# Chapter 30

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# 第十九章 與時間有關之電磁場

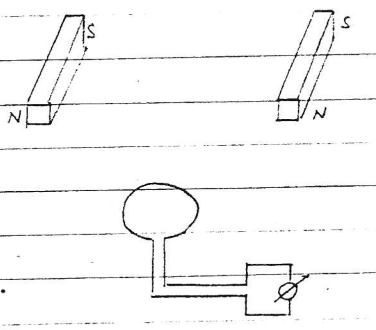
## 第一節 法那第一亨利定律

1. 簡介 我們在此節中將討論法那第一亨利定律及其應用

2. 基本觀念

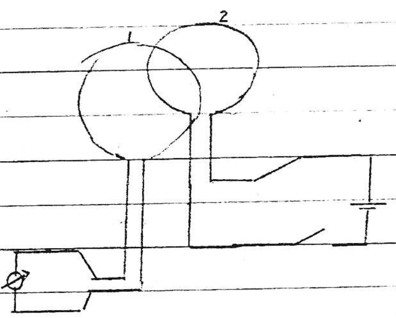
當電磁場能隨時間改變時，會產生與靜電，靜磁場完全不同的效應。

我們首先討論一些實驗結果




(1) 將一永久磁鐵在一線圈前通過時我們發現與線圈相聯之電流計的指針先向一邊偏移。在磁鐵經過中央以後則向反方向偏移。

(2) 若將磁鐵固定，而將線圈在其前面移動，同樣的電流計之指針同樣的向一邊偏移，然後向反方向偏移。




(3) 兩線圈如左圖所示，當我們將第二線圈之開關閉上時則與第一線圈相聯之電流計之指針向一方偏移。將開關打開時則向反方向偏移。但當開關閉上一長時間後則第一線圈上無電流通過。

以上的實驗顯示在線圈附近磁場的改變會在線圈上產生一電動勢而使得在線圈上有電流通過。我們可以用以下的實驗來證明磁場改變引起的是電動勢而非直接引起電流，改變線圈的電阻 $R$ 時，其線圈上感應的電流與電阻成反比。

法那第一亨利定律[1,2,3]

$$V_\varepsilon = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot\hat{u}_N\, dS \qquad (1)$$

此處 $V_\varepsilon$ 是一線圈上之電動勢，$\Phi_B$ 是通過線圈所包圍面積之磁通量

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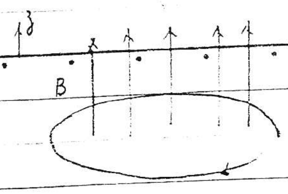
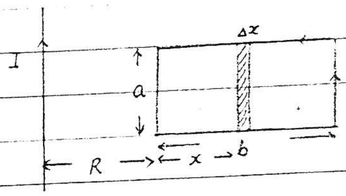



若 $\vec{B}$ 沿 $\hat{z}$ 方向增加則在線圈上電動勢之方向則為順時鐘方向.

我們現在舉例來說明

一無窮長導線上有電流 I 流過，在其附近有一邊長為 a 及 b 之長方形之線圈以速度 $v$ 沿與導線垂直的方向進行，求在此一線圈上感應的電動勢




取線圈上電動勢之方向如圖所示(反時鐘方向)

則 $\hat{u}_N$ 之方向按右手定則為向外

在離導線 $R+x$ 處 由電流 I 所引起的磁感應之大小為

$$|\vec{B}| = \frac{\mu_0 I}{2\pi(R+x)} \tag{2}$$

其方向為向內.

因此通過斜線部分之磁動量為

$$-\frac{\mu_0 I}{2\pi(R+x)}\, a\,\Delta x \tag{3}$$

通過線圈的磁動量為

$$\Phi_B = \int \vec{B}\cdot\hat{u}_N\, dS = -\frac{\mu_0 I}{2\pi}\int_{x=0}^{x=b}\frac{a\,dx}{R+x} \tag{4}$$

$$= -\frac{\mu_0 I a}{2\pi}\ln\left(1+\frac{b}{R}\right) \tag{5}$$

此式中 $R$ 是時間的函數, $\frac{dR}{dt} = v$ (6)

$$\text{因此}\quad -\frac{d}{dt}\Phi_B = -\frac{\mu_0 I ab}{2\pi R(R+b)}\, v \tag{7}$$

$$\text{由一式得}\quad V_E = -\frac{\mu_0 I ab}{2\pi R(R+b)}\, v \tag{8}$$

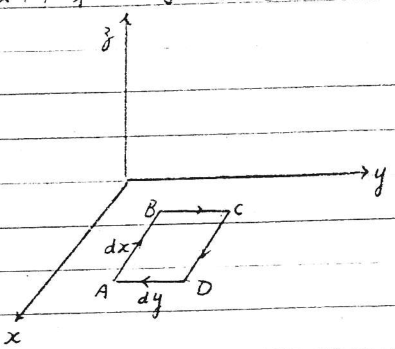
此處之負號表示實際上電動勢之方向與我們原來選擇的相反，也即是電動勢是沿順時鐘方向.

感應電流之方向可利用冷次定律來決定.

冷次定律[4] 感應電流之方向是抵制產生此改變之方向。

如上例中感應電流是由於向內之磁通量減少所引起，若感應電流為順時鐘方向時則會引起向內之磁場，若感應電流為反時鐘方向則會引起一向外之磁場，冷次定律於是告訴我們感應電流應是順時鐘的方向

法那第一亨利定律之微分形式




$$\oint \vec{\mathcal{E}} \cdot d\vec{l}$$

$$= \left( \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} \right) dx\, dy \tag{9}$$

$$\int \vec{B} \cdot \hat{u}_N \, ds = B_z \, dx\, dy \tag{10}$$

利用法那第一亨利定律得

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = - \frac{\partial B_z}{\partial t} \tag{11}$$

同理得

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = - \frac{\partial B_x}{\partial t} \tag{12}$$

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = - \frac{\partial B_y}{\partial t} \tag{13}$$

將 (11), (12), (13) 式合併得

$$\nabla \times \vec{\mathcal{E}} = - \frac{\partial \vec{B}}{\partial t} \tag{14}$$

3. 討論

(1) 此一定律仍是電磁學中之主要定律之一 是法那第由實驗中所發現

(2) 此一公式中主要的觀念是磁通量，要利用此一公式必須首先能定義磁通量也即是必須能定義由綫圈所圍成之面積

(3) 通過一綫圈磁動量改變可來自 (i) 磁場的變化 (ii) 將綫圈在一不均勻磁場中

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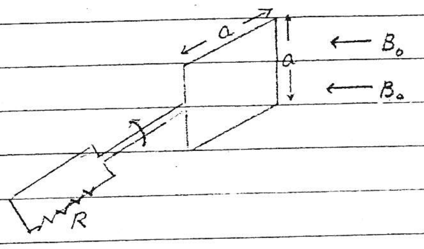
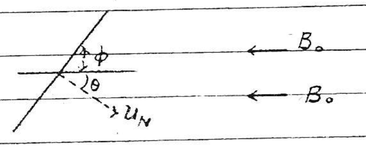
運動，如以下的例子所示 (3) 改變線圈的面積

(4) 此一定律仍是由實驗中所得來，同時也與能量守恒有關，此一定律只適用於感應電流

4. 應用

(1) 發電機 法那第－亨利定律提供了我們將動能轉換為電能的方法




通過一線圈之磁通量 $= B_0 A \cos\theta = B_0 A \sin\phi$ (15)

若有 N 個線圈則通過此一面積之總磁通量為 $\Phi = N B_0 A \sin\phi$ (16)

若此 N 線圈在磁場中以等角速度轉動則 $\phi = \omega t + \phi_0$ 若 $t=0$ 時 $\phi = 0$

則 $\Phi = N B_0 A \sin\omega t$ (17)

利用法那第－亨利定律則所產生之電動勢

$$V_{emf} = -\frac{d\Phi}{dt} = -N B_0 A \omega \cos\omega t \tag{18}$$

在電阻上通過之電流為

$$I = \frac{V_{emf}}{R} = -\frac{N B_0 A \omega}{R} \cos\omega t \tag{19}$$

在時間 t 消耗之功率為

$$P(t) = I^2 R = \frac{N^2 B_0^2 A^2 \omega^2}{R^2} \cos^2\omega t \cdot R$$

$$= \frac{N^2 B_0^2 A^2 \omega^2}{R} \cos^2\omega t \tag{20}$$

其平均功率為

$$\langle P \rangle = \frac{1}{T}\int_0^T P(t)\, dt \qquad T = \frac{2\pi}{\omega} \tag{21}$$

$$= \frac{N^2 B_0^2 A^2 \omega^2}{2R} \tag{22}$$

在電阻中消耗之能量是來自我們使線圈在磁場中轉動所需加之能量

此 $N$ 線圈之總磁矩之大小為 $|\vec{m}| = NIA$ (23)

$\vec{m}$ 之方向為沿 $\hat{a}_N$ 之方向

所以它受一力矩 $\vec{\tau} = \vec{m} \times \vec{B}_o$ (24)

要使此線圈保持等速度運轉我們必須外加一力矩 $\vec{\tau}' = -\vec{\tau}$ (25)

$$|\vec{\tau}'| = |\vec{m}| B_o \sin\theta = |\vec{m}| B_o \cos\phi$$

$$= NIAB_o \cos\omega t$$

$$= N \frac{NB_oA\omega\cos\omega t}{R} AB_o \cos\omega t$$

$$= \frac{N^2B_o^2A^2\omega}{R} \cos^2\omega t \qquad (26)$$

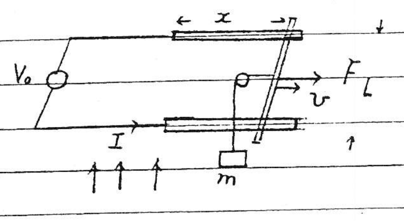
由於 $dW = |\vec{\tau}'| d\phi$ (27)

$$P\,(\text{外加之功率}) = |\vec{\tau}'| \frac{d\phi}{dt} = |\vec{\tau}'| \omega \qquad (28)$$

$$= \frac{N^2B_o^2A^2\omega^2}{R} \cos^2\omega t \qquad (29)$$

與 (20) 式完全符合。

(2) 線型馬達（簡單之起重機）




當有電流 $I$ 通過 $L$ 段時（沿反時鐘方向）

則該段線受之力為

$F = ILB_o$ 向右 (30)

當 $F = mg$ 時則保持平衡 (31)

當線以 $v$ 之速度向右進行時，通過其線圈之磁通量

$\Phi = B_o x L$ (32)

而 $\frac{d\Phi}{dt} = B_o L v$ (33)

因此產生一感應電動勢

$$V_{emf} = -\frac{d\Phi}{dt} = -BLv \qquad (34)$$

$$RI = V_0 + V_{emf} = V_0 - BLv \qquad (35)$$

也即是 $$I = \frac{V_0 - BLv}{R} \qquad (36)$$

平衡狀況要求 $$mg = ILB_0$$

$$= \left(\frac{V_0 - BLv}{R}\right) L B_0 \qquad (37)$$

所以 $$v = \frac{1}{B_0 L}\left(V_0 - \frac{Rmg}{B_0 L}\right) \qquad I = \frac{mg}{BL} \qquad (38)$$

我們現在從功率的觀念來看此一問題

$$F = mg = ILB_0$$

所以 $$I = \frac{mg}{B_0 L} \qquad (39)$$

由電池供應之功率 $P_1 = V_0 I = V_0 mg / B_0 L \qquad (40)$

對外作之功率 $P_2 = mgv + RI^2$ ← 熱能

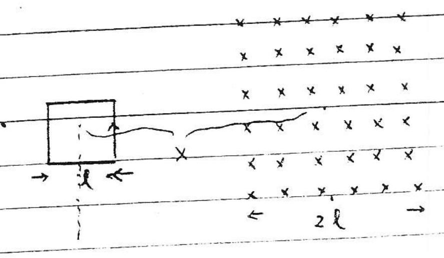
$$= mg \frac{1}{B_0 L}\left(V_0 - \frac{Rmg}{B_0 L}\right) + R\left(\frac{mg}{B_0 L}\right)^2$$

$$= V_0 \frac{mg}{B_0 L} \qquad (41)$$

顯然地 $P_1 = P_2$ 與能量守恒之要求相符。

此一馬達之效率 $$= \frac{mgv}{P_1} \qquad (42)$$

(3)




當 $x = -\frac{3}{2}l$ 時開始進入磁場，當 $x = -\frac{l}{2}$ 時完全進入磁場 $d\Phi = -B\,dx\,l$ $\hat{u}_N$ 向外，$\vec{B}$ 向內

$$V_{emf} = Blv \text{（反時鐘方向）} \qquad (43)$$

在 $x = -\frac{l}{2}$ 到 $x = \frac{l}{2}$ 之間線圈完全在磁場內 $d\Phi = 0$，因此 $V_{emf} = 0$ (44)

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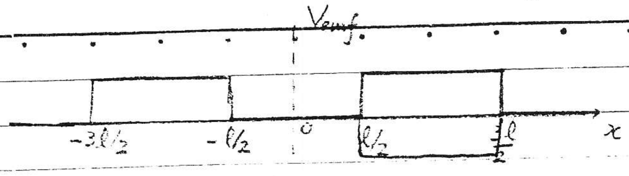
由 $x=\frac{l}{2}$ 到 $x=\frac{3l}{2}$ 線圈離開磁場，$V_{emf} = -Blv$ (順時鐘方向) (45)



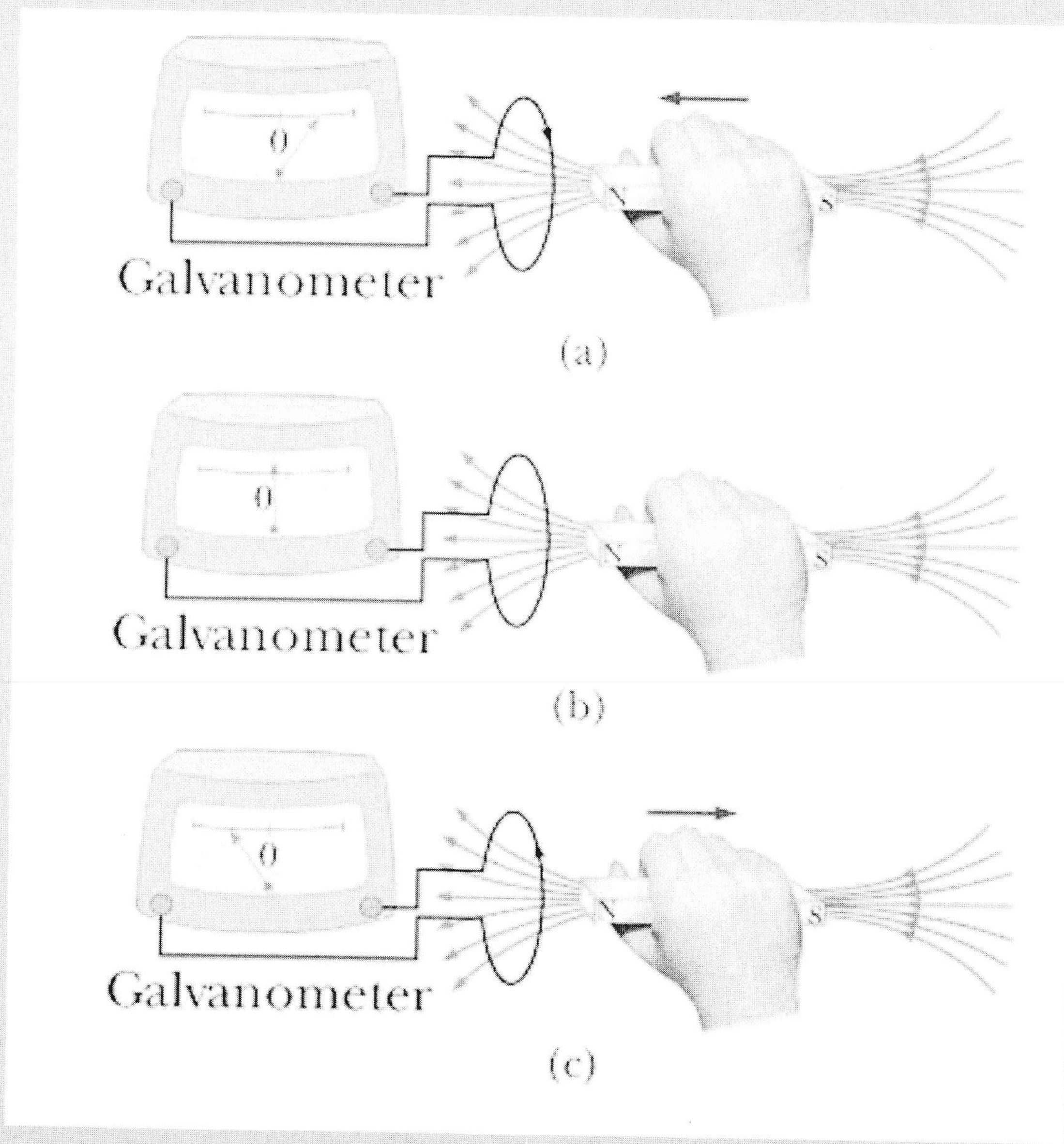
# This Time: Faraday's Law

Fourth (Final) Maxwell's Equation

(but we still have to go back and add another term to Ampere's Law!)

Underpinning of Much Technology

# Electromagnetic Induction



# Movie and Visualization: Induction

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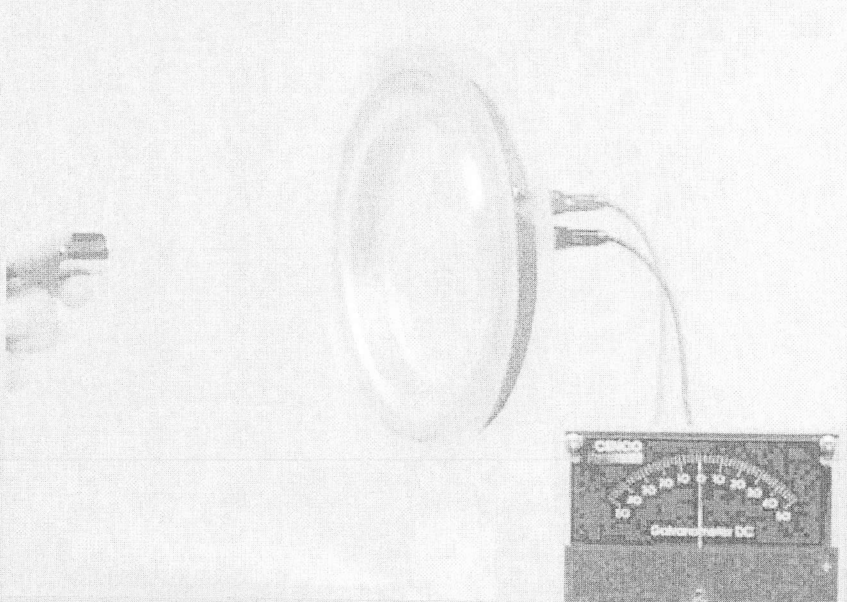
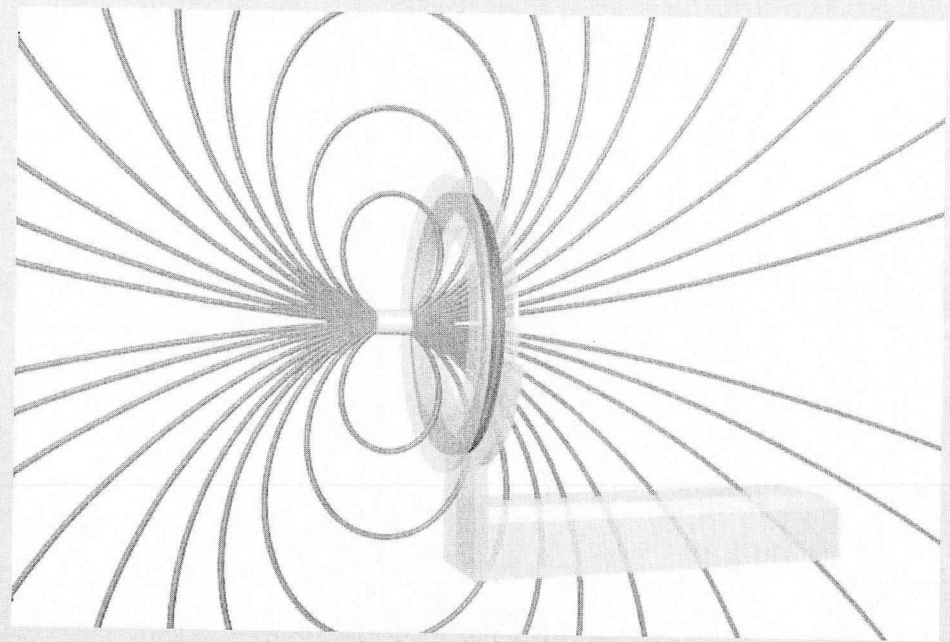




Lenz's Law says that the flux tries to remain the same, so the field lines get "hung up" at the coil.

# Faraday's Law of Induction

$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$

A changing magnetic flux *induces* an EMF

# What is EMF?

$$\mathcal{E} = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

Looks like potential. It's a "driving force" for current

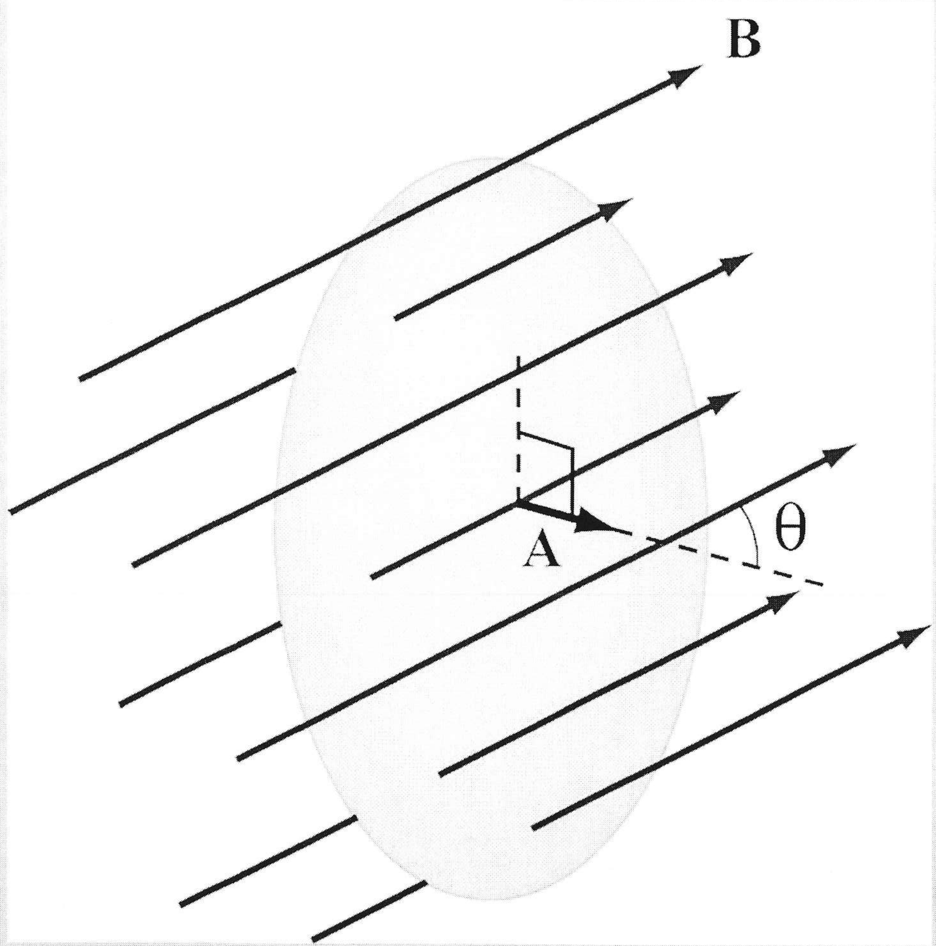
# Faraday's Law of Induction

$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$

A changing magnetic flux
*induces* an EMF

# Magnetic Flux Thru Wire Loop

Analogous to Electric Flux (Gauss' Law)




(1) Uniform **B**

$$\Phi_B = B_{\perp} A = BA\cos\theta = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

(2) Non-Uniform **B**

$$\boxed{\mathbf{\Phi}_B = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}}$$

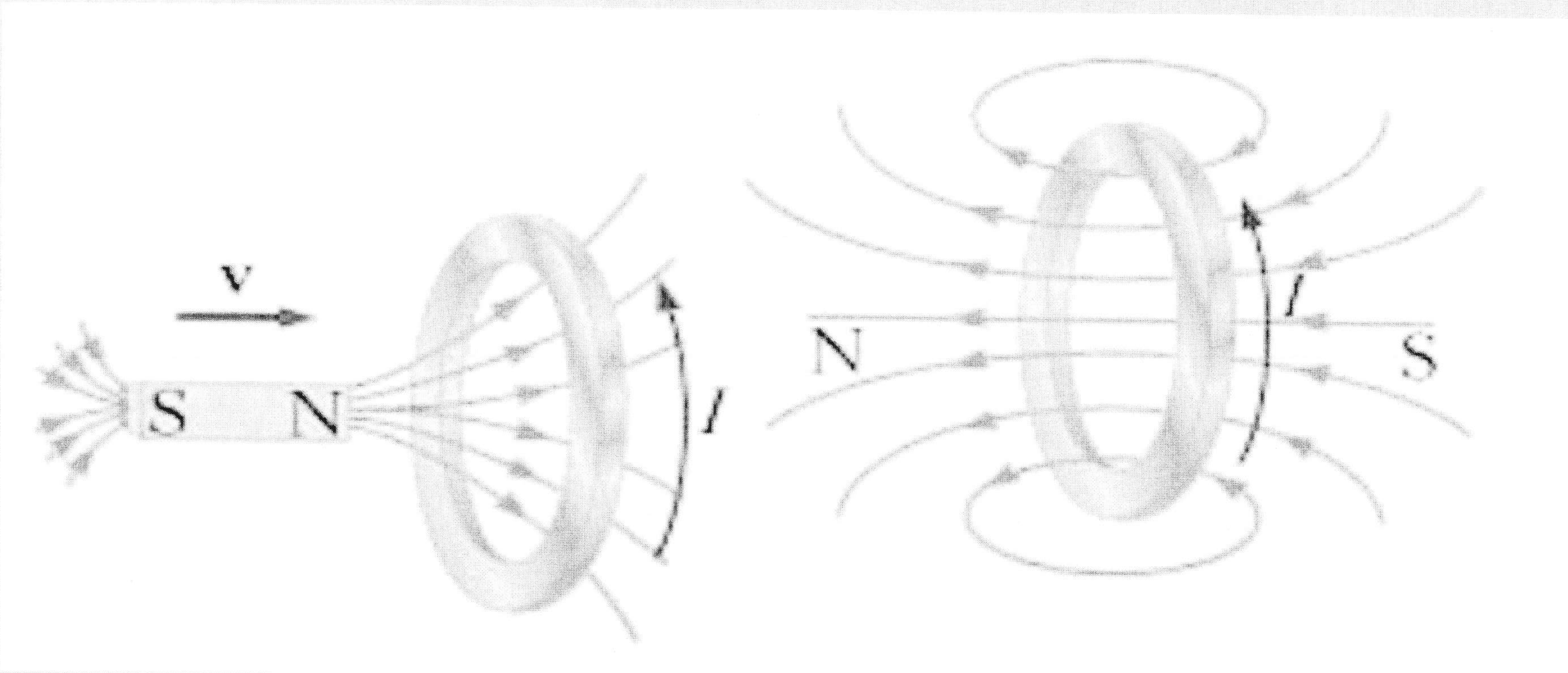
# Faraday's Law of Induction

$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$

A changing magnetic flux
*induces* an EMF

# Minus Sign? Lenz's Law

Induced EMF is in direction that ***opposes*** the change in flux that caused it



# Faraday's Law of Induction

$$\boxed{\mathcal{E} = -N\,\frac{d\Phi_B}{dt}}$$

A changing magnetic flux *induces* an EMF

# Ways to Induce EMF

$$\boxed{\mathcal{E} = -N\frac{d}{dt}\left(BA\cos\theta\right)}$$

Quantities which can vary with time:

- Magnitude of B
- Area A enclosed by the loop
- Angle θ between B and loop normal

# Ways to Induce EMF

$$\boxed{\mathcal{E} = -N \frac{d}{dt}\left( BA\cos\theta \right)}$$

Quantities which can vary with time:

- **Magnitude of B**
- Area A enclosed by the loop
- Angle $\theta$ between B and loop normal

# Ways to Induce EMF

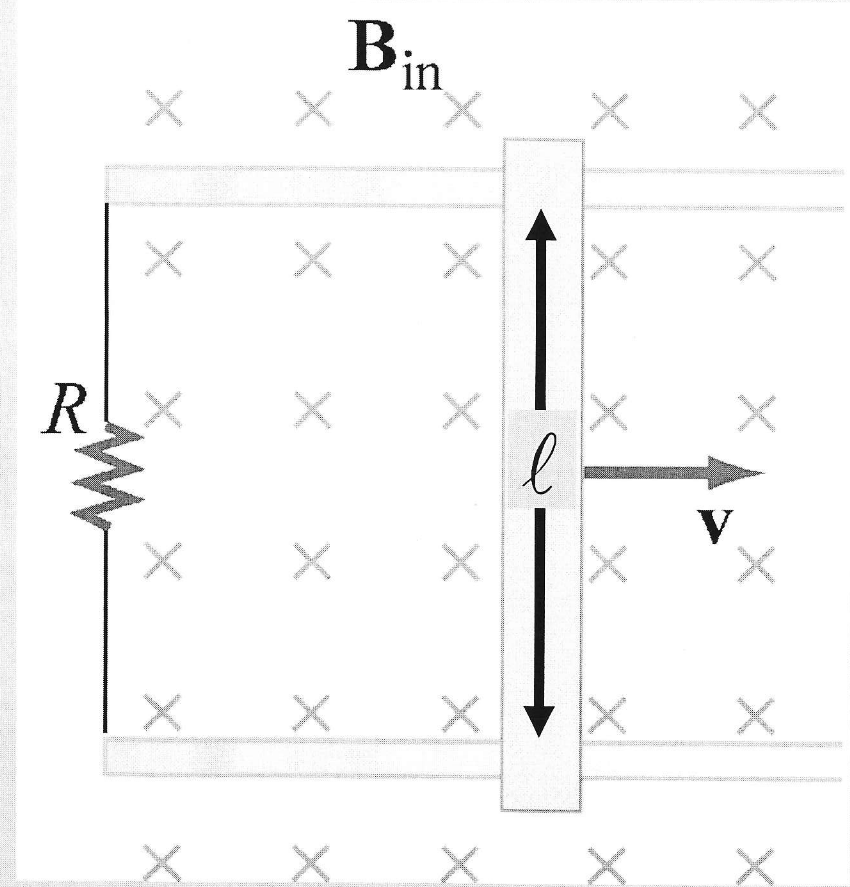
$$\boxed{\mathcal{E} = -N\frac{d}{dt}\left(BA\cos\theta\right)}$$

Quantities which can vary with time:

- Magnitude of B **e.g. Falling Magnet**
- **Area A enclosed by the loop**
- Angle θ between B and loop normal

# Group Problem: Changing Area

Conducting rod pulled along two conducting rails in a uniform magnetic field B at constant velocity v




1. Direction of induced current?
2. Direction of resultant force?
3. Magnitude of EMF?
4. Magnitude of current?
5. Power externally supplied to move at constant v?

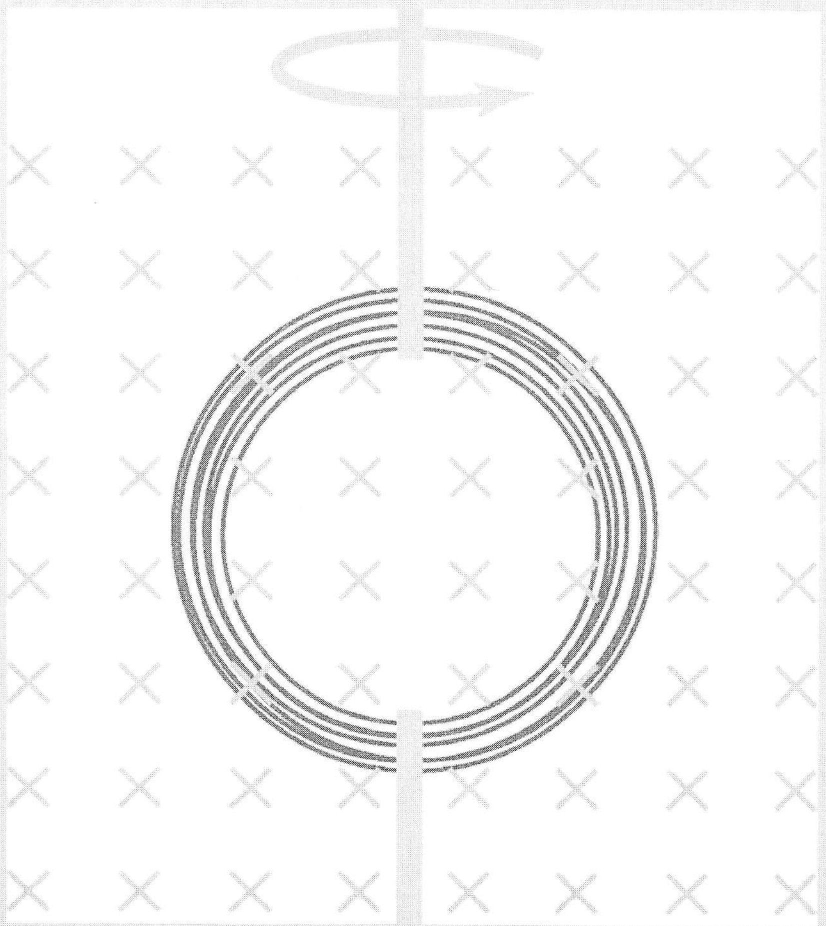
# Ways to Induce EMF

$$\boxed{\mathcal{E} = -N\frac{d}{dt}\left(BA\cos\theta\right)}$$

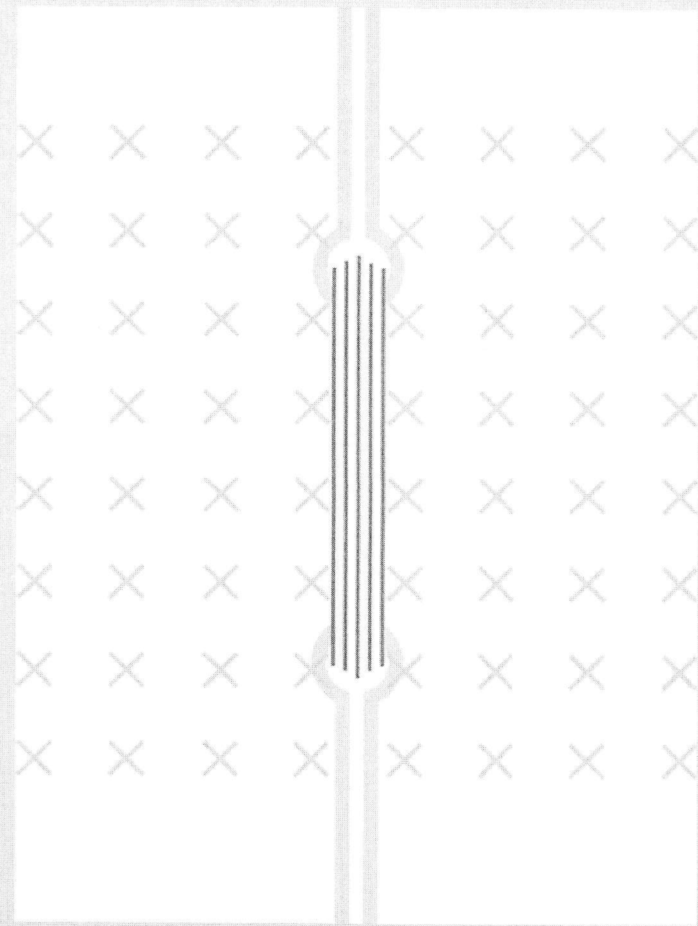
Quantities which can vary with time:

- Magnitude of B **e.g. Moving Coil & Dipole**
- Area A enclosed **e.g. Sliding bar**
- **Angle $\theta$ between B and loop normal**

# Changing Angle

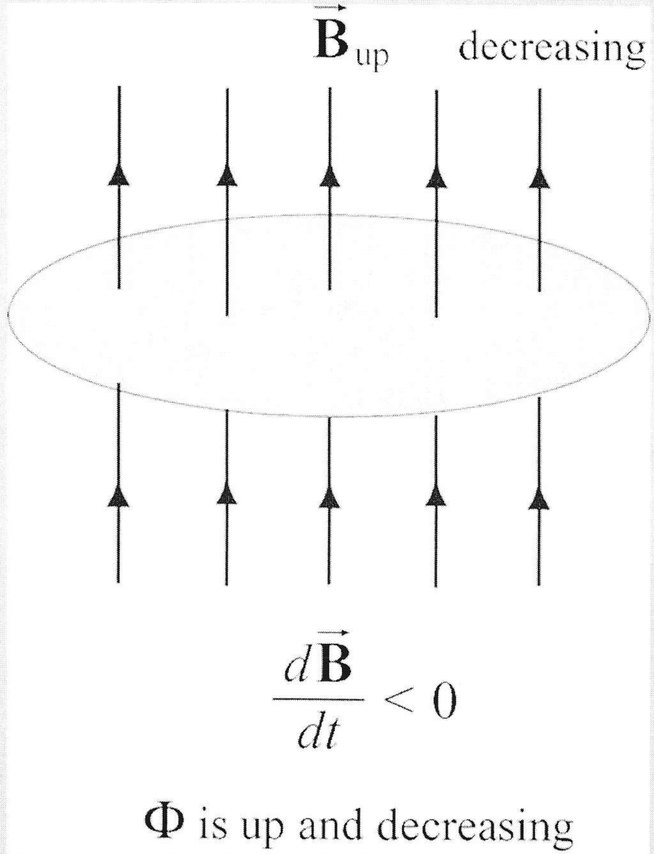


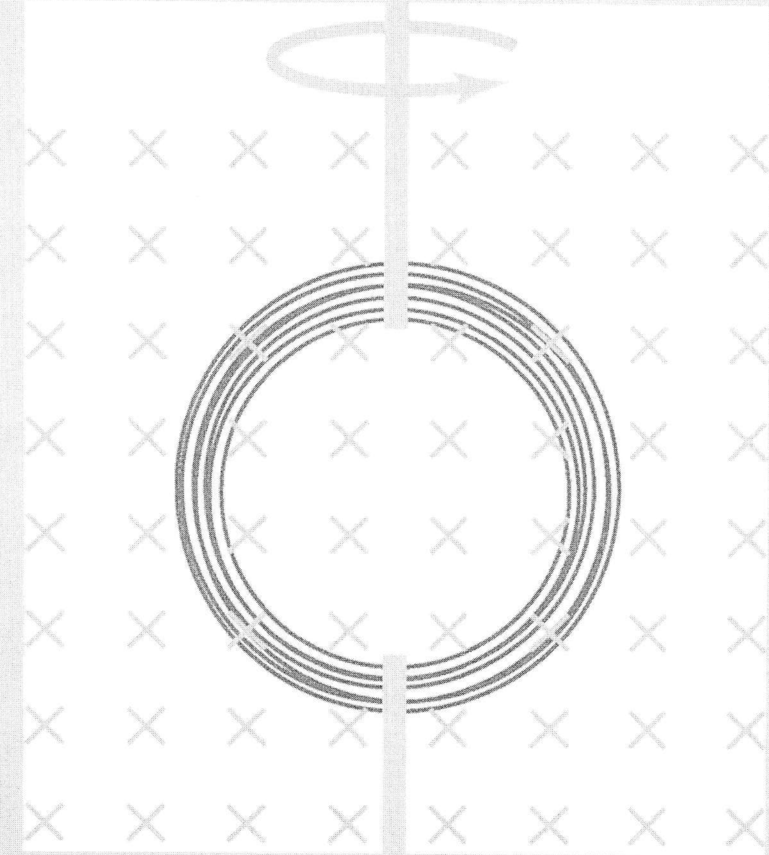
$$\Phi_B = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} = BA$$



$$\Phi_B = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} = 0$$

# Applets that show these 3 cases



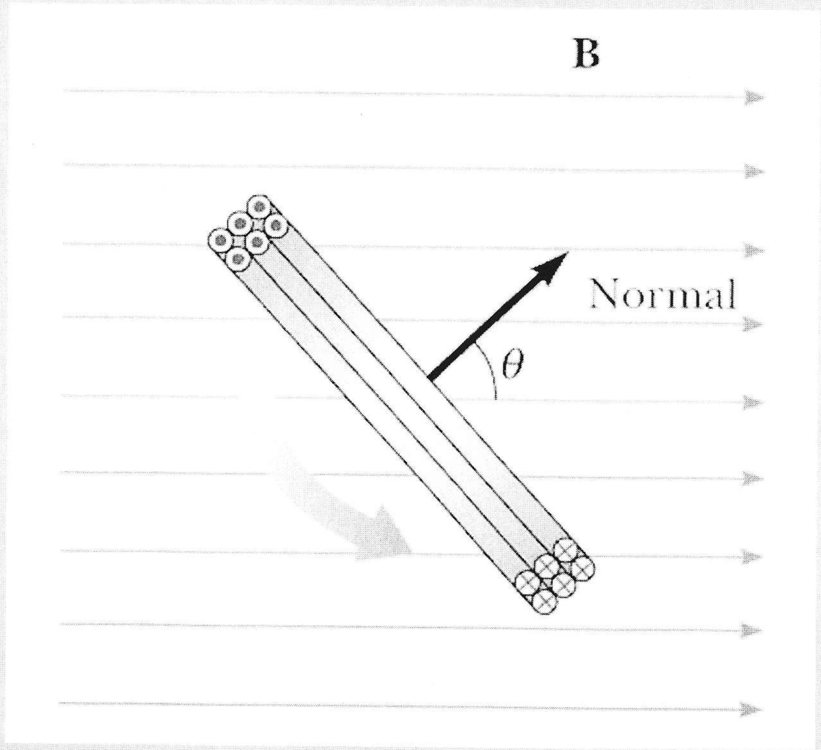
http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/faraday/13-faradayapp02/13-faradayapp02_320.html

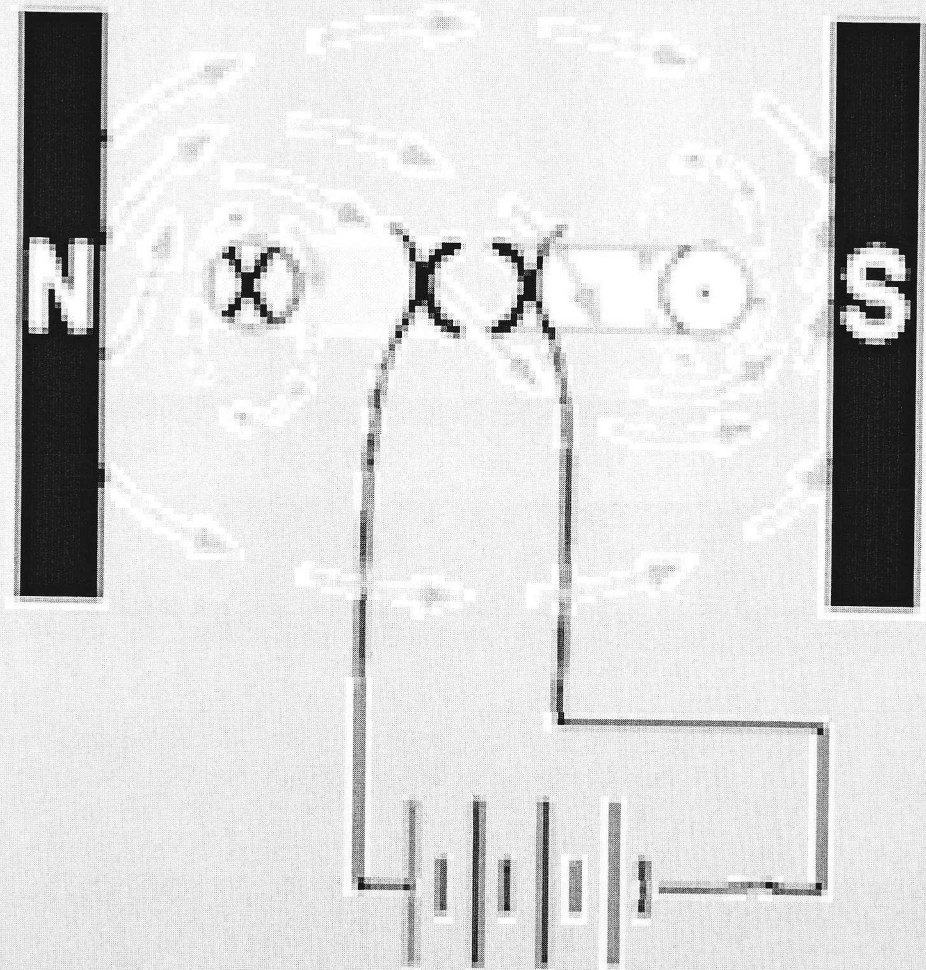
# Faraday's Law
# The last of the Maxwell's Equations (Kind of, still need one more term in Ampere's Law)

# Technology

**<u>Many</u>** Applications of
Faraday's Law

# DC Motor (magnetostatics)



# Maxwell's Equations

## Creating Electric Fields

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0}$$ (Gauss's Law)

$$\oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$ (Faraday's Law)

## Creating Magnetic Fields

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$ (Magnetic Gauss's Law)

$$\oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{enc}$$ (Ampere's Law)

# Chapter 10

# Faraday's Law of Induction

# Faraday's Law of Induction

## 10.1 Faraday's Law of Induction

The electric fields and magnetic fields considered up to now have been produced by stationary charges and moving charges (currents), respectively. Imposing an electric field on a conductor gives rise to a current which in turn generates a magnetic field. One could then inquire whether or not an electric field could be produced by a magnetic field. In 1831, Michael Faraday discovered that, by varying magnetic field with time, an electric field could be generated. The phenomenon is known as electromagnetic induction. Figure 10.1.1 illustrates one of Faraday's experiments.

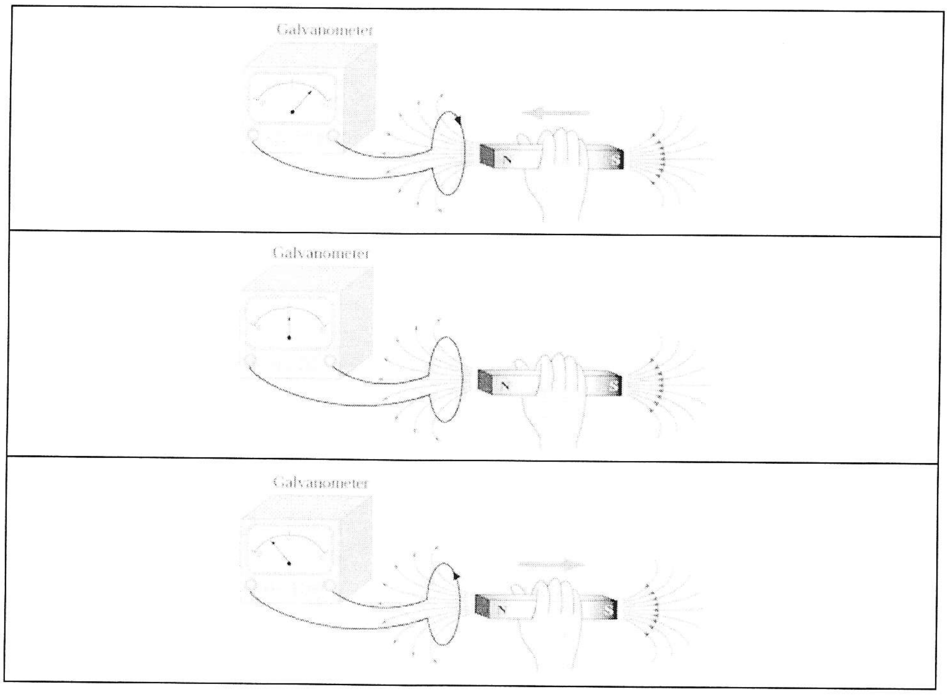



**Figure 10.1.1** Electromagnetic induction

Faraday showed that no current is registered in the galvanometer when bar magnet is stationary with respect to the loop. However, a current is induced in the loop when a relative motion exists between the bar magnet and the loop. In particular, the galvanometer deflects in one direction as the magnet approaches the loop, and the opposite direction as it moves away.

Faraday's experiment demonstrates that an electric current is induced in the loop by changing the magnetic field. The coil behaves as if it were connected to an emf source. Experimentally it is found that the induced emf depends on the rate of change of magnetic flux through the coil.

### 10.1.1 Magnetic Flux

Consider a uniform magnetic field passing through a surface *S*, as shown in Figure 10.1.2 below:

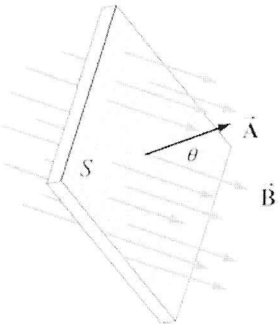



**Figure 10.1.2** Magnetic flux through a surface

Let the area vector be $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$, where $A$ is the area of the surface and $\hat{\mathbf{n}}$ its unit normal. The magnetic flux through the surface is given by

$$\Phi_B = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} = BA\cos\theta \qquad (10.1.1)$$

where $\theta$ is the angle between $\vec{\mathbf{B}}$ and $\hat{\mathbf{n}}$. If the field is non-uniform, $\Phi_B$ then becomes

$$\Phi_B = \iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \qquad (10.1.2)$$

The SI unit of magnetic flux is the weber (Wb):

$$1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2$$

Faraday's law of induction may be stated as follows:

> The induced emf $\varepsilon$ in a coil is proportional to the negative of the rate of change of magnetic flux:
>
> $$\varepsilon = -\frac{d\Phi_B}{dt} \qquad (10.1.3)$$

For a coil that consists of $N$ loops, the total induced emf would be $N$ times as large:

$$\varepsilon = -N\frac{d\Phi_B}{dt} \qquad (10.1.4)$$

Combining Eqs. (10.1.3) and (10.1.1), we obtain, for a spatially uniform field $\vec{\mathbf{B}}$,

$$\varepsilon = -\frac{d}{dt}(BA\cos\theta) = -\left(\frac{dB}{dt}\right)A\cos\theta - B\left(\frac{dA}{dt}\right)\cos\theta + BA\sin\theta\left(\frac{d\theta}{dt}\right) \qquad (10.1.5)$$

Thus, we see that an emf may be induced in the following ways:

(i) by varying the magnitude of $\vec{\mathbf{B}}$ with time (illustrated in Figure 10.1.3.)

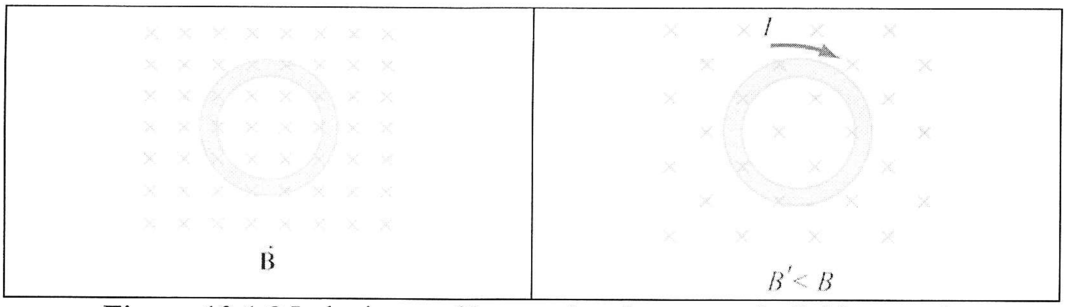



**Figure 10.1.3** Inducing emf by varying the magnetic field strength

(ii) by varying the magnitude of $\vec{\mathbf{A}}$, i.e., the area enclosed by the loop with time (illustrated in Figure 10.1.4.)

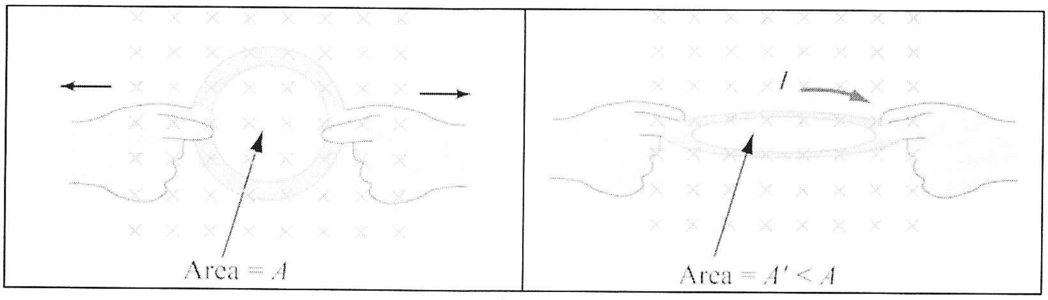



**Figure 10.1.4** Inducing emf by changing the area of the loop

(iii) varying the angle between $\vec{\mathbf{B}}$ and the area vector $\vec{\mathbf{A}}$ with time (illustrated in Figure 10.1.5.)

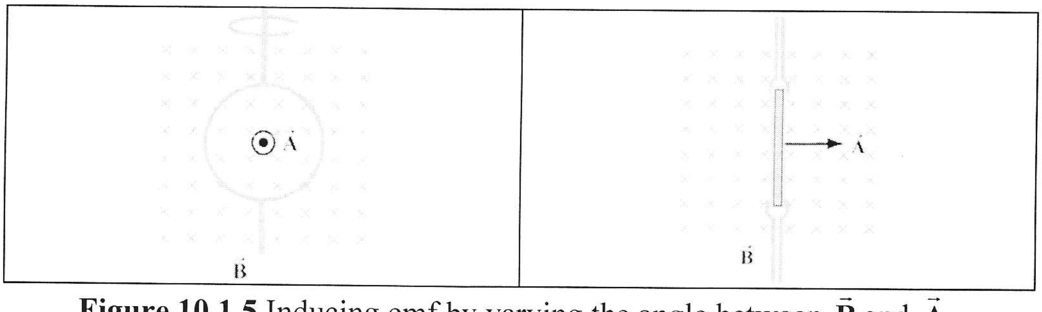



**Figure 10.1.5** Inducing emf by varying the angle between $\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}$.

### 10.1.2 Lenz's Law

The direction of the induced current is determined by Lenz's law:

| |
|---|
| The induced current produces magnetic fields which tend to oppose the change in magnetic flux that induces such currents. |

To illustrate how Lenz's law works, let's consider a conducting loop placed in a magnetic field. We follow the procedure below:

1. Define a positive direction for the area vector $\vec{\mathbf{A}}$.

2. Assuming that $\vec{\mathbf{B}}$ is uniform, take the dot product of $\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}$. This allows for the determination of the sign of the magnetic flux $\Phi_B$.

3. Obtain the rate of flux change $d\Phi_B / dt$ by differentiation. There are three possibilities:

$$\frac{d\Phi_B}{dt}: \begin{cases} > 0 & \Rightarrow \text{induced emf } \varepsilon < 0 \\ < 0 & \Rightarrow \text{induced emf } \varepsilon > 0 \\ = 0 & \Rightarrow \text{induced emf } \varepsilon = 0 \end{cases}$$

4. Determine the direction of the induced current using the right-hand rule. With your thumb pointing in the direction of $\vec{\mathbf{A}}$, curl the fingers around the closed loop. The induced current flows in the same direction as the way your fingers curl if $\varepsilon > 0$, and the opposite direction if $\varepsilon < 0$, as shown in Figure 10.1.6.

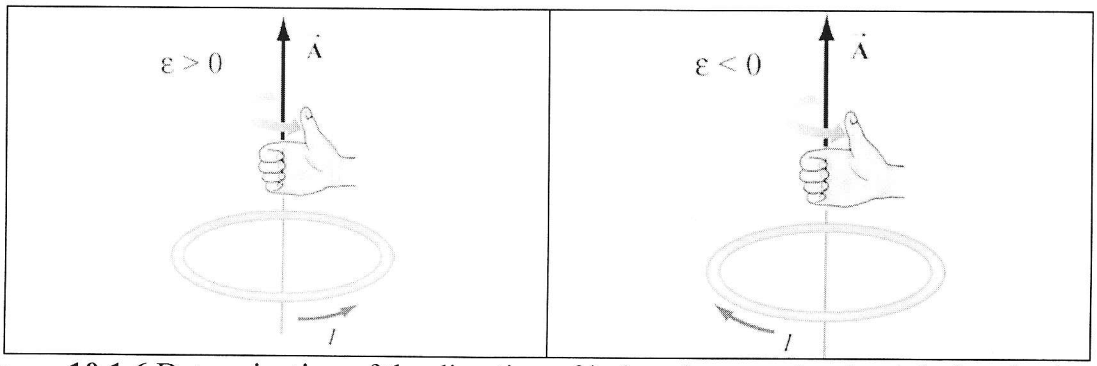



**Figure 10.1.6** Determination of the direction of induced current by the right-hand rule

In Figure 10.1.7 we illustrate the four possible scenarios of time-varying magnetic flux and show how Lenz's law is used to determine the direction of the induced current $I$.

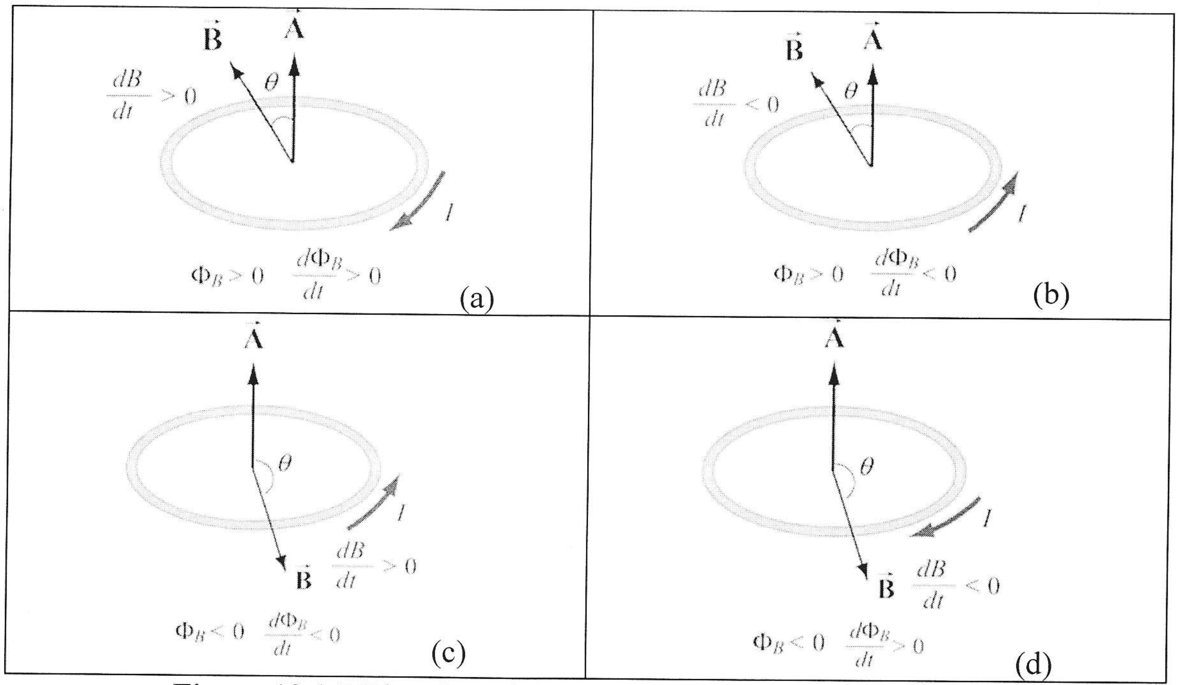



**Figure 10.1.7** Direction of the induced current using Lenz's law

The above situations can be summarized with the following sign convention:

| $\Phi_B$ | $d\Phi_B / dt$ | $\varepsilon$ | $I$ |
|---|---|---|---|
| + | + | − | − |
| | − | + | + |
| − | + | − | − |
| | − | + | + |

The positive and negative signs of $I$ correspond to a counterclockwise and clockwise currents, respectively.

As an example to illustrate how Lenz's law may be applied, consider the situation where a bar magnet is moving toward a conducting loop with its north pole down, as shown in Figure 10.1.8(a). With the magnetic field pointing downward and the area vector $\vec{\mathbf{A}}$ pointing upward, the magnetic flux is negative, *i.e.*, $\Phi_B = -BA < 0$, where $A$ is the area of the loop. As the magnet moves closer to the loop, the magnetic field at a point on the loop increases ($dB/dt > 0$), producing more flux through the plane of the loop. Therefore, $d\Phi_B / dt = -A(dB/dt) < 0$, implying a positive induced emf, $\varepsilon > 0$, and the induced current flows in the counterclockwise direction. The current then sets up an induced magnetic field and produces a *positive* flux to counteract the change. The situation described here corresponds to that illustrated in Figure 10.1.7(c).

Alternatively, the direction of the induced current can also be determined from the point of view of magnetic force. Lenz's law states that the induced emf must be in the direction that opposes the change. Therefore, as the bar magnet approaches the loop, it experiences a repulsive force due to the induced emf. Since like poles repel, the loop must behave as if it were a bar magnet with its north pole pointing up. Using the right-hand rule, the direction of the induced current is counterclockwise, as view from above. Figure 10.1.8(b) illustrates how this alternative approach is used.

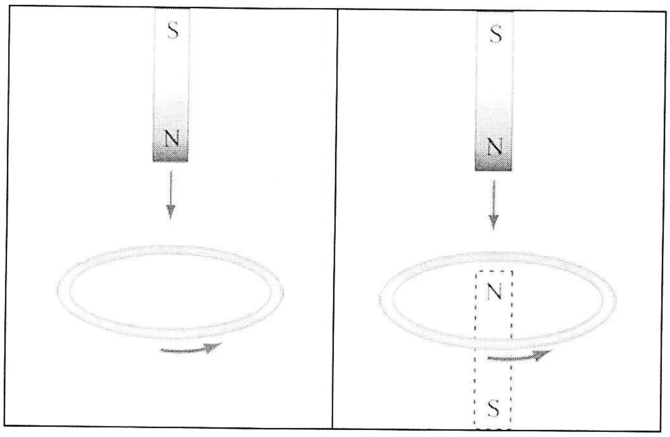



**Figure 10.1.8** (a) A bar magnet moving toward a current loop. (b) Determination of the direction of induced current by considering the magnetic force between the bar magnet and the loop

## 10.2 Motional EMF

Consider a conducting bar of length $l$ moving through a uniform magnetic field which points into the page, as shown in Figure 10.2.1. Particles with charge $q > 0$ inside experience a magnetic force $\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ which tends to push them upward, leaving negative charges on the lower end.

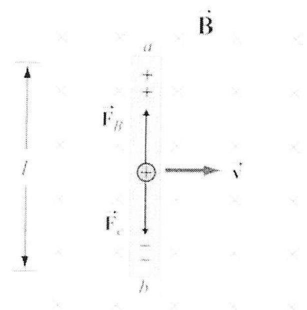



**Figure 10.2.1** A conducting bar moving through a uniform magnetic field

The separation of charge gives rise to an electric field $\vec{\mathbf{E}}$ inside the bar, which in turn produces a downward electric force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}}$. At equilibrium where the two forces cancel, we have $qvB = qE$, or $E = vB$. Between the two ends of the conductor, there exists a potential difference given by

$$V_{ab} = V_a - V_b = \varepsilon = El = Blv \tag{10.2.1}$$

Since $\varepsilon$ arises from the motion of the conductor, this potential difference is called the motional emf. In general, motional emf around a closed conducting loop can be written as

$$\varepsilon = \oint (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{s}} \tag{10.2.2}$$

where $d\vec{\mathbf{s}}$ is a differential length element.

Now suppose the conducting bar moves through a region of uniform magnetic field $\vec{\mathbf{B}} = -B\hat{\mathbf{k}}$ (pointing into the page) by sliding along two frictionless conducting rails that are at a distance $l$ apart and connected together by a resistor with resistance $R$, as shown in Figure 10.2.2.

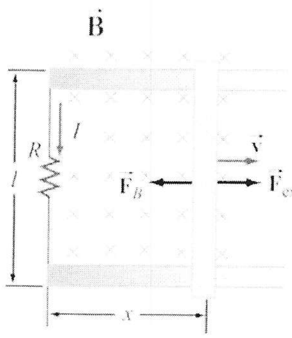



**Figure 10.2.2** A conducting bar sliding along two conducting rails

Let an external force $\vec{\mathbf{F}}_{\text{ext}}$ be applied so that the conductor moves to the right with a constant velocity $\vec{\mathbf{v}} = v\hat{\mathbf{i}}$. The magnetic flux through the closed loop formed by the bar and the rails is given by

$$\Phi_B = BA = Blx \tag{10.2.3}$$

Thus, according to Faraday's law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(Blx) = -Bl\frac{dx}{dt} = -Blv \tag{10.2.4}$$

where $dx/dt = v$ is simply the speed of the bar. The corresponding induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{Blv}{R} \tag{10.2.5}$$

and its direction is counterclockwise, according to Lenz's law. The equivalent circuit diagram is shown in Figure 10.2.3:

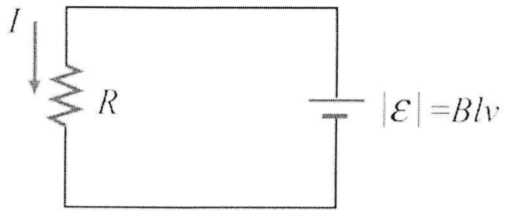



**Figure 10.2.3** Equivalent circuit diagram for the moving bar

The magnetic force experienced by the bar as it moves to the right is

$$\vec{\mathbf{F}}_B = I(l\,\hat{\mathbf{j}}) \times (-B\,\hat{\mathbf{k}}) = -IlB\hat{\mathbf{i}} = -\left(\frac{B^2l^2v}{R}\right)\hat{\mathbf{i}} \tag{10.2.6}$$

which is in the opposite direction of $\vec{\mathbf{v}}$. For the bar to move at a constant velocity, the net force acting on it must be zero. This means that the external agent must supply a force

$$\vec{\mathbf{F}}_{\text{ext}} = -\vec{\mathbf{F}}_B = +\left(\frac{B^2l^2v}{R}\right)\hat{\mathbf{i}} \tag{10.2.7}$$

The power delivered by $\vec{\mathbf{F}}_{\text{ext}}$ is equal to the power dissipated in the resistor:

$$P = \vec{\mathbf{F}}_{\text{ext}} \cdot \vec{\mathbf{v}} = F_{\text{ext}} v = \left(\frac{B^2l^2v}{R}\right)v = \frac{(Blv)^2}{R} = \frac{\varepsilon^2}{R} = I^2R \tag{10.2.8}$$

as required by energy conservation.

From the analysis above, in order for the bar to move at a constant speed, an external agent must constantly supply a force $\vec{\mathbf{F}}_{\text{ext}}$. What happens if at $t = 0$, the speed of the rod is $v_0$, and the external agent stops pushing? In this case, the bar will slow down because of the magnetic force directed to the left. From Newton's second law, we have

$$F_B = -\frac{B^2l^2v}{R} = ma = m\frac{dv}{dt} \tag{10.2.9}$$

or

$$\frac{dv}{v} = -\frac{B^2l^2}{mR}dt = -\frac{dt}{\tau} \tag{10.2.10}$$

where $\tau = mR / B^2 l^2$. Upon integration, we obtain

$$v(t) = v_0 e^{-t/\tau} \tag{10.2.11}$$

Thus, we see that the speed decreases exponentially in the absence of an external agent doing work. In principle, the bar never stops moving. However, one may verify that the total distance traveled is finite.

## 10.3 Induced Electric Field

In Chapter 3, we have seen that the electric potential difference between two points $A$ and $B$ in an electric field $\vec{\mathbf{E}}$ can be written as

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \tag{10.3.1}$$

When the electric field is conservative, as is the case of electrostatics, the line integral of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ is path-independent, which implies $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$.

Faraday's law shows that as magnetic flux changes with time, an induced current begins to flow. What causes the charges to move? It is the induced emf which is the work done per unit charge. However, since magnetic field can do not work, as we have shown in Chapter 8, the work done on the mobile charges must be electric, and the electric field in this situation cannot be conservative because the line integral of a conservative field must vanish. Therefore, we conclude that there is a non-conservative electric field $\vec{\mathbf{E}}_{\text{nc}}$ associated with an induced emf:

$$\varepsilon = \oint \vec{\mathbf{E}}_{\text{nc}} \cdot d\vec{\mathbf{s}} \tag{10.3.2}$$

Combining with Faraday's law then yields

$$\boxed{\oint \vec{\mathbf{E}}_{\text{nc}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}} \tag{10.3.3}$$

The above expression implies that a changing magnetic flux will induce a non-conservative electric field which can vary with time. It is important to distinguish between the induced, non-conservative electric field and the conservative electric field which arises from electric charges.

As an example, let's consider a uniform magnetic field which points *into* the page and is confined to a circular region with radius $R$, as shown in Figure 10.3.1. Suppose the magnitude of $\vec{\mathbf{B}}$ increases with time, *i.e.*, $dB/dt > 0$. Let's find the induced electric field everywhere due to the changing magnetic field.

Since the magnetic field is confined to a circular region, from symmetry arguments we choose the integration path to be a circle of radius $r$. The magnitude of the induced field $\vec{\mathbf{E}}_{\text{nc}}$ at all points on a circle is the same. According to Lenz's law, the direction of $\vec{\mathbf{E}}_{\text{nc}}$ must be such that it would drive the induced current to produce a magnetic field opposing the change in magnetic flux. With the area vector $\vec{\mathbf{A}}$ pointing *out* of the page, the magnetic flux is negative or inward. With $dB/dt > 0$, the inward magnetic flux is increasing. Therefore, to counteract this change the induced current must flow counterclockwise to produce more outward flux. The direction of $\vec{\mathbf{E}}_{\text{nc}}$ is shown in Figure 10.3.1.

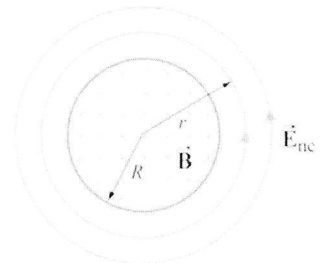



**Figure 10.3.1** Induced electric field due to changing magnetic flux

Let's proceed to find the magnitude of $\vec{\mathbf{E}}_{\text{nc}}$. In the region $r < R$, the rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\left(\vec{\mathbf{B}}\cdot\vec{\mathbf{A}}\right) = \frac{d}{dt}(-BA) = -\left(\frac{dB}{dt}\right)\pi r^2 \tag{10.3.4}$$

Using Eq. (10.3.3), we have

$$\oint \vec{\mathbf{E}}_{\text{nc}}\cdot d\vec{\mathbf{s}} = E_{\text{nc}}\left(2\pi r\right) = -\frac{d\Phi_B}{dt} = \left(\frac{dB}{dt}\right)\pi r^2 \tag{10.3.5}$$

which implies

$$E_{\text{nc}} = \frac{r}{2}\frac{dB}{dt} \tag{10.3.6}$$

Similarly, for $r > R$, the induced electric field may be obtained as

$$E_{\text{nc}}\left(2\pi r\right) = -\frac{d\Phi_B}{dt} = \left(\frac{dB}{dt}\right)\pi R^2 \tag{10.3.7}$$

or

$$E_{\text{nc}} = \frac{R^2}{2r}\frac{dB}{dt} \tag{10.3.8}$$

A plot of $E_{\text{nc}}$ as a function of $r$ is shown in Figure 10.3.2.

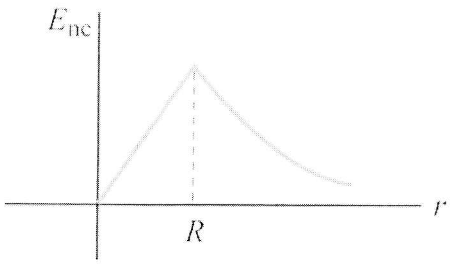



**Figure 10.3.2** Induced electric field as a function of $r$

## 10.4 Generators

One of the most important applications of Faraday's law of induction is to generators and motors. A generator converts mechanical energy into electric energy, while a motor converts electrical energy into mechanical energy.

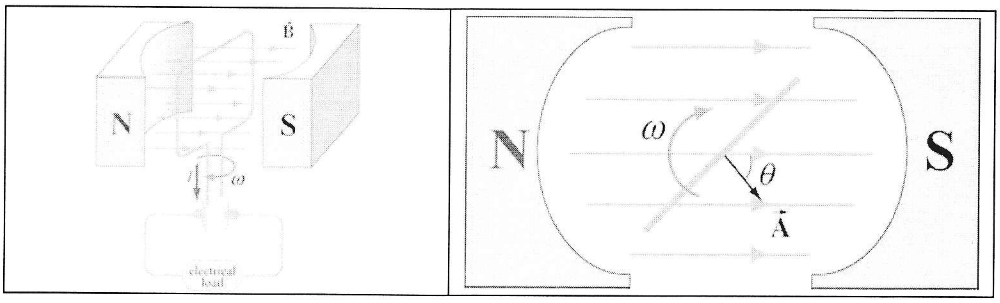



**Figure 10.4.1** (a) A simple generator. (b) The rotating loop as seen from above.

Figure 10.4.1(a) is a simple illustration of a generator. It consists of an $N$-turn loop rotating in a magnetic field which is assumed to be uniform. The magnetic flux varies with time, thereby inducing an emf. From Figure 10.4.1(b), we see that the magnetic flux through the loop may be written as

$$\Phi_B = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} = BA\cos\theta = BA\cos\omega t \tag{10.4.1}$$

The rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = -BA\omega \sin \omega t \qquad (10.4.2)$$

Since there are $N$ turns in the loop, the total induced emf across the two ends of the loop is

$$\varepsilon = -N\frac{d\Phi_B}{dt} = NBA\omega \sin \omega t \qquad (10.4.3)$$

If we connect the generator to a circuit which has a resistance $R$, then the current generated in the circuit is given by

$$I = \frac{|\varepsilon|}{R} = \frac{NBA\omega}{R}\sin \omega t \qquad (10.4.4)$$

The current is an alternating current which oscillates in sign and has an amplitude $I_0 = NBA\omega / R$. The power delivered to this circuit is

$$P = I|\varepsilon| = \frac{(NBA\omega)^2}{R}\sin^2 \omega t \qquad (10.4.5)$$

On the other hand, the torque exerted on the loop is

$$\tau = \mu B \sin\theta = \mu B \sin \omega t \qquad (10.4.6)$$

Thus, the mechanical power supplied to rotate the loop is

$$P_m = \tau\omega = \mu B\omega \sin \omega t \qquad (10.4.7)$$

Since the dipole moment for the $N$-turn current loop is

$$\mu = NIA = \frac{N^2A^2B\omega}{R}\sin \omega t \qquad (10.4.8)$$

the above expression becomes

$$P_m = \left(\frac{N^2A^2B\omega}{R}\sin \omega t\right)B\omega \sin \omega t = \frac{(NAB\omega)^2}{R}\sin^2 \omega t \qquad (10.4.9)$$

As expected, the mechanical power put in is equal to the electrical power output.

## 10.5 Eddy Currents

We have seen that when a conducting loop moves through a magnetic field, current is induced as the result of changing magnetic flux. If a solid conductor were used instead of a loop, as shown in Figure 10.5.1, current can also be induced. The induced current appears to be circulating and is called an *eddy current*.

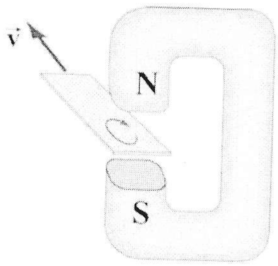



**Figure 10.5.1** Appearance of an eddy current when a solid conductor moves through a magnetic field.

The induced eddy currents also generate a magnetic force that opposes the motion, making it more difficult to move the conductor across the magnetic field (Figure 10.5.2).

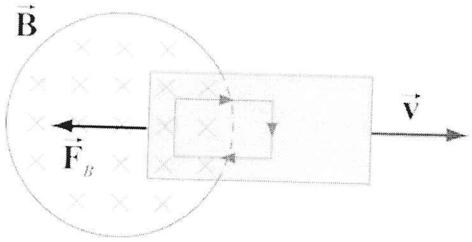



**Figure 10.5.2** Magnetic force arising from the eddy current that opposes the motion of the conducting slab.

Since the conductor has non-vanishing resistance $R$, Joule heating causes a loss of power by an amount $P = \varepsilon^2 / R$. Therefore, by increasing the value of $R$, power loss can be reduced. One way to increase $R$ is to laminate the conducting slab, or construct the slab by using gluing together thin strips that are insulated from one another (see Figure 10.5.3a). Another way is to make cuts in the slab, thereby disrupting the conducting path (Figure 10.5.3b).

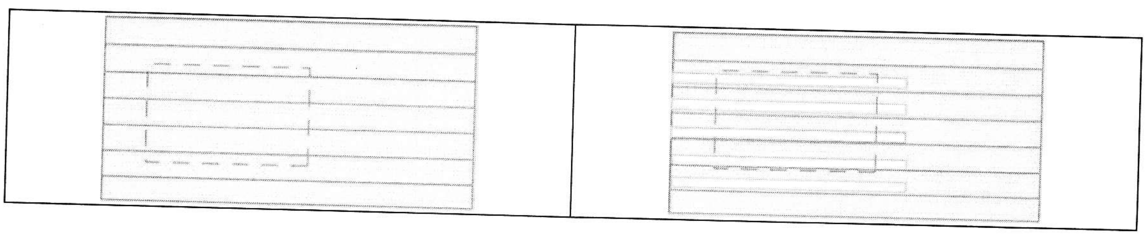


**Figure 10.5.3** Eddy currents can be reduced by (a) laminating the slab, or (b) making cuts on the slab.

There are important applications of eddy currents. For example, the currents can be used to suppress unwanted mechanical oscillations. Another application is the magnetic braking systems in high-speed transit cars.

## 10.6 Summary

- The **magnetic flux** through a surface $S$ is given by

$$\Phi_B = \iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

- **Faraday's law** of induction states that the induced emf $\varepsilon$ in a coil is proportional to the negative of the rate of change of magnetic flux:

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

- The direction of the induced current is determined by **Lenz's law** which states that the induced current produces magnetic fields which tend to oppose the changes in magnetic flux that induces such currents.

- A **motional emf** $\varepsilon$ is induced if a conductor moves in a magnetic field. The general expression for $\varepsilon$ is

$$\varepsilon = \oint (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{s}}$$

  In the case of a conducting bar of length $l$ moving with constant velocity $\vec{\mathbf{v}}$ through a magnetic field which points in the direction perpendicular to the bar and $\vec{\mathbf{v}}$, the induced emf is $\varepsilon = -Bvl$.

- An induced emf in a stationary conductor is associated with a **non-conservative electric field** $\vec{\mathbf{E}}_{\text{nc}}$:

$$\varepsilon = \oint \vec{\mathbf{E}}_{\text{nc}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

## 10.7 Appendix: Induced Emf and Reference Frames

In Section 10.2, we have stated that the general equation of motional emf is given by

$$\varepsilon = \oint (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{s}}$$

where $\vec{\mathbf{v}}$ is the velocity of the length element $d\vec{\mathbf{s}}$ of the *moving* conductor. In addition, we have also shown in Section 10.4 that induced emf associated with a *stationary* conductor may be written as the line integral of the non-conservative electric field:

$$\varepsilon = \oint \vec{\mathbf{E}}_{\text{nc}} \cdot d\vec{\mathbf{s}}$$

However, whether an object is moving or stationary actually depends on the reference frame. As an example, let's examine the situation where a bar magnet is approaching a conducting loop. An observer $O$ in the rest frame of the loop sees the bar magnet moving toward the loop. An electric field $\vec{\mathbf{E}}_{\text{nc}}$ is induced to drive the current around the loop, and a charge on the loop experiences an electric force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}}_{\text{nc}}$. Since the charge is at rest according to observer $O$, no magnetic force is present. On the other hand, an observer $O'$ in the rest frame of the bar magnet sees the loop moving toward the magnet. Since the conducting loop is moving with a velocity $\vec{\mathbf{v}}$, a motional emf is induced. In this frame, $O'$ sees the charge $q$ moving with a velocity $\vec{\mathbf{v}}$, and concludes that the charge experiences a magnetic force $\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$.

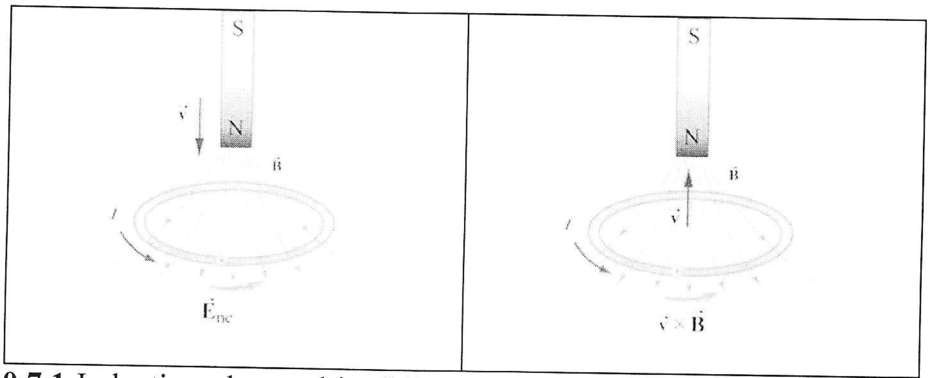



**Figure 10.7.1** Induction observed in different reference frames. In (a) the bar magnet is moving, while in (b) the conducting loop is moving.

Since the event seen by the two observer is the same except the choice of reference frames, the force acting on the charge must be the same, $\vec{\mathbf{F}}_e = \vec{\mathbf{F}}_B$, which implies

$$\vec{\mathbf{E}}_{\text{nc}} = \vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{10.7.1}$$

In general, as a consequence of relativity, an electric phenomenon observed in a reference frame $O$ may appear to be a magnetic phenomenon in a frame $O'$ that moves at a speed $v$ relative to $O$.

## 10.8 Problem-Solving Tips: Faraday's Law and Lenz's Law

In this chapter we have seen that a changing magnetic flux induces an emf:

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

according to Faraday's law of induction. For a conductor which forms a closed loop, the emf sets up an induced current $I = |\varepsilon|/R$, where $R$ is the resistance of the loop. To compute the induced current and its direction, we follow the procedure below:

1. For the closed loop of area $A$ on a plane, define an area vector $\vec{\mathbf{A}}$ and let it point in the direction of your thumb, for the convenience of applying the right-hand rule later. Compute the magnetic flux through the loop using

$$\Phi_B = \begin{cases} \vec{\mathbf{B}}\cdot\vec{\mathbf{A}} & (\vec{\mathbf{B}} \text{ is uniform}) \\ \iint \vec{\mathbf{B}}\cdot d\vec{\mathbf{A}} & (\vec{\mathbf{B}} \text{ is non-uniform}) \end{cases}$$

Determine the sign of $\Phi_B$.

2. Evaluate the rate of change of magnetic flux $d\Phi_B/dt$. Keep in mind that the change could be caused by

(i) changing the magnetic field $dB/dt \neq 0$,
(ii) changing the loop area if the conductor is moving ($dA/dt \neq 0$), or
(iii) changing the orientation of the loop with respect to the magnetic field ($d\theta/dt \neq 0$).

Determine the sign of $d\Phi_B/dt$.

3. The sign of the induced emf is the opposite of that of $d\Phi_B/dt$. The direction of the induced current can be found by using Lenz's law discussed in Section 10.1.2.

## 10.9 Solved Problems

### 10.9.1 Rectangular Loop Near a Wire

An infinite straight wire carries a current $I$ is placed to the left of a rectangular loop of wire with width $w$ and length $l$, as shown in the Figure 10.9.1.

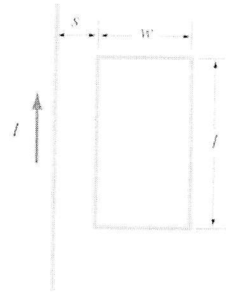



**Figure 10.9.1** Rectangular loop near a wire

(a) Determine the magnetic flux through the rectangular loop due to the current *I*.

(b) Suppose that the current is a function of time with $I(t) = a + bt$, where $a$ and $b$ are positive constants. What is the induced emf in the loop and the direction of the induced current?

**Solutions:**

(a) Using Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \tag{10.9.1}$$

the magnetic field due to a current-carrying wire at a distance $r$ away is

$$B = \frac{\mu_0 I}{2\pi r} \tag{10.9.2}$$

The total magnetic flux $\Phi_B$ through the loop can be obtained by summing over contributions from all differential area elements $dA = l\, dr$:

$$\Phi_B = \int d\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 I l}{2\pi} \int_s^{s+w} \frac{dr}{r} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{s+w}{s}\right) \tag{10.9.3}$$

Note that we have chosen the area vector to point *into* the page, so that $\Phi_B > 0$.

(b) According to Faraday's law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\frac{\mu_0 I l}{2\pi} \ln\left(\frac{s+w}{s}\right)\right] = -\frac{\mu_0 l}{2\pi} \ln\left(\frac{s+w}{s}\right) \cdot \frac{dI}{dt} = -\frac{\mu_0 b l}{2\pi} \ln\left(\frac{s+w}{s}\right) \tag{10.9.4}$$

where we have used $dI/dt = b$.

The straight wire carrying a current $I$ produces a magnetic flux into the page through the rectangular loop. By Lenz's law, the induced current in the loop must be flowing *counterclockwise* in order to produce a magnetic field out of the page to counteract the increase in inward flux.

### 10.9.2 Loop Changing Area

A square loop with length $l$ on each side is placed in a uniform magnetic field pointing into the page. During a time interval $\Delta t$, the loop is pulled from its two edges and turned into a rhombus, as shown in the Figure 10.9.2. Assuming that the total resistance of the loop is $R$, find the average induced current in the loop and its direction.

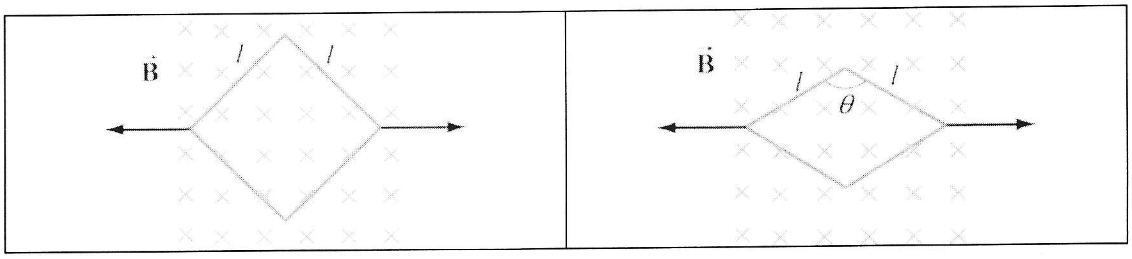



**Figure 10.9.2** Conducting loop changing area

**Solution:**

Using Faraday's law, we have

$$\varepsilon = -\frac{\Delta \Phi_B}{\Delta t} = -B\left(\frac{\Delta A}{\Delta t}\right) \tag{10.9.5}$$

Since the initial and the final areas of the loop are $A_i = l^2$ and $A_f = l^2 \sin\theta$, respectively (recall that the area of a parallelogram defined by two vectors $\vec{l}_1$ and $\vec{l}_2$ is $A = |\vec{l}_1 \times \vec{l}_2| = l_1 l_2 \sin\theta$), the average rate of change of area is

$$\frac{\Delta A}{\Delta t} = \frac{A_f - A_i}{\Delta t} = -\frac{l^2(1-\sin\theta)}{\Delta t} < 0 \tag{10.9.6}$$

which gives

$$\varepsilon = \frac{Bl^2(1-\sin\theta)}{\Delta t} > 0 \tag{10.9.7}$$

Thus, the average induced current is

$$I = \frac{\varepsilon}{R} = \frac{Bl^2(1-\sin\theta)}{\Delta t R} \tag{10.9.8}$$

Since $(\Delta A/\Delta t) < 0$, the magnetic flux into the page decreases. Hence, the current flows in the clockwise direction to compensate the loss of flux.

### 10.9.3 Sliding Rod

A conducting rod of length $l$ is free to slide on two parallel conducting bars as in Figure 10.9.3.

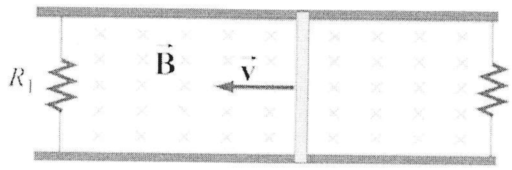



**Figure 10.9.3** Sliding rod

In addition, two resistors $R_1$ and $R_2$ are connected across the ends of the bars. There is a uniform magnetic field pointing into the page. Suppose an external agent pulls the bar to the left at a constant speed $v$. Evaluate the following quantities:

(a) The currents through both resistors;

(b) The total power delivered to the resistors;

(c) The applied force needed for the rod to maintain a constant velocity.

**Solutions:**

(a) The emf induced between the ends of the moving rod is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -Blv \tag{10.9.9}$$

The currents through the resistors are

$$I_1 = \frac{|\varepsilon|}{R_1}, \quad I_2 = \frac{|\varepsilon|}{R_2} \tag{10.9.10}$$

Since the flux into the page for the left loop is decreasing, $I_1$ flows clockwise to produce a magnetic field pointing into the page. On the other hand, the flux into the page for the right loop is increasing. To compensate the change, according to Lenz's law, $I_2$ must flow counterclockwise to produce a magnetic field pointing out of the page.

(b) The total power dissipated in the two resistors is

$$P_R = I_1|\varepsilon| + I_2|\varepsilon| = (I_1 + I_2)|\varepsilon| = \varepsilon^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = B^2 l^2 v^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{10.9.11}$$

(c) The total current flowing through the rod is $I = I_1 + I_2$. Thus, the magnetic force acting on the rod is

$$F_B = IlB = |\varepsilon| lB\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = B^2 l^2 v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{10.9.12}$$

and the direction is to the right. Thus, an external agent must apply an equal but opposite force $\vec{\mathbf{F}}_{\text{ext}} = -\vec{\mathbf{F}}_B$ to the left in order to maintain a constant speed.

Alternatively, we note that since the power dissipated in the resistors must be equal to $P_{\text{ext}}$, the mechanical power supplied by the external agent. The same result is obtained since

$$P_{\text{ext}} = \vec{\mathbf{F}}_{\text{ext}} \cdot \vec{\mathbf{v}} = F_{\text{ext}} v \tag{10.9.13}$$

### 10.9.4 Moving Bar

A conducting rod of length $l$ moves with a constant velocity $\vec{\mathbf{v}}$ perpendicular to an infinitely long, straight wire carrying a current $I$, as shown in the Figure 10.9.4. What is the emf generated between the ends of the rod?

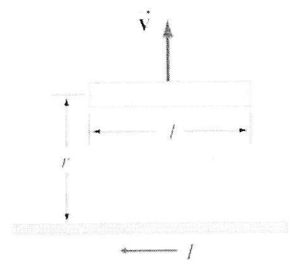



**Figure 10.9.4** A bar moving away from a current-carrying wire

**Solution:**

From Faraday's law, the motional emf is

$$|\varepsilon| = Blv \tag{10.9.14}$$

where $v$ is the speed of the rod. However, the magnetic field due to the straight current-carrying wire at a distance $r$ away is, using Ampere's law:

$$B = \frac{\mu_0 I}{2\pi r} \tag{10.9.15}$$

Thus, the emf between the ends of the rod is given by

$$|\varepsilon| = \left(\frac{\mu_0 I}{2\pi r}\right) lv \tag{10.9.16}$$

### 10.9.5 Time-Varying Magnetic Field

A circular loop of wire of radius $a$ is placed in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field, as shown in Figure 10.9.5.

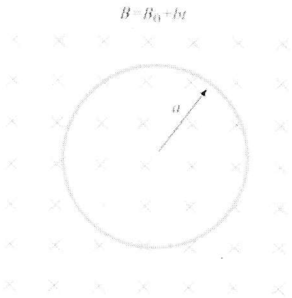



**Figure 10.9.5** Circular loop in a time-varying magnetic field

The magnetic field varies with time according to $B(t) = B_0 + bt$, where $B_0$ and $b$ are positive constants.

(a) Calculate the magnetic flux through the loop at $t = 0$.

(b) Calculate the induced emf in the loop.

(c) What is the induced current and its direction of flow if the overall resistance of the loop is $R$?

(d) Find the power dissipated due to the resistance of the loop.

**Solution:**

(a) The magnetic flux at time $t$ is given by

$$\Phi_B = BA = (B_0 + bt)(\pi a^2) = \pi(B_0 + bt)a^2 \tag{10.9.17}$$

where we have chosen the area vector to point *into* the page, so that $\Phi_B > 0$. At $t = 0$, we have

$$\Phi_B = \pi B_0 a^2 \tag{10.9.18}$$

(b) Using Faraday's Law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -A\frac{dB}{dt} = -(\pi a^2)\frac{d(B_0 + bt)}{dt} = -\pi b a^2 \tag{10.9.19}$$

(c) The induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{\pi b a^2}{R} \tag{10.9.20}$$

and its direction is counterclockwise by Lenz's law.

(d) The power dissipated due to the resistance $R$ is

$$P = I^2 R = \left(\frac{\pi b a^2}{R}\right)^2 R = \frac{(\pi b a^2)^2}{R} \tag{10.9.21}$$

### 10.9.6 Moving Loop

A rectangular loop of dimensions $l$ and $w$ moves with a constant velocity $\vec{\mathbf{v}}$ away from an infinitely long straight wire carrying a current $I$ in the plane of the loop, as shown in Figure 10.9.6. Let the total resistance of the loop be $R$. What is the current in the loop at the instant the near side is a distance $r$ from the wire?

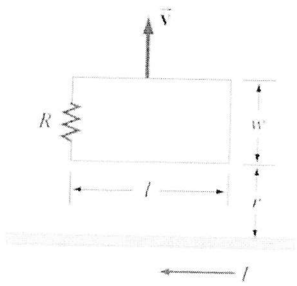



**Figure 10.9.6** A rectangular loop moving away from a current-carrying wire

**Solution:**

The magnetic field at a distance $s$ from the straight wire is, using Ampere's law:

$$B = \frac{\mu_0 I}{2\pi s} \tag{10.9.22}$$

The magnetic flux through a differential area element $dA = l\,ds$ of the loop is

$$d\Phi_B = \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 I}{2\pi s} l\, ds \tag{10.9.23}$$

where we have chosen the area vector to point *into* the page, so that $\Phi_B > 0$. Integrating over the entire area of the loop, the total flux is

$$\Phi_B = \frac{\mu_0 I l}{2\pi} \int_r^{r+w} \frac{ds}{s} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{r+w}{r}\right) \tag{10.9.24}$$

Differentiating with respect to *t*, we obtain the induced emf as

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 I l}{2\pi}\frac{d}{dt}\left(\ln\frac{r+w}{r}\right) = -\frac{\mu_0 I l}{2\pi}\left(\frac{1}{r+w} - \frac{1}{r}\right)\frac{dr}{dt} = \frac{\mu_0 I l}{2\pi}\frac{wv}{r(r+w)} \tag{10.9.25}$$

where $v = dr/dt$. Notice that the induced emf can also be obtained by using Eq. (10.2.2):

$$\begin{aligned} \varepsilon &= \oint (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{s}} = vl\left[B(r) - B(r+w)\right] = vl\left[\frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi(r+w)}\right] \\ &= \frac{\mu_0 I l}{2\pi}\frac{vw}{r(r+w)} \end{aligned} \tag{10.9.26}$$

The induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{\mu_0 I l}{2\pi R}\frac{vw}{r(r+w)} \tag{10.9.27}$$

## 10.10 Conceptual Questions

1. A bar magnet falls through a circular loop, as shown in Figure 10.10.1

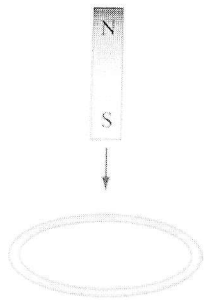



**Figure 10.10.1**

(a) Describe qualitatively the change in magnetic flux through the loop when the bar magnet is above and below the loop.

(b) Make a qualitative sketch of the graph of the induced current in the loop as a function of time, choosing $I$ to be positive when its direction is counterclockwise as viewed from above.

2. Two circular loops $A$ and $B$ have their planes parallel to each other, as shown in Figure 10.10.2.

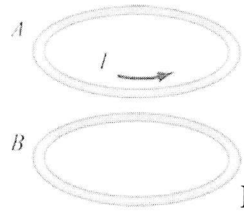



**Figure 10.10.2**

Loop $A$ has a current moving in the counterclockwise direction, viewed from above.

(a) If the current in loop $A$ decreases with time, what is the direction of the induced current in loop $B$? Will the two loops attract or repel each other?

(b) If the current in loop $A$ increases with time, what is the direction of the induced current in loop $B$? Will the two loops attract or repel each other?

3. A spherical conducting shell is placed in a time-varying magnetic field. Is there an induced current along the equator?

4. A rectangular loop moves across a uniform magnetic field but the induced current is zero. How is this possible?

Maxwell Equation, (First Visit)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (1) \qquad \nabla \cdot \vec{B} = 0 \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3) \qquad \nabla \times \vec{B} = \mu_0 \vec{J} \quad (4)$$

$\vec{E}, \vec{B}, \rho, \vec{J}$ are function of $(\vec{r}, t)$

Faraday's Law: $\vec{E}, \vec{B}$ are related

Charge Conservation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad (5)$$

$$\underbrace{\nabla \cdot (\nabla \times \vec{B})}_{=0} = \mu_0 \nabla \cdot \vec{J}$$

↳ vector identity

inconsistent with the continuity equation

if $\frac{\partial \rho}{\partial t} = 0$

By symmetry ⇒ add a term $a \frac{\partial \vec{E}}{\partial t}$

$a$ ↓ constant

$$\nabla \times B = \mu_0 \vec{J} + a \frac{\partial \vec{E}}{\partial t}$$

⇓

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + a \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$a \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right) = a \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = a \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \mu_0 \left[ \nabla \cdot \vec{J} + \frac{a}{\epsilon_0 \mu_0} \frac{\partial \rho}{\partial t} \right] = 0$$

⇓

continuity equation

if

$a = \epsilon_0 \mu_0$

Maxwell displacement current

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad (4)'$$

(1), (2), (3), (4) form the Maxwell equations

Note [ $\nabla \cdot (\nabla \times \vec{V}) = 0$
$\downarrow$
vector identity ]

Define $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$
$\downarrow$
velocity of light
$\downarrow$
Exercise check the dimensionality
and
the value

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \qquad (4)''$$

In general, the second term is $\ll$ first
term in the above term
$\downarrow$
Ampere were not be
able to measure

Maxwell equations not only is consistent
with the continuity equation
but includes the continuity equations

(4)' is "derived" from theoretical
consideration.

Maxwell equation $\Rightarrow$ EM wave propagates with velocity $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ in vacuum

$$\nabla \cdot \vec{E} = 0 , \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 , \quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{E}) \underset{\text{vector identity}}{=} \nabla(\overbrace{\nabla \cdot \vec{E}}^{=0}) - \nabla^2 \vec{E}$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t}$$

Exercise

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

Thus the proof.

Eureka!

遇有新發現時的

勝利歡呼

(阿基米德)