x_0,y_0,0)} (x - x_0)\, dx$$

average out to be zero

$$= (B_x(x_0, y_0, 0) - \frac{1}{2} \left.\frac{\partial B_x}{\partial y}\right|_{(x_0,y,0)} \Delta y)\,\Delta x$$

Third part (along $y = y_0 + \frac{\Delta y}{2}$) gives a contribution of

$$= -(B_x(x_0, y, 0) + \frac{1}{2} \left.\frac{\partial B_x}{\partial y}\right|_{(x_0,y_0,0)} \Delta y)\,\Delta x$$

$d\vec{r} = -dx\,\hat{i}$

$$\Rightarrow (1) + (3) = -\frac{\partial B_x}{\partial y} \Delta y \Delta x$$

Similarily, $(2) + (4) = \frac{\partial B_y}{\partial x} \Delta x \Delta y$

$$\Rightarrow \oint_C \vec{B} \cdot d\vec{r} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \underbrace{\Delta x \Delta y}_{\text{area element}}$$

$$(\nabla \times \vec{B})_z \, dS_z$$

For a loop not necessarily lying on the z plane, the result can be generalized to

$$\oint \vec{B} \cdot d\vec{r} = \nabla \times \vec{B} \cdot d\vec{S} = (\nabla \times \vec{B})_n \Delta S$$

normal to the surface area

where
$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

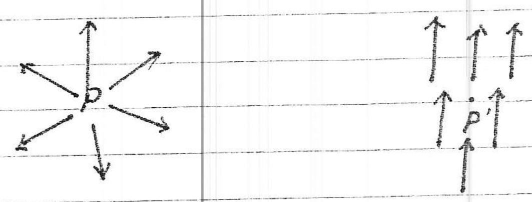
$$= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{i} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{k}$$

$$\Rightarrow (\nabla \times \vec{B})_n = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \vec{B} \cdot d\vec{r}$$

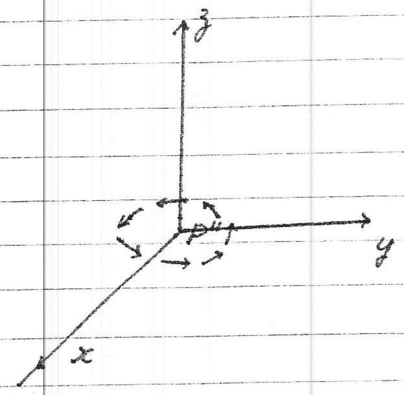
Operational definition of $\nabla \times \vec{B}$

$\nabla \times \vec{B}$ is a vector whose magnitude is the maximum net circulation of $\vec{B}$, $\oint \vec{B} \cdot d\vec{r}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.

$\nabla \times \vec{B}$ at a point is a measure of how much the vectors curl around the point in question.



The curl of the vector field around both P and P' are zero



In above figure, the curl of the vector field around P'' is non-zero and is pointing along the z direction.

Example: The curl of velocity field of surface water at a point of whirlpool is non-zero.

Float a cork with toothpick pointing out radially, if it start to turn, then it has been placed at a point of non-zero curl.

Generalize to the finite element case.

(i) Divide the surface into adjoining surface elements

(ii) Note: at the common line, the contributions to the line integral are equal in magnitude and opposite in sign

$\Rightarrow$ they cancel.

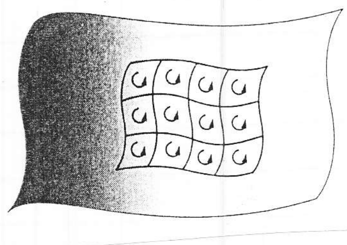
$$\underbrace{\oint_C}_{\text{around the external boundary}} \vec{B} \cdot d\vec{r} = \underbrace{\int_S}_{\text{any open surface bounded by the curve } C} (\nabla \times \vec{B}) \cdot d\vec{S}$$

Stoke's theorem.

From Stoke's theorem, if $\nabla \times \vec{B} = 0$, then $\oint \vec{B} \cdot d\vec{r} = 0$

$\Rightarrow$ $\int_a^b \vec{B} \cdot d\vec{r}$ is independent of the path

$\Rightarrow$ $\vec{B}$ is conservative field.



Lecture 8

The Fundamental Theorems for Vector Integral Calcul

The fundamental theorem for gradient

$$\int_a^b (\nabla f)\cdot d\vec{r} = f(b) - f(a)$$

$\Rightarrow \int_a^b (\nabla f)\cdot d\vec{r}$ is independent of path taken from a to b

$\Rightarrow \oint (\nabla f)\cdot d\vec{r}$

If a force $\vec{F}$ can be expressed in terms of a gradient $\vec{F} = \nabla f$, then $\vec{F}$ is conservative force.

The converse is also valid

Example: For electrostatic case $\oint \vec{E}\cdot d\vec{r} = 0$

$\Longrightarrow \vec{E} = -\nabla V$

The fundamental theorem for divergence

$$\int_{\text{volume}} (\nabla\cdot\vec{V})\, dv = \oint_{\text{surface}} \vec{V}\cdot d\vec{S}$$

↓

Gauss' theorem.

Geometrical interpretation

$\oint_{\text{surface}} \vec{V}\cdot d\vec{S}$

↓

flux of $\vec{V}$ through the surface

↓

Surface integral of $\vec{V}$ over some surface

If $\vec{V} \rightarrow$ velocity of incompressible fluid, then

$\oint \vec{V}\cdot d\vec{S}$ = amount of fluid passing through the surface per unit time

$\nabla\cdot\vec{V} \rightarrow$ source of $\vec{V}$

↓

faucets

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The physical meaning of divergence theorem

$\int$ (All faucets in the volume) $= \int$ (flow out through the surface)

Example : For electrostatics

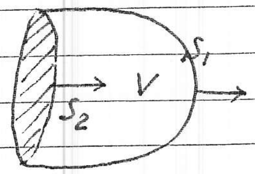
$$\oint \vec{E} \cdot d\vec{S} = \int \frac{\rho}{\epsilon_0} \, dV = \frac{Q_{inside}}{\epsilon_0}$$

The fundamental theorem for curls

$$\int_{\text{surface}} (\nabla \times \vec{V}) \cdot d\vec{S} = \oint_{\substack{\text{boundary}\\\text{line}}} \vec{V} \cdot d\vec{r}$$

Stoke's theorem

$\Rightarrow \int_{\text{surface}} (\nabla \times \vec{V}) \cdot d\vec{S}$ depends only on the boundary line, not on the particular surface used



$$\int_{S_1} (\nabla \times \vec{V}) \cdot d\vec{S} = \int_{S_2} (\nabla \times \vec{V}) \cdot d\vec{S}$$

Corollary $\oint (\nabla \times \vec{V}) \cdot d\vec{S} = 0$ for any closed surface.

(i) For volume V, the closed surface is formed by S_1 and $-S_2$

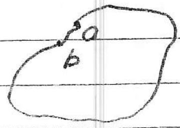
(ii) Since the boundary line, like the mouth of a ballon, shrinks down to a point.

Example: Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{inside}$$

Some simple applications

(1)



Let $a \to b$

$$\oint (\nabla f) \cdot d\vec{r} = 0 \quad \text{(theorem for gradient)}$$

$$\underbrace{\oint (\nabla f) \cdot d\vec{r}}_{\substack{\| \\ 0 \\ \text{for any loop}}} = \int_{surface} [\nabla \times (\nabla f)] \cdot d\vec{S}$$

$\Rightarrow \int_{surface} [\nabla \times (\nabla f)] \cdot d\vec{S} = 0$ for any surface

$\Rightarrow$ another way to prove $\nabla \times (\nabla f) = 0$

(2) $\oint (\nabla \times \vec{V}) \cdot d\vec{S} = \int_{volume} \nabla \cdot (\nabla \times \vec{V})\, dV$

Gauss' theorem

0 from the corollary given above

$\Rightarrow \int_{volume} \nabla \cdot (\nabla \times \vec{V})\, dV = 0$ for any volume

$\Rightarrow$ another way to prove $\nabla \cdot (\nabla \times \vec{V}) = 0$

Integration by Parts

$$\int_a^b f \frac{dg}{dx}\, dx \underset{\downarrow}{=} -\int_a^b g\left(\frac{df}{dx}\right) dx + fg\Big|_a^b$$

a technique used often in calculus

$$\nabla\cdot(f\vec{A}) \underset{\downarrow}{=} f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

one of the vector identity

$$\int \nabla\cdot(f\vec{A})\, d\tau = \int f(\nabla\cdot\vec{A})\,d\tau + \int \vec{A}\cdot(\nabla f)\, d\tau$$

$\updownarrow$ divergence theorem

$$\oint f\vec{A}\cdot d\vec{a}$$

$$\Rightarrow \int_V f(\nabla\cdot\vec{A})\,d\tau = -\int \vec{A}\cdot(\nabla f)\,d\tau + \oint_S f\vec{A}\cdot d\vec{a}$$

Note the similar behavior.

This is a useful formula for this course.