Chapter 34

Maxwell equation

Potential Formalism

Gauge ~~T~~Transform

Gauge Invariance

Wave Equation.

Next Topics

EM wave

Maxwell Equation, (First Visit)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (1) \qquad \nabla \cdot \vec{B} = 0 \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3) \qquad \nabla \times \vec{B} = \mu_0 \vec{J} \quad (4)$$

$\vec{E}, \vec{B}, \rho, \vec{J}$ are function of $(\vec{r}, t)$

Faraday's Law: $\vec{E}, \vec{B}$ are related

Charge Consveration

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (5)$$

$$\underbrace{\nabla \cdot (\nabla \times \vec{B})}_{=0} = \mu_0 \nabla \cdot \vec{J}$$

↳ vector identity

inconsistent with the continuity equation

if $\frac{\partial \rho}{\partial t} = 0$

By symmetry ⇒ add a term $a \frac{\partial \vec{E}}{\partial t}$

a ↓ constant

$$\nabla \times B = \mu_0 \vec{J} + a \frac{\partial \vec{E}}{\partial t}$$

$$\Downarrow$$

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + a \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$a \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right) = a \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = a \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \mu_0 \left[\nabla \cdot \vec{J} + \frac{a}{\epsilon_0 \mu_0} \frac{\partial \rho}{\partial t} \right] = 0$$

$$\Downarrow$$

continuity equation

if

$a = \epsilon_0 \mu_0$

Maxwell displacement current

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad (4)'$$

(1), (2), (3), (4) form the Maxwell equations

Note [$\nabla \cdot (\nabla \times \vec{V}) = 0$
↓
vector identity]

Define $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$
↓
velocity of light
↓
Exercise check the dimensionality
and
the value

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad (4)''$$

In general, the second term is << first term in the above term
↓
Ampere were not be able to measure

Maxwell equations not only is consistent with the continuity equation but includes the continuity equations

(4)' is "derived" from theoretical consideration.

Maxwell equation $\Rightarrow$ EM wave propagates with velocity $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ in vacuum

$$\nabla \cdot \vec{E} = 0 , \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 , \quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{E}) \underset{\text{vector identity}}{=} \nabla(\overbrace{\nabla \cdot \vec{E}}^{=0}) - \nabla^2 \vec{E}$$

$$\begin{aligned} &\parallel \\ &\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) \\ &\parallel \\ &-\frac{\partial}{\partial t} \nabla \times \vec{B} \\ &\parallel \\ &-\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \end{aligned}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t}$$

Exercise

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

Thus the proof.

Eureka!
遇有新發現時的
勝利歡呼
(阿基米德)

Table 15-1

FALSE IN GENERAL (true only for statics)	TRUE ALWAYS
$F = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r^2}$ (Coulomb's law)	$F = q(E + v \times B)$ (Lorentz force) $\rightarrow \nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Gauss' law)
$\nabla \times E = 0$ $E = -\nabla\phi$ $E(1) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\rho(2)e_{12}}{r_{12}^2}\,dV_2$ For conductors, $E = 0$, $\phi =$ constant. $Q = CV$	$\rightarrow \nabla \times E = -\dfrac{\partial B}{\partial t}$ (Faraday's law) $E = -\nabla\phi - \dfrac{\partial A}{\partial t}$ In a conductor, E makes currents.
$c^2\nabla \times B = \dfrac{j}{\epsilon_0}$ (Ampere's law) $B(1) = \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2) \times e_{12}}{r_{12}^2}\,dV_2$	$\rightarrow \nabla \cdot B = 0$ (No magnetic charges) $B = \nabla \times A$ $\rightarrow c^2\nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}$
$\nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$ (Poisson's equation) $\nabla^2 A = -\dfrac{j}{\epsilon_0 c^2}$ with $\nabla \cdot A = 0$	$\nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}$ and $\nabla^2 A - \dfrac{1}{c^2}\dfrac{\partial^2 A}{\partial t^2} = -\dfrac{j}{\epsilon_0 c^2}$ with $c^2\nabla \cdot A + \dfrac{\partial\phi}{\partial t} = 0$
$\phi(1) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(2)}{r_{12}}\,dV_2$ $A(1) = \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2)}{r_{12}}\,dV_2$	$\phi(1, t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(2, t')}{r_{12}}\,dV_2$ and $A(1, t) = \dfrac{1}{4\pi\epsilon_0 c^2}\displaystyle\int \dfrac{j(2, t')}{r_{12}}\,dV_2$ with $t' = t - \dfrac{r_{12}}{c}$
$U = \frac{1}{2}\displaystyle\int \rho\phi\,dV + \frac{1}{2}\int j \cdot A\,dV$	$U = \displaystyle\int \left(\dfrac{\epsilon_0}{2}E \cdot E + \dfrac{\epsilon_0 c^2}{2}B \cdot B\right) dV$

The equations marked by an arrow ($\rightarrow$) are Maxwell's equations.

Maxwell Equation

Page 15-15 of Feynman's Lecture

in the next few lecture
We shall fill in the gaps

Exercise, write the equation in appropriate units

Start with the statics

$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\nabla\phi$

$\nabla \times \vec{E} = \nabla \times (-\nabla\phi) = 0$ ← vector identity

$\vec{E} = -\nabla\phi$ automatically satisfies $\nabla \times \vec{E} = 0$

$\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$

$\Rightarrow \nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$ Poisson's equation

$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}$

$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$

$c^2 \nabla \times \vec{B} = \dfrac{\vec{J}}{\epsilon_0}$

ϕ, $\vec{A}$

$\vec{A}$ → vector potential

ϕ → scalar

vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$$

$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$$

Choose $\nabla \cdot \vec{A} = 0$ (choose Coulomb gauge)

Freedom of choosing the gauge ⇒ gauge invariance

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\chi$$

$$\nabla \times \vec{A} = \vec{B}$$

$$\nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla\chi)$$

$$= \nabla \times \vec{A} = \vec{B}$$

$\vec{A}$, $\vec{A}'$ gives the same $\vec{B}$ ⇒ $\vec{A}$ is not unique

⇒ we have many $\vec{A}$ to choose from.

Example see Feynman's notes

$\nabla \cdot \vec{A} \neq 0$ make a transformation $\vec{A} \to \vec{A}'$

$\nabla \cdot (\vec{A} + \nabla \chi) = 0$ $\nabla \cdot \vec{A}' = 0$

$\nabla \cdot \vec{A} + \nabla^2 \chi = 0$

$\downarrow$

$\nabla^2 \chi = \nabla \cdot \vec{A}$

$\Rightarrow$ we can always choose $\vec{A}$ among those satify the gauge conditions

With $\nabla \cdot \vec{A} = 0$

$\Rightarrow$ $-c^2 \nabla^2 \vec{A} = \dfrac{\vec{J}}{\epsilon_0}$

$\downarrow$

the structure is the same as the Poisson equation

$\phi(1)$ is given as $\phi(1) = \int \dfrac{\rho(2)}{r_{12}} dV_2$

$\Downarrow$

$A(1)$ is given as $\vec{A}(1) = \int \dfrac{J(2)}{r_{12}} dV_2$

$$U = \frac{1}{2} \int \rho \phi \, dV + \frac{1}{2} \int \vec{J} \cdot \vec{A} \, dV.$$

Comments. Note the symmetry. "beauty"

Now we go to the General case

$\boxed{\vec{E} = -\nabla \phi - \dfrac{\partial \vec{A}}{\partial t}}$ $\underline{\underline{\vec{B} = \nabla \times \vec{A}}}$

$\nabla \times \vec{E} = -\nabla \times (\nabla \phi) - \nabla \times \left(\dfrac{\partial \vec{A}}{\partial t} \right)$

$= 0 - \dfrac{\partial}{\partial t} \nabla \times \vec{A} = -\dfrac{\partial \vec{B}}{\partial t}$

$\phi, \vec{A}$ scalar and vector field potential.

$$\text{Maxwell Equation}\begin{cases} \nabla\cdot\vec{B} = 0 & ① \\ \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t} & ② \\ \nabla\cdot\vec{E} = \dfrac{1}{\epsilon_0}\rho & ③ \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & ④ \end{cases}$$

①, ④ homogeneous

②, ③ inhomogeneous

Compare with Feynman's equation.

Solve ①, ④ once for all

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla\times\vec{A} \qquad Ⓐ$$

Scalar and vector potential

$$\nabla\times\vec{E} = \nabla\times(-\nabla\phi) - \frac{\partial}{\partial t}(\nabla\times\vec{A})$$
$$= -\frac{\partial\vec{B}}{\partial t}$$

Solve the second equation

$$\nabla\cdot\vec{B} = 0$$

Now we substitute the equation Ⓐ into ②, ③

$$\nabla\cdot\vec{E} = \nabla\cdot\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right)$$
$$\frac{\rho}{\epsilon_0} = -\nabla^2\phi - \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) \qquad Ⓑ1$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$
$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$
$$\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \qquad Ⓑ2$$

Gauge transformation

$$\begin{cases}\vec{A}' = \vec{A} + \nabla\lambda \\ \phi' = \phi - \dfrac{\partial\lambda}{\partial t}\end{cases}$$

$\Downarrow$

leaves $\vec{E}$, $\vec{B}$ unchanged

$$\nabla\cdot\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

$$= -\nabla\phi' - \frac{\partial\vec{A}'}{\partial t}$$

$$= -\nabla\left[\phi - \frac{\partial\lambda}{\partial t}\right)] - \frac{\partial}{\partial t}(A + \nabla\lambda]$$

$$= -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{E} = -\nabla\phi' - \frac{\partial\vec{A}'}{\partial t} = -\underline{\nabla}\left(\phi - \underline{\frac{\partial\lambda}{\partial t}}\right) - \underline{\frac{\partial}{\partial t}}(\vec{A} + \underline{\nabla\lambda})$$

$$\overset{?}{=} -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

$\downarrow$

yes

$$\vec{B} = \nabla\times A' = \nabla\times(\vec{A} + \nabla\lambda)$$

$\vec{E}$, $\vec{B}$ is unchanged under gauge transformation obviously, they are still freedom in choosing the gauge condition.

In electrodynamics, the most popular gauge condition is the Lorentz gauge.

$$\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0$$

B1

$$\frac{\rho}{\epsilon_0} = -\nabla^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = \Box^2\phi$$

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$\downarrow$

d' Alembertian

(B 2) reduces to

$$\nabla^2 \vec{A} - \underset{\underset{\frac{1}{c^2}}{\shortparallel}}{\mu_0 \epsilon_0} \frac{\partial^2 \vec{A}}{\partial t} = -\mu_0 \vec{J}$$

Note the form is very "beautiful"

↓

Now we can undertand the page in Feynman's paper

From Maxwell equation to

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{J}{\epsilon_0 c^2}$$

$\downarrow$

with $c^2 \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0$

$\downarrow$

Lorentz gauge

$(\vec{r}, t) \xrightarrow{\text{take time}} (\vec{r}', t')$

for the field to propagate

The explaination of ~~the~~ page 15-15

$\downarrow$

Summary of E and M

Maxwell equation (no charge current at infinite)

Maxwell equation $\Rightarrow$ EM wave

no current
or
charge

$$\nabla \cdot \vec{E} = 0 , \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Notice the symmetry

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\overset{0}{\cancel{\nabla \cdot \vec{E}}}) - \nabla^2 \vec{E}$$

$$= \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t}$$

$$\boxed{\nabla^2 \vec{E}}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x$$

Similarly

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

wave equation

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Euroka!

Hertz

$$\nabla \cdot (\nabla \times \vec{v}) = 0$$

emphasize the generalization from statics

↓

time dependence