Chapter 33

A-C Circuits

$$V_0 = L\frac{dI}{dt} + RI + \frac{1}{C}Q$$

$$I = \frac{dQ}{dt}$$

$$V_0 = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C}$$

⇒ initial conditions must be given

↓

electromagnetic oscillation

$$F_0 - \lambda\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

↓ friction force ↓ restoring force

$$F_0 = m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx$$

⇒ initial conditions must be provided.

EM oscillation	Mechanical oscillation
Q	x
L	m
R	λ
$\frac{1}{C}$	k
V_0	F_0
$I = \frac{dQ}{dt}$	$v = \frac{dx}{dt}$

$V_0 = R = 0$	$F = \lambda = 0$
$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$ ↓	$m\frac{d^2x}{dt^2} + kx = 0$
LC oscillation	simple harmonic oscillation
$Q(t) = A\sin(\omega t + \alpha)$ $\omega = \sqrt{\frac{1}{LC}}$	$x(t) = A\sin(\omega t + \alpha)$ $\omega = \sqrt{\frac{k}{m}}$

A, α are to be determined by initial conditions

Energy consideration

$L\frac{dI}{dt} + \frac{Q}{C} = 0$	$m\frac{d^2x}{dt^2} + kx = 0$
$I = \frac{dQ}{dt}$	$v = \frac{dx}{dt}$
$LI\frac{d}{dt}I + \frac{1}{C}\frac{dQ}{dt}Q = 0$	$mv\frac{dv}{dt} + k\frac{dx}{dt}x = 0$
$\frac{d}{dt}\left[\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}\right] = 0$ magnetic energy, electric energy electromagnetic energy conservation	$\frac{d}{dt}\left[\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right] = 0$ kinetic energy, potential energy mechanical energy

Example : At $t = 0$ $Q = Q_0$, $I = 0$

$$Q(t) = A\sin(\omega t + \alpha), \quad I(t) = A\omega\cos(\omega t + \alpha)$$

$$\Rightarrow \begin{array}{l} Q_0 = A\sin\alpha \\ 0 = A\omega\cos\alpha \end{array} \Rightarrow A = Q_0, \ \sin\alpha = 1, \ \cos\alpha = 0$$

$$\Rightarrow \begin{array}{l} Q(t) = Q_0\cos\omega t \\ I(t) = -Q_0\sin\omega t \end{array}$$

$$U_E = \frac{1}{2C}Q^2 = \frac{Q_0^2}{2}\cos^2\omega t$$

$$U_M = \frac{1}{2}LI^2 = \frac{1}{2}LQ_0^2\omega^2\sin^2\omega t = \frac{Q_0^2}{2C}\sin^2\omega t$$

$$\Rightarrow U_{EM} = \frac{Q_0^2}{2C} = \text{constant} \Rightarrow \text{conserved.}$$

$V_0 = 0$ $\qquad\qquad$ $F_0 = 0$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0 \qquad m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0$$

$\downarrow$

damped oscillation.

$$Q(t) = e^{-\frac{R}{2L}t}u(t)$$

$$\frac{dQ}{dt} = e^{-\frac{R}{2L}t}\frac{du}{dt} + (-\frac{R}{2L})e^{-\frac{R}{2L}t}u(t)$$

$$\frac{d^2Q}{dt^2} = e^{-\frac{R}{2L}}\frac{d^2u}{dt} + (-\frac{R}{2L})e^{-\frac{R}{2L}t}\frac{du}{dt}$$
$$+ (-\frac{R}{2L})^2 e^{-\frac{R}{2L}t}u(t) + (-\frac{R}{2L})e^{-\frac{R}{2L}t}\frac{du}{dt}$$

$$= e^{-\frac{R}{2L}t}[\frac{d^2u}{dt^2} - \frac{R}{L}\frac{du}{dt} + (\frac{R}{2L})^2u]$$

Substitute back into the differential equation

$$Le^{-\frac{R}{2L}t}[\frac{d^2u}{dt^2} - \frac{R}{L}\frac{du}{dt} + (\frac{R}{2L})^2u] + Re^{-\frac{R}{2L}t}[\frac{du}{dt} - \frac{R}{2L}u] + \frac{e}{C}^{-\frac{R}{2L}t}u = 0$$

$$\Rightarrow \frac{d^2u}{dt^2} + [\frac{1}{LC} - (\frac{R}{2L})^2]u = 0$$

(i) $\frac{1}{LC} - (\frac{R}{2L})^2 = \omega'^2 > 0$

$$\frac{d^2u}{dt^2} + \omega'^2u = 0$$

$u(t) = A\sin(\omega't + \alpha')$; A, α' to be determined by initial conditions $\Rightarrow$ $Q(t) = e^{-\frac{R}{2L}t}A\sin(\omega't + \alpha')$

(ii) $(\frac{R}{2L})^2 - \frac{1}{LC} = \beta^2 > 0$

$$\frac{d^2u}{dt^2} - \beta^2u = 0$$

$u(t) = Ae^{\beta t} + Be^{-\beta t}$; A, B to be determined by initial conditions $\Rightarrow$ $Q(t) = e^{-\frac{R}{2L}t}(Ae^{\beta t} + Be^{-\beta t})$

(iii) $\frac{1}{LC} = (\frac{R}{2L})^2$

$$\frac{d^2u}{dt^2} = 0$$

$u(t) = A + Bt$; A, B to be determined by the initial conditions.

Energy consideration

$$L \frac{d^2Q}{dt^2} + RI + \frac{1}{c} Q = 0$$

$$L \frac{d}{dt} I + IR + \frac{1}{c} Q = 0$$

$$LI \frac{d}{dt} I + \frac{1}{c} \frac{dQ}{dt} Q = - I^2 R$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} L I^2 + \frac{1}{2} \frac{Q^2}{c} \right) = \underset{\downarrow}{-} I^2 R$$

decrease in electromagnetic energy
per unit time
⇒ heat energy produced per
unit time.

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{c}Q = V_0 \sin\omega t$$

$$Q_s(t) = A\cos(\omega t - \alpha)$$

$$\frac{dQ_s}{dt} = -A\omega\sin(\omega t - \alpha)$$

$$\frac{d^2Q_s}{dt^2} = -A\omega^2\cos(\omega t - \alpha)$$

$$-LA\omega^2\cos(\omega t - \alpha) - RA\omega\sin(\omega t - \alpha) + \frac{1}{c}A\cos(\omega t - \alpha)$$

$$= V_0 \sin\omega t$$

$$\Rightarrow -LA\omega^2[\cos\omega t\cos\alpha + \sin\omega t\sin\alpha]$$

$$-RA\omega[\sin\omega t\cos\alpha - \cos\omega t\sin\alpha]$$

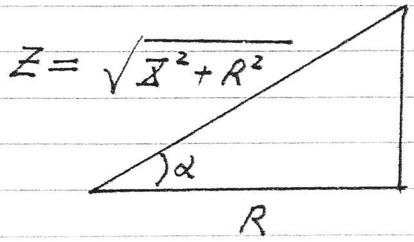
$$+\frac{1}{c}A[\cos\omega t\cos\alpha + \sin\omega t\sin\alpha] = V_0\sin\omega t$$

Coefficient of $\cos\omega t$

$$-LA\omega^2\cos\alpha + RA\omega\sin\alpha + \frac{A}{c}\cos\alpha = 0$$

$$-L\omega + R\tan\alpha + \frac{1}{c\omega} = 0$$

$$R\tan\alpha = L\omega - \frac{1}{c\omega}. \Rightarrow \tan\alpha = \frac{X}{R}$$



$$X = L\omega - \frac{1}{c\omega}$$

$$\sin\alpha = \frac{X}{Z}$$

$$\cos\alpha = \frac{R}{Z}$$

Coefficient of $\sin\omega t$

$$-LA\omega^2\frac{X}{Z} - RA\omega\frac{R}{Z} + \frac{1}{C}A\frac{X}{Z} = V_0$$

$$\Rightarrow \quad A\omega\left[-L\omega\frac{X}{Z} + \frac{1}{C\omega}\frac{X}{Z} - R\frac{R}{Z}\right] = V_0$$

$$\Rightarrow \quad A\omega\left[-\frac{X^2}{Z} - \frac{R^2}{Z}\right] = V_0$$

$$\Rightarrow \quad -A\omega Z = V_0$$

$$\Rightarrow \quad A = -\frac{V_0}{\omega Z}$$

$Q_s(t) = -\frac{V_0}{\omega Z}\cos(\omega t - \alpha)$ with

$$\tan\alpha = \frac{X}{R}$$

is a solution of $L\frac{d^2Q_s}{dt^2} + R\frac{dQ_s}{dt} + \frac{1}{C}Q_s = V_0\sin\omega t$

but clearly it may not satisfy the initial conditio

$Q_h(t) = e^{-\frac{R}{2L}t}u(t)$ is a solution of

$$L\frac{d^2Q_h}{dt^2} + R\frac{dQ_h}{dt} + \frac{Q_h}{C} = 0$$

For example $\frac{1}{LC} - \left(\frac{R}{2L}\right)^2 > 0$

$$Q_h(t) = A\sin(\omega' t + \alpha')\cdot e^{-\frac{R}{2L}t}$$

with arbitrary A, α' constants are solutions of

$$L\frac{d^2Q_h}{dt^2} + R\frac{dQ_h}{dt} + \frac{1}{C}Q_h = 0$$

Clearly $Q_h(t) + Q_s(t) \equiv Q(t)$ is a solutions

of $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0\sin\omega t$

$Q(t) = -\frac{V_0}{\omega Z}\cos(\omega t - \alpha) + e^{-\frac{R}{2L}t}A\sin(\omega' t +$

are solutions of $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0\sin\omega t$

A, α' are arbitrary constants to be fixed to meet the initial

$\Rightarrow$ solution of the problem.

With $t \gg \frac{2L}{R}$, the last terms can be neglected

$$\Rightarrow \quad Q(t) \simeq -\frac{V_0}{\omega Z} \cos(\omega t - \alpha)$$

$$\Downarrow$$

$$I(t) = \frac{V_0}{Z} \sin(\omega t - \alpha)$$

$$P(t) = V(t)\, I(t)$$

$$= V_0 \sin \omega t \cdot \frac{V_0}{Z} \sin(\omega t - \alpha)$$

$$= \frac{V_0^2}{Z} \sin^2 \omega t \cos\alpha - \frac{V_0^2}{Z} \sin\omega t \cos\omega t \sin\alpha$$

$$\langle p \rangle = \frac{1}{T} \int_0^T P(t)\, dt$$

$$= \frac{1}{2} \frac{V_0^2}{Z} \cos\alpha$$

$$= \frac{1}{2} \frac{V_0^2}{Z} \frac{R}{Z}$$

$$= \frac{1}{2} V_0^2 \frac{R}{\sqrt{R^2 + (L\omega - \frac{1}{\omega C})^2}^{\,2}}$$

Maximum occur at $L\omega - \frac{1}{\omega C} = 0$

$$\Rightarrow \quad \omega = \underset{\substack{\downarrow \\ \text{resonance condition.}}}{\frac{1}{\sqrt{LC}}}$$

Maximum energy transfer to the system per unit time.

Resonance is a very important phenomenon in physics.

Energy Consideration

$$\frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C^2}\right) = -I^2R + V(t)I$$

↓

energy put in by
the external potential
per unit time.

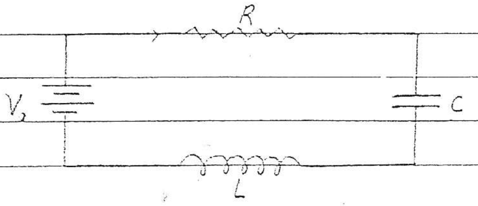
分類：
編號： 1
總號：

第三節 電振盪及交流電路

1. 簡介 在本節中我們將討論電振盪及最簡單之交流電路並特別強調與力學中振盪的相似處

2. 基本觀念

在圖一中之 LRC 電路中, 克希何夫之第二定則可寫成



$$V_0 = IR + L\frac{dI}{dt} + \frac{Q}{C} \qquad (1)$$

由於

$$I = \frac{dQ}{dt} \qquad (2)$$

上式可寫成

$$V_0 = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q \qquad (3)$$

此公式的形式與強迫阻尼振盪器之公式相似, 其公式為

$$F_0 = m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx \qquad (4)$$

此處 x 是離平衡點之距離.

比較 (3), (4) 兩式我們獲得以下的對應

電振盪		力振盪
Q	↔	x
L	↔	m
R	↔	λ
$\frac{1}{C}$	↔	k
V_0	↔	F_0
$I = \frac{dQ}{dt}$	↔	$v = \frac{dx}{dt}$

當 $V_0 = R = 0$ 時, 則第(3)式簡化成

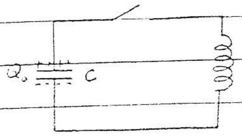
$$L\frac{d^2Q}{dt^2} = -\frac{1}{C}Q \qquad (5)$$

此一公式之解為 $Q(t) = A\sin(\omega t + \alpha)$, $\omega = \sqrt{\frac{1}{LC}}$, 而 A, α 仍是取決於開始之情況, 也即是其情況與簡諧運動完全相似.

此一對應更可延伸至能量的觀念上。在電容中之電能為 $\frac{1}{2}\frac{1}{C}Q^2$ 對應於 $\frac{1}{2}kx^2$ 也即是位能。在電感應中之磁能為 $\frac{1}{2}LI^2$ 對應於 $\frac{1}{2}mv^2$，也即是動能。此一系統之總能量為 $\frac{1}{2}\frac{1}{C}Q^2+\frac{1}{2}LI^2$ 守恒，与在簡諧運動中之總能量 $\frac{1}{2}mv^2+\frac{1}{2}kx^2$ 守恒相似。

電振盪	力振盪
電能 $=\frac{1}{2}\frac{1}{C}Q^2$	位能 $=\frac{1}{2}kx^2$
磁能 $=\frac{1}{2}LI^2$	動能 $=\frac{1}{2}mv^2$
在振盪過程中 電能 ↔ 磁能	在振盪過程中 位能 ↔ 動能
振盪頻率 $=\frac{1}{2\pi}\frac{1}{\sqrt{LC}}$	振盪頻率 $=\frac{1}{2\pi}\frac{1}{\sqrt{m/k}}$

例.



$t=0 \quad Q=Q_0 \;,\; I=0$

$$\text{則}\quad Q(t)=Q_0\cos\omega t \qquad \omega=\sqrt{\frac{1}{LC}} \tag{6}$$

$$I(t)=-Q_0\omega\sin\omega t \tag{7}$$

$$u_E=\frac{1}{2C}Q^2=\frac{Q_0^2}{2C}\cos^2\omega t \tag{8}$$

$$u_M=\frac{1}{2}LI^2=\frac{1}{2}LQ_0^2\omega^2\sin^2\omega t=\frac{Q_0^2}{2C}\sin^2\omega t \tag{9}$$

所以此一系統之總能量 $=\frac{Q_0^2}{2C}$ 守恒

當 $V_0=0$ 則第(3)式可寫成

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=0 \tag{10}$$

此一公式對應於減幅振盪.

$$\frac{d^2Q}{dt^2}+2\frac{R}{2L}\frac{dQ}{dt}+\frac{1}{LC}Q=0 \tag{11}$$

令 $Q(t)=e^{-\frac{R}{2L}t}u(t)$，代入上式得

$$\frac{d^2u(t)}{dt^2}+\left[\frac{1}{LC}-\left(\frac{R}{2L}\right)^2\right]u(t)=0 \tag{12}$$

(i)當 $\frac{1}{LC}-\left(\frac{R}{2L}\right)^2>0$ 時

$$u(t)=A\sin(\omega' t+\alpha) \qquad \omega'=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2} \tag{13}$$

A 与 α 則取決於最初的情況

(ii) 當 $\frac{1}{LC} - (\frac{R}{2L})^2 < 0$ 時

$$u(t) = Ae^{\alpha t} + Be^{-\alpha t} , \quad \alpha = \sqrt{(\frac{R}{2L})^2 - (\frac{1}{LC})} \tag{14}$$

A. B 也是由最初情況所決定

(iii) 當 $\frac{1}{LC} - (\frac{R}{2L})^2 = 0$ 時

$$\frac{d^2u}{dt^2} = 0 \tag{15}$$

也即是 $u(t) = A + Bt$, A 与 B 取決於最初的情況 (16)

簡單的交流電路 $V = V_0 \sin \omega t$

將上式代入第(3)式得

$$V_0 \sin \omega t = L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q \tag{17}$$

此一公式的形式与強迫振盪相似。

$Q = Q_r + Q_s$

Q_r satisfy the homogeneous equation

transient

將上式對 t 微分得

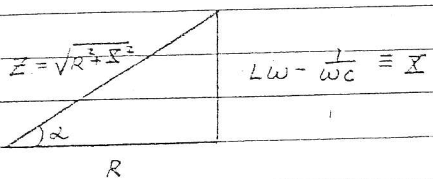
$$L \frac{dI}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \omega V_0 \cos \omega t \tag{18}$$

$$I = I_0 \sin(\omega t - \alpha) \tag{19}$$

$\begin{cases} \text{Satisfy the equation} \\ \text{Satisfied ~~the~~ initial condition} \end{cases}$

I_0. α 滿足以下之關係式

Phasor



$$I_0 = \frac{V_0}{Z} \tag{20}$$

而 $Q = -\frac{I_0}{\omega} \cos(\omega t - \alpha)$ (21)

我們定義

$$V_R = RI , \quad V_L = L \frac{dI}{dt} , \quad V_C = \frac{1}{C} Q . \tag{22}$$

則　$V = V_0 \sin\omega t$

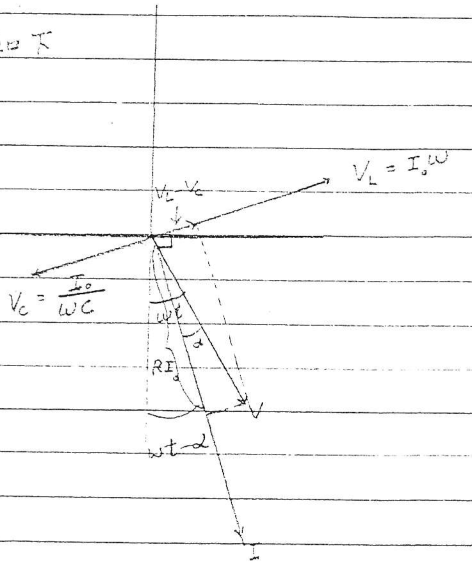
$V_R = RI_0 \sin(\omega t - \alpha)$

$V_L = L\,I_0\,\omega \cos(\omega t - \alpha)$

$V_C = -\frac{I_0}{\omega C} \cos(\omega t - \alpha)$　　　　$I = I_0 \sin(\omega t - \alpha)$

$\begin{matrix}\sin\omega t\\ \cos\omega t\end{matrix}$ 之係數

可用圖形表示如下



由外加電池傳遞至此一 LRC 電路之功率為

$P = VI = V_0 \sin\omega t\, I_0 \sin(\omega t - \alpha)$

$\quad = I_0 V_0 \cos\alpha \sin^2\omega t - I_0 V_0 \sin\alpha \sin\omega t \cos\omega t$

$\langle p \rangle = \frac{1}{T}\int_0^T P\,dt$

此處　$T = \frac{2\pi}{\omega}$ 是振盪週期

$\langle p \rangle = \frac{1}{2} I_0 V_0 \cos\alpha$

因此當 $\alpha = 0$ 時 $\langle p \rangle = \langle p \rangle_{max}$，此時 $\omega = \frac{1}{\sqrt{LC}}$，$\langle p \rangle_{max} = \frac{1}{2} I_0 V_0$

$= \frac{1}{2} I_0 R^2$

Chapter

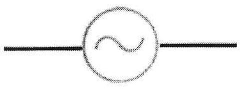
Chapter 33

Alternating-Current Circuits

Alternating-Current Circuits

12.1 AC Sources

In Chapter 10 we learned that changing magnetic flux can induce an emf according to Faraday's law of induction. In particular, if a coil rotates in the presence of a magnetic field, the induced emf varies sinusoidally with time and leads to an alternating current (AC), and provides a source of AC power. The symbol for an AC voltage source is



An example of an AC source is

$$V(t) = V_0 \sin \omega t \tag{12.1.1}$$

where the maximum value V_0 is called the *amplitude*. The voltage varies between V_0 and $-V_0$ since a sine function varies between +1 and −1. A graph of voltage as a function of time is shown in Figure 12.1.1.

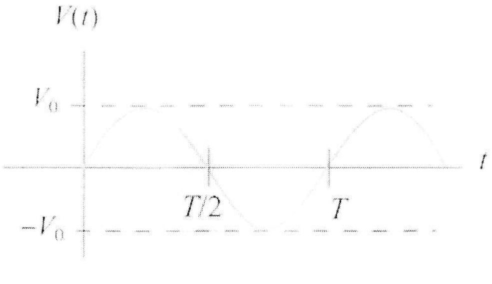


Figure 12.1.1 Sinusoidal voltage source

The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t' = t + T$ where T is the *period*. The *frequency*, f, defined as $f = 1/T$, has the unit of inverse seconds (s^{-1}), or hertz (Hz). The angular frequency is defined to be $\omega = 2\pi f$.

When a voltage source is connected to an *RLC* circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called driven or forced oscillations.

After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current, written as

$$I(t) = I_0 \sin(\omega t - \phi) \qquad (12.1.2)$$

will oscillate with the same frequency as the voltage source, with an amplitude I_0 and phase ϕ that depends on the driving frequency.

12.2 Simple AC circuits

Before examining the driven *RLC* circuit, let's first consider the simple cases where only one circuit element (a resistor, an inductor or a capacitor) is connected to a sinusoidal voltage source.

12.2.1 Purely Resistive load

Consider a purely resistive circuit with a resistor connected to an AC generator, as shown in Figure 12.2.1. (As we shall see, a purely resistive circuit corresponds to infinite capacitance $C = \infty$ and zero inductance $L = 0$.)

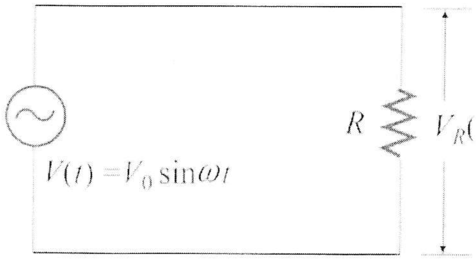


Figure 12.2.1 A purely resistive circuit

Applying Kirchhoff's loop rule yields

$$V(t) - V_R(t) = V(t) - I_R(t)R = 0 \qquad (12.2.1)$$

where $V_R(t) = I_R(t)R$ is the instantaneous voltage drop across the resistor. The instantaneous current in the resistor is given by

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_{R0}\sin\omega t}{R} = I_{R0}\sin\omega t \qquad (12.2.2)$$

where $V_{R0} = V_0$, and $I_{R0} = V_{R0}/R$ is the maximum current. Comparing Eq. (12.2.2) with Eq. (12.1.2), we find $\phi = 0$, which means that $I_R(t)$ and $V_R(t)$ are in phase with each other, meaning that they reach their maximum or minimum values at the same time. The time dependence of the current and the voltage across the resistor is depicted in Figure 12.2.2(a).

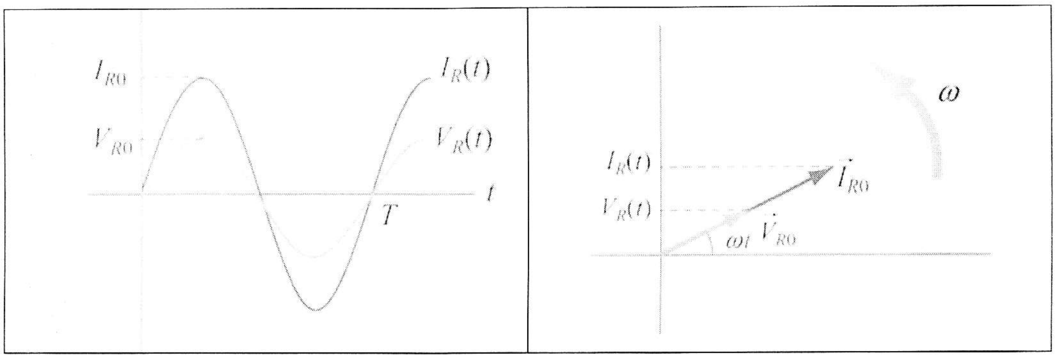


Figure 12.2.2 (a) Time dependence of $I_R(t)$ and $V_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $I_R(t)$ and $V_R(t)$ can also be represented with a phasor diagram, as shown in Figure 12.2.2(b). A phasor is a rotating vector having the following properties:

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.

(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

We shall denote a phasor with an arrow above it. The phasor $\vec{V}_{R0}$ has a constant magnitude of V_{R0}. Its projection along the vertical direction is $V_{R0}\sin\omega t$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\vec{I}_{R0}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period can be obtained as:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\; dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T}\; dt = 0 \qquad (12.2.3)$$

This average vanishes because

$$\langle \sin\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\; dt = 0 \qquad (12.2.4)$$

Similarly, one may find the following relations useful when averaging over one period:

$$\begin{aligned}\langle\cos\omega t\rangle &= \frac{1}{T}\int_0^T \cos\omega t\; dt = 0 \\ \langle\sin\omega t\cos\omega t\rangle &= \frac{1}{T}\int_0^T \sin\omega t\cos\omega t\; dt = 0 \\ \langle\sin^2\omega t\rangle &= \frac{1}{T}\int_0^T \sin^2\omega t\; dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} \\ \langle\cos^2\omega t\rangle &= \frac{1}{T}\int_0^T \cos^2\omega t\; dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}\end{aligned} \qquad (12.2.5)$$

From the above, we see that the average of the square of the current is non-vanishing:

$$\langle I_R^2(t)\rangle = \frac{1}{T}\int_0^T I_R^2(t)dt = \frac{1}{T}\int_0^T I_{R0}^2 \sin^2\omega t\; dt = I_{R0}^2\frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}I_{R0}^2 \qquad (12.2.6)$$

It is convenient to define the root-mean-square (rms) current as

$$I_{\text{rms}} = \sqrt{\langle I_R^2(t)\rangle} = \frac{I_{R0}}{\sqrt{2}} \qquad (12.2.7)$$

In a similar manner, the rms voltage can be defined as

$$V_{\text{rms}} = \sqrt{\langle V_R^2(t)\rangle} = \frac{V_{R0}}{\sqrt{2}} \qquad (12.2.8)$$

The rms voltage supplied to the domestic wall outlets in the United States is $V_{\text{rms}} = 120\text{ V}$ at a frequency $f = 60\text{ Hz}$.

The power dissipated in the resistor is

$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R \qquad (12.2.9)$$

from which the average over one period is obtained as:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{R0}^2R = I_{\text{rms}}^2R = I_{\text{rms}}V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} \qquad (12.2.10)$$

12.2.2 Purely Inductive Load

Consider now a purely inductive circuit with an inductor connected to an AC generator, as shown in Figure 12.2.3.

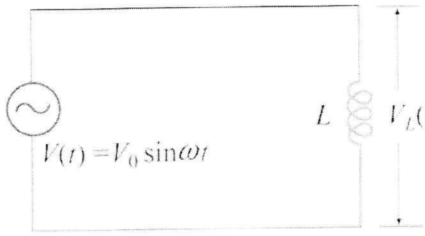


Figure 12.2.3 A purely inductive circuit

As we shall see below, a purely inductive circuit corresponds to infinite capacitance $C=\infty$ and zero resistance $R=0$. Applying the modified Kirchhoff's rule for inductors, the circuit equation reads

$$V(t)-V_L(t)=V(t)-L\frac{dI_L}{dt}=0 \qquad (12.2.11)$$

which implies

$$\frac{dI_L}{dt}=\frac{V(t)}{L}=\frac{V_{L0}}{L}\sin\omega t \qquad (12.2.12)$$

where $V_{L0}=V_0$. Integrating over the above equation, we find

$$I_L(t)=\int dI_L=\frac{V_{L0}}{L}\int\sin\omega t\; dt=-\left(\frac{V_{L0}}{\omega L}\right)\cos\omega t=\left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t-\frac{\pi}{2}\right) \qquad (12.2.13)$$

where we have used the trigonometric identity

$$-\cos\omega t=\sin\left(\omega t-\frac{\pi}{2}\right) \qquad (12.2.14)$$

for rewriting the last expression. Comparing Eq. (12.2.14) with Eq. (12.1.2), we see that the amplitude of the current through the inductor is

$$I_{L0}=\frac{V_{L0}}{\omega L}=\frac{V_{L0}}{X_L} \qquad (12.2.15)$$

where

$$X_L=\omega L \qquad (12.2.16)$$

is called the *inductive reactance*. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

By comparing Eq. (12.2.14) to Eq. (12.1.2), we also find the phase constant to be

$$\phi = +\frac{\pi}{2} \tag{12.2.17}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.4 below.

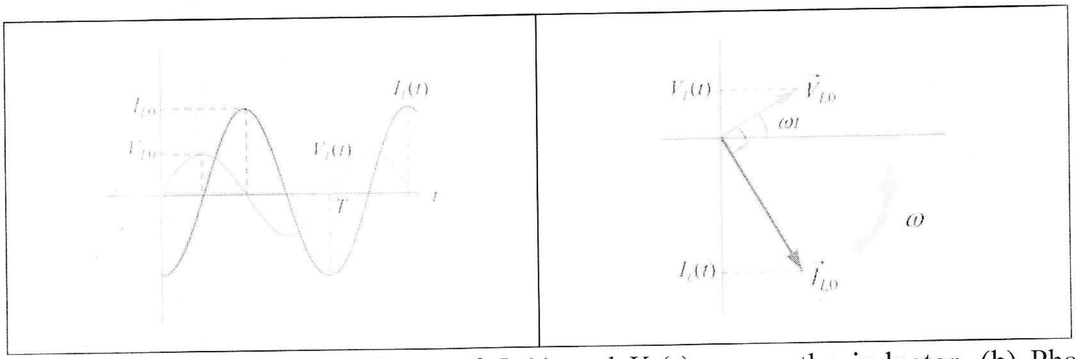


Figure 12.2.4 (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\phi = \pi/2$; it reaches its maximum value after $V_L(t)$ does by one quarter of a cycle. Thus, we say that

The current lags voltage by $\pi/2$ in a purely inductive circuit

12.2.3 Purely Capacitive Load

In the purely capacitive case, both resistance R and inductance L are zero. The circuit diagram is shown in Figure 12.2.5.

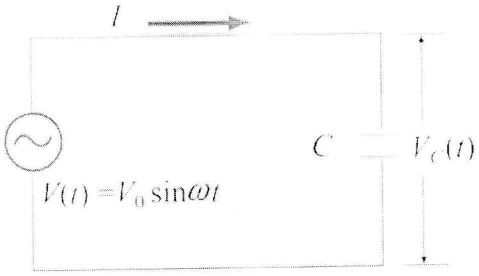


Figure 12.2.5 A purely capacitive circuit

Again, Kirchhoff's voltage rule implies

$$V(t)-V_C(t)=V(t)-\frac{Q(t)}{C}=0 \qquad (12.2.18)$$

which yields

$$Q(t)=CV(t)=CV_C(t)=CV_{C0}\sin\omega t \qquad (12.2.19)$$

where $V_{C0}=V_0$. On the other hand, the current is

$$I_C(t)=+\frac{dQ}{dt}=\omega CV_{C0}\cos\omega t=\omega CV_{C0}\sin\left(\omega t+\frac{\pi}{2}\right) \qquad (12.2.20)$$

where we have used the trigonometric identity

$$\cos\omega t=\sin\left(\omega t+\frac{\pi}{2}\right) \qquad (12.2.21)$$

The above equation indicates that the maximum value of the current is

$$I_{C0}=\omega CV_{C0}=\frac{V_{C0}}{X_C} \qquad (12.2.22)$$

where

$$X_C=\frac{1}{\omega C} \qquad (12.2.23)$$

is called the *capacitance reactance*. It also has SI units of ohms and represents the effective resistance for a purely capacitive circuit. Note that X_C is inversely proportional to both C and ω, and diverges as ω approaches zero.

By comparing Eq. (12.2.21) to Eq. (12.1.2), the phase constant is given by

$$\phi=-\frac{\pi}{2} \qquad (12.2.24)$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.6 below.

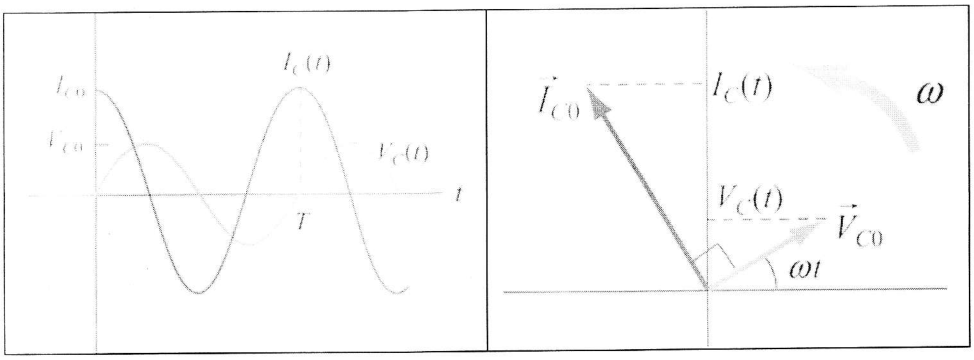


Figure 12.2.6 (a) Time dependence of $I_C(t)$ and $V_C(t)$ across the capacitor. (b) Phasor diagram for the capacitive circuit.

Notice that at $t = 0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\phi = \pi / 2$). Thus, we say that

The current leads the voltage by $\pi/2$ in a capacitive circuit

12.3 The *RLC* Series Circuit

Consider now the driven series *RLC* circuit shown in Figure 12.3.1.

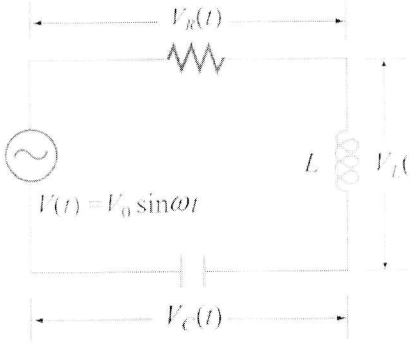


Figure 12.3.1 Driven series *RLC* Circuit

Applying Kirchhoff's loop rule, we obtain

$$V(t) - V_R(t) - V_L(t) - V_C(t) = V(t) - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0 \qquad (12.3.1)$$

which leads to the following differential equation:

$$L\frac{dI}{dt}+IR+\frac{Q}{C}=V_0\sin\omega t \tag{12.3.2}$$

Assuming that the capacitor is initially uncharged so that $I=+dQ/dt$ is proportional to the *increase* of charge in the capacitor, the above equation can be rewritten as

$$\boxed{L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V_0\sin\omega t} \tag{12.3.3}$$

One possible solution to Eq. (12.3.3) is

$$Q(t)=Q_0\cos(\omega t-\phi) \tag{12.3.4}$$

where the amplitude and the phase are, respectively,

$$\begin{aligned}Q_0&=\frac{V_0/L}{\sqrt{(R\omega/L)^2+(\omega^2-1/LC)^2}}=\frac{V_0}{\omega\sqrt{R^2+(\omega L-1/\omega C)^2}}\\&=\frac{V_0}{\omega\sqrt{R^2+(X_L-X_C)^2}}\end{aligned} \tag{12.3.5}$$

and

$$\tan\phi=\frac{1}{R}\left(\omega L-\frac{1}{\omega C}\right)=\frac{X_L-X_C}{R} \tag{12.3.6}$$

The corresponding current is

$$I(t)=+\frac{dQ}{dt}=I_0\sin(\omega t-\phi) \tag{12.3.7}$$

with an amplitude

$$I_0=-Q_0\omega=-\frac{V_0}{\sqrt{R^2+(X_L-X_C)^2}} \tag{12.3.8}$$

Notice that the current has the same amplitude and phase at all points in the series *RLC* circuit. On the other hand, the instantaneous voltage across each of the three circuit elements *R*, *L* and *C* has a different amplitude and phase relationship with the current, as can be seen from the phasor diagrams shown in Figure 12.3.2.

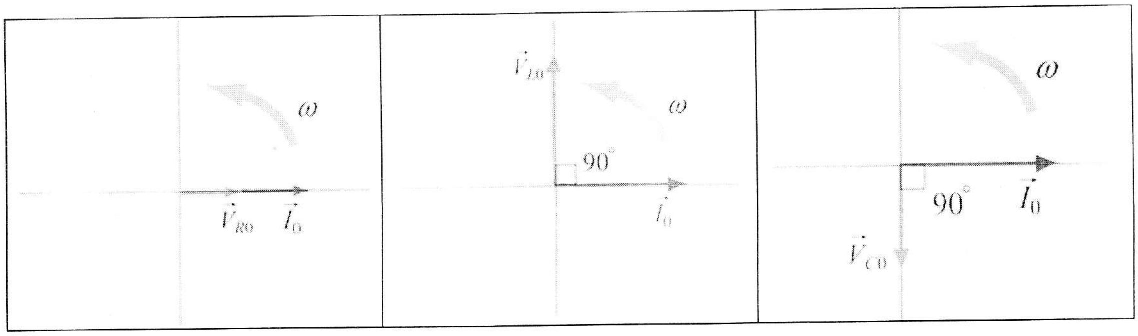


Figure 12.3.2 Phasor diagrams for the relationships between current and voltage in (a) the resistor, (b) the inductor, and (c) the capacitor, of a series *RLC* circuit.

From Figure 12.3.2, the instantaneous voltages can be obtained as:

$$\begin{aligned} V_R(t) &= I_0 R \sin \omega t = V_{R0} \sin \omega t \\ V_L(t) &= I_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_{L0} \cos \omega t \\ V_C(t) &= I_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -V_{C0} \cos \omega t \end{aligned} \qquad (12.3.9)$$

where

$$V_{R0} = I_0 R, \qquad V_{L0} = I_0 X_L, \qquad V_{C0} = I_0 X_C \qquad (12.3.10)$$

are the amplitudes of the voltages across the circuit elements. The sum of all three voltages is equal to the instantaneous voltage supplied by the AC source:

$$V(t) = V_R(t) + V_L(t) + V_C(t) \qquad (12.3.11)$$

Using the phasor representation, the above expression can also be written as

$$\vec{V}_0 = \vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0} \qquad (12.3.12)$$

as shown in Figure 12.3.3 (a). Again we see that current phasor $\vec{I}_0$ leads the capacitive voltage phasor $\vec{V}_{C0}$ by $\pi/2$ but lags the inductive voltage phasor $\vec{V}_{L0}$ by $\pi/2$. The three voltage phasors rotate counterclockwise as time passes, with their relative positions fixed.

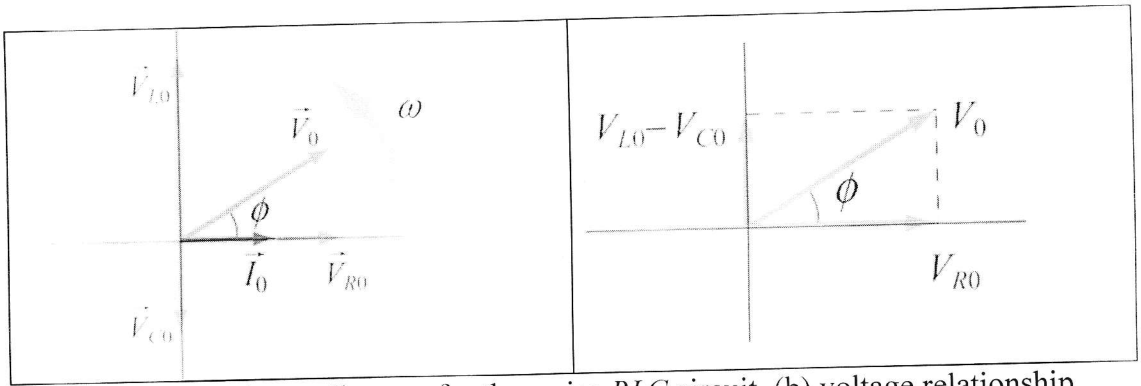


Figure 12.3.3 (a) Phasor diagram for the series *RLC* circuit. (b) voltage relationship

The relationship between different voltage amplitudes is depicted in Figure 12.3.3(b). From the Figure, we see that

$$\begin{aligned} V_0 &= |\vec{V}_0| = |\vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0}| = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} \\ &= \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2} \\ &= I_0 \sqrt{R^2 + (X_L - X_C)^2} \end{aligned} \qquad (12.3.13)$$

which leads to the same expression for I_0 as that obtained in Eq. (12.3.7).

It is crucial to note that the maximum amplitude of the AC voltage source V_0 is not equal to the sum of the maximum voltage amplitudes across the three circuit elements:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0} \qquad (12.3.14)$$

This is due to the fact that the voltages are not in phase with one another, and they reach their maxima at different times.

12.3.1 Impedance

We have already seen that the inductive reactance $X_L = \omega L$ and capacitance reactance $X_C = 1/\omega C$ play the role of an effective resistance in the purely inductive and capacitive circuits, respectively. In the series *RLC* circuit, the effective resistance is the *impedance*, defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad (12.3.15)$$

The relationship between Z, X_L and X_C can be represented by the diagram shown in Figure 12.3.4:

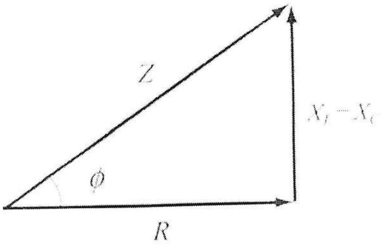


Figure 12.3.4 Diagrammatic representation of the relationship between Z, X_L and X_C.

The impedance also has SI units of ohms. In terms of Z, the current may be rewritten as

$$I(t) = \frac{V_0}{Z}\sin(\omega t - \phi) \qquad (12.3.16)$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C.

Using Eq. (12.3.6) for the phase ϕ and Eq. (12.3.15) for Z, we may readily recover the limits for simple circuit (with only one element). A summary is provided in Table 12.1 below:

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \frac{1}{\omega C}$	$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$	$Z = \sqrt{R^2 + (X_L - X_C)^2}$
purely resistive	R	0	∞	0	0	0	R
purely inductive	0	L	∞	X_L	0	$\pi/2$	X_L
purely capacitive	0	0	C	0	X_C	$-\pi/2$	X_C

Table 12.1 Simple-circuit limits of the series *RLC* circuit

12.3.2 Resonance

Eq. (12.3.15) indicates that the amplitude of the current $I_0 = V_0 / Z$ reaches a maximum when Z is at a minimum. This occurs when $X_L = X_C$, or $\omega L = 1/\omega C$, leading to

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (12.3.17)$$

The phenomenon at which I_0 reaches a maximum is called a resonance, and the frequency ω_0 is called the resonant frequency. At resonance, the impedance becomes $Z = R$, the amplitude of the current is

$$I_0 = \frac{V_0}{R} \tag{12.3.18}$$

and the phase is

$$\phi = 0 \tag{12.3.19}$$

as can be seen from Eq. (12.3.5). The qualitative behavior is illustrated in Figure 12.3.5.

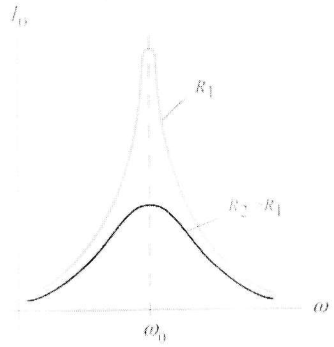


Figure 12.3.5 The amplitude of the current as a function of ω in the driven *RLC* circuit.

12.4 Power in an AC circuit

In the series *RLC* circuit, the instantaneous power delivered by the AC generator is given by

$$\begin{aligned} P(t) = I(t)V(t) &= \frac{V_0}{Z}\sin(\omega t - \phi)\cdot V_0 \sin\omega t = \frac{V_0^2}{Z}\sin(\omega t - \phi)\sin\omega t \\ &= \frac{V_0^2}{Z}\left(\sin^2\omega t\cos\phi - \sin\omega t\cos\omega t\sin\phi\right) \end{aligned} \tag{12.4.1}$$

where we have used the trigonometric identity

$$\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi \tag{12.4.2}$$

The time average of the power is

$$
\begin{aligned}
\langle P(t)\rangle &= \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin^2 \omega t\cos\phi\, dt - \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin\omega t\cos\omega t\sin\phi\, dt \\
&= \frac{V_0^2}{Z}\cos\phi\left\langle \sin^2 \omega t\right\rangle - \frac{V_0^2}{Z}\sin\phi\left\langle \sin\omega t\cos\omega t\right\rangle \\
&= \frac{1}{2}\frac{V_0^2}{Z}\cos\phi
\end{aligned} \tag{12.4.3}
$$

where Eqs. (12.2.5) and (12.2.7) have been used. In terms of the rms quantities, the average power can be rewritten as

$$
\langle P(t)\rangle = \frac{1}{2}\frac{V_0^2}{Z}\cos\phi = \frac{V_{\mathrm{rms}}{}^2}{Z}\cos\phi = I_{\mathrm{rms}}V_{\mathrm{rms}}\cos\phi \tag{12.4.4}
$$

The quantity $\cos\phi$ is called the *power factor*. From Figure 12.3.4, one can readily show that

$$
\cos\phi = \frac{R}{Z} \tag{12.4.5}
$$

Thus, we may rewrite $\langle P(t)\rangle$ as

$$
\langle P(t)\rangle = I_{\mathrm{rms}}V_{\mathrm{rms}}\left(\frac{R}{Z}\right) = I_{\mathrm{rms}}\left(\frac{V_{\mathrm{rms}}}{Z}\right)R = I_{\mathrm{rms}}^2 R \tag{12.4.6}
$$

In Figure 12.4.1, we plot the average power as a function of the driving angular frequency ω.

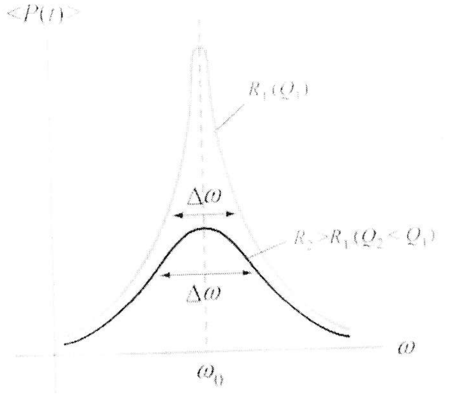


Figure 12.4.1 Average power as a function of frequency in a driven series *RLC* circuit.

We see that $\langle P(t)\rangle$ attains the maximum when $\cos\phi = 1$, or $Z = R$, which is the resonance condition. At resonance, we have

$$\langle P\rangle_{\max} = I_{\mathrm{rms}}V_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}^2}{R} \qquad (12.4.7)$$

12.4.1 Width of the Peak

The peak has a line width. One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the driving angular frequency such that the power is equal to half its maximum power at resonance. This is called *full width at half maximum*, as illustrated in Figure 12.4.2. The width $\Delta\omega$ increases with resistance *R*.

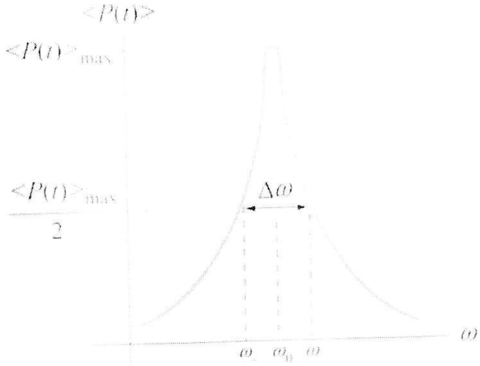


Figure 12.4.2 Width of the peak

To find $\Delta\omega$, it is instructive to first rewrite the average power $\langle P(t)\rangle$ as

$$\langle P(t)\rangle = \frac{1}{2}\frac{V_0^2 R}{R^2 + (\omega L - 1/\omega C)^2} = \frac{1}{2}\frac{V_0^2 R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2} \qquad (12.4.8)$$

with $\langle P(t)\rangle_{\max} = V_0^2/2R$. The condition for finding $\omega_\pm$ is

$$\frac{1}{2}\langle P(t)\rangle_{\max} = \langle P(t)\rangle\Big|_{\omega_\pm} \quad \Rightarrow \quad \frac{V_0^2}{4R} = \frac{1}{2}\frac{V_0^2 R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2}\Bigg|_{\omega_\pm} \qquad (12.4.9)$$

which gives

$$(\omega^2 - \omega_0^2)^2 = \left(\frac{R\omega}{L}\right)^2 \qquad (12.4.10)$$

Taking square roots yields two solutions, which we analyze separately.

case 1: Taking the positive root leads to

$$\omega_+^2-\omega_0^2=+\frac{R\omega_+}{L} \qquad (12.4.11)$$

Solving the quadratic equation, the solution with positive root is

$$\omega_+=\frac{R}{2L}+\sqrt{\left(\frac{R}{4L}\right)^2+\omega_0{}^2} \qquad (12.4.12)$$

Case 2: Taking the negative root of Eq. (12.4.10) gives

$$\omega_-^2-\omega_0^2=-\frac{R\omega_-}{L} \qquad (12.4.13)$$

The solution to this quadratic equation with positive root is

$$\omega_-=-\frac{R}{2L}+\sqrt{\left(\frac{R}{4L}\right)^2+\omega_0{}^2} \qquad (12.4.14)$$

The width at half maximum is then

$$\Delta\omega=\omega_+-\omega_-=\frac{R}{L} \qquad (12.4.15)$$

Once the width $\Delta\omega$ is known, the quality factor Q (not to be confused with charge) can be obtained as

$$\boxed{Q=\frac{\omega_0}{\Delta\omega}=\frac{\omega_0 L}{R}} \qquad (12.4.16)$$

Comparing the above equation with Eq. (11.8.17), we see that both expressions agree with each other in the limit where the resistance is small, and $\omega'=\sqrt{\omega_0^2-(R/2L)^2}\approx\omega_0$.

12.5 Transformer

A transformer is a device used to increase or decrease the AC voltage in a circuit. A typical device consists of two coils of wire, a primary and a secondary, wound around an iron core, as illustrated in Figure 12.5.1. The primary coil, with N_1 turns, is connected to alternating voltage source $V_1(t)$. The secondary coil has N_2 turns and is connected to a "load resistance" R_2. The way transformers operate is based on the principle that an alternating current in the primary coil will induce an alternating emf on the secondary coil due to their mutual inductance.

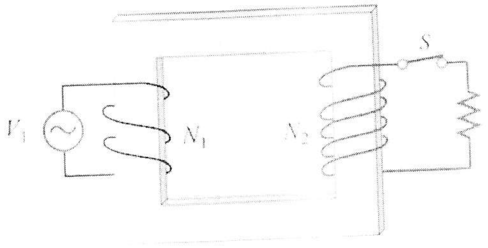


Figure 12.5.1 A transformer

In the primary circuit, neglecting the small resistance in the coil, Faraday's law of induction implies

$$V_1 = -N_1 \frac{d\Phi_B}{dt} \qquad (12.5.1)$$

where Φ_B is the magnetic flux through one turn of the primary coil. The iron core, which extends from the primary to the secondary coils, serves to increase the magnetic field produced by the current in the primary coil and ensure that nearly all the magnetic flux through the primary coil also passes through each turn of the secondary coil. Thus, the voltage (or induced emf) across the secondary coil is

$$V_2 = -N_2 \frac{d\Phi_B}{dt} \qquad (12.5.2)$$

In the case of an ideal transformer, power loss due to Joule heating can be ignored, so that the power supplied by the primary coil is completely transferred to the secondary coil:

$$I_1 V_1 = I_2 V_2 \qquad (12.5.3)$$

In addition, no magnetic flux leaks out from the iron core, and the flux Φ_B through each turn is the same in both the primary and the secondary coils. Combining the two expressions, we are lead to the transformer equation:

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}} \qquad (12.5.4)$$

By combining the two equations above, the transformation of currents in the two coils may be obtained as:

$$I_1 = \left(\frac{V_2}{V_1}\right) I_2 = \left(\frac{N_2}{N_1}\right) I_2 \qquad (12.5.5)$$

Thus, we see that the ratio of the output voltage to the input voltage is determined by the *turn ratio* N_2 / N_1. If $N_2 > N_1$, then $V_2 > V_1$, which means that the output voltage in the second coil is greater than the input voltage in the primary coil. A transformer with $N_2 > N_1$ is called a *step-up* transformer. On the other hand, if $N_2 < N_1$, then $V_2 < V_1$, and the output voltage is smaller than the input. A transformer with $N_2 < N_1$ is called a *step-down* transformer.

12.6 Parallel *RLC* Circuit

Consider the parallel *RLC* circuit illustrated in Figure 12.6.1. The AC voltage source is $V(t) = V_0 \sin \omega t$.

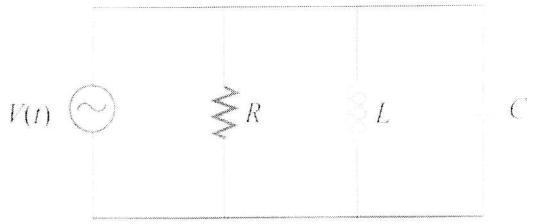


Figure 12.6.1 Parallel *RLC* circuit.

Unlike the series *RLC* circuit, the instantaneous voltages across all three circuit elements *R*, *L*, and *C* are the same, and each voltage is in phase with the current through the resistor. However, the currents through each element will be different.

In analyzing this circuit, we make use of the results discussed in Sections 12.2 – 12.4. The current in the resistor is

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin \omega t = I_{R0} \sin \omega t \qquad (12.6.1)$$

where $I_{R0} = V_0 / R$. The voltage across the inductor is

$$V_L(t) = V(t) = V_0 \sin \omega t = L\frac{dI_L}{dt} \qquad (12.6.2)$$

which gives

$$I_L(t) = \int_0^t \frac{V_0}{L}\sin \omega t' dt' = -\frac{V_0}{\omega L}\cos \omega t = \frac{V_0}{X_L}\sin\left(\omega t - \frac{\pi}{2}\right) = I_{L0}\sin\left(\omega t - \frac{\pi}{2}\right) \qquad (12.6.3)$$

where $I_{L0} = V_0 / X_L$ and $X_L = \omega L$ is the inductive reactance.

Similarly, the voltage across the capacitor is $V_C(t) = V_0 \sin \omega t = Q(t)/C$, which implies

$$I_C(t) = \frac{dQ}{dt} = \omega C V_0 \cos \omega t = \frac{V_0}{X_C} \sin\left(\omega t + \frac{\pi}{2}\right) = I_{C0} \sin\left(\omega t + \frac{\pi}{2}\right) \tag{12.6.4}$$

where $I_{C0} = V_0 / X_C$ and $X_C = 1/\omega C$ is the capacitive reactance.

Using Kirchhoff's junction rule, the total current in the circuit is simply the sum of all three currents.

$$\begin{aligned} I(t) &= I_R(t) + I_L(t) + I_C(t) \\ &= I_{R0} \sin \omega t + I_{L0} \sin\left(\omega t - \frac{\pi}{2}\right) + I_{C0} \sin\left(\omega t + \frac{\pi}{2}\right) \end{aligned} \tag{12.6.5}$$

The currents can be represented with the phasor diagram shown in Figure 12.6.2.

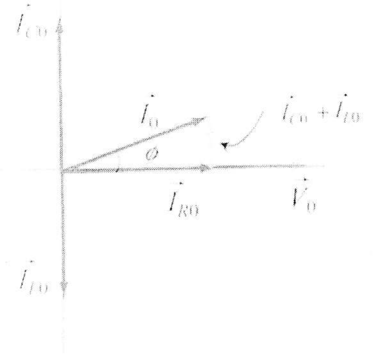


Figure 12.6.2 Phasor diagram for the parallel *RLC* circuit

From the phasor diagram, we see that

$$\vec{I}_0 = \vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0} \tag{12.6.6}$$

and the maximum amplitude of the total current, I_0, can be obtained as

$$\begin{aligned} I_0 &= |\vec{I}_0| = |\vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0}| = \sqrt{I_{R0}^2 + (I_{C0} - I_{L0})^2} \\ &= V_0 \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = V_0 \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \end{aligned} \tag{12.6.7}$$

Note however, since $I_R(t)$, $I_L(t)$ and $I_C(t)$ are not in phase with one another, I_0 is not equal to the sum of the maximum amplitudes of the three currents:

$$I_0 \neq I_{R0} + I_{L0} + I_{C0} \qquad (12.6.8)$$

With $I_0 = V_0 / Z$, the (inverse) impedance of the circuit is given by

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \qquad (12.6.9)$$

The relationship between Z, R, X_L and X_C is shown in Figure 12.6.3.

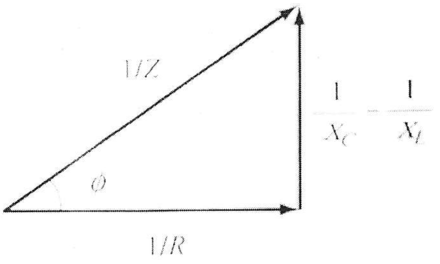


Figure 12.6.3 Relationship between Z, R, X_L and X_C in a parallel *RLC* circuit.

From the figure or the phasor diagram shown in Figure 12.6.2, we see that the phase can be obtained as

$$\tan\phi = \left(\frac{I_{C0} - I_{L0}}{I_{R0}}\right) = \frac{\frac{V_0}{X_C} - \frac{V_0}{X_L}}{\frac{V_0}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right) = R\left(\omega C - \frac{1}{\omega L}\right) \qquad (12.6.10)$$

The resonance condition for the parallel *RLC* circuit is given by $\phi = 0$, which implies

$$\frac{1}{X_C} = \frac{1}{X_L} \qquad (12.6.11)$$

The resonant frequency is

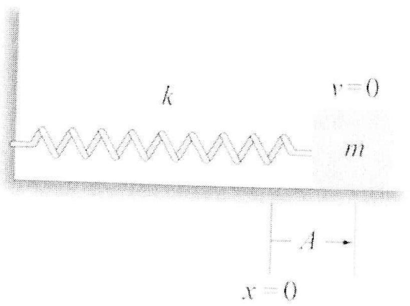
$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (12.6.12)$$

which is the same as for the series *RLC* circuit. From Eq. (12.6.9), we readily see that $1/Z$ is minimum (or Z is maximum) at resonance. The current in the inductor exactly

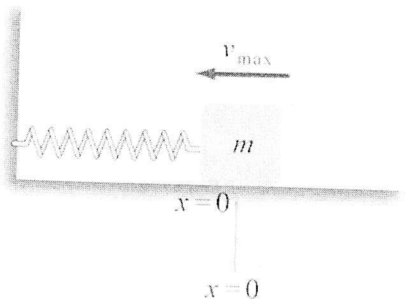
LC Circuits
Mass on a Spring:
Simple Harmonic Motion
(Demonstration)

Mass on a Spring

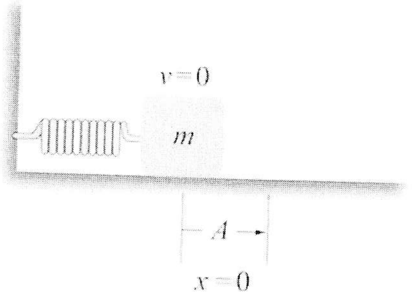
(1)



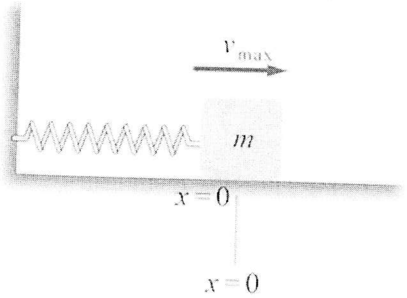
(2)



(3)



(4)



What is Motion?

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Simple Harmonic Motion

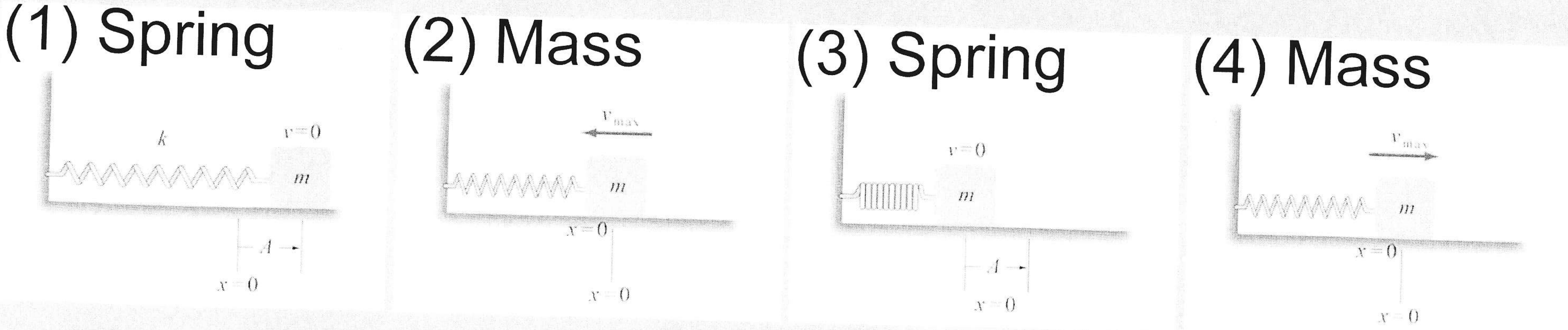
$$x(t) = x_0 \cos(\omega_0 t + \phi)$$

x_0: Amplitude of Motion

ϕ: Phase (time offset)

$$\omega_0 = \sqrt{\frac{k}{m}} = \text{Angular frequency}$$

Mass on a Spring: Energy



(1) Spring (2) Mass (3) Spring (4) Mass

$$x(t) = x_0 \cos(\omega_0 t + \phi) \qquad x'(t) = -\omega_0 x_0 \sin(\omega_0 t + \phi)$$

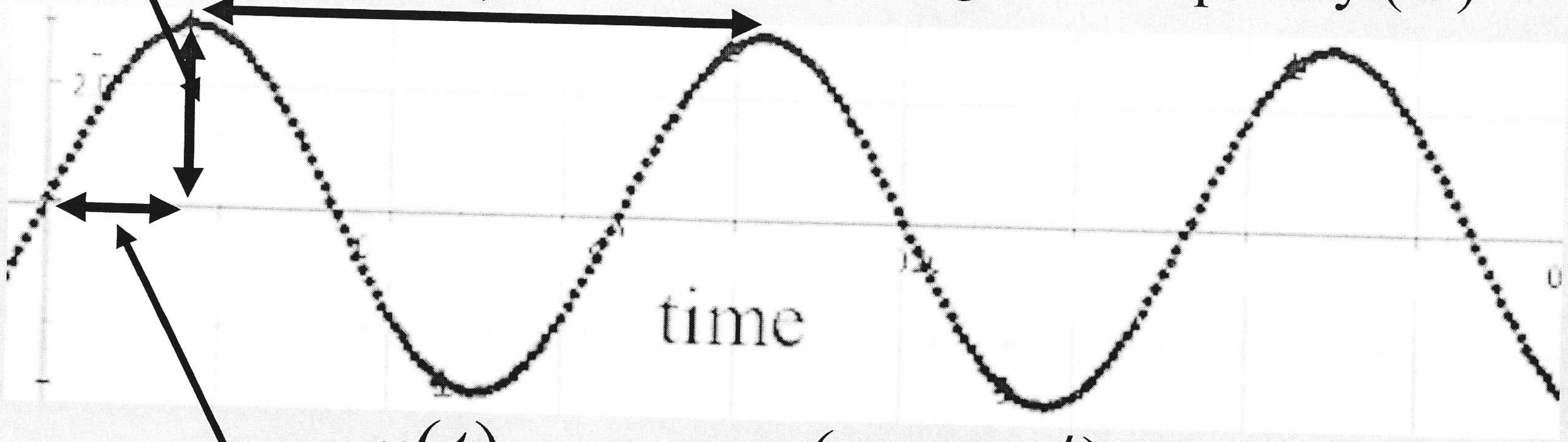
Energy has 2 parts: (Mass) Kinetic and (Spring) Potential

$$K = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 = \frac{1}{2} k x_0{}^2 \sin^2(\omega_0 t + \phi)$$

$$U_s = \frac{1}{2} k x^2 = \frac{1}{2} k x_0{}^2 \cos^2(\omega_0 t + \phi)$$

Energy sloshes back and forth

Simple Harmonic Motion



$$\text{Period}\ (T) = \frac{1}{\text{frequency}\ (f)}$$

$$= \frac{2\pi}{\text{angular frequency}\ (\omega)}$$

Amplitude (x_0)

$$x(t) = x_0 \cos(\omega_0 t - \phi)$$

$$\text{Phase Shift}\ (\varphi) = \frac{\pi}{2}$$

Electronic Analog:
LC Circuits

Analog: LC Circuit

Mass doesn't like to accelerate

Kinetic energy associated with motion

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}; \quad E = \frac{1}{2}mv^2$$

Inductor doesn't like to have current change

Energy associated with current

$$\varepsilon = -L\frac{dI}{dt} = -L\frac{d^2q}{dt^2}; \quad E = \frac{1}{2}LI^2$$

Analog: LC Circuit

Spring doesn’t like to be compressed/extended

Potential energy associated with compression

$$F = -kx; \quad E = \frac{1}{2}kx^2$$

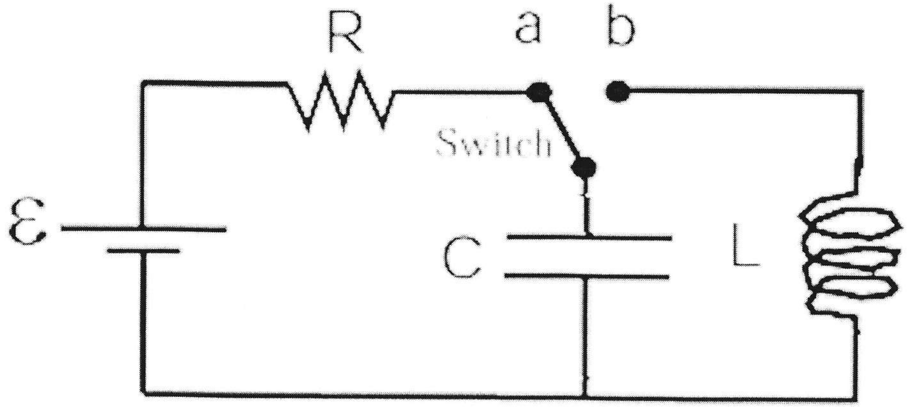
Capacitor doesn’t like to be charged (+ or -)

Energy associated with stored charge

$$\varepsilon = \frac{1}{C}q; \quad E = \frac{1}{2}\frac{1}{C}q^2$$

$$\boxed{F \rightarrow \varepsilon;\; x \rightarrow q;\; v \rightarrow I;\; m \rightarrow L;\; k \rightarrow C^{-1}}$$

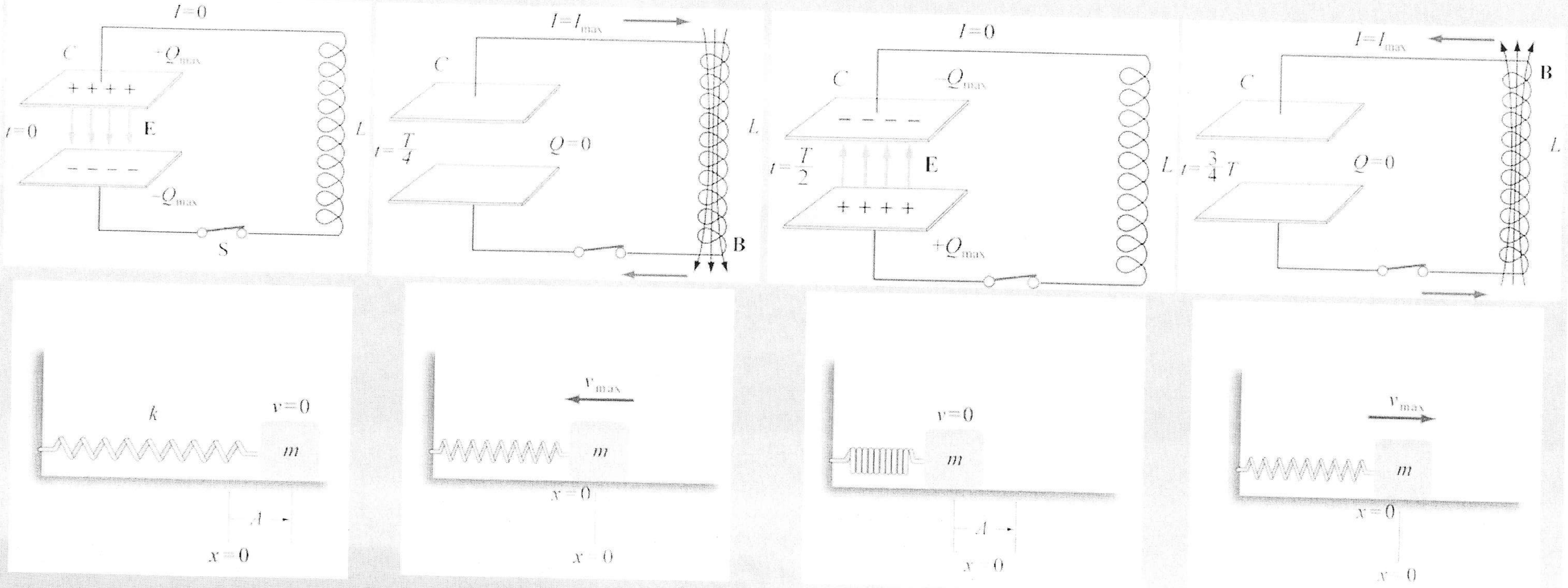
LC Circuit



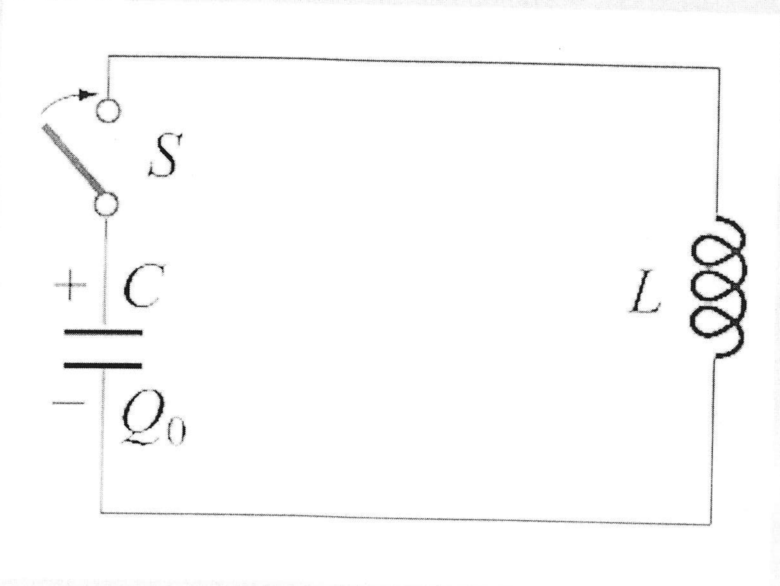
1. Set up the circuit above with capacitor, inductor, resistor, and battery.
2. Let the capacitor become fully charged.
3. Throw the switch from a to b
4. What happens?

LC Circuit

It undergoes simple harmonic motion, just like a mass on a spring, with trade-off between charge on capacitor (Spring) and current in inductor (Mass)



LC Circuit



$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \quad ; \quad I = -\frac{dQ}{dt}$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

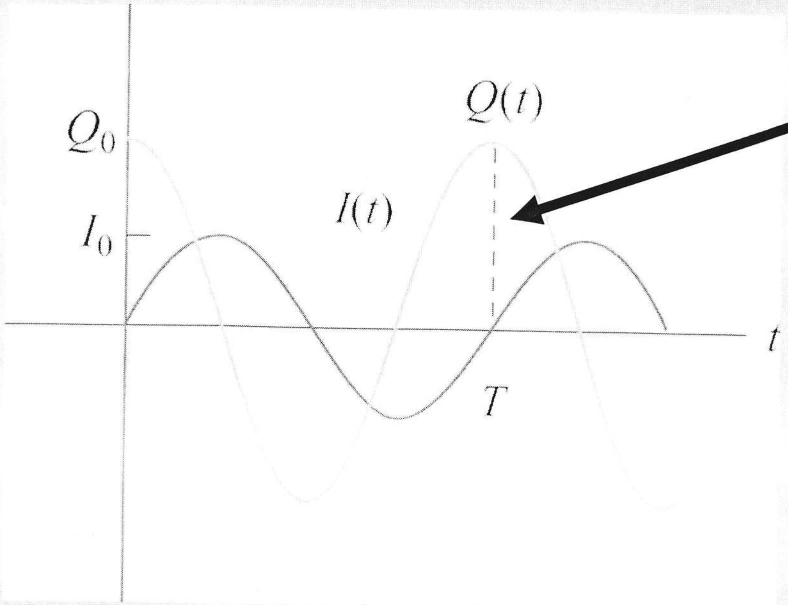
Simple Harmonic Motion

$$Q(t) = Q_0 \cos(\omega_0 t + \phi) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

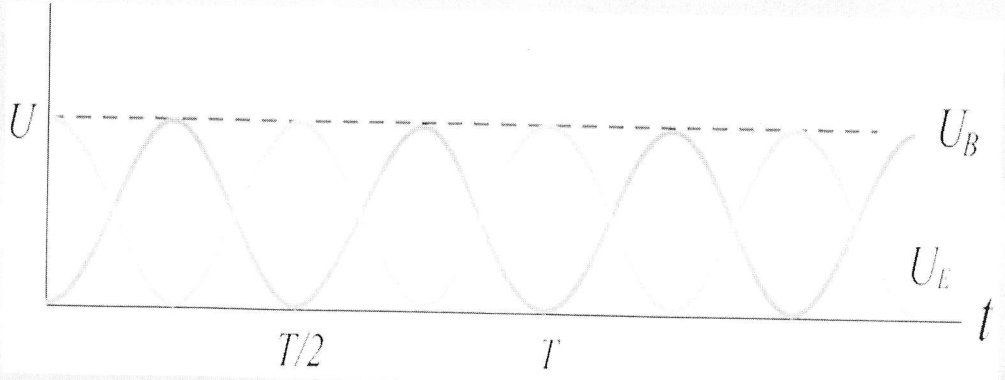
Q_0: Amplitude of Charge Oscillation

ϕ: Phase (time offset)

LC Oscillations: Energy



Notice relative phases

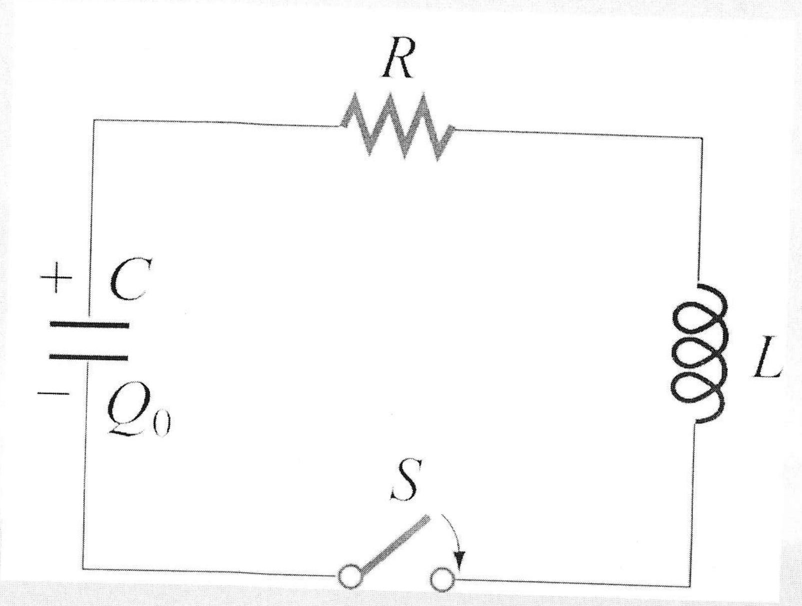


$$U_E = \frac{Q^2}{2C} = \left(\frac{{Q_0}^2}{2C}\right)\cos^2\omega_0 t \qquad U_B = \frac{1}{2}LI^2 = \frac{1}{2}L{I_0}^2\sin^2\omega_0 t = \left(\frac{{Q_0}^2}{2C}\right)\sin^2\omega_0 t$$

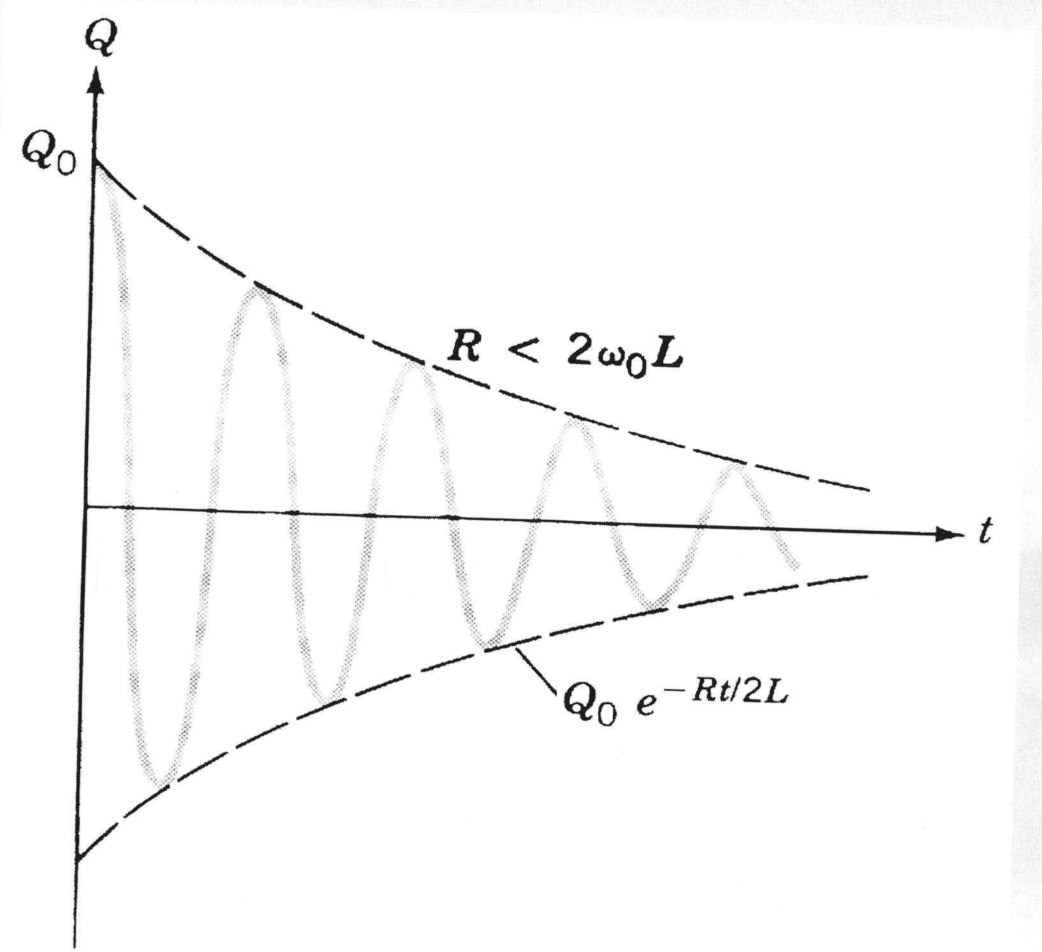
$$U = U_E + U_B = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{{Q_0}^2}{2C}$$

Total energy is conserved !!

Damped LC Oscillations



Resistor dissipates energy and system rings down over time



Also, frequency decreases: $\omega' = \sqrt{\omega_0{}^2 - \left(\frac{R\square}{2L}\right)^2}$