Chapter 34 (Continue)

Maxwell's Equation

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{Continuity Equation.}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Electrostatics

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (A)$$

$$\nabla \times \vec{E} = 0$$

Remark: $\vec{E}$ and $\vec{B}$ are independent

$$\nabla \times \vec{E} = -\nabla \phi \quad \Longleftrightarrow \quad \nabla \times \vec{E} = 0$$

ϕ is the scalar potential

$$\nabla \times (-\nabla \phi) = -\nabla \times (\nabla \phi) = 0$$

vector identity

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla \cdot (-\nabla \phi) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \qquad \text{Poisson Equation}$$

Advantage of using ϕ the scalar potential

ϕ is a scalar function

$\vec{E}$ can be obtained from ϕ by differentiation

Instead solving A (which write it out $\downarrow$ 4 equation)

Writing $\vec{E} = -\nabla\phi$, then $\nabla \times \vec{E} = 0$ is automatically satisfied

Note $\nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$ with localized charge we choose $\phi(\vec{r}) \to 0$ as $|\vec{r}| \to \infty$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{|\vec{r}-\vec{r}\,'|}$$

$\downarrow$

energy density

Magnetostatics

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} = \mu_0 \vec{J}$

$\nabla \cdot \vec{B} = 0 \Leftrightarrow \vec{B} = \nabla \times \vec{A}$

$\downarrow$

vector identity

$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) \overset{\downarrow}{=} 0$

With $\vec{B} = \nabla \times \vec{A}$
Equation $\nabla \cdot \vec{B} = 0$ is automatically satisfied.

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A})$$
$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

If we can choose $\nabla \cdot \vec{A}$, then the above equation reduces to

$$\nabla^2 \vec{A} = \mu_0 \vec{J}$$

This equation has three components
↓
each components have exactly the same form as that of the Poisson equation

The rest follows starightforwardly.

The problem now is to show we can always choose $\nabla \cdot \vec{A} = 0$

- $\vec{A}$ is not unique.

In classical physics, $\vec{A}$ is just a tool for calculating $\vec{B}$

Two vector potentials $\vec{A}$, $\vec{A}'$ as just as good if both give the same $\vec{B}$
↓
namely, $\vec{A}$ is not unique
↓
there is large set of A's that work as well.

We can use any of them. to solve the problem.

If $\vec{A}$ is a vector potential, then under a transformation

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \chi$$

↳ any scalar function

$$\nabla \times \vec{A} = \nabla \times \vec{A}'$$

Proof $\quad \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla \chi) = \nabla \times \vec{A}$

$$[\nabla \times (\nabla \chi) = 0]$$

This is a huge number of $\vec{A}$

$\downarrow$

we can impose extra condition

such as

$\nabla \cdot \vec{A}$.

If $\vec{A}$ is found, but $\nabla \cdot \vec{A} \neq 0$

then we can always find $\vec{A}' = \vec{A} + \nabla \chi$

and impose $\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \chi = 0$

by choosing a χ such that the condition

$\nabla \cdot \vec{A}' = 0$ is satisfied

$\vec{A}$, $\vec{A}'$ will give the same $\vec{B}$

$\Rightarrow$ We can always find a vector potential that gives the correct $\vec{B}$ and also satisfied the condition $\nabla \cdot \vec{A} = 0$

With the $\nabla \cdot \vec{A} = 0$ condition

$\Rightarrow$ $\nabla^2 \vec{A} = \mu_0 \vec{J}$

the rest is straightforward.

Remark:

$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \chi$ gauge transformation

Physics is unchange under gauge transformation

$\Rightarrow$ gauge invariance.

The freedom of imposed extra condition on $\vec{A}$ such as $\nabla \cdot \vec{A} = 0$ is known as gauge freedom.

Choosing an extra condition on $\nabla \cdot \vec{A} = 0$ to work with is known as fixing the gauge (condition)

$\nabla \cdot \vec{A} = 0$ is known as the Coulomb gauge

$\downarrow$

usually used in static case.

$$\nabla^2 A = \mu_0 \vec{J}$$

$\vec{J}$ is localized
we shall choose $\vec{A}(\vec{r}) \to 0$
as $|\vec{r}| \to \infty$

$\downarrow$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')\, d\tau'}{|\vec{r} - \vec{r}'|}$$

We can show that if $\nabla \cdot \vec{J} = 0$ then the $\vec{A}(\vec{r})$ gived above satisfies the Coulomb gauge condition $\nabla \cdot \vec{A} = 0$

Next we shall extend this discussion to the general case.

Go back to the Maxwell's Equation for the time independent case

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)$$

$$\nabla \cdot \vec{B} = 0 \quad (3)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \quad (4)$$

All quantities are function of $\vec{r}, t$

(2), (3) are known as homogeneous equations, i.e., independent of $\rho, \vec{J}$

$\downarrow$

we should solve them once for all

$\downarrow$

vector and scalar potential

$$\nabla\cdot\vec{B}=0 \quad\Rightarrow\quad \vec{B}=\nabla\times\vec{A} \qquad (A)$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{A}) = -\nabla\times\left(\frac{\partial}{\partial t}\vec{A}\right)$$

$$\nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\nabla\phi$$

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \qquad (B)$$

With the introduction of ϕ, $\vec{A}$

- the $\vec{E}$, $\vec{B}$ can be calculated by differentiation.
- equations (2), (3) of the Maxwell are automatically satisfied.

Substitute the equations (A), (B) into equations (1), (4), we obtain

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla\cdot\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right) = \frac{1}{\epsilon_0}\rho \qquad (C)$$

$$\downarrow$$

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \frac{-1}{\epsilon_0}\rho$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\underbrace{\nabla\times(\nabla\times\vec{A})}_{=\ \nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}} = \mu_0\vec{J} - \mu_0\epsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J} \qquad (D)$$

Equation (D) can be written as

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t} = -\mu_0 \vec{J}$$

$$\left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{A} = -\mu_0 \vec{J} \qquad (E)$$

$$\Downarrow$$

$$\Box \equiv \left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right)$$

$$\downarrow$$

d' Alembertian

$$\Box \vec{A} = -\mu_0 \vec{J}$$

Equation (C)

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = \frac{1}{\epsilon_0} \rho$$

$$\downarrow$$

$$\frac{\partial}{\partial t} \left(-\epsilon_0 \mu_0 \frac{\partial \phi}{\partial t} \right) = \frac{1}{\epsilon_0} \rho$$

$$\nabla^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = \frac{-1}{\epsilon_0} \rho$$

$$\Downarrow$$

$$\Box \phi = -\frac{\rho}{\epsilon_0} \iff \left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \phi = -\frac{\rho}{\epsilon_0} \qquad (F)$$

Remark:

With this choice,

$\vec{A}, \phi$ are decoupled

and

have the same form.

The solutions are given as follows

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|} d\tau'$$

$$\vec{A}(r, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{} d\tau'$$

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

Maxwell's Equations in "Vacuum"

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Note: The symmetry

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

The importance of the displacement current

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Since $\nabla \cdot \vec{E} = 0$

$$\underline{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$$

$$= \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Since $\nabla \cdot \vec{B} = 0$

$$\underline{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

$$\underline{\epsilon_0 \mu_0 = \frac{1}{c^2}}$$

These are the basic equations for EM waves

1. One dimensional wave (harmonic)

$$y(x,t) = \underset{\substack{\downarrow \\ \text{amplitude}}}{A} \sin(kx - \omega t)$$

$$= A \sin k\left(x - \frac{\omega}{k} t\right)$$

$$= A \sin k(x - v t)$$

Wave is propagating with velocity v along x

If y is in the direction of propagation then it is called longitudinal wave.
If y is perpendicular to the direction of propagation then it is called transverse wave

$$y(x, t+\Delta t) = A \sin k(x - v\Delta t - vt)$$

$$= y(x - v\Delta t, t)$$

↓

the wave is propagating with velocity v along the x direction.

2. Wavelength, frequency

$$y(x+\lambda, t) = A \sin[k(x+\lambda) - \omega t]$$

λ ↳ wavelength

$$= y(x,t)$$

$\Rightarrow k\lambda = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$ — k is known as the wave number

$$y(x, t+T) = y(x,t)$$

T ↓ period

$$\Rightarrow \omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\nu\,(f)$$

ν ↓ frequency

$$v = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu$$

↓

very useful result.

3. Wave equation

$$\frac{\partial y}{\partial t} = A(-\omega)\cos(kx-\omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = A(-\omega^2)\sin(kx-\omega t)$$

$$\frac{\partial y}{\partial x} = Ak\cos(kx-\omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2\sin(kx-\omega t)$$

$$\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{\omega^2}{k^2} = v^2$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

↓

the wave equation

4. The wave equation is linear
   $\Rightarrow$ if y_1, y_2 are solutions of the wave equation, then $\alpha y_1 + \beta y_2$ (α, β are constants) is also a solution of the wave equation
   $\Rightarrow$ solutions of the wave equation are superposition of harmonic waves.

5. Three dimensional wave

$k \longrightarrow \vec{k}$

↳ wave vector, direction is along the direction of propagation

$x \longrightarrow \vec{r} = (x, y, z)$

6. Electromagnetic wave.

Propagation of the $\vec{E}(x,y,z;t)$ and $\vec{B}(x,y,z;t)$

Note:

(i) Electromagnetic wave is a vector wave.

(ii) It satisfies the Maxwell equations.

A harmonic electromagnetic wave is given by

$$\vec{E}(x,y,z;t) = \vec{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)$$

$\downarrow$

constant vector (E_{0x}, E_{0y}, E_{0z})

$$\vec{B}(x,y,z;t) = \vec{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)$$

constant (B_{0x}, B_{0y}, B_{0z})

Written in compact form

$$\vec{E} = \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

$$\vec{B} = \vec{B}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$$

7. Harmonic wave and Maxwell equation

Want to find the conditions that $\vec{E}, \vec{B}$ must satisfy in order for the electromagnetic wave to be propagating in vacuum.

$\downarrow$

must satisfy the free Maxwell equation

(i) $\nabla\cdot\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = 0$

$$E_x = E_{0x} \sin[k_x x + k_y y + k_z z - \omega t]$$

$$= E_{0x} \sin[\vec{k}\cdot\vec{r} - \omega t)$$

$$\frac{\partial E_x}{\partial x} = E_{0x} k_x \cos[k_x x + k_y y + k_z z - \omega t)$$

$$= E_{0x} k_x \cos[\vec{k}\cdot\vec{r} - \omega t]$$

Use the same method, we find

$$\frac{\partial E_y}{\partial y} = E_{0y} k_y \cos[\vec{k}\cdot\vec{r} - \omega t]$$

$$\frac{\partial E_z}{\partial z} = E_{0z} k_z \cos[\vec{k}\cdot\vec{r} - \omega t]$$

Combine these $\Rightarrow$ $E_{0x}k_x + E_{0y}k_y + E_{0z}k_z = 0$

$\Rightarrow$ $\vec{E}_0\cdot\vec{k} = 0 \Rightarrow \vec{E}\cdot\vec{k} = 0$ $\quad\vec{E}$ field is transverse.

(ii) Same argument $\Rightarrow \vec{B}_0 \cdot \vec{k} = 0 \Rightarrow \vec{B} \cdot \vec{k} = 0$

$\Rightarrow \vec{B}$ field is transverse.

(iii) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Look at the x component

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

Follow the same procedure as in last section, one finds

$$\frac{\partial E_z}{\partial y} = E_{oz} k_y \cos[\vec{k} \cdot \vec{r} - \omega t]$$

$$\frac{\partial E_y}{\partial z} = E_{oy} k_z \cos[\vec{k} \cdot \vec{r} - \omega t]$$

$$\frac{\partial B_x}{\partial t} = -B_{ox}\, \omega \cos[\vec{k} \cdot \vec{r} - \omega t]$$

Put it together

$$\Rightarrow \quad E_{oz} k_y - E_{oy} k_z = \omega B_x$$

Work out the other components and combine them into vector form

$$\Rightarrow \quad \vec{E}_o \times \vec{k} = \omega \vec{B}_o$$

$\Rightarrow \vec{E}, \vec{B}, \vec{k}$ are mutually $\perp$

$$\Rightarrow \quad |E_o||k| = \omega |B_o|$$

$$|E_o| = \frac{\omega}{k} |B_o|$$

$$= \underset{\downarrow}{c}\, |B_o|$$

velocity of light in vacuum.

(iv) In vacuum $\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Note: no current

Use the same procedure as last section

$$\Rightarrow \vec{B}_0 \times \vec{k} = \omega \mu_0 \epsilon_0 \vec{E}_0$$

$$|B_0||k| = \omega \mu_0 \epsilon_0 |E_0| = \mu_0 \epsilon_0 \frac{\omega^2}{k} B_0$$

$$\Rightarrow \underbrace{\frac{\omega^2}{k^2}}_{= C^2} = \frac{1}{\mu \epsilon}$$

$$\Rightarrow C = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

"Eurco" !

The most important discovery of the 19th century.

8. Energy transfer

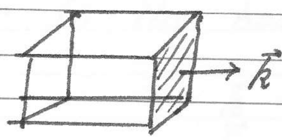
The most important usage of the electromagnetic wave is energy transfer

Energy transfer across a unit surface ($\perp$ to the direction of propagating) per unit time is given by

$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

↳ Poynting vector

Clearly $\vec{S}$ is along $\vec{k}$, the direction of wave propagation $\leftrightarrow$ direction of energy transfer



energy cross the surface in time Δt

$= U_{em} \cdot \Delta V$

ΔV ↳ $A c \Delta t$

U_{em} ↓ electromagnetic energy density

$\Rightarrow$ Energy transfer per unit area per unit time

$= U_{em} c$

9. Intensity

$$\underset{\text{intensity}}{I} = \overline{S^2} = \frac{1}{c\mu_0}|\vec{E}_0|^2 \, \overline{\sin^2(kx-\omega t)}$$

$$= \frac{1}{2\mu_0 c}|E_0|^2 = \frac{1}{\mu_0 c}|\underbrace{\tfrac{1}{\sqrt{2}}\vec{E}_0}_{E_{rms.}}|^2$$

· In harmonic electromagnetic wave, it can be easily shown that $U_E = U_m$

magnetic energy density

electric energy density.

$$I = \frac{P_s}{4\pi r^2}$$

P_s → power at the source

$4\pi r^2$ → geometrical factor for spherical wave

Clearly I is closely related to $\vec{E}^2_{rms.}$

10. Radiation pressure

$$\underset{\text{momentum transfer}}{\Delta p} = \frac{\Delta U}{c}$$

ΔU → energy transfer.

If an electromagnetic wave is absorbed at a surface, the surface receives a radiation pressure

$$\frac{\Delta p}{A\,\Delta t} = \frac{\Delta U}{A\,dt\,c} = \frac{I}{c}$$

If an electromagnetic wave is reflected at a surface the momentum transfer is twice as the absorbing case, the surface receives a radiation pressure of $\frac{2I}{c}$.

11. Polarization

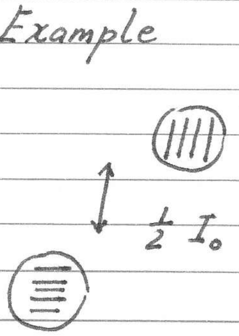
Electromagnetic wave is a vector field, the direction of the $\vec{E}$ field specifies its polarization.

. Polarizing sheet: only allow electromagnetic wave with certain polarization component through.

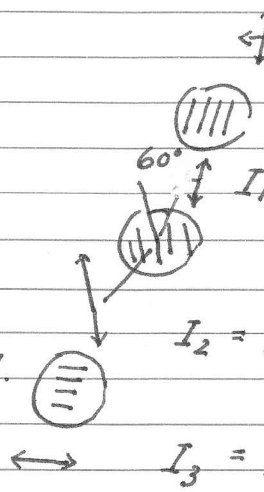
$$I = I_0 \cos^2\theta \quad \text{(Malu's law)}$$

Unpolarized wave, after going through polarizing sheets of any orientation, the intensity will be $\frac{1}{2}$ of its original value.

Example



no electromgnetic wave is transmitted.



$I_1 = \frac{1}{2} I_0$

$I_2 = \frac{1}{2} I_0 \cos 60^\circ$

$$I_3 = \frac{1}{2} I_0 \cos^2 60^\circ \cos^2 30^\circ = 0.094\, I_0$$

$\downarrow$

there is electromagnetic wave transmitted in this case.

- We are discussed linearly polarized electromagnetic wave here
  $\Rightarrow$ Sometimes circularly polarized electromagnetic wave is used

  Circularly polarized electromagnetic waves are linear combination of the linearly polarized electromagnetic wave.

- Key concept here is principle of superposition.
  $\Rightarrow$ help in the understanding of wave mechanics (quantum mechanics)

- Geometrical optics will be combined into materials to be discussed in next chapter.