# Chapter 1

## Special Relativity

### 1. Galilean transformations and classical mechanics

Physical event : happens at a point in space and at an instant in time.

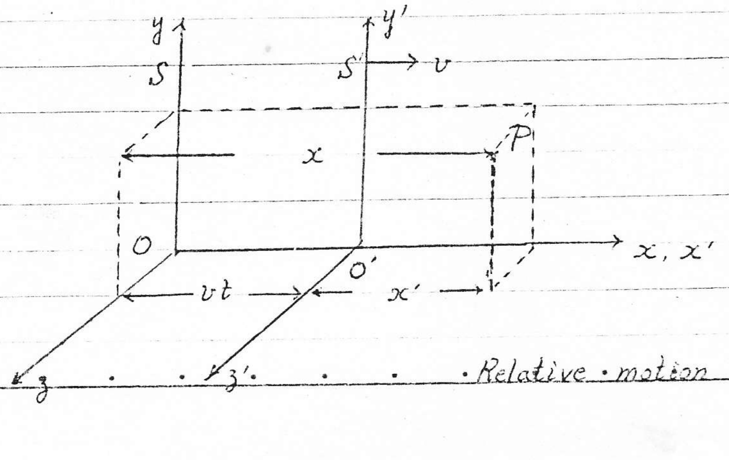
Specify an event by four (space - time) measurements in a particular frame of reference.

The same event observed from a different reference frame would, in general by four different numbers.

Inertial frame : a frame of reference in which the law of inertia (Newton's first law) holds, i.e., a frame of reference in which a body not under the influence of forces will move with a constant velocity.

Special relativity deals with the description of events by observers in inertial frames.

The objects whose motions we study may be accelerating with respect to such frames but the frames themselves are unaccelerated.




$S'$ is moving at a constant velocity $v$ with respect to $S$

$\Leftrightarrow$ $S$ is moving with constant velocity $-v$ with respect to $S'$

Relative motion to be along $x, x'$ axis.

At $t = t'$ the two frames of reference coincide.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

These equations are known as Galilean transformation.

$$x' = x - vt$$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$
$\parallel \leftarrow$ since $t = t'$
$$\frac{dx'}{dt'}$$

$$u_x' = u_x - v$$
$$u_y' = u_y$$
$$u_z' = u_z$$

These equations are known as classical velocity addition theorem.

$$\frac{du_x'}{dt} = \frac{du_x}{dt}$$
$\parallel \leftarrow t = t'$
$$\frac{du_x'}{dt'}$$

$$a_x' = a_x$$
$$a_y' = a_y$$
$$a_z' = a_z$$

$\Rightarrow$ the acceleration of a particle is the same in all reference frame which move relative to one another with constant velocity.

Consider the force provided by the interaction between two bodies

Suppose it is a function only of their seperation

$$\Rightarrow \quad \vec{F}_{12} = \vec{F}_{12}' \longrightarrow$$ force exerted on body 2 by body 1 in System S'

$\vec{F}_{12}$ $\hookrightarrow$ force exerted on body 2 by body 1 in system S

Newtonian dynamics $\Rightarrow$ mass is constant $m_2 = m_2'$

$$\vec{F}_{12} = m_2 \vec{a}_2 \Rightarrow \vec{F}_{12}' = m_2' \vec{a}_2'$$

$\Rightarrow$ Newton's laws do not change when we make a Galilean transformation.

Momentum conservation. For an isolated system of particles, the total linear momentum of the system will remain constant

Consider a collision of two balls as view from frame $S$ and from $S'$

Conservation of momentum as applied in $S$

$$m_1 \vec{V}_{1i} + m_2 \vec{V}_{2i} = m_1 \vec{V}_{1f} + m_2 \vec{V}_{2f}$$

According to classical velocity addition theorem, one can show that

$$m_1 \vec{V}_{1i}' + m_2 \vec{V}_{2i}' = m_1 \vec{V}_{1f}' + m_2 \vec{V}_{2f}'$$

Thus, if the momentum is conserved in one inertial frame then under a Galilean transformation, we find it is conserved in all inertial frames.

Principle of relativity: All laws of mechanics will be the same in all inertial frames.

## 2. The Galilean transformation and electromagnetic theory

Maxwell equations are not invariant under Galilean transformation

Maxwell equations $\Rightarrow$ existence of electromagnetic wave

19 th century physicists $\Rightarrow$ existence of a propagating medium known as ether

Electromagnetic equations, in the form presented by Maxwell, were assumed to be valid in a reference frame at rest with respect to the ether.

zero density, perfect transparency.
have elastic properties to substain the vibrations

A solution of these equation $\Rightarrow$ velocity of light $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ $\sim 3 \times 10^{10}$ cm/sec.

In a new frame of reference $S'$ moving with constant velocity $\vec{v}$ respect to ether frame $\Rightarrow$ an observer would measure a different speed of light

$$\vec{V}_{\substack{\text{light with respect} \\ \text{to } S'}} = \underbrace{\vec{V}_{\substack{\text{light with respect} \\ \text{to ether}}}}_{= c} - \vec{V}_{\substack{\text{new frame } S' \text{ with} \\ \text{respect to ether}}}$$

Summary.

Assumptions: (a) the validity of Newton's law
(b) the validity of Maxwell's equation
(c) the validity of Galilean transformation

$\Rightarrow$ (i) all inertial frames of reference are equivalent as far as mechanical phenomena were concerned.

(ii) in regard to electromagnetic phenomena, they are not equivalent ; there were only one frame of reference , the ether frame , in which the velocity of light = c.

Three possibilities.

(i) A relativity exists for mechanics , but not for EM

In EM there is a preferred inertial frame : the ether frame

Galilean transformation is correct ⇒ we would be able to located the ether frame.

(ii) A relativity principle exists both for mechanics and for EM but the EM laws as given by Maxwell are not correct.

Galilean transformation is correct ⇒ have to be able to

(α) show deviation from Maxwell's EM laws and

(β) reformulate the EM laws

(iii) A relativity principle exists both for mechanics and for EM , but the laws of mechanics as given by Newton are not corrent ⇒ have to be able to (α) show deviations from Newtonian mechanics , (β) reformulate the mechanical laws , (γ) find the correct transformation laws which are consistent with classical EM and the new mechanics

## 3. The Michelson-Morley experiment

Designed to demonstrate the existence of the ether frame, and to determine the motion of the earth with respect to that frame.

Earth travels in a roughly circular orbit with a period of one year and with orbital velocity $\sim 3 \times 10^6$ cm/sec $= 30$ km/sec.

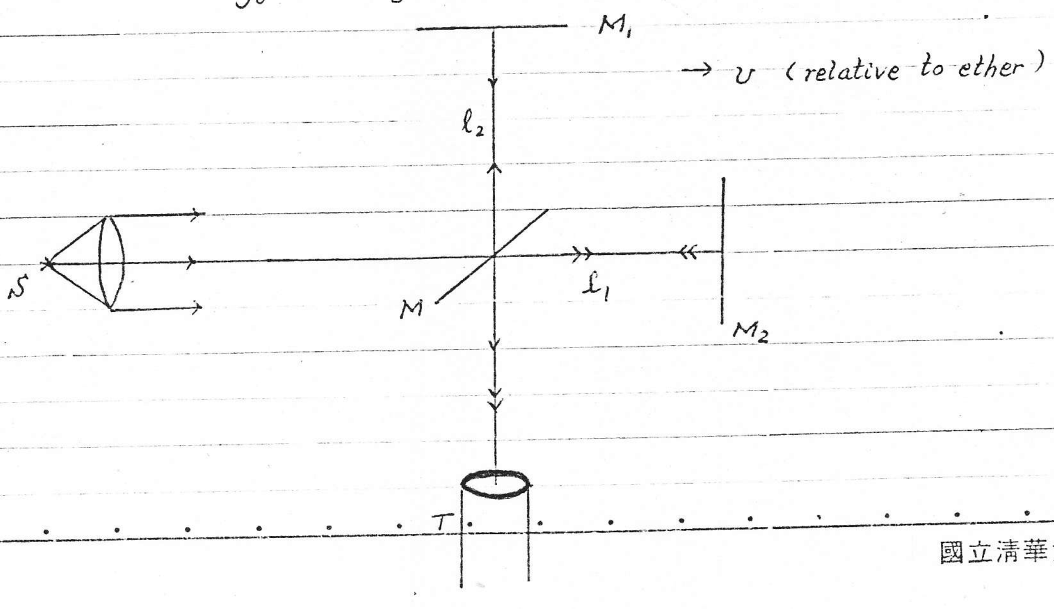
Assume the ether frame was at rest with the respect to the center of mass of our solar system

Basic idea of Michelson-Morley experiment:

Measure the velocity of light as seen from a frame of reference fixed with respect to the earth, in two perpendicular directions

$$\vec{V}_{\text{light with respect to new frame}} = \vec{V}_{\text{light with respect to ether}} - \vec{V}_{\text{new frame with respect to ether}}$$

$\Rightarrow$ the apparent velocity of light with respect to the earth would be different for light traveling in different directions.



The beam of light (plane waves, or parallel rays) from the laboratory source $S$ (fixed with respect to the instrument) is split by the partially silvered mirror into two coherent beams.

Beam 1 (i) being transmitted through $M$
(ii) is reflected back to $M$ by mirror $M_1$
(iii) the returning beam 1 is partially reflected by $M$ back to a telescope at $T$

Beam 2 (i) being reflected off $M$
(ii) is reflected back to $M$ by mirror $M_2$
(iii) the returning beam 2 is partially transmitted by $M$ to a telescope at $T$

$\Rightarrow$ two beams interfere at $T$.

The interference is constructive or destructive depending on the phase difference of the beams.

Compute the phase difference between the beam 1 and 2

This difference can arise from
(i) different path lengths traveled $l_1$ and $l_2$
(ii) the different speed of travel with respect to the instrument because of the "ether" wind $v$
↳ the relative velocity of the instrument and the ether rest system.

The time for beam 1 to travel from $M$ to $M_1$ and back

The calculation is done in the rest system of the instrument.

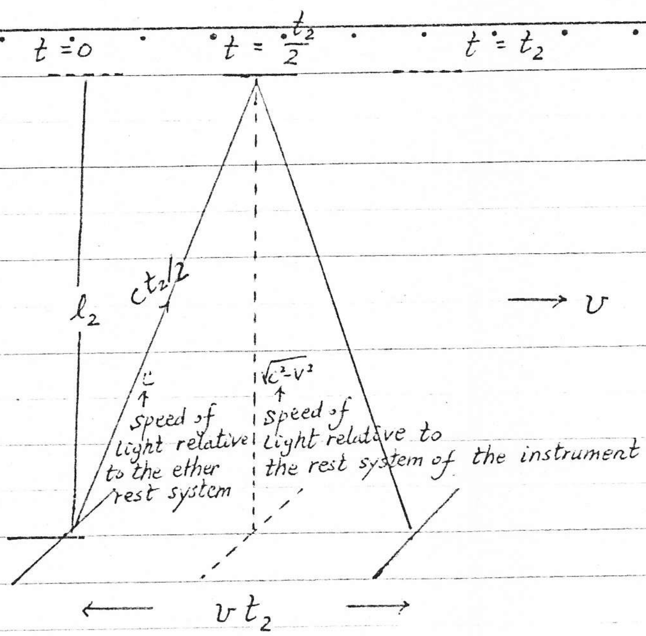
The velocity of the light during $M$ to $M_1$ is $c-v$
The absolute velocity of the light during $M_1$ to $M$ is $c+v$

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = l_1\left(\frac{2c}{c^2-v^2}\right)$$

$$= \frac{2l_1}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

The time for beam 2 to travel from $M$ to $M_2$ and back

The calculation will be done in the ether rest system




$$2\left[\, l_2^2 + \left(\frac{v t_2}{2}\right)^2 \right]^{1/2} = c t_2$$

$$4 l_2^2 + v^2 t_2^2 = c^2 t_2^2$$

$$t_2 = \frac{2 l_2}{\sqrt{c^2 - v^2}} = \frac{2 l_2}{c} \frac{1}{\sqrt{1 - v^2/c^2}}$$

The calculation of $t_2$ is made in the ether rest frame
of $t_1$ is made in the rest frame of the instrument

Because time is an absolute in classical physics $\Rightarrow$ this is perfectly acceptable classically

$$\Delta t = t_2 - t_1$$

$$= \frac{2}{c}\left[ \frac{l_2}{\sqrt{1 - v^2/c^2}} - \frac{l_1}{1 - v^2/c^2} \right]$$

Instrument rotated by $90^\circ$ $\quad$ $l_1 \rightarrow$ cross-stream length, $l_2 \rightarrow$ downstream length

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left[ \frac{l_2}{1 - v^2/c^2} - \frac{l_1}{\sqrt{1 - v^2/c^2}} \right]$$

$\Rightarrow$ the rotation changes the differences by

$$\Delta t' - \Delta t = \frac{2}{c}\left[ \frac{l_2 + l_1}{1 - v^2/c^2} - \frac{l_2 + l_1}{\sqrt{1 - v^2/c^2}} \right]$$

$$\simeq \frac{2}{c}(l_1 + l_2)\left\{ 1 + \frac{v^2}{c^2} - 1 - \frac{1}{2}\frac{v^2}{c^2} \right\} = \left(\frac{l_1 + l_2}{c}\right)\frac{v^2}{c^2}$$

using binomial expansion and dropping terms higher than second order

If the optical path difference between the beams changes by one wavelength

$\Rightarrow$ there will be a shift of one fringe across the crosshairs of the viewing telescope.

$$\Delta N = \frac{\Delta t' - \Delta t}{T} \simeq \frac{l_1 + l_2}{cT}\frac{v^2}{c^2} = \frac{l_1 + l_2}{\lambda}\frac{v^2}{c^2}$$

$$( T = \frac{1}{\nu} = \lambda / c )$$

$l_1 + l_2 = 22m$, $\lambda = 5.5 \times 10^{-7}\ m$, $\frac{v}{c} = 10^{-4}$

$\Rightarrow \Delta N = 0.4$

Observed $\Delta N < 0.005$

$\Rightarrow$ Null result.

Experiments were made :

(i) day and light $\rightarrow$ as the earth spins about its axis
(ii) all seasons of the year $\rightarrow$ as the earth rotates about the sun.
$\Rightarrow$ still dull effect.

Michelson - Morley experiment showed that the velocity of light is the same when measured along two $\perp$ axes in a reference frame which, presumably, is moving relative to the ether rest frame in different directions at different time of the year.

One way to interpret the null result of the Michelson- Morley experiment $\Rightarrow$ the measured speed of light is the same, that is c, in every inertial system.

# 7. Lorentz transformation.

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Linearity $\Leftarrow$ homogenity

$x$ axis $\Rightarrow$ $x'$ axis

$y=0,\ z=0 \Rightarrow y'=0,\ z'=0$

$$y' = a_{22}y + a_{23}z$$

$$z' = a_{32}y + a_{33}z$$

$x-y$ plane $\Rightarrow$ $x'-y'$ plane

$z=0 \Rightarrow z'=0$

$x-z$ plane $\Rightarrow$ $x'-z'$ plane

$y=0 \Rightarrow y'=0$

$$y' = a_{22}y$$

rod at rest in $S$ length $=1$ along $y$ axis

$S' \rightarrow a_{22}$

rod at rest in $S'$ length must $=1$ along $y'$ axis

$S \rightarrow \frac{1}{a_{22}}$

$$\frac{1}{a_{22}^2} = 1 \Rightarrow a_{22} = 1$$

$$y' = y$$

Similiar argument $\Rightarrow$ $z' = z$

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Symmetry (isotropy) $\Rightarrow$ $a_{12} = a_{13} = a_{42} = a_{43} = 0$

$O'$ moving with velocity $v$ in $S$

$x' = 0 \quad \Rightarrow \quad x - vt$

$$\Rightarrow \quad \begin{aligned} x' &= a_{11}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= a_{41}x + a_{44}t \end{aligned}$$

Constant of velocity of light

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$\downarrow$$

$$a_{11}^2 (x - vt)^2 + y^2 + z^2 = c^2 (a_{41}x + a_{44}t)^2$$

$$(a_{11}^2 - c^2 a_{41}^2)\, x^2 + y^2 + z^2 - 2(v a_{11}^2 + c^2 a_{41} a_{44})\, xt$$

$$= (c^2 a_{44}^2 - v^2 a_{11}^2)\, t^2$$

$$\begin{cases} a_{11}^2 - c^2 a_{41}^2 = 1 \\ v a_{11}^2 + c^2 a_{41} a_{44} = 0 \\ c^2 a_{44}^2 - v^2 a_{11}^2 = c^2 \end{cases}$$

$$a_{44} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{11} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{41} = -\frac{v}{c^2} \Big/ \sqrt{1 - v^2/c^2}$$

$$\Rightarrow \quad x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$$

This is Lorentz transformation

It can be shown that

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}$$

That is the Lorentz transformation from $S'$ to $S$.

This equation is identical in form with the above equations except that

$v \to -v$ as required.

As $\frac{v}{c} \ll 1$

$$\left.\begin{array}{llll} x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}} & \longrightarrow & x' = x - vt \\ y' = y & \longrightarrow & y' = y \\ z' = z & \longrightarrow & z' = z \\ t' = \left(t - \dfrac{vx}{c^2}\right) \Big/ \sqrt{1 - \dfrac{v^2}{c^2}} & \longrightarrow & t' = t \end{array}\right\} \text{classical Galilean transformation equations}$$

## 8. The relativity of simultaneity

$t = t'$ $\Rightarrow$ absolute time (universal time)
↳ basic premise of Newtonian mechanics.

If different inertial observers disagree as to whether two events are simultaneous $\Rightarrow$ certainly not have a universal time scale.

Simultaneity in a single frame of reference

(i) Events occur at the same place : no problem.

(ii) Events occur at different location.

×A clock A ×B clock B

Synochronization.

(α) Require two clocks so that they always read the same time as seen by observer A. Problem : the velocity of light is finite.

(β) Set the two clocks to read the same time and then move them to the positions where the events occur. Problem : we do not know ahead of time (and therefore cannot assume) that the motion of the clocks will not affect their readings and time-keeping ability.

(γ) Put the clock into positions and synchronize them by means of signals. Signals ↔ EM wave

×A ×C ×B

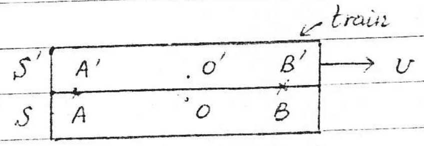
C is medpoint of A, B
A, B set clock to $t = 0$ when the turned-on light reached them.

Time of events $\Rightarrow$ measured by the clock whose location coincides with that of the event.

Events occuring at two different places in the frame must be called simultaneous when the clocks at the respective places record the same time for them.

Suppose one inertial observer finds two sperated events to be simultaneous. Will these same events be measured as simultaneous by an observer on another inertial frame which is moving with speed $v$ with respect to the first?

The answer is "No" $\Rightarrow$ simultaneity is not independent of the frame of references used to describe events




$A, A'$ $B, B'$ coincide
explosions leaves marks $A'$, $B'$ on the train

$O$ is midpoint of $A, B$
$O'$ is midpoint of $A', B'$

Signals from $A, A'; B, B'$ reaches $O$ at the same time
$\Rightarrow$ $A, A'; B, B'$ is simultaneous in $S$ frame.

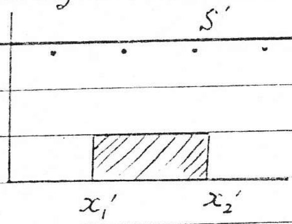
Signal from $B', B$ reaches $O'$ before it reaches $O$
Signal from $A' A$ reach $O'$ after it reaches $O$
$\Rightarrow$ $A, A'$ ; $B, B'$ is not simultaneous in $S'$ frame
Events $B, B'$ occurs early than $A, A'$ in the $S'$ system.

Signals from $A, A'$; $B, B'$ reaches $O'$ at the same time
$\Rightarrow$ $A, A'$; $B, B'$ is simultaneous in $S'$ system
Signal from $B, B'$ reaches $O'$ before it reaches $O$
Signal from $A, A'$ reaches $O'$ after it reaches $O$
$\Rightarrow$ Event $B, B'$ occur later than $A, A'$ in the $S$ system

$\Rightarrow$ relativity of simultaneity

$\Rightarrow$ time interval measurement is also relative.

9. The Lorentz - FitzGerald contraction




rod is at rest

S system $(x_2, t)$ , $(x_1, t)$

$t$ require to measure the length at the same time

$$x_2' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$$

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

$L_0$ = length of the rod in its own rest system.

$$= x_2' - x_1'$$

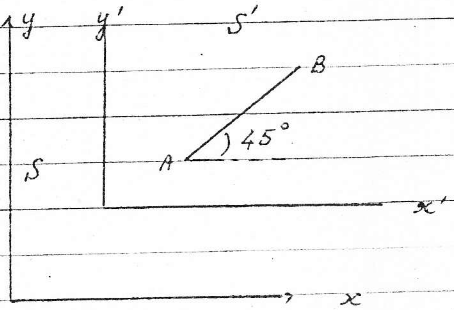
$$= \frac{1}{\sqrt{1 - v^2/c^2}} (x_2 - x_1)$$

$(x_2 - x_1) = L$ = length of the rod measured in the inertial frame of reference moving with velocity $v$ with respect to the frame of reference in which the rod is at rest.

$$L = \sqrt{1 - \frac{v^2}{c^2}}\ L_0$$

The Lorentz - FitzGerald contraction occurs only in the direction of the relative motion. if $v$ is parallel to $x$, the $y$ and $z$ direction dimensions of a moving object are the same in both S and S'

Example




$L' = L_0 = 1.5\ m$

$\theta' = 45°$

$\frac{v}{c} = 0.98$

$$L_y = L_y' = L' \sin\theta'$$

$$L_x = \sqrt{1 - \frac{v^2}{c^2}}\ L_x'$$

$$= \sqrt{1 - \frac{v^2}{c^2}}\ L' \cos\theta'$$

$$L = \sqrt{L_x^2 + L_y^2}$$

$$\tan\theta = \frac{L_y}{L_x} = \frac{\tan\theta'}{\sqrt{1-v^2/c^2}}$$

Put in number $L = 1.08\ m$ , $\theta = 78.7°$

Visual appearance of rapidly moving objects.

(i) Lorentz - FitzGerald contraction.

(ii) Light reaching the camera (or eye) from the more distant parts of the object was emitted earlier than that coming from the nearer parts.

( Physics Today 18 (9) : 24 , 1960 )

rotated in orientation as well as change in shape.

10. Time dilation.

In a reference frame $S'$ moving with velocity $v$ with respect to $S$

Event 1 occurs at $t_1'$, $x'$ ⇒ note: occur at the same point
2 occurs at $t_2'$, $x'$

In a frame of reference $S$

Event 1 will occur at $t_1 = \dfrac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-v^2/c^2}}$

Event 2 will occur at $t_2 = \dfrac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-v^2/c^2}}$

$$\Rightarrow \quad \underset{\displaystyle \Delta t}{\underset{\downarrow}{t_2 - t_1}} = \frac{t_2' - t_1'}{\sqrt{1-v^2/c^2}}$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}}$$

Muon decay ($\mu$ is not a meson, it is a lepton)

$\Delta t_0 = 2 \times 10^{-6}$ sec moving with $v = 0.998\,c$ in the reference
↑ $\mu$'s mean lifetime in its rest system
frame of the earth

$$\Delta t = \frac{1}{\sqrt{1-v^2/c^2}}\Delta t_0$$

$$= 31.6 \times 10^{-6} \text{ sec}$$

↳ mean lifetime in earth rest system.

distance travelled in the reference $y = 0.998 \times 3 \times 10^{8}$ m/sec

$\times\ 31.7 \times 10^{-6}$ sec $= 9500$ m

Without time dilation effect it will travel a distance $y = 0.998 \times 3 \times 10^{8}$ m/sec

$\times\ 2 \times 10^{-6}$ sec $= 600$ m

Example The Crab Nebula is $\sim 6.7 \times 10^{11}$ miles from us and emits cosmic radiation. Some of the emitted particles are neutron which, when at rest, decay with a lifetime of .932 seconds. With what

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velocity must the neutrons be emitted in order to arrive at the earth?

$$\Delta t = \frac{1}{\sqrt{1-\frac{v^2}{c}}} \underset{9\,\overset{''}{3}2\ \text{sec}}{\Delta t_0}$$

$$= \frac{d}{v} \qquad d \sim 6.7 \times 10^{11} \text{ miles}$$

Solve for $v \Rightarrow v = 0.99999996\, c$

# Chapter 2 Relativistic Mechanics

## 1. Relativistic addition of velocities

Galilean transformation

$$\vec{u}' = \vec{u} - \vec{v}$$

Lorentz transformation

$$x' = (x - vt) / \sqrt{1 - v^2/c^2} \quad \Rightarrow \quad dx' = \frac{dx - v\,dt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad \Rightarrow \quad dy' = dy$$

$$z' = z \quad \Rightarrow \quad dz' = dz$$

$$t' = (t - \frac{v}{c^2} x) / \sqrt{1 - v^2/c^2} \quad \Rightarrow \quad dt' = \frac{dt - \frac{v}{c^2} dx}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad V_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2} dx} = \frac{V_x - v}{1 - \frac{v}{c^2} V_x}$$

$$\Rightarrow \quad V_y' = \frac{dy'}{dt'} = \frac{dy}{dt - \frac{v}{c^2} dx} \sqrt{1 - v^2/c^2} = \frac{V_y}{1 - \frac{v}{c^2} V_x} \sqrt{1 - v^2/c^2}$$

$$\Rightarrow \quad V_z' = \frac{dz'}{dt'} = \frac{V_z}{1 - \frac{v}{c^2} V_x} \sqrt{1 - v^2/c^2}$$

(i) As $v \to 0$ ( or $c \to \infty$ ), then it reduces to the results given by Galilean transformation

(ii) It can be easily shown that the inverse transformation is given by

$$V_x = \frac{V_x' + v}{1 + \frac{v V_x'}{c^2}}$$

$$V_y = \frac{V_y' \sqrt{1 - v^2/c^2}}{1 + \frac{v V_x'}{c^2}}$$

$$V_z = \frac{V_z' \sqrt{1 - v^2/c^2}}{1 + \frac{v V_x'}{c^2}}$$

That is, it can be obtained from the velocity transformation equations by

(α) $v \to -v$ (β) exchange the primed and unprimed quantities.

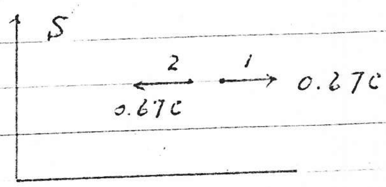
(iii) If $V_x' = c$, then

$$V_x = \frac{c+v}{1+\frac{v}{c^2}\cdot c} = c$$

Both observers determine the same value for the speed of light.

as required by the postulate of relativity

Example.




Electron 1 moving toward right ⇒ $S'$ system.

Lab. system $S$ system

$v = 0.67C$ ( $S'$ relative to $S$ )

$V_x$ = velocity of 2 in the system $S$

$= -0.67c$

$V_x'$ = velocity of 2 in the system $S'$

$$= \frac{-0.67c - 0.67c}{1 + \frac{(0.67c)^2}{c^2}}$$

$= -0.92c. < c.$

Note: Galilean transformation ⇒ $1.34c$.

2. Acceleration transformation.

$$V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}V_x}$$

$$dV_x' = \frac{dV_x}{1-\frac{v}{c^2}V_x} + (V_x - v)\frac{-1}{(1-\frac{v}{c^2}V_x)^2}\left(-\frac{v}{c^2}\right)dV_x$$

$$= \frac{1 - \frac{v}{c^2}V_x + \frac{v}{c^2}V_x - \frac{v^2}{c^2}}{(1-\frac{v}{c^2}V_x)^2}\, dV_x$$

$$= \frac{(1 - v^2/c^2)}{(1-\frac{v}{c^2}V_x)^2}\, dV_x$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$$

$$dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1-v^2/c^2}}$$

$$a_x' = \frac{dV_x'}{dt'}$$

$$= \frac{(1-v^2/c^2)^{3/2} \quad dV_x}{(1-\frac{v}{c^2}V_x)^2\left[dt - \frac{v}{c^2}dx\right]}$$

$$= \frac{(1-v^2/c^2)^{3/2}}{(1-\frac{v}{c^2}V_x)^3}\, a_x$$

$$V_y' = \frac{V_y}{1-\frac{v}{c^2}V_x}\sqrt{1-v^2/c^2}$$

$$dV_y' = \sqrt{1-v^2/c^2}\left[\frac{dV_y}{1-\frac{v}{c^2}V_x} + V_y\frac{-1}{(1-\frac{v}{c^2}V_x)^2}\left(-\frac{v}{c^2}\right)dV_x\right]$$

$$a_y' = \frac{dV_y'}{dt'}$$

$$= \sqrt{1-v^2/c^2}\;\frac{dV_y \quad \sqrt{1-v^2/c^2}}{(1-\frac{v}{c^2}V_x)\left[dt - \frac{v}{c^2}dx\right]}$$

$$+ \sqrt{1-v^2/c^2}\;\frac{V_y\frac{v}{c^2}dV_x \quad \sqrt{1-v^2/c^2}}{(1-\frac{v}{c^2}V_x)^2\left[dt - \frac{v}{c^2}dx\right]}$$

$$= \frac{(1-\frac{v^2}{c^2})\,a_y}{(1-\frac{v}{c^2}V_x)^2} + \frac{(1-\frac{v^2}{c^2})\,\frac{vV_y}{c^2}\,a_x}{(1-\frac{v}{c^2}V_x)^{3/2}}$$

Use similiar method

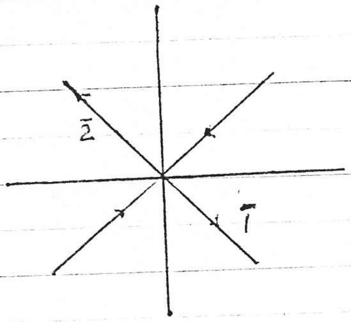
$$a_z' = \frac{(1-\frac{v^2}{c^2})\,a_z}{(1-\frac{v}{c^2}V_x)^2} + \frac{(1-\frac{v^2}{c^2})\,\frac{vV_z}{c^2}\,a_x}{(1-\frac{v}{c^2}V_x)^{3/2}}$$

Not only it is not invariant, but the expressions for it are in general cumbersome and moreover its different components transform in different ways $\Rightarrow$ acceleration is a quantity of limited and questionable value in special relativity.

# Relativistic mass

Consider an elastic collision between two equal-mass particles.

Look at the collision in the $S$ - system




$-$ refers after collision

Before collision

$$\begin{cases} V_x^{(1)} = a, & V_y^{(1)} = b, & V_z^{(1)} = 0 \\ V_x^{(2)} = -a, & V_y^{(2)} = -b, & V_z^{(2)} = 0 \end{cases}$$

After collision

$$\begin{cases} \bar{V}_x^{(1)} = a, & \bar{V}_y^{(1)} = -b, & \bar{V}_z^{(1)} = 0 \\ \bar{V}_x^{(2)} = -a, & \bar{V}_y^{(2)} = b, & \bar{V}_z^{(2)} = 0 \end{cases}$$

(Note, the speeds of all the particles are the same.)

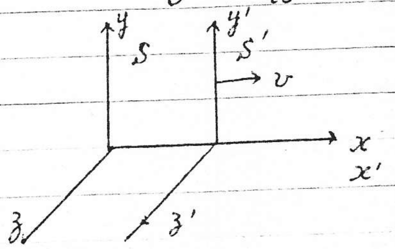
If mass is a constant and $m_1 = m_2$, then in system $S$ momentum conservation law is obviously valid.

$S \to S^*$, Lorentz transformation is used

$$V_x^* = \frac{V_x - v}{1 - V_x v/c^2}, \quad V_y^* = \frac{V_y}{\gamma(1 - V_x v/c^2)}, \quad V_z^* = \frac{V_z}{\gamma(1 - V_x v/c^2)}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$v$ is the relative velocity between $S'$ and $S$




Take the special case $v = a$, then in $S^*$ system.

Before collision

$$\begin{cases} V_x^{(1)*} = 0, & V_y^{(1)*} = \dfrac{b}{(1 - a^2/c^2)^{1/2}}, & V_z^{(1)*} = 0 \\ V_x^{(2)*} = -\dfrac{2a}{1 + a^2/c^2}, & V_y^{(2)*} = \dfrac{-b(1 - a^2/c^2)^{1/2}}{1 + a^2/c^2}, & V_z^{(2)*} = 0 \end{cases}$$

After collision

$$\begin{cases} \bar{V}_x^{(1)*} = 0, & \bar{V}_y^{(1)*} = \dfrac{-b}{(1 - a^2/c^2)^{1/2}}, & \bar{V}_z^{(1)*} = 0 \\ \bar{V}_x^{(2)*} = -\dfrac{2a}{1 + a^2/c^2}, & \bar{V}_y^{(2)*} = \dfrac{b(1 - a^2/c^2)^{1/2}}{(1 + a^2/c^2)}, & \bar{V}_z^{(2)*} = 0 \end{cases}$$

If $\vec{P} = m\vec{v}$ with $m$ = constant, then momentum conservation is valid in system $S$ but is not valid in system $S'$.

Modify $\vec{P} = \mu(v)\vec{v}$, then momentum conservation laws is still valid in system $S$

↑ speed of the particle

Question: what is the form of $\mu(v)$ that will make the momentum conservation law is also valid in system $S^*$?

In $S^*$ system

Before collision

$$P_x^* = \mu(V_1^*)V_x^{(1)*} + \mu(V_2^*)V_x^{(2)*} = \mu(V_2^*)\frac{-2a}{1+a^2/c^2} \quad (1)$$

$$P_y^* = \mu(V_1^*)V_y^{(1)*} + \mu(V_2^*)V_y^{(2)*}$$

$$= \mu(V_1^*)\frac{b}{(1-a^2/c^2)^{1/2}} + \mu(V_2^*)\frac{-b(1-a^2/c^2)^{1/2}}{1+a^2/c^2} \quad (2)$$

After collision

$$\bar{P}_x^* = \mu(\bar{V}_1^*)\bar{V}_x^{(1)*} + \mu(\bar{V}_2^*)\bar{V}_x^{(2)*} = \mu(\bar{V}_2^*)\left(-\frac{2a}{1+a^2/c^2}\right) \quad (3)$$

$$\bar{P}_y^* = \mu(\bar{V}_1^*)\bar{V}_y^{(1)*} + \mu(\bar{V}_2^*)\bar{V}_y^{(2)*}$$

$$= \mu(\bar{V}_1^*)\frac{-b}{(1-a^2/c^2)^{1/2}} + \mu(\bar{V}_2^*)\frac{b(1-a^2/c^2)^{1/2}}{1+a^2/c^2} \quad (4)$$

Momentum conservation in system $S^*$ requires

(i) (1) = (3)

Since $V_2^* = \bar{V}_2^*$, the above equation is automatically satisfied.

(ii) (2) = (4) $\Rightarrow$ $\mu(V_1^*)\frac{b}{(1-a^2/c^2)^{1/2}} + \mu(V_2^*)\frac{-b(1-a^2/c^2)^{\frac{1}{2}}}{1+a^2/c^2}$

$$= \mu(\bar{V}_1^*)\frac{-b}{(1-a^2/c^2)^{1/2}} + \mu(\bar{V}_2^*)\frac{b(1-a^2/c^2)^{1/2}}{1+a^2/c^2}.$$

$V_1^* = \bar{V}_1^*$, $V_2^* = \bar{V}_2^*$

Therefore (2) = (4) requires

$$\mu(V_1^*)\ \frac{1}{(1-a^2/c^2)^{1/2}} = \mu(V_2^*)\ \frac{(1-a^2/c^2)^{\frac{1}{2}}}{1+a^2/c^2}$$

As $b \to 0 \quad \Rightarrow \quad V_1^* \to 0\ , \quad V_2^* \to \dfrac{2a}{1+a^2/c^2}$

$$\mu(0) = \mu(V_2^*)\ \frac{1-a^2/c^2}{1+a^2/c^2}$$

In this case

$$1-\frac{(V_2^*)^2}{c^2} = 1-\frac{(2a)^2/c^2}{(1+a^2/c^2)^2} = \left[\frac{1-a^2/c^2}{1+a^2/c^2}\right]^2$$

$$\Rightarrow \quad \mu(0) = \mu(V_2^*)\ \sqrt{1-\left(\frac{V_2^*}{c}\right)^2}$$

That is $\quad m(V) = \dfrac{m_0}{\sqrt{1-V^2/c^2}}$

$$\vec{P} = \frac{m_0 \vec{V}}{\sqrt{1-V^2/c^2}}$$

4. Relativistic dynamics

In relativistic mechanics

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\ \frac{m_0\vec{V}}{\sqrt{1-V^2/c^2}}$$

Motion of high speed charged particles in $\vec{E}$, $\vec{B}$ fields

$$\underbrace{q(\vec{E}+\vec{v}\times\vec{B})}_{\text{Lorentz force}} = \frac{d}{dt}\ \frac{m_0\vec{V}}{\sqrt{1-V^2/c^2}}$$

The above equations are verified by numerous experiments.

Example: A particle in a constant electric field.

Choose $\vec{E} = E\,\hat{i}$

Initial conditions $\quad t=0: \quad \vec{r}=0$

$\qquad\qquad\qquad\qquad\qquad\qquad \vec{v}=0$

Equation of motion

$$\frac{dp_x}{dt} = qE \quad \Rightarrow \quad \frac{d}{dt}\ \frac{m v_x}{\sqrt{1-\frac{v^2}{c^2}}} = qE$$

It can be easily shown that $v_y = 0$, $v_z = 0$ $\left( \frac{d}{dt} \frac{m_0 v_y}{\sqrt{1-v^2/c^2}} = 0 \right)$
$p_y$ = constant

We have an one dimensional problem, we shall use $v = v_x$

$$\frac{d}{dt} \frac{m_0 v}{\sqrt{1-v^2/c^2}} = qE$$

$$\frac{d}{dt} \frac{v}{\sqrt{1-v^2/c^2}} = \frac{qE}{m_0} \equiv \underset{\text{constant}}{a_0}$$

$$LHS = \frac{1}{\sqrt{1-v^2/c^2}} \frac{dv}{dt} + v\left(-\frac{1}{2}\right) \frac{\left(-\frac{2v}{c^2}\right)\frac{dv}{dt}}{(1-v^2/c^2)^{3/2}}$$

$$= \frac{1}{(1-v^2/c^2)^{3/2}} \frac{dv}{dt} \left[ 1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right]$$

Therefore $\underset{\overset{\parallel}{a}}{\frac{dv}{dt}} = a_0 \left(1 - \frac{v^2}{c^2}\right)^{3/2}$

(i) $v \to 0$ $a \to a_0$ as it should.
(ii) $v \to c$ $a \to 0$ as it should.

From $\frac{d}{dt} \frac{v}{\sqrt{1-v^2/c^2}} = a_0$

$$\frac{v}{\sqrt{1-v^2/c^2}} = a_0 t + \alpha$$

$t = 0$, $v = 0 \Rightarrow \alpha = 0$

$$\frac{v}{\sqrt{1-v^2/c^2}} = a_0 t$$ relate $v \to t$

Solve $v$ in terms of $t$

$$v = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}$$

(i) $v \ll c$ $v = a_0 t$ → agrees with non-relativistic result.
(ii) $t \to \infty$ $v \to c$.

$$\frac{dx}{dt} = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}$$

Integrate and use $t = 0$, $x = 0$ to determine the integration constant

$$\Rightarrow \quad x = \frac{c^2}{a_0} \left[ \sqrt{1 + a_0^2 t^2/c^2} - 1 \right]$$

$$\frac{a_0}{c^2} x + 1 = \sqrt{1 + \frac{a_0^2 t^2}{c^2}}$$

$$\Rightarrow \quad x = \frac{1}{2} a_o t^2 - \frac{a_o x^2}{2c^2}$$

Relation between $v$ and $x$:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt} = a_o\left(1-\frac{v^2}{c^2}\right)^{3/2}$$

$$\frac{v\,dv}{(1-v^2/c^2)^{3/2}} = a_o\,dx$$

Integrate both side of the equation, remembering when $x=0$, $v=0$

$$v^2 = 2a_o x\left[\frac{1+\frac{a_o x}{2c^2}}{\left(1+\frac{a_o x}{c^2}\right)^2}\right]$$

Example A charged particle in a constant magnetic field.

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B}$$

$$\vec{p} = \frac{m_o}{\sqrt{1-v^2/c^2}}\,\vec{v}$$

$$\frac{d}{dt}\,\vec{p}\cdot\vec{p} = 2\vec{p}\cdot\frac{d\vec{p}}{dt} = 2\vec{p}\cdot(q\,\vec{v}\times\vec{B}) = 0$$

since $\vec{p}\parallel\vec{v}$

$$|\vec{p}| = \text{constant} \Rightarrow |\vec{v}| = \text{constant}$$

particle moves in a circle with radius $\rho$.

The equation of motion

$$\frac{m_o}{\sqrt{1-v^2/c^2}}\;\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B}$$

$$m \qquad \frac{v^2}{\rho} \qquad q\,\omega_c\,\rho\,B$$

$$\omega_c^2\rho$$

$$\omega_c = \frac{qB}{m}$$

$$\rho = \frac{v}{\omega_c} = \frac{mv}{qB} = \frac{p}{qB}$$

From measuring $\rho$ $\Rightarrow$ momentum of the charged particle

5. Kinetic energy

Work - energy relation

$$K.E. = \int_{u=0}^{u=u} \vec{F}\cdot d\vec{l}$$

We shall treat one dimensional case

[ Newtonian : $K.E. = \int F\,dx = \int m_0 \frac{dv}{dt} dx = \int_0^u m_0 v\,dv$

$= \frac{1}{2} m_0 u^2$ ]

Relativistic case

$$\Delta K.E. = \int F\,dx = \int \frac{d}{dt}(mv) \underset{\substack{\parallel \\ v\,dt}}{dx}$$

$$= \int m_0 v \frac{dv}{dt} \cdot \frac{1}{(1-v^2/c^2)^{3/2}} dt$$

Claim. $\frac{d}{dt}\left[\frac{m_0 c^2}{(1-v^2/c^2)^{1/2}}\right]$

$$= -\frac{1}{2}\frac{m_0 c^2 (-\frac{1}{c^2} 2v \frac{dv}{dt})}{(1-v^2/c^2)^{3/2}}$$

$$= m_0 v \frac{dv}{dt} \frac{1}{(1-v^2/c^2)^{3/2}}$$

$$K.E. = \int \frac{d}{dt}\left(\frac{m_0 c^2}{(1-v^2/c^2)^{1/2}}\right) dt$$

$$= \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2$$

$$K.E. + m_0 c^2 = E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = mc^2$$

For $v \ll c$

$$K.E. \underset{\substack{\uparrow \\ \text{use binomial expansion}}}{\cong} m_0 c^2 \left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \cdots\right] - m_0 c^2$$

$\sim \frac{1}{2} m_0 v^2$

↳ agrees with the Newtonian result.

It can be easily shown that

$$E^2 - (pc)^2 = (m_0 c^2)^2$$

$$\Rightarrow \quad E = c\sqrt{p^2 + m_0^2 c^2}$$

$$\frac{dE}{dp} = \frac{pc}{\sqrt{p^2 + m_0^2 c^2}} = \frac{pc^2}{E}$$

$$= v \qquad \left(\frac{E}{c^2} = m, \quad p = mv\right)$$

$\vec{a}$ ( in relativistic theory) :

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

$$m = \frac{E}{c^2}$$

$$\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d}{dt}(K.E. + m_0 c^2) = \frac{1}{c^2}\frac{d}{dt}(K.E.)$$

$$\frac{d(K.E)}{dt} = \frac{\vec{F}\cdot d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$$

Therefore $$\vec{F} = m\underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}} + \vec{v}\cdot\frac{1}{c^2}\vec{F}\cdot\vec{v}$$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} - \frac{v}{mc^2}(\vec{F}\cdot\vec{v})$$

$\Rightarrow$ In general $\vec{a}$ is not parallel to the force in relativity

2. Space - time.

Interval between events

S system : an event is specified by $(x_1, y_1, z_1, t_1)$ ← event 1

observer O  second event is specified by $(x_2, y_2, z_2, t_2)$ ← event 2.

Define the interval between two events

$$S_{12}^2 \equiv c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

S′ system : event 1 is specified by $(x_1', y_1', z_1', t_1')$

observer O′  event 2 is specified by $(x_2', y_2', z_2', t_2')$

$$S_{12}'^2 = c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

Use ; Lorentz transformation , we can prove (by direct substitution)

$$S_{12}^2 = S_{12}'^2$$

Even though O and O′ might find significantly different values of the time elapsed between two events and their spatial seperation , the interval is the same for both , it is an invariant common to all observers in inertial frames.

$S_{12}^2 > 0$ the interval is said to be time-like

(i) it is possible to find an inertial frame in which the two events occur at same place.

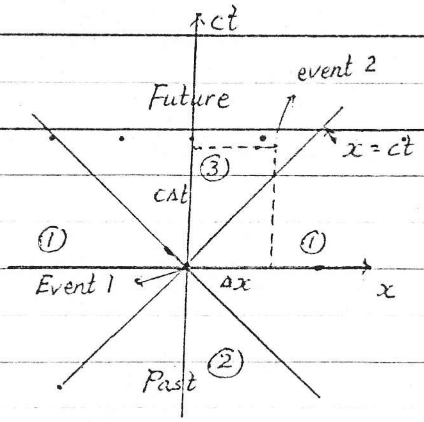
(ii) it is impossible to find an inertial frame in which the two events occur at the same time (simultaneously) since in such $S_{12}^2$ would be negative contrary to the condition $S_{12}^2 > 0$

$S_{12}^2 < 0$ the interval is said to be space-like

(i) it is possible to find an inertial frame in which two events are simultaneous

(ii) it is impossible to find a frame in which the events occur at the same place

$S_{12}^2 = 0$ the interval is said to be light-like




S system :

Event 1 is at the origin

Event 2 can be related causally to event 1 provided that a signal travelling slower than the velocity of light can connect these events

$c\,\Delta t > |\Delta x| \Rightarrow (\Delta S)^2 > 0$ time-like $\Rightarrow$ events 2 must lie within the light cones bounded by $x = \pm ct$

A physical object or a signal can get from a point in region 2 to the event 1 by moving along at a speed $< c$.

Events in region 2 can affect event 1, can have an inflence on it from the past.

Events in the region 3 can be affected by event 1.
$\Rightarrow$ this is the world whose future can be affected by event 1
$\Rightarrow$ future.

Region 1 $\quad c\,\Delta t < |\Delta x| \Rightarrow$ space-like.
Events in this region can neither influence event 1 nor can be influenced by event 1.

Events lie on the boundaries of the lightcone are connected to event 1 by light signal only.

Path of a particle in space-time is called its world line. Since it moves with speed $< c \Rightarrow$ world light of a particle must lie within its light cone.

Rotation in two dimension

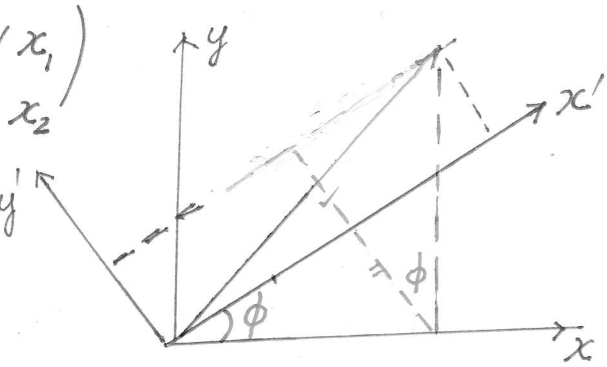
$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$x'^2 + y'^2 = x^2 + y^2$$

$$x_1' = a_{11} x_1 + a_{12} x_2$$
$$x_2' = a_{21} x_1 + a_{22} x_2$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{R} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$




$$x_i' = \sum_{i=1}^{2} a_{ij} x_j$$

Length is perserved

$$\Rightarrow \sum a_{ij} a_{ik} = \delta_{jk}$$

$$\sum \tilde{a}_{ji} a_{ik} = \delta_{jk}$$

$$\tilde{R} R = I$$

R is orthogonal

Easily, it can be expand to higher dimension

$$\tilde{R} R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ +\sin\phi\cos\phi \mp \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (R)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\tilde{R} R = 1$$

If $$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = R\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

then $\vec{V}$ is a vector

$\vec{F}$ is a vector

$$\vec{F} = m\vec{a}$$

$$R\vec{F} = mR\vec{a}$$

$$\vec{F}' = m\vec{a}'$$

$\Rightarrow$ $\vec{F} = m\vec{a}$ is invariant under rotation.

13. Four vectors :

Let $w = ict$, then

$$-S_{12}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2$$

May interprete $x, y, z, w$ as cartesian coordinates in a four dimensional Euclidean space

$\sqrt{-S_{12}^2} \longleftrightarrow$ the "distance" between $(x_1, y_1, z_1, w_1)$ and $(x_2, y_2, z_2, w_2)$

displacement $\longrightarrow$ four vector $(x_2-x_1,\ y_2-y_1,\ z_2-z_1,\ w_2-w_1)$

$$(x, y, z, w) \rightarrow \underbrace{\mathbf{r}}_{\text{radius four-vector}} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} + w\,\hat{l}$$

$$r^2 = \underbrace{\mathbf{r}\cdot\mathbf{r}}_{\text{scalar}} = x^2 + y^2 + z^2 + w^2$$

Lorentz transformation

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{i\,v/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\,v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Minkowski

Lorentz transformation $\Longleftrightarrow$ transformation of a four-vector under a rotation of the $x$ and $w$ axes in the four dimensional $x, y, z, w$ space through an imaginary angle $\phi = \tan^{-1} i\frac{v}{c}$

This can be seen by comparing

$$x' = x\cos\phi + w\sin\phi \quad , \quad w' = -x\sin\phi + w\cos\phi$$

$$x' = \frac{1}{\sqrt{1-v^2/c^2}}\,x + i\,\frac{v/c}{\sqrt{1-v^2/c^2}}\,w \quad , \quad w' = \frac{-i\frac{v}{c}}{\sqrt{1-v^2/c^2}}\,x + \frac{1}{\sqrt{1-v^2/c^2}}\,w$$

$$\cos\phi = \frac{1}{\sqrt{1-v^2/c^2}} \quad , \quad \sin\phi = i\,\frac{v/c}{\sqrt{1-v^2/c^2}} .$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + w^2 = \text{invariant}$$

$\underset{\sim}{A} = (A_x, A_y, A_z, A_w)$ is a four vector, it must transform to $\underset{\sim}{A}' = (A_x', A_y', A_z', A_w')$ in going from $S$ to $S'$ in the same way as the radius four vector, i.e.,

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \\ A_w' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{i\, v/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\, v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \\ A_w \end{pmatrix}$$

If $\underset{\sim}{A} = (A_x, A_y, A_z, A_w)$, $\underset{\sim}{B} = (B_x, B_y, B_z, B_w)$ are four vectors then $\underset{\sim}{A} \cdot \underset{\sim}{B} = A_x B_x + A_y B_y + A_z B_z + A_w B_w$ is the scalar product of these two four vectors. $\underset{\sim}{A} \cdot \underset{\sim}{B}$ is a scalar invariant, i.e., the same for all inertial frames.

In next chapter, we shall show $(p_x, p_y, p_z, \frac{iE}{c})$ is a four-vector.

分類：
編號：　2-11
總號：

6. Energy-momentum four vector

Define $c^2(d\tau)^2 = c^2\,(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$

$= \text{invariant}$ ( can be shown using Lorentz transformation ).

Three dimensional space

A quantity that is invariant under rotation $\Rightarrow$ scalar. e.g., $\underset{\uparrow \text{charge}}{e}$ , $t$

$d\tau \Rightarrow$ is a Lorentz scalar

$\underset{\sim}{u} = \dfrac{d\underset{\sim}{r}}{d\tau}$ is a four vector

$= \dfrac{d\underset{\sim}{r}}{dt}\dfrac{dt}{d\tau}$

$= \left(\dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k} + \underset{\updownarrow\, ic}{\dfrac{dw}{dt}}\hat{l}\right)\dfrac{1}{\sqrt{1-V^2/c^2}}$

Thus $\underset{\sim}{P} = \dfrac{m_0}{\sqrt{1-V^2/c^2}}\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k} + ic\,\hat{l}\right)$ is a 4 vector

$\dfrac{im_0c}{\sqrt{1-V^2/c^2}} = \dfrac{i}{c}\dfrac{m_0c^2}{\sqrt{1-V^2/c^2}} = \dfrac{iE}{c}$

$\Rightarrow \left(P_x, P_y, P_z, \dfrac{iE}{c}\right)$ is a four-vector.

It is known as energy-momentum four-vector.

Consequences.

(i) $\begin{pmatrix} P_x' \\ P_y' \\ P_z' \\ \frac{iE'}{c} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{i\frac{v}{c}}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\frac{v}{c}}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}\begin{pmatrix} P_x \\ P_y \\ P_z \\ \frac{iE}{c} \end{pmatrix}$

$v$: velocity of $S'$ relative to $S$

$P_x' = \dfrac{P_x - \frac{v}{c^2}E}{\sqrt{1-v^2/c^2}}$

$P_y' = P_y$

$P_z' = P_z$

$\dfrac{iE'}{c} = \dfrac{1}{\sqrt{1-v^2/c^2}}\left[-i\dfrac{v}{c}P_x + \dfrac{iE}{c}\right]$

$E' = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(E - vP_x\right)$

These represent how energy momentum transform under Lorentz transformation

(ii) $\quad p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = \underset{\substack{\uparrow \\ \text{invariant}}}{m_0^2 c^2}$

(iii) $\quad (\vec{p}_1, \frac{iE_1}{c})$, $(\vec{p}_2, \frac{iE_2}{c})$ are four vector

$\Rightarrow (\vec{p}_1 + \vec{p}_2, \; i\frac{E_1}{c} + i\frac{E_2}{c})$ is again a four-vector

$\Rightarrow (\vec{p}_1 + \vec{p}_2)^2 - \frac{(E_1 + E_2)^2}{c^2}$ is invariant under Lorentz transformation

## 7 Relativistic energy momentum conservation.

When particles interact or decay in an isolated system, the laws of conservation of both momentum and energy are implicit in the relation.

$$\left(\sum_{i=1}^{N_i} \underset{\sim}{p_i}\right)_{initial} = \left(\sum_{i=1}^{N_f} \underset{\sim}{p_i}\right)_{final}$$

where $\underset{\sim}{p_i}$ is the four-momentum for the ith particle.

Threshold energy:

A reaction $\quad 1 + 2 \rightarrow \bar{1} + \bar{2} + \bar{3} + \bar{4}$

$\quad m_1 \quad m_2 \quad \bar{m}_1 \quad \bar{m}_2 \quad \bar{m}_3 \quad \bar{m}_4 \quad$ (rest masses)

1 $\longrightarrow$ $\quad$ 2

$\vec{P}_1 = p\hat{k}$ $\qquad \vec{P}_2 = 0$

$E_1 = \sqrt{p^2c^2 + m_1^2c^4}$ $\qquad E_2 = m_2c^2$

Ask: What is the minimum value of $p$ in order the above reaction can occur?

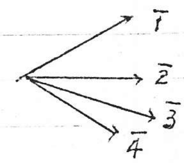
This minimum momentum $\Rightarrow$ threshold momentum $P_{min} \equiv P_{th}$

Minimum energy $\Rightarrow$ threshold energy $E_{th} = \sqrt{P_{th}^2c^2 + m_1^2c^4}$

In the lab. system.

Before collision


After collision




Energy momentum conservation:

$$\sqrt{p^2c^2 + m_1^2c^4} + m_2c^2 = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \bar{E}_4 \qquad \bar{E}_i = \sqrt{\bar{P}_i^2c^2 + \bar{m}_i^2c^4}$$

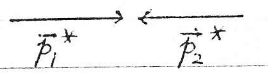
$$p\hat{k} = \vec{\bar{P}}_1 + \vec{\bar{P}}_2 + \vec{\bar{P}}_3 + \vec{\bar{P}}_4$$

It is difficult to see what is the minimum value of $P$ in order to satisfy all above equations

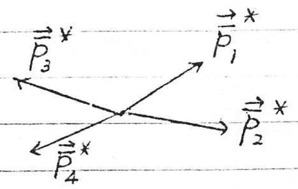
( Note $\vec{\bar{p}}_i$ cannot be all equal to zero )

Look at the problem in the center of mass system.

Before collision.




After collision




Energy momentum conservation

$$\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = \vec{\bar{p}}_1^{\,*} + \vec{\bar{p}}_2^{\,*} + \vec{\bar{p}}_3^{\,*} + \vec{\bar{p}}_4^{\,*} = 0$$

$$E_1^* + E_2^* = \sqrt{\bar{p}_1^{*2}c^2 + \bar{m}_1^2c^4} + \sqrt{\bar{p}_2^{*2}c^2 + \bar{m}_2^2c^4} + \sqrt{\bar{p}_3^{*2}c^2 + \bar{m}_3^2c^4} + \sqrt{\bar{p}_4^{*2}c^2 + \bar{m}_4^2c^4}$$

It is obvious that $(E_1^* + E_2^*)_{min} = (\bar{m}_1 + \bar{m}_2 + \bar{m}_3 + \bar{m}_4)c^2$
( $\vec{\bar{p}}_i$'s can all be zero)

Relation between $E_{th}$ and $(E_1^* + E_2^*)_{min}$

Use invarant:

$$\underbrace{(\vec{p}_1 + \vec{p}_2)^2 - \frac{(E_1+E_2)^2}{c^2}}_{\text{lab system}} = \underbrace{(\vec{p}_1^{\,*} + \vec{p}_2^{\,*})^2 - \frac{(E_1^*+E_2^*)^2}{c^2}}_{\text{c.m. system}}$$

$$p_{th}^2 - \frac{\left(\sqrt{p_{th}^2c^2 + m_1^2c^4} + m_2c^2\right)^2}{c^2} = -\frac{1}{c^2}(E_1^* + E_2^*)^2_{min}$$

$$c^2p_{th}^2 + m_1^2c^4 + 2E_{th}m_2c^2 + m_2^2c^4 - p_{th}^2c^2 = (\bar{m}_1 + \bar{m}_2 + \bar{m}_3 + \bar{m}_4)^2c^4$$

$$E_{th} = \left[(\bar{m}_1 + \bar{m}_2 + \bar{m}_3 + \bar{m}_4)^2c^2 - m_1^2c^2 - m_2^2c^2\right] / 2m_2$$

Example: $p + p \to p + p + p + \bar{p}$

$$m_1 = m_2 = \bar{m}_1 = \bar{m}_2 = \bar{m}_3 = \bar{m}_4 = m_p = \text{rest mass of proton.}$$

$$\therefore E_{th} = 7m_pc^2 \simeq 6.566\ \text{GeV}$$

A $\pi^{\circ}$ meson moving in the $x$ direction with a kinetic energy in the lab. frame equal to its rest energy decays into two photons. In the meson rest system these photons are emitted in the positive and negative $y'$ direction. Find the energies of the two photons in the meson rest system, and the energies and directions of propagation of the two photons in the lab. system.

Lab. system : $S$ system
Meson rest system $S'$ system.

$S'$ system

$P'_{x_1} = P'_{x_2} = 0 \qquad P'_{z_1} = P'_{z_2} = 0 \qquad$ (Given)

$P'_{y_1} = -P'_{y_2} = P$ (From momentum conservation.

$E_1 = pc \quad , \quad E_2 = pc \quad$ (Photon has zero mass)

$2pc = \underbrace{m_{\pi^\circ}}_{\text{rest mass of } \pi^\circ} c^2 \qquad$ (Energy conservation)

$$\Rightarrow \quad P'_{y_1} = \frac{m_\pi c}{2} \quad , \quad P'_{y_2} = -\frac{m_\pi c}{2}$$
$$E'_1 = \frac{m_\pi c^2}{2} \quad , \quad E'_2 = \frac{m_\pi c^2}{2}$$

Relative velocity between $S$ and $S'$ system.

$$E = \underbrace{K.E.}_{\text{given}} + m_{\pi^\circ} c^2 = 2 m_{\pi^\circ} c^2 = \gamma m_{\pi^\circ} c^2$$

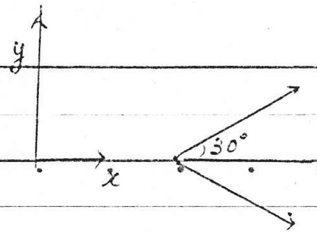
$\gamma = 2 \quad \Rightarrow \quad v = \frac{\sqrt{3}}{2} c$

↳ velocity of $S'$ system relative to the $S$ system. along $+x$ direction

$$P_{x_1} = \gamma\left(P'_{x_1} + v\frac{E'_1}{c^2}\right) = \frac{\sqrt{3}}{2} m_\pi c \quad , \quad P_{y_1} = \frac{1}{2} m_\pi c$$

$$\tan\theta_1 = \frac{P_{y_1}}{P_{x_1}} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta_1 = 30^\circ$$

$$P_{x_2} = \gamma\left(P'_{x_2} + v\frac{E'_2}{c^2}\right) = \frac{\sqrt{3}}{2} m_\pi c \quad , \quad P_{y_2} = -\frac{1}{2} m_\pi c .$$



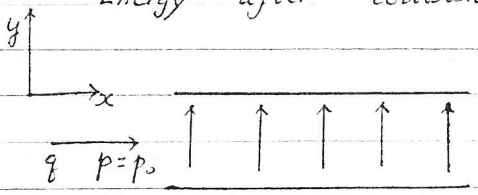

$$p_1 = \left[(p_{x1})^2 + (p_{y1})^2\right]^{1/2} = m_\pi c \quad , \quad E_1 = m_\pi c^2$$

$$p_2 = \left[(p_{x2})^2 + (p_{y2})^2\right]^{1/2} = m_\pi c \quad , \quad E_2 = m_\pi c^2$$

In the $S$ system

Energy before collision $= 2m_\pi c^2$
Energy after collision $= E_1 + E_2 = 2m_\pi c^2$

$\rangle$ energy is conserved.




A charged particle travels through an electric field $\mathcal{E}$

$$\frac{dp_x}{dt} = 0 \quad \Rightarrow \quad p_x = p_0$$

$$\frac{dp_y}{dt} = q\mathcal{E} \quad \Rightarrow \quad p_y = q\mathcal{E}t$$

$$\begin{aligned} E &= \gamma m_0 c^2 \\ \vec{p} &= \gamma m_0 \vec{v} \end{aligned} \quad \Rightarrow \quad \vec{v} = \frac{\vec{p}c^2}{E}$$

$$\begin{aligned} E &= \sqrt{\vec{p}^2c^2 + m_0^2c^4} \\ &= \sqrt{p_0^2c^2 + (q\mathcal{E}t)^2c^2 + m_0^2c^4} \\ &= \sqrt{E_0^2 + (q\mathcal{E}tc)^2} \end{aligned}$$

$$v_x = \frac{p_0 c^2}{\sqrt{E_0^2 + (q\mathcal{E}tc)^2}} \quad , \quad v_y = \frac{(q\mathcal{E}t)c^2}{\sqrt{E_0^2 + (q\mathcal{E}tc)^2}} \Rightarrow \tan\theta = \frac{v_y}{v_x} = \frac{q\mathcal{E}t}{p_0}$$

$$v_x = \frac{dx}{dt}$$

$$\int_0^L dx = \int_0^{t_L} \frac{p_0 c^2}{\sqrt{E_0^2 + (q\mathcal{E}tc)^2}}\, dt$$

$$L = \frac{p_0 c}{q\mathcal{E}} \sinh^{-1} \frac{q\mathcal{E}t_L c}{E_0}$$

$$\Rightarrow \quad t_L = \frac{E_0}{q\mathcal{E}c} \sinh \frac{q\mathcal{E}L}{p_0 c}$$

Cerenkov radiation

Charged particle moves through a medium with refractive index $n$ with a speed $v > \frac{c}{n}$

$\Rightarrow$ coherent light is emitted $\leftarrow$ Cerenkov radiation.

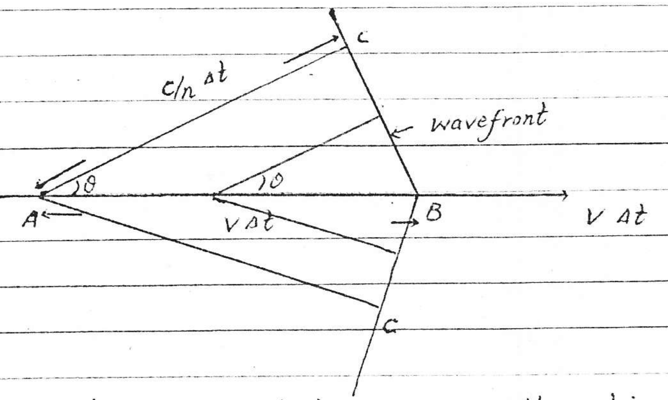
$\updownarrow$

analogous to acoustic shock wave

Discovery . Cerenkov 1934

Explanation (on the basis of classical electromagnetic theory)

given by Frank and Tamm.




A charged particle moving through a dielectric medium has an accompanying electric field $\Rightarrow$ polarizes the atoms

follow the wavefront of the pulse $\Rightarrow$ reradiate.

scattered radiation

$v < \frac{c}{n}$ scattered radiation from all segments of the track interfere destructively.

$v > \frac{c}{n}$ scattered radiation from all points on the track can be in phase in certain directions.

Coherent wavefront is propagated through the medium in the direction AC which satisfies the condition

$$\cos\theta = \frac{c/n}{v} = \frac{c}{nv}$$

Energy loss from Cerenkov radiation $\ll$ ionization energy loss, bremsstrahlung energy loss.

Importances. (i) the light is concentrated in the visible and near ultraviolet. (ii) it appears at a well defined angle and with well defined polarization, (iii) the Cerenkov pulse is sharp ( $\ll 10^{-10}$ sec )

$\Rightarrow$ useful in detecting and measuring the speed of high energy charged particles.